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Partial Entropy and Bundle-Like Entropy for Topological Dynamical Systems

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Abstract

Entropy is an important notion for understanding the complexity of dynamical systems. Several important entropy-like invariants based on the preimage structure for noninvertible maps have been defined and studied by some authors. In this chapter, following the idea of Hurley, we first further study the relationship among the topological entropy, pseudo-orbit, and preimage entropies for topological dynamical systems from the view of localization. Secondly, two entropy-like invariants, which are called the partial entropy and bundle-like entropy, for nonautonomous discrete dynamical systems are introduced. A relationship between the topological entropy and such two entropies is established.

1. Introduction

By a *topological dynamical system*, we mean a pair (X, T), where X is a compact metric space with a metric d and T is a continuous surjective map from X to itself [1]. An important notion for understanding the complexity of dynamical systems is topological entropy, which was first introduced by Adler et al. [2] in 1965, and later Dinaburg [3] and Bowen [4] gave two equivalent definitions on a metric space by using separated sets and spanning sets. Roughly speaking, topological entropy measures the maximal exponential growth rate of orbits for an arbitrary topological dynamical system.

When a considered mapping T is invertible, it is well-known that T and the inverse mapping T^{-1} have the same topological entropy. However, when the map T is not invertible, the "inverse" is set-valued, yielding the iterated preimage set $T^{-n}(x) = \{z \in X : T^n z = x\}$ of a point $x \in X$ which is in general a set rather than a point, so different ways of "extending the procedure into the past" lead to several new entropy-like invariants for non-invertible maps.

In 1991, Langevin and Walczak [5] regard the "inverse" as a relation and formulate a notion of entropy for this relation (analogous to the entropy of a foliation [6]), based on distinguishing points by means of the structure of their "preimage trees," which is called preimage relation entropy. The interested reader can see [7] or [8] for more details on this invariant. Later, several important entropy-like invariants based on the preimage structure for non-invertible maps, such as pointwise preimage

entropies, preimage branch entropy [1, 8–10], partial preimage entropy, conditional preimage entropy [11], etc., have been introduced, and their relationships with topological entropy have been established. To learn more about the results related to the preimage entropy for noninvertible maps, one can see [12–23].

The local entropy theory for topological dynamical systems started in the early 1990s with the work of Blanchard (see [24, 25]). Nowadays this theory has become a very interesting topic in the field of dynamical systems and has also proven to be fundamental to many other related fields. For example, Blanchard defined the notion of entropy pairs and used it to obtain a disjointness theorem [26]. The notion of entropy pairs can also be used to show the existence of the maximal zero-entropy factor, called the topological Pinsker factor, for any topological dynamical system [25]. In order to gain a better understanding of the topological version of a K-system, the theory of entropy tuples [27–29] was developed. To learn more about the theory related to the local entropy, we refer the interested reader to see the survey paper [30] and references therein.

We remark that in reality, it is difficult to find a real orbit in the system, but a pseudo-orbit can be used to approximate the real orbit, and so there have been a lot of applications in many fields. Since the works of Bowen [31] and Conley [32], pseudo-orbits have proved to be a powerful tool in dynamical systems. For instance, Hammel et al. [33, 34] have investigated the role of pseudo-orbits in computer simulations of certain dynamical systems; Barge and Swanson [35] made use of pseudo-orbits to study rotation sets of circle and annulus maps. Also, a remarkable result by Misiurewicz [36] showed that the topological entropy can be computed by measuring the exponential growth rate of the numbers of pseudo-orbits (related results can see [37]). In [1], Hurley showed that the point entropy with pseudo-orbits that is defined by replacing inverse orbit segments by inverse pseudo-orbit segments in the definition of pointwise preimage entropy is in fact equal to the topological entropy.

In this chapter, following Hurley [1] we further study the preimage entropy for topological dynamical system from the view of localization. In Section 2, we consider the calculation of topological entropy for open covers from pseudo-orbits (Theorem 2.3). In Section 3, we investigate the relationship among the topological entropy for open covers and several preimage entropy invariants, which is viewed as the local version of the Hurley inequality (Theorem 3.1). In Section 4, we show that the topological entropy for open covers can be computed by measuring the exponential growth rate of the number of pseudo-orbits that end at a particular point (Theorems 4.2 and 4.3).

A nonautonomous discrete dynamical system is a natural generalization of a classical dynamical system; its dynamics is determined by a sequence of continuous self-maps $f_n : X_n \rightarrow X_{n+1}$, which defined on a sequence on compact metric spaces (X_n, d_n). The topological entropy of nonautonomous discrete dynamical systems was introduced by Kolyada and Snoha [38]. In Section 5, following the idea of [1, 39], we introduce two entropy-like invariants, which are called the partial entropy and bundle-like entropy, for nonautonomous discrete dynamical systems, and study the relationship among them and the topological entropy (Theorems 5.2, 5.3, and 5.5).

2. Topological entropy and pseudo-orbits

2.1 Topological entropy via open covers

Topological entropy was defined originally by Adler et al. [2] for continuous maps on compact topological spaces. Let (X, T) be a topological dynamical system. A finite *open cover* of X is a finite family of open sets whose union is X. Denoted by

$\mathcal{C}_X^o$ is the set of finite open covers of X. Given two open covers $\mathcal{U}, \mathcal{V} \in \mathcal{C}_X^o$, $\mathcal{U}$ is said to be *finer than* $\mathcal{V}$ ($\mathcal{U} \succcurlyeq \mathcal{V}$) if each element of $\mathcal{U}$ is contained in some element of $\mathcal{V}$. Let $\mathcal{U} \vee \mathcal{V} = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$. It is clear that $\mathcal{U} \vee \mathcal{V} \succcurlyeq \mathcal{U}$ and $\mathcal{U} \vee \mathcal{V} \succcurlyeq \mathcal{V}$.

Let $\mathcal{U} \in \mathcal{C}_X^o$. For two nonnegative integers $M \leq N$, denoted by $\mathcal{U}_M^N = \vee_{n=M}^N T^{-n} \mathcal{U}$, where $T^{-n} \mathcal{U} = \{T^{-n}(U) : U \in \mathcal{U}\}$ for all positive integers n. For any $K \subset X$, define $N(\mathcal{U}|K)$ as the minimal cardinality of any subcovers of $\mathcal{U}$ that covers K. We write $N(\mathcal{U}|X)$ simply by $N(\mathcal{U})$. The *topological entropy of $\mathcal{U}$ with respect to T* is defined by

$$h_{\text{top}}(T, \mathcal{U}) = \lim_{n \to \infty} \frac{1}{n} \log N(\mathcal{U}_0^{n-1}) = \inf_{n \geq 1} \frac{1}{n} \log N(\mathcal{U}_0^{n-1}). \tag{1}$$

The *topological entropy of T* is

$$h_{\text{top}}(T) = \sup_{\mathcal{U} \in \mathcal{C}_X^o} h_{\text{top}}(T, \mathcal{U}). \tag{2}$$

2.2 Separated sets, spanning sets, and topological entropy

In this subsection, we recall two equivalent definitions, which are given by Dinaburg [3] and Bowen [4]. Let (X, T) be a topological dynamical system. Given a nonempty subset K of X, for any $\epsilon > 0$ and $n \in \mathbb{N}$, a subset E of K is called an (n, ϵ)-*separated set* of K if any $x \neq y \in E$ implies $d_n(x, y) \geq \epsilon$, where

$$d_n(x, y) := \max_{0 \leq i \leq n-1} d(T^i x, T^i y).$$

Denote the maximal cardinality of any (n, ϵ)-separated subset of K by $s(n, \epsilon, K)$. A subset F of K is called an (n, ϵ)-*spanning set* of K, if for any $x \in K$, there exists $y \in F$ with $d_n(x, y) < \epsilon$. Denote the minimal cardinality of any (n, ϵ)-spanning set for K by $r(n, \epsilon, K)$.

The following lemma is well-known, and its proof is not difficult, so we omit its detail proof.

Lemma 2.1. *Let (X, T) be a topological dynamical system. For any subset K of X and any integer $n \geq 1$, we have the following properties*:

1. $r(n, \epsilon, K) \leq s(n, \epsilon, K) \leq r(n, \epsilon/2, K)$ *for all $\epsilon > 0$.*

2. $N(\mathcal{U}_0^{n-1}|K) \leq r(n, \delta, K)$ *for any $n \in \mathbb{N}$ and any $\mathcal{U} \in \mathcal{C}_X^o$ with the Lebesgue number 2δ.*

3. $s(n, \epsilon, K) \leq N(\mathcal{U}_0^{n-1}|K)$ *for any $\mathcal{U} \in \mathcal{C}_X^o$ with $\text{diam}(\mathcal{U}) < \epsilon$.*

By Lemma 2.1, we obtain directly the following result.

Theorem 2.2. (see [3, 4, 40]). *Let (X, T) be a topological dynamical system. Then*

$$h_{\text{top}}(T) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s(n, \epsilon, X) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log r(n, \epsilon, X).$$

2.3 Topological entropy via pseudo-orbits

Let (X, d) be a compact metric space. Denote X^n as the n-fold Cartesian product of X ($n \geq 1$). Fixing a positive number ϵ, a subset $E \subset X^n$ is said to be (n, ϵ)-*separated* if for any two distinct points $\tilde{x} = (x_0, x_1, \cdots, x_{n-1}), \tilde{y} = (y_0, y_1, \cdots, y_{n-1}) \in E$, there is a $0 \leq i \leq n-1$ such that $d(x_i, y_i) > \epsilon$. By the compactness of X, any (n, ϵ)-separated set

is finite. If $Z \subset X^n$ is a nonempty subset, then we denote the maximal cardinality of any (n, ϵ)-separated subset of Z by $s(n, \epsilon, Z)$.

Let $Z \subset X^n$ be a nonempty subset. A subset $F \subset Z$ is called (n, ϵ)-*panning* for Z if for each $\tilde{z} = (z_0, z_1, \cdots, z_{n-1}) \in Z$, there is a $\tilde{y} = (y_0, y_1, \cdots, y_{n-1}) \in F$ with $d(z_i, y_i) < \epsilon$ for every $0 \leq i \leq n - 1$. We denote the minimal cardinality of any (n, ϵ)-spanning subset of Z by $r(n, \epsilon, Z)$.

For each positive integer $n \geq 1$, we let O_n denote the set of all orbit segments of length n, that is,

$$O_n = \{(x, Tx, \cdots, T^{n-1}x) \in X^n : x \in X\}.$$

Note that a point $\tilde{w} = (x, Tx, \cdots, T^{n-1}x) \in O_n$ is uniquely determined by its initial point $x \in X$. Thus, we have

$$h_{\text{top}}(T) = \lim_{\epsilon \to 0} \lim_{\alpha \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s(n, \epsilon, O_n)$$

$$= \lim_{\epsilon \to 0} \lim_{\alpha \to 0} \limsup_{n \to \infty} \frac{1}{n} \log r(n, \epsilon, O_n).$$

Topological entropy has been characterized by Misiurewicz [36] and Barge and Swanson [37] in terms of growth rates of pseudo-orbits. Let (X, T) be a topological dynamical system. For $\alpha > 0$, an α-*pseudo-orbit* for T of length n is a point $\tilde{x} = (x_0, x_1, \cdots, x_{n-1}) \in X^n$ with the property that $d(T(x_{j-1}), x_j) < \alpha$ for all $1 \leq j \leq n - 1$. Let $\Psi_n(\alpha) \subset X^n$ denote all α-pseudo-orbits of length n. It was shown in [36, 37] that

$$h_{\text{top}}(T) = \lim_{\epsilon \to 0} \lim_{\alpha \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s(n, \epsilon, \Psi_n(\alpha))$$

$$= \lim_{\epsilon \to 0} \lim_{\alpha \to 0} \limsup_{n \to \infty} \frac{1}{n} \log r(n, \epsilon, \Psi_n(\alpha)).$$

In the following, we will show that the topological entropy for an open cover can be characterized by pseudo-orbits. Before proceeding, let us first introduce a definition of pseudo-orbit entropy via open covers. Let (X, T) be a topological dynamical system. For each integer $n \geq 1$ and $\mathcal{U} \in \mathcal{C}_X^o$, we define an open cover $\mathcal{U}^n$ of the product space X^n by

$$\mathcal{U}^n = \{U_0 \times U_1 \times \cdots \times U_{n-1} : U_j \in \mathcal{U} \text{ for each } j = 0, 1, ..., n - 1\},$$

where

$$U_0 \times U_1 \times \cdots \times U_{n-1} = \{(u_0, u_1, ..., u_{n-1}) : u_j \in U_j \text{ for each } j = 0, 1, ..., n - 1\}.$$

Given $\alpha > 0$, it is not hard to see that $a_n = N(\mathcal{U}^n | \Psi_n(\alpha))$ is a nonnegative sub-additive sequence, that is, $a_{n+m} \leq a_n + a_m$ for all positive integers n and m. The α-*pseudo-orbit entropy* of $\mathcal{U}$ is then defined by

$$h_\Psi(T, \mathcal{U}, \alpha) = \lim_{n \to \infty} \frac{1}{n} \log N(\mathcal{U}^n | \Psi_n(\alpha)) = \inf_{n \geq 1} \frac{1}{n} \log N(\mathcal{U}^n | \Psi_n(\alpha)), \tag{3}$$

and the *pseudo-orbit entropy* of $\mathcal{U}$ is defined by

$$h_\Psi(T, \mathcal{U}) = \lim_{\alpha \to 0} h_\Psi(T, \mathcal{U}, \alpha). \tag{4}$$

Theorem 2.3. *Let (X, T) be a topological dynamical system. If $\mathcal{U} \in \mathcal{C}_X^o$, then we have*

$$h_{\text{top}}(T, \mathcal{U}) = h_\Psi(T, \mathcal{U}). \qquad (5)$$

Proof. To prove (5), it suffices to note that $h_\Psi(T, \mathcal{U}, \alpha_1) \leq h_\Psi(T, \mathcal{U}, \alpha_2)$ whenever $\alpha_1 < \alpha_2$ and $\inf_{0 < \alpha \leq 1} N(\mathcal{U}^n | \Psi_n(\alpha)) = N(\mathcal{U}^n | O_n) = N(\mathcal{U}_0^{n-1})$. Thus, we have

$$h_\Psi(T, \mathcal{U}) = \lim_{\alpha \to 0} h_\Psi(T, \mathcal{U}, \alpha)$$

$$= \inf_{0 < \alpha \leq 0} \inf_{n \geq 1} \frac{1}{n} \log N(\mathcal{U}^n | \Psi_n(\alpha))$$

$$= \inf_{n \geq 1} \inf_{0 < \alpha \leq 1} \frac{1}{n} \log N(\mathcal{U}^n | \Psi_n(\alpha))$$

$$= \inf_{n \geq 1} \frac{1}{n} \log N(\mathcal{U}_0^{n-1}) = h_{\text{top}}(T, \mathcal{U}).$$

This completes the proof of the theorem. □

Remark 2.4. Combining (2) and (5), we have

$$h_{\text{top}}(T) = \sup_{\mathcal{U} \in \mathcal{C}_X^o} h_\Psi(T, \mathcal{U}).$$

On the other hand, let us define $h_\Psi(T) = \sup_{\mathcal{U} \in \mathcal{C}_X^o} h_\Psi(T, \mathcal{U})$, which is called the *pseudo-orbit entropy* of T. Using the same techniques of topological entropy (see Lemma 2.1), we can easily show that

$$h_\Psi(T) = \lim_{\epsilon \to 0} \lim_{\alpha \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s(n, \epsilon, \Psi_n(\alpha))$$

$$= \lim_{\epsilon \to 0} \lim_{\alpha \to 0} \limsup_{n \to \infty} \frac{1}{n} \log r(n, \epsilon, \Psi_n(\alpha)).$$

So, it is in fact to give a simpler proof of Theorem 1 of [37] by Theorem 2.3.

3. Pointwise preimage entropies for open covers and local Hurley's inequality

When T is not invertible, one can ask about growth rates of inverse images $f^{-n}(x)$. In this section we describe two ways of doing this, which were introduced by Hurley in [1].

3.1 Preimage branch entropy

Let (X, T) be a topological dynamical system. Given $x \in X$ let $T_n(x)$ denote the *tree of inverse images of x up to order n*, which is defined by

$$T_n(x) = \{(z_0, z_1, \cdots, z_n) : z_n = x \text{ and } z_j = T(z_{j-1}) \text{ for all } 1 \leq j \leq n\}.$$

Each $(z_0, z_1, \cdots, z_n) \in T_n(x)$ is called a *branch* of $T_n(x)$, and its *length* is n. Note that every branch of $T_n(x)$ ends with x. Let $\mathcal{T}_n = \cup_{x \in X} T_n(x)$; we define a metric on $\mathcal{T}_n$ as follows: suppose that $\tilde{z} = (z_0, z_1, \cdots, z_n)$ and $\tilde{w} = (w_0, w_1, \cdots, w_n)$ are two branches of the length n, the *branch distance* between them is defined as

$$d_{B,\,n}(\tilde{z}, \tilde{w}) = \max_{0 \le j \le n} d(z_j, w_j).$$

Let $O_n = \{T_n(x) : x \in X\}$. Given two trees $T_n(x)$ and $T_n(y)$ in O_n, the *branch Hausdorff* distance between them, $d_{bH}(T_n(x), T_n(y))$ is the usual Hausdorff metric based upon $d_{B,\,n}$; that is,

$$d_{bH}(T_n(x), T_n(y)) = \max\left\{ \max_{\tilde{z} \in T_n(x)} \min_{\tilde{w} \in T_n(y)} d_{B,\,n}(\tilde{z}, \tilde{w}), \max_{\tilde{w} \in T_n(y)} \min_{\tilde{z} \in T_n(x)} d_{B,\,n}(\tilde{z}, \tilde{w}) \right\}.$$

Note that $d_{bH}(T_n(x), T_n(y)) < \epsilon$ if and only if each branch of either tree is $d_{B,\,n}$ within ϵ of at least one branch of the other tree. Two trees $T_n(x)$ and $T_n(y)$ in O_n are said to be d_{bH}-(n, ϵ)-*separated* if $d_{bH}(T_n(x), T_n(y)) < \epsilon$, that is, there is a branch $\tilde{z}$ of one of the trees with the property that $d_{B,\,n}(\tilde{z}, \tilde{w}) > \epsilon$ for all branches $\tilde{w}$ of the other tree. Let $t(n, \epsilon)$ denote the maximum cardinality of any d_{bH}-(n, ϵ)-separated sets of O_n. Define the entropy by

$$h_b(T) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log t(n, \epsilon),$$

which is called the *preimage branch entropy* of T.

3.2 Pointwise preimage entropies

Let us recall two non-invertible invariants defined by Hurley [1] in 1995. Hurley's invariants are about the maximum rate of dispersal of the preimage sets of individual points, which are called *pointwise preimage entropies* in [8]. The difference between these two invariants is when the maximization takes place:

$$h_p(T) = \sup_{x \in X} \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s(n, \epsilon, T^{-n}(x))$$

$$= \sup_{x \in X} \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log r(n, \epsilon, T^{-n}(x)),$$

$$h_m(T) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \sup_{x \in X} s(n, \epsilon, T^{-n}(x))$$

$$= \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \sup_{x \in X} r(n, \epsilon, T^{-n}(x)).$$

It is clear that $h_p(T) \le h_m(T)$, and in [18] the authors constructed an example for which $h_p(T) < h_m(T)$. In addition, Hurley established the following relationships among preimage branch entropy, pointwise preimage entropy, and topological entropy (see [1], Theorem 3.1):

$$h_m(T) \le h_{\text{top}}(T) \le h_m(T) + h_b(T).$$

We call it the Hurley inequality.

3.3 Local Hurley's inequality

In this subsection, we mainly study the relationship among the topological entropy for open covers and several preimage entropy invariants, which is viewed

as the local version of the Hurley inequality. To do it, we first introduced a definition of preimage entropy via open covers.

Let (X, T) be a topological dynamical system. Given $\mathcal{U} \in \mathcal{C}_X^o$, define two *pointwise preimage entropies of* $\mathcal{U}$ with respect to T by

$$h_p(T, \mathcal{U}) = \sup_{x \in X} \limsup_{n \to \infty} \frac{1}{n} \log N\left(\mathcal{U}_0^{n-1} | T^{-n}(x)\right)$$

and

$$h_m(T, \mathcal{U}) = \limsup_{n \to \infty} \frac{1}{n} \log \left(\sup_{x \in X} N(\mathcal{U}_0^{n-1} | T^{-n}(x)) \right).$$

Theorem 3.1. (Local Hurley's inequality). *Let* (X, T) *be a topological dynamical system. If* $\mathcal{U} \in \mathcal{C}_X^o$, *then we have*

$$h_p(T, \mathcal{U}) \le h_m(T, \mathcal{U}) \le h_{\text{top}}(T, \mathcal{U}) \le h_m(T, \mathcal{U}) + h_b(T).$$

Proof. It is obvious that $N\left(\mathcal{U}_0^{n-1} | T^{-n}(x)\right) \le N\left(\mathcal{U}_0^{n-1}\right)$ for every $x \in X$ and every integer $n \ge 1$. So that $h_p(T, \mathcal{U}) \le h_m(T, \mathcal{U}) \le h_{\text{top}}(T, \mathcal{U})$. Now we show the last inequality $h_{\text{top}}(T, \mathcal{U}) \le h_m(T, \mathcal{U}) + h_b(T)$.

Let $\epsilon > 0$ be a Lebesgue number of $\mathcal{U}$. Fix $n \ge 1$, and let Y denote a $d_{bH}\text{-}(n, \epsilon/3)$-separated set of O_n with cardinality $t(n, \epsilon/3)$. Let Z denote the set of all root points of trees in Y, where the root point of the tree $T_n(x)$ is x. For each $z \in Z$, let $\mathcal{V}(z, \mathcal{U})$ be a subcover of $\mathcal{U}_0^{n-1}$ with cardinality $N\left(\mathcal{U}_0^{n-1} | T^{-n}(z)\right)$ that covers $T^{-n}(z)$, and let

$$\mathcal{V} = \bigcup_{z \in Z} \mathcal{V}(z, \mathcal{U}).$$

We claim that $\mathcal{V}$ is an open cover of X.

In fact, let $x \in X$ be given and let $w = f^n(x)$. Since Y is a $d_{bH}\text{-}(n, \epsilon/3)$-separated set of O_n with maximal cardinality, there is a tree $T_n(y) \in Y$ such that $d_{bH}(T_n(w), T_n(y)) < \epsilon/3$. Now we consider the branch $\tilde{w}$ of $T_n(w)$ begins with x, i.e., $\tilde{w} = \left(x, f(x), \cdots, f^{n-1}(x), f^n(x) = w\right) \in T_n(w)$. Then there exists a branch $\tilde{y} = \left(y_0, y_1, \cdots, y_n = y\right) \in T_n(y)$ such that $d_{B, n}(\tilde{w}, \tilde{y}) < \epsilon/3$. This means that $d\left(T^j(y_0), T^j(x)\right) < \epsilon/3$ for each $0 \le j \le n$. Thus, there exists $V \in \mathcal{V}(y, \mathcal{U})$ such that $x \in V$. This yields the claim that $\mathcal{V}$ is an open cover of X. So that $N\left(\mathcal{U}_0^{n-1}\right) \le |\mathcal{V}|$, where $|\mathcal{V}|$ denotes the cardinality of $\mathcal{V}$. Using the claim, we have

$$N\left(\mathcal{U}_0^{n-1}\right) \le |\mathcal{V}| \le \sum_{z \in Z} |\mathcal{V}(z, \mathcal{U})| = \sum_{z \in Z} N\left(\mathcal{U}_0^{n-1} | T^{-n}(z)\right)$$

$$\le |Z| \cdot \left(\sup_{x \in X} N(\mathcal{U}_0^{n-1} | T^{-n}(x)) \right)$$

$$= |Y| \cdot \left(\sup_{x \in X} N(\mathcal{U}_0^{n-1} | T^{-n}(x)) \right) = t(n, \epsilon/3) \cdot \left(\sup_{x \in X} N(\mathcal{U}_0^{n-1} | T^{-n}(x)) \right).$$

So that,

$$h_{\text{top}}(T, \mathcal{U}) = \lim_{n \to \infty} \frac{1}{n} \log N(\mathcal{U}_0^{n-1})$$

$$\leq \limsup_{n \to \infty} \frac{1}{n} \left[\log t(n, \epsilon/3) + \log \left(\sup_{x \in X} N(\mathcal{U}_0^{n-1} | T^{-n}(x)) \right) \right]$$

$$\leq \limsup_{n \to \infty} \frac{1}{n} \log t(n, \epsilon/3) + \limsup_{n \to \infty} \frac{1}{n} \log \left(\sup_{x \in X} N(\mathcal{U}_0^{n-1} | T^{-n}(x)) \right)$$

$$= \limsup_{n \to \infty} \frac{1}{n} \log t(n, \epsilon/3) + h_m(T, \mathcal{U}) \leq h_b(T) + h_m(T, \mathcal{U}).$$

This completes the proof of the theorem. □

We remark that Theorem 3.1 generalizes the classical Hurley's inequality given in [26, Theorem 3.1]. A direct consequence of Theorem 3.1 is.

Corollary 3.2. (Hurley's inequality). *Let (X, T) be a topological dynamical system. Then we have*

$$h_p(T) \leq h_m(T) \leq h_{\text{top}}(T) \leq h_m(T) + h_b(T). \tag{6}$$

Proof. It follows directly from Lemma 2.1 that

$$h_p(T) = \sup_{\mathcal{U} \in \mathcal{C}_X^o} h_p(T, \mathcal{U}) \quad \text{and} \quad h_m(T) = \sup_{\mathcal{U} \in \mathcal{C}_X^o} h_p(T, \mathcal{U}). \tag{7}$$

Thus, combining (2), (7), and Theorem 3.1 gives (6). □

4. Point entropy for open covers with pseudo-orbits

In [1], Hurley considered pseudo-orbits for inverse images and showed that the topological entropy can be characterized in terms of growth rates of pseudo-orbits that end at a particular point. Let (X, T) be a topological dynamical system. Recall that if $\alpha > 0$, then an α-pseudo-orbit $(x_0, x_1, ..., x_{n-1}) \in X^n$ is an approximate orbits segment for T in the sense that $d(T(x_j), x_{j+1}) < \alpha$ for all $0 \leq j \leq n - 1$.

For each $x \in X$, let $\Psi_n(\alpha, x) \subset X^n$ denote the set of all α-pseudo-orbits of length n that end at x, i.e., an element of $\Psi_n(\alpha, x)$ is an α-pseudo-orbit $(y_0, y_1, \cdots, y_{n-1})$ with $y_{n-1} = x$. It was shown in [1], (Propositions 4.2 and 4.3) that

$$h_{\text{top}} = \lim_{\epsilon \to 0} \lim_{\alpha \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \left(\max_{x \in X} s(n, \epsilon, \Psi_n(\alpha, x)) \right)$$

$$= \sup_{x \in X} \lim_{\epsilon \to 0} \lim_{\alpha \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s(n, \epsilon, \Psi_n(\alpha, x)). \tag{8}$$

In either formula $s(n, \epsilon, \Psi_n(\alpha, x))$ can be replaced by $r(n, \epsilon, \Psi_n(\alpha, x))$.

In the following, we will show that the topological entropy for an open cover can be characterized by pseudo-orbits for inverse images. Before proceeding, let us consider the following definitions, which use the notation introduced in Section 2.3.

Let (X, T) be a topological dynamical system. For each integer $n \geq 1$, $\mathcal{U} \in \mathcal{C}_X^o$, and $\alpha > 0$, we define

$$N_{\max}(n, \mathcal{U}, \alpha) = \max_{x \in X} N(\mathcal{U}^n | \Psi_n(\alpha, x)). \tag{9}$$

Clearly,

$$N(\mathcal{U}^n|\Psi_n(\alpha, x)) \leq N_{\max}(n, \mathcal{U}, \alpha) \leq N(\mathcal{U}^n|\Psi_n(\alpha)) \tag{10}$$

for every $x \in X$. In addition, by the compactness of X, there is some point $y \in X$ such that

$$N(\mathcal{U}^n|\Psi_n(\alpha, y)) = N_{\max}(n, \mathcal{U}, \alpha).$$

Lemma 4.1. *Let (X, T) be a topological dynamical system and $\mathcal{U} \in \mathcal{C}_X^o$. Suppose that $\varepsilon > 0$ is a Lebesgue number of $\mathcal{U}$ and $0 < \alpha < \varepsilon/4$. Then there is a constant $K = K(\alpha)$ such that for every $n \geq 1$,*

$$N(\mathcal{U}^n|\Psi_n(\alpha)) \leq K \cdot N_{\max}(n, \mathcal{U}, \alpha). \tag{11}$$

Proof. Let $\{x_1, x_2, \cdots, x_K\}$ be a finite α-dense subset of X, i.e., $\bigcup_{i=1}^n B(x_i, \alpha) = X$, where $B(x_i, \alpha) = \{z \in X : d(x_i, z) < \alpha\}$. For each $1 \leq i \leq K$, let $\mathcal{V}_i$ be a subcover of $\mathcal{U}^n$ that covers $\Psi_n(\alpha, x_i)$ with cardinality $N(\mathcal{U}^n|\Psi_n(\alpha, x_i))$. Define $\mathcal{V} = \bigcup_{i=1}^K \mathcal{V}_i$. Clearly, $|\mathcal{V}| \leq \sum_{i=1}^K |\mathcal{V}_i| \leq K \cdot N_{\max}(n, \mathcal{U}, \alpha)$. So, to complete the proof of the lemma, it suffices to show $\mathcal{V}$ is a subcover of $\mathcal{U}^n$ that covers $\Psi_n(\alpha)$.

In fact, let $\tilde{y} = (y_0, y_1, \cdots, y_{n-1})$ be an α-pseudo-orbit. Since $\{x_1, x_2, \cdots, x_K\}$ is an α-dense subset of X, there is some x_i satisfying $d(T(y_{n-2}), x_i) < \alpha$. This implies $\tilde{z} = (z_0, z_1, \cdots, z_{n-2}, z_{n-1}) = (y_0, y_1, \cdots, y_{n-2}, x_i)$ is an α-pseudo-orbit ending at x_i. Since $\mathcal{V}_i$ is a subcover of $\mathcal{U}^n$ that covers $\Psi_n(\alpha, x_i)$, there is some $V \in \mathcal{V}_i$ such that $\tilde{z} \in V$. Since $z_j = y_j$ for all $0 \leq j \leq n - 2$ and ε is the Lebesgue number of $\mathcal{U}$, in order to show that $\tilde{y} \in V$, we need only to show that $d(y_{n-1}, x_i) < \varepsilon/2$; this is obviously, as $d(y_{n-1}, x_i) \leq d(y_{n-1}, T(y_{n-2})) + d(T(y_{n-2}), x_i) < 2\alpha < \varepsilon/2$. $\qquad\square$

Theorem 4.2. *Let (X, T) be a topological dynamical system. If $\mathcal{U} \in \mathcal{C}_X^o$, then we have*

$$h_{\text{top}}(T, \mathcal{U}) = \lim_{\alpha \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \left(\sup_{x \in X} N(\mathcal{U}^n|\Psi_n(\alpha, x)) \right). \tag{12}$$

Proof. Combining (10) and (11), we have

$$N_{\max}(n, \mathcal{U}) \leq N(\mathcal{U}^n|\Psi_n(\alpha)) \leq K \cdot N_{\max}(n, \mathcal{U})$$

for each fixed $0 < \alpha < \varepsilon/4$ and all $n \geq 1$, where ε is a Lebesgue number of $\mathcal{U}$ and $K = K(\alpha)$ in Lemma 4.1 is independent of n. This implies that

$$\limsup_{n \to \infty} \frac{1}{n} \log N_{\max}(n, \mathcal{U}, \alpha) = \limsup_{n \to \infty} \frac{1}{n} \log N(\mathcal{U}^n|\Psi(\alpha)) \tag{13}$$

for all positive number $0 < \alpha < \varepsilon/2$. Thus, we have

$$
\begin{aligned}
h_{\text{top}}(T, \mathcal{U}) &= \lim_{\alpha \to 0} \lim_{n \to \infty} \frac{1}{n} \log N(\mathcal{U}^n|\Psi_n(\alpha)) \quad \text{(by Theorem 2.3)} \\
&= \lim_{\alpha \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(\mathcal{U}^n|\Psi_n(\alpha)) \\
&= \lim_{\alpha \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N_{\max}(n, \mathcal{U}, \alpha) \quad \text{(by (4.6))} \\
&= \lim_{\alpha \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \left(\sup_{x \in X} N(\mathcal{U}^n|\Psi_n(\alpha, x)) \right) \quad \text{(by (4.1))}
\end{aligned}
$$

This completes the proof. □

Theorem 4.3. *Let (X, T) be a topological dynamical system. If $\mathcal{U} \in C_X^o$, then we have*

$$h_{\text{top}}(T, \mathcal{U}) = \sup_{x \in X} \lim_{\alpha \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(\mathcal{U}^n | \Psi_n(\alpha, x)). \tag{14}$$

Proof. It follows directly from (10) and (12) that

$$h_{\text{top}}(T, \mathcal{U}) = \lim_{\alpha \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \left(\sup_{x \in X} N(\mathcal{U}^n | \Psi_n(\alpha, x)) \right)$$

$$\geq \sup_{x \in X} \lim_{\alpha \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(\mathcal{U}^n | \Psi_n(\alpha, x)).$$

Now we start to prove the converse inequality.

Note that for the given $n \geq 1$ and $\alpha > 0$, there is a point $y(n, \mathcal{U}, \alpha) \in X$ such that

$$N(\mathcal{U}^n | \Psi_n(\alpha, y(n, \mathcal{U}, \alpha))) = \max_{x \in X} N(\mathcal{U}^n | \Psi_n(\alpha, x)).$$

Taking a sequence of integers $n_i = n_i(\alpha) \to \infty$ such that

$$\limsup_{n \to \infty} \frac{1}{n} \log N(\mathcal{U}^n | \Psi_n(\alpha, y(n, \mathcal{U}, \alpha))) = \lim_{i \to \infty} \frac{1}{n_i} \log N(\mathcal{U}^{n_i} | \Psi_{n_i}(\alpha, y(n_i, \mathcal{U}, \alpha))).$$

By restricting to a subsequence, we can assume without loss of generality that the sequence $y_i(\alpha) = y(n_i, \mathcal{U}, \alpha)$ converses to a limit $q(\alpha)$.

Let ϵ be a Lebesgue number of $\mathcal{U}$. If $0 < \beta < \epsilon/4$ and $d(y_i(\alpha), q(\alpha)) < \beta$, then $\mathcal{V}$ is a subcover of $\mathcal{U}^n$ that covers $\Psi_n(\alpha, y_i(\alpha))$ whenever $\mathcal{V}$ is a subcover of $\mathcal{U}^n$ that covers $\Psi_n(\alpha + \beta, q(\alpha))$. This implies that

$$N(\mathcal{U}^n | \Psi_n(\alpha + \beta, q(\alpha))) \geq N(\mathcal{U}^n | \Psi_n(\alpha, y_i(\alpha))) \tag{15}$$

whenever $d(y_i(\alpha), q(\alpha)) < \beta$.

Now we choose a sequence $\alpha_j \to 0$ such that $q(\alpha_j)$ converges to some point $q \in X$. Similar to the proof as above we have

$$N(\mathcal{U}^n | \Psi_n(\alpha_j + 2\beta, q)) \geq N(\mathcal{U}^n | \Psi_n(\alpha_j, q(\alpha_j))) \tag{16}$$

whenever $d(q(\alpha_j), q) < \beta$. Combining inequalities (15) and (16), one has

$$N(\mathcal{U}^n | \Psi_n(\alpha_j + 2\beta, q)) \geq N(\mathcal{U}^n | \Psi_n(\alpha_j, y_i(\alpha_j))) \tag{17}$$

whenever $d(y_i(\alpha_j), q(\alpha_j)) < \beta$ and $d(q(\alpha_j), q) < \beta$. If j is a fixed integer with $d(q(\alpha_j), q) < \beta$, then (17) holds for all sufficiently large integers i. Thus,

$$\limsup_{n \to \infty} \frac{1}{n} \log N(\mathcal{U}^n | \Psi_n(\alpha_j + 2\beta, q))$$

$$\geq \limsup_{i \to \infty} \frac{1}{n_i} \log N(\mathcal{U}^{n_i} | \Psi_{n_i}(\alpha_j + 2\beta, q)) \tag{18}$$

$$\geq \lim_{i \to \infty} \frac{1}{n_i} \log N(\mathcal{U}^{n_i} | \Psi_{n_i}(\alpha_j, y_i(\alpha_j)))$$

$$= \limsup_{n \to \infty} \frac{1}{n} \log \left(\max_{x \in X} N(\mathcal{U}^n | \Psi_n(\alpha_j), x) \right).$$

Now let $j \to \infty$ and use the fact that both sides (18) are nonincreasing as α decreases to conclude that

$$\limsup_{n \to \infty} \frac{1}{n} \log N(\mathcal{U}^n | \Psi_n(3\beta, q))$$

$$\geq \lim_{j \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(\mathcal{U}^n | \Psi_n(\alpha_j + 2\beta, q))$$

$$\geq \lim_{j \to \infty} \limsup_{n \to \infty} \frac{1}{n} \log \left(\max_{x \in X} N(\mathcal{U}^n | \Psi_n(\alpha_j), x) \right) \qquad (19)$$

$$= \inf_{\alpha > 0} \limsup_{n \to \infty} \frac{1}{n} \log \left(\max_{x \in X} N(\mathcal{U}^n | \Psi_n(\alpha, x)) \right)$$

$$= \lim_{\alpha \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \left(\max_{x \in X} N(\mathcal{U}^n | \Psi_n(\alpha), x) \right).$$

Therefore, combining (12) and (19), we have

$$h_{\mathrm{top}}(T, \mathcal{U}) = \lim_{\alpha \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \left(\max_{x \in X} N(\mathcal{U}^n | \Psi_n(\alpha), x) \right)$$

$$\leq \inf_{\beta > 0} \limsup_{n \to \infty} \frac{1}{n} \log N(\mathcal{U}^n | \Psi_n(3\beta, q))$$

$$= \inf_{\alpha > 0} \limsup_{n \to \infty} \frac{1}{n} \log N(\mathcal{U}^n | \Psi_n(\alpha, q)) \qquad (20)$$

$$= \lim_{\alpha \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(\mathcal{U}^n | \Psi_n(\alpha, q))$$

$$\leq \sup_{x \in X} \lim_{\alpha \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(\mathcal{U}^n | \Psi_n(\alpha, x)).$$

This completes the proof. $\square$

5. Partial entropy and bundle-like entropy for nonautonomous discrete dynamical systems

In [38, 41], topological entropy for certain nonautonomous discrete dynamical system was defined and studied. In this section, we study the topological entropy for nonautonomous discrete dynamical systems by introducing two entropy-like invariants called the partial entropy and bundle-like entropy as being motivated by the idea of [1, 39].

5.1 Topological entropy for nonautonomous discrete dynamical systems

Let **X** be a collection of countable infinitely many compact metric space (X_i, d_i) and **F** be a collection of countable infinite many continuous maps $f_i : X_i \to X_{i+1}$, $i = 1, 2, \cdots$. Then the pair $(\mathbf{X}, \mathbf{F})$ is called a *nonautonomous discrete dynamical system*.

For any integer $n \geq 1$, we define a metric $\tilde{d}_n$ on $\prod_{i=1}^{n} X_i$ as follows: for any two points $\tilde{x}_n = (x_1, x_2, \cdots, x_n), \tilde{y}_n = (y_1, y_2, \cdots, y_n) \in \prod_{i=1}^{n} X_i$,

$$\tilde{d}_n(\tilde{x}_n, \tilde{y}_n) = \max_{1 \le i \le n} d_i(x_i, y_i).$$

Fixing an integer $n \ge 1$ and a positive number ϵ. A subset Z of $\prod_{i=1}^n X_i$ is called $\tilde{d}_n$-(n, ϵ)-*separated* if for any two distinct points $\tilde{x}_n, \tilde{y}_n \in Z$ we have $\tilde{d}_n(\tilde{x}_n, \tilde{y}_n) > \epsilon$. Denote the maximal cardinality of any $\tilde{d}_n$-separated subset of Z by $s(n, \epsilon, Z)$. A subset $W \subset Z$ is called $\tilde{d}_n$-(n, ϵ)-*spanning* for Z if for each $\tilde{z}_n \in Z$, there is a $\tilde{w}_n \in W$ such that $\tilde{d}_n(\tilde{z}_n, \tilde{w}_n) < \epsilon$. Denote the minimal cardinality of any $\tilde{d}_n$-spanning subset of Z by $r(n, \epsilon, Z)$.

The following result is trivial, so we omit its detail proof.

Lemma 5.1. *Suppose that n is a positive integer and Z is a nonempty subset of $\prod_{i=1}^n X_i$. Then for each $\epsilon > 0$, we have*

$$r(n, \epsilon, Z) \le s(n, \epsilon, Z) \le r(n, \epsilon/2, Z).$$

For each $n \ge 1$ let Z_n be a nonempty subset of $\prod_{i=1}^n X_i$. Then it follows immediately from Lemma 5.1 that

$$\lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log r(n, \epsilon, Z_n) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s(n, \epsilon, Z_n). \qquad (21)$$

Given a nonautonomous discrete dynamical system $(\mathbf{X}, \mathbf{F})$, denoted by $O_{n, \mathbf{F}}$ or O_n for short the set of all orbit segments of length n for each $n \ge 1$, i.e.,

$$O_n = O_{n, \mathbf{F}} := \{(x_1, x_2, \cdots, x_n) : x_1 \in X_1 \text{ and } x_{i+1} = f_i(x_i), i = 1, 2, \cdots, n-1\}.$$

Then the common limit in (21) by taking $Z_n = O_n$ is defined to be the *topological entropy of $(\mathbf{X}, \mathbf{F})$*, written $h_{\text{top}}(\mathbf{X}, \mathbf{F})$ or $h_{\text{top}}(\mathbf{F})$ for short if there is no confusion.

5.2 Partial entropy and bundle-like entropy

Let $(\mathbf{X}, \mathbf{F})$ be a nonautonomous discrete dynamical system. A collection $\mathcal{P} = (\mathcal{P}_i : i \ge 1)$ is said to be a *cover* of $\mathbf{X}$ if each $\mathcal{P}_i$ covers X_i, respectively. We now define two entropies, partial entropy and bundle-like entropy, for $(\mathbf{X}, \mathbf{F})$ relative to $\mathcal{P}$.

For any integer $n \ge 1$ and $D \in \mathcal{P}_n$, let $W_n(D) \subset \prod_{i=1}^n X_i$ denote the set of all orbit segments of length that end at some point $x_n \in D$, i.e.,

$$W_n(D) = \{(x_1, x_2, \cdots, x_n) \in O_n : x_n \in D\}.$$

Put $s_{\max, \mathcal{P}_n}(n, \epsilon) = \sup_{D \in \mathcal{P}_n} s(n, \epsilon, W_n(D))$. Define the entropy by

$$h_{p, \mathcal{P}}(\mathbf{X}, \mathbf{F}) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s_{\max, \mathcal{P}_n}(n, \epsilon),$$

which is called the *partial entropy of $(\mathbf{X}, \mathbf{F})$ relative to $\mathcal{P}$* and written shortly by $h_{p, \mathcal{P}}(\mathbf{F})$ if there is no confusion.

Let $O_{n, \mathcal{P}_n} = \{W_n(D) : D \in \mathcal{P}_n\}$. For any two elements, $W_n(D)$ and $W_n(E)$ of $O_{n, \mathcal{P}_n}$, denoted by $d_H(W_n(D), W_n(E))$, the usual Hausdorff metric between them is based upon metric $\tilde{d}_n$ of $\prod_{i=1}^n X_i$ defined as before and by $s(n, \epsilon, O_{n, \mathcal{P}_n})$ the maximum cardinality of any d_H-(n, ϵ)-separated subset of $O_{n, \mathcal{P}_n}$. Define the entropy by

$$h_{b,\,\mathcal{P}}(\mathbf{X}, \mathbf{F}) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s(n, \epsilon, O_{n,\,\mathcal{P}_n}),$$

which is called the *bundle-like entropy of* $(\mathbf{X}, \mathbf{F})$ *relative to* $\mathcal{P}$ and written shortly by $h_{b,\,\mathcal{P}}(\mathbf{F})$ if there is no confusion.

Also, we have the spanning set versions of definitions of $h_{p,\,\mathcal{P}}(\mathbf{F})$ and $h_{b,\,\mathcal{P}}(\mathbf{F})$, respectively.

5.3 Some relationships between $h_{\text{top}}(\mathbf{F})$ and $h_{p,\,\mathcal{P}}(\mathbf{F})$

Theorem 5.2. *Let* $(\mathbf{X}, \mathbf{F})$ *be a nonautonomous discrete dynamical system, and* $\mathcal{P} = (\mathcal{P}_i : i \geq 1)$ *be a cover of* $\mathbf{X}$. *Then we have*

$$h_{p,\,\mathcal{P}}(\mathbf{F}) \leq h_{\text{top}}(\mathbf{F}) \leq h_{b,\,\mathcal{P}}(\mathbf{F}) + h_{p,\,\mathcal{P}}(\mathbf{F}).$$

Proof. Note that $s_{\max,\,\mathcal{P}_n}(n, \epsilon) \leq s(n, \epsilon, O_n)$ for any cover $\mathcal{P}$ of $\mathbf{X}$ and any $\epsilon > 0$. Then the former inequality is obtained. Now we show the later one. If $h_{b,\,\mathcal{P}}(\mathbf{F}) = \infty$, then there is nothing to prove. Now assuming $h_{b,\,\mathcal{P}}(\mathbf{F}) < \infty$.

Fixing a sufficiently small $\epsilon > 0$ and an integer $n \geq 1$, let Y be a d_H-(n, ϵ)-separated subset of $O_{n,\,\mathcal{P}_n}$ with cardinality $s(n, \epsilon, O_{n,\,\mathcal{P}_n})$. For each $W_n(D) \in Y$, let $M(D)$ be a $\tilde{d}_n$-(n, ϵ)-separated subset of $W_n(D)$ with cardinality $s(n, \epsilon, W_n(D))$. Put $M = \cup_{W_n(D) \in Y} M(D)$. We claim that M is a $\tilde{d}_n$-$(n, 3\epsilon)$-spanning subset of O_n.

In fact, for any $x = (x_1, x_2, \cdots, x_n) \in O_n$, since Y is a d_H-(n, ϵ)-separated subset of $O_{n,\,\mathcal{P}_n}$ with maximum cardinality and $\mathcal{P}_n$ covers X_n, there is an $E \in \mathcal{P}_n$ with $x_n \in E$ and a $W_n(D) \in Y$ such that $d_H(W_n(D), W_n(E)) \leq \epsilon$. Then it follows that there is a $y = (y_1, y_2, \cdots, y_n) \in W_n(D)$ such that $\tilde{d}_n(x, y) \leq \epsilon$. Also note that $M(D)$ is a $\tilde{d}_n$-(n, ϵ)-separated subset of $W_n(D)$ with maximum cardinality; there is a $z \in M(D)$ such that $\tilde{d}_n(y, z) \leq \epsilon$. Hence we have

$$\tilde{d}_n(x, z) \leq \tilde{d}_n(x, y) + \tilde{d}_n(y, z) < 3\epsilon.$$

This yields the claim that M is a $\tilde{d}_n$-(n, ϵ)-spanning subset of O_n. So we have $r(n, 3\epsilon, O_n) \leq |M|$, where M denotes the cardinality of M. Using the claim we have

$$r(n, 3\epsilon, O_n) \leq |M| \leq |Y| \cdot \max\{|M(D)| : W_n(D) \in Y\}$$
$$\leq s(n, \epsilon, O_{n,\,\mathcal{P}_n}) \cdot s_{\max,\,\mathcal{P}_n}(n, \epsilon).$$

Taking limits as the requirements of the related definitions of entropies establishes the desired inequality. This completes the proof. □

Let $\mathcal{P}(\delta)$ be a finite cover of a compact metric space X consisting of open balls with radius less than some $\delta > 0$. Write $\mathbf{F}_X = \{f_i : f_i : X \to X \text{ is continous}, i \geq 1\}$ and $\mathcal{P}_X(\delta) = (\mathcal{P}(\delta), \mathcal{P}(\delta), \cdots)$.

Theorem 5.3.

$$h_{\text{top}}(\mathbf{F}_X) = h_{p,\,\mathcal{P}_X(\delta)}(\mathbf{F}_X) = \lim_{\epsilon \to 0} \lim_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s_{\max,\,\mathcal{P}(\delta)}(n, \epsilon).$$

Proof. Note that $\lim_{n \to \infty} \frac{1}{n} \log |\mathcal{P}(\delta)| = 0$. Then, by Theorem 5.2, we have the former equality. Now we show the later equality.

Clearly, $s(n, \epsilon, O_n) \geq s_{\max,\,\mathcal{P}_X(\delta)}(n, \epsilon)$ for any $\delta > 0$, so we have

$$\limsup_{n\to\infty} \log s(n, \epsilon, O_n) \geq \lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{n} \log s_{\max, \mathcal{P}_X(\delta)}(n, \epsilon).$$

This implies

$$h_{\text{top}}(\mathbf{F}_X) \geq \lim_{\epsilon\to 0} \lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{n} \log s_{\max, \mathcal{P}_X(\delta)}(n, \varepsilon). \tag{22}$$

On the other hand, from the proof of Theorem 5.2, it follows that

$$r(n, 3\epsilon, O_n) \leq s\big(n, \epsilon, O_{n, \mathcal{P}(\delta)}\big) \cdot s_{\max, \mathcal{P}_X(\delta)}(n, \epsilon)$$

for any integer $n \geq 1$, any sufficiently small $\epsilon > 0$ and any $\delta > 0$. Noting that $s\big(n, \epsilon, O_{n, \mathcal{P}(\delta)}\big) \leq |\mathcal{P}(\delta)|$ for any integer $n \geq 1$, then we have

$$\limsup_{n\to\infty} \frac{1}{n} \log r(n, 3\epsilon, O_n) \leq \lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{n} \log s_{\max, \mathcal{P}_X(\delta)}(n, \epsilon).$$

This implies

$$h_{\text{top}}(\mathbf{F}_X) \leq \lim_{\epsilon\to 0} \lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{n} \log s_{\max, \mathcal{P}_X(\delta)}(n, \epsilon). \tag{23}$$

Thus, combining (22) and (23) gets the later equality. This completes the proof. □

Remark 5.4. The first equality of Theorem 5.3 is in fact a simpler version of Theorem 7.6 of [40] (a useful result for calculating the classical topological entropy) when restricting to the autonomous discrete dynamical systems.

Given a nonautonomous discrete dynamical system $(\mathbf{X}, \mathscr{F})$, when does $h_{\text{top}}(\mathbf{F}) = h_{p, \mathcal{P}}(\mathbf{F})$ for any cover $\mathcal{P}$ of $\mathbf{X}$? The following theorem gives an answer to this question.

Theorem 5.5. *Let* $(\mathbf{X}, \mathbf{F})$ *be a nonautonomous discrete dynamical system. Then* $h_{\text{top}}(\mathbf{F}) = h_{p, \mathcal{P}}(\mathbf{F})$ *for any cover* $\mathcal{P}$ *of* $\mathbf{X}$ *if the following conditions hold:*

(1) *For each integer* $i \geq 1$, *there exists* $\delta_i > 0$ *such that* $d_{i+1}\big(f_i(x), f_i(y)\big) \geq d_i(x, y)$ *whenever* $d_i(x, y) \leq \delta_i$ *for* $x, y \in X_i$.

(2) *For each integer* $i \geq 1$, *every* $x \in X_{i+1}$ *has an open neighborhood* U_x *whose preimage* $f_i^{-1}(U_x)$ *is an union of disjoint open sets on each of which* f_i *is a homeomorphism.*

(3) $\limsup_{n\to\infty} \frac{1}{n} \log N(\epsilon_n, X_n) = 0$ *for every monotonic decreasing sequence* $\{\epsilon_n\}$ *with* $\lim_{n\to\infty} \epsilon = 0$, *where each* $N(\epsilon_n, X_n)$ *denotes the minimal cardinality of the open cover of* X *consisting of open* ϵ_n-*ball for the compact metric space* X_n.

Proof. It suffices to show that $h_{b, \mathcal{P}}(\mathbf{F}) = 0$ for any cover $\mathcal{P}$ of $\mathbf{X}$ by Theorem 5.2. Let $\mathcal{P}_{\max} = \big\{\mathcal{P}_{i, \max} : i \geq 1\big\}$ be the cover of $\mathbf{X}$ in which each $\mathcal{P}_{i, \max}$ cover X_i consisting to singletons of X_i, i.e., $\mathcal{P}_{i, \max} = \{\{z\} : z \in X_i\}$. It is easy to see that $h_{b, \mathcal{P}}(\mathbf{F}) \leq h_{b, \mathcal{P}_{\max}}(\mathbf{F})$ for any cover $\mathcal{P}$ of $\mathbf{X}$. So from Theorem 5.2, it follows that what we want to prove is $h_{b, \mathcal{P}_{\max}}(\mathbf{F}) = 0$.

For each $n \geq 2$, by condition (1), there exists a $\delta_{n-1} > 0$ such that

$$d_n\big(f_{n-1}(x), f_{n-1}(y)\big) \geq d_{n-1}(x, y)$$

for any $x, y \in X_{n-1}$ whenever $d_{n-1}(x, y) \leq \delta_{n-1}$. Also, by condition (2) and the compactness of X_n, there exists an $\epsilon_n > 0$ such that the ϵ_n-ball $B(x_n, \epsilon_n)$ about any point $x_n \in X_n$ has preimage $f_{n-1}^{-1}(B(x_n, \epsilon))$ equals the union of disjoint open sets of

diameter less than δ_{n-1}. Then we get a sequence $\{\epsilon_n\}$. Furthermore, we can take ϵ_n such that $\{\epsilon_n\}$ is monotonic decreasing sequence and $\lim_{n\to\infty} \epsilon_n = 0$.

Now, given $y_n \in X_n$ and $\tilde{x} = (x_1, x_2, \cdots, x_n) \in W_n(B(y_n, \epsilon_n,))$, we want to find a point $\tilde{y} = (y_1, y_2, \cdots, y_n) \in O_n$ with $\tilde{d}_n(\tilde{x}, \tilde{y}) = d_n(x_n, y_n)$ and then $\tilde{d}_n(\tilde{x}, \tilde{y}) < \epsilon_n$. In fact, for $1 < k < n$, we can easily find a point $(y_k, \cdots, y_n) \in \prod_{i=k}^{n} X_i$ with $d_j(x_j, y_j) \le \epsilon_j$ and $d_{j+1}(x_{j+1}, y_{j+1}) \ge d_j(x_j, y_j)$, for $j = n-1, n-2, \cdots, k$. Let V be the piece of $f_{k-1}^{-1}(B(x_k, \epsilon_k))$ with $x_{k-1} \in V$. Since $y_k \in B(x_k, \epsilon_k)$, there is a unique point $y_{k-1} \in V \cap f_{k-1}^{-1}(y_k)$ such that $d_{k-1}(x_{k-1}, y_{k-1}) < \delta_{k-1}$. Then we have

$$d_{k-1}(x_{k-1}, y_{k-1}) \le d_k(x_k, y_k) \le d_n(x_n, y_n) < \epsilon_n < \epsilon_{k-1}.$$

This argument shows that $r(n, \epsilon_n, O_{n, comma \max}) \le N(\epsilon_n, X_n)$. Thus, by condition (3), we get

$$\limsup_{n\to\infty} \frac{1}{n} \log r(n, \epsilon_n, O_{n, comma \max}) \le \limsup_{n\to\infty} \frac{1}{n} \log N(\epsilon_n, X_n) = 0.$$

For any sufficiently small $\epsilon > 0$, there exists $N > 0$ such that $\epsilon_n < \epsilon$ for any $n \ge N$. Then we have $r(n, \epsilon, O_{n, \mathcal{P}_{n, \max}}) \le r(n, \epsilon_n, O_{n, \mathcal{P}_{n, \max}})$ and hence $h_{b, \mathcal{P}_{\max}}(\mathbf{F}) = 0$. This completes the proof. $\square$

6. Conclusion

Several important entropy-like invariants based on the preimage structure for non-invertible maps have been defined and studied by some authors. In this chapter, we first further study the preimage entropy for topological dynamical system from the view of localization. We show that the topological entropy for an open cover can be characterized by pseudo-orbits (Theorems 2.3, 4.2, and 4.3). We also establish an inequality relating the topological entropy for open covers and several preimage entropy invariants, which is viewed as the local version of the Hurley's inequality (Theorem 3.1). Finally, we discuss the topological entropy for nonautonomous discrete dynamical systems by introducing two entropy-like invariants called the partial entropy and bundle-like entropy. We establish some relationships among such two invariants and the topological entropy (Theorem 5.2, 5.3, and 5.5).

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