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# Cylindrical Surface Wave: Revisiting the Classical Biot's Problem

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## Abstract

The problem on a surface harmonic elastic wave propagating along the free surface of cylindrical cavity in the direction of cavity axis is considered. In the case of isotropic medium, this is the classical Biot's problem of 1952. First, the Biot pioneer work is revisited: the analytical part of Biot's findings is shown in the main fragments. The features are using two potentials and representation of solution by Macdonald functions of different indexes. Then the new direct generalization of Biot's problem on the case of transversely isotropic medium within the framework of linear theory of elasticity is proposed. Transition to the transverse isotropy needs some novelty—necessity of using the more complex representations of displacements through two potentials. Finally, a generalization of Biot's problem on the case of isotropic and transversely isotropic media in the framework of linearized theory of elasticity with allowance for initial stresses is stated. This part repeats briefly the results of A.N. Guz with co-authors of 1974. The main features are using the linearized theory of elasticity and one only potential. All three parts are shown as analytical study up to the level when the numerical methods have to be used.

**Keywords:** surface harmonic cylindrical wave, classical Biot's problem, generalization to the case of transversely isotropic medium

## 1. Introduction

Note first that the seismic waves include mainly the primary and secondary body waves and different kinds of surface waves. This chapter is devoted to one kind of surface waves. The problem is stated as follows: the infinite medium with cylindrical circular cavity having the symmetry axis  $Oz$  and constant radius is analyzed. An attenuating in depth of medium surface harmonic wave propagates along the cavity surface in direction  $Oz$ . In this case, the problem becomes mathematically the axisymmetric one. This problem is solved by Biot in 1952 [1] with assumption that the medium is isotropic. The context of this chapter includes four parts. The subchapter 1 "Introduction" is the standard one. The subchapter 2 is named: "Main Stages of Solving the Classical Biot's Problem on Surface Wave along Cylindrical Cavity." Here, the analytical part of Biot's findings is shown in the main fragments. The features are using two potentials and representation of solution by Macdonald functions of different indexes. The subchapter 3 "Direct Generalization of Biot's Problem on the Case of Transversely Isotropic Media within the

Framework of Linear Theory of Elasticity” contains the new approach to the classical Biot’s problem and represents the direct generalization of this problem that uses the Biot’s scheme of analysis. Transition to the case of transverse isotropy needs some novelty—necessity of using the more complex representations of displacements through two potentials. The subchapter 4 “Generalization of Biot’s Problem on the case of Isotropic and Transversely Isotropic Media within the framework of Linearized Theory of Elasticity with Allowance for Initial Stresses” repeats briefly the results of A.N. Guz with co-authors (1974). They considered a generalization of the Biot’s problem on the case of elastic media with allowance for the initial stresses. The main features are using the linearized theory of elasticity, one only potential, and Macdonald function of one index.

## 2. Main stages of solving the classical Biot’s problem on surface wave along a cylindrical cavity

### 2.1 Statement of problem and main equations in potentials

A cylindrical system of coordinates  $Or\vartheta z$  is chosen, and a harmonic wave is considered that has the phase variable  $\sigma = k(z - vt)$ , unknown wave number  $k = (\omega/v)$ , unknown phase velocity  $v$ , and arbitrary (but given) frequency  $\omega$  and amplitude  $A$ . It is supposed that the wave propagates in an infinite medium with cylindrical cavity of constant radius  $r_0$  in the direction of vertical coordinate  $z$  and possibly attenuates in the direction of radial coordinate  $r$ . In this linear statement and in assumption that deformations are small, the problem is axisymmetric, and deformations are described by two displacements ( $u_r(r, z, t), u_\varphi(r, z, t) = 0$  and  $u_z(r, z, t)$ ) and two Lamé equations of the form

$$\frac{C_{11} - C_{12}}{2} \left( \Delta_{rz} u_r - \frac{1}{r^2} u_r \right) + \frac{C_{11} + C_{12}}{2} \left( u_{r,r} + \frac{1}{r} u_r + u_{z,z} \right)_{,r} = \rho u_{r,tt}, \quad (1)$$

$$\frac{1}{2} (C_{11} - C_{12}) \Delta_{rz} u_z + \frac{1}{2} (C_{11} + C_{12}) \left( u_{r,r} + \frac{1}{r} u_r + u_{z,z} \right)_{,z} = \rho u_{z,tt}, \quad (2)$$

or

$$(\lambda + 2\mu) \left( u_{r,rr} + \frac{1}{r} u_{r,r} - \frac{1}{r^2} u_r + u_{z,rz} \right) + \mu (u_{r,zz} - u_{z,rz}) = \rho u_{r,tt} \quad (3)$$

$$(\lambda + 2\mu) \left( u_{r,rz} + \frac{1}{r} u_{r,z} + u_{z,zz} \right) - \mu \left[ \frac{1}{r} (u_{r,z} - u_{z,r}) + (u_{r,rz} - u_{z,rr}) \right] = \rho u_{z,tt}. \quad (4)$$

Further the potentials  $\Phi(r, z, t)$ ,  $\Psi(r, z, t)$  are introduced

$$u_r = \Phi_{,r} - \Psi_{,z}, \quad u_z = \Phi_{,z} + \Psi_{,r} + (1/r)\Psi. \quad (5)$$

When Eq. (5) is substituted into Eqs. (3) and (4), then two uncoupled linear wave equations are obtained:

$$\Delta_{rz} \Phi - (1/v_L)^2 \Phi_{,tt} = 0, \quad (6)$$

$$\Delta_{rz} \Psi - (1/r^2) \Psi - (1/v_T)^2 \Psi_{,tt} = 0. \quad (7)$$

Here the standard notations of Laplace operator  $\Delta_{rz}$  and velocities of longitudinal and transverse waves in isotropic elastic medium  $v_L = \sqrt{(\lambda + 2\mu)/\rho}$ ,  $v_T = \sqrt{\mu/\rho}$  are used.

## 2.2 Solving the wave equations in the form of Macdonald functions

The solution of Eqs. (6) and (7) is found in the form of harmonic waves in the direction of vertical coordinate:

$$\Phi(r, z, t) = \Phi^*(r)e^{i(kz - \omega t)}, \quad \Psi(r, z, t) = \Psi^*(r)e^{i(kz - \omega t)}, \quad (8)$$

$$\Phi(r, z, t) = \Phi^*(r) \cos k(z - vt), \quad \Psi(r, z, t) = \Psi^*(r) \sin k(z - vt).$$

A substitution of representations (8) into the wave Eqs. (6) and (7) gives the equations relative to the unknown amplitudes  $\Phi^*(r)$ ,  $\Psi^*(r)$

$$\Phi^*_{,rr} + (1/r)\Phi^*_{,r} - (k^2 - k_L^2)\Phi^* = 0.$$

$$\left( \Phi^*_{,rr} + (1/r)\Phi^*_{,r} - k^2 \left( 1 - (v/v_L)^2 \right) \Phi^* = 0 \right), \quad (9)$$

$$\begin{aligned} \Psi^*_{,rr} - (1/r)\Psi^*_{,r} - [k^2 - k_T^2 + (1/r^2)]\Psi^* &= 0 \\ \left( \Psi^*_{,rr} - (1/r)\Psi^*_{,r} - \left\{ k^2 \left[ 1 - (v/v_T)^2 \right] + (1/r^2) \right\} \Psi^* = 0 \right) \end{aligned} \quad (10)$$

These equations correspond to the Bessel equation for Macdonald functions  $K_\lambda(x)$  (modified Bessel functions of the second kind [2–4])

$$y'' + (1/x)y' - [1 + (\lambda^2/x^2)]y = 0 \quad (11)$$

More exactly, Eqs. (9) and (10) have the solutions in the form of Macdonald functions, if the conditions.

$$k^2 - k_L^2 > 0, k^2 - k_T^2 > 0 \left( k^2 \left( 1 - (v/v_L)^2 \right) > 0, k^2 \left( 1 - (v/v_T)^2 \right) > 0 \right) \quad (12)$$

are fulfilled. According to conditions (12), the wave number of cylindrical wave must be real, and the wave velocity must be less of the velocities of classical longitudinal and transverse plane waves.

Further the wave Eqs. (9) and (10) are considered separately. The first equation is written in the form

$$\Phi^*_{,rr} + (1/r)\Phi^*_{,r} - m_L^2 \Phi^* = 0 \quad m_L = k \sqrt{\left( 1 - (v/v_L)^2 \right)} \quad (13)$$

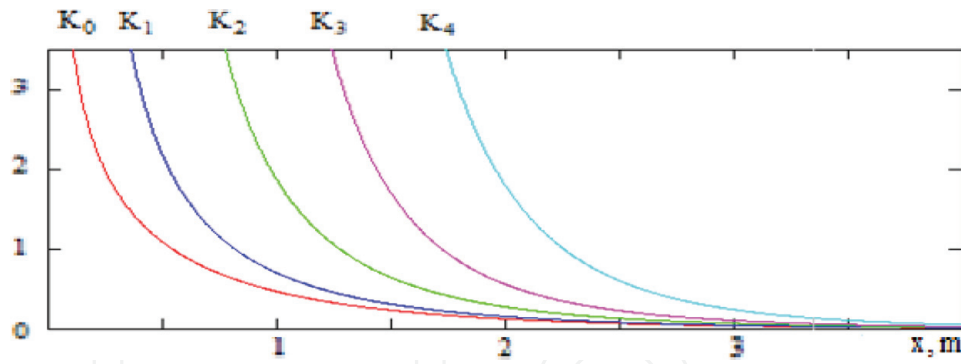
This equation has the solution in the form of Macdonald function:

$$\Phi^*(r) = A_\Phi K_0(m_L r) \quad (14)$$

of zeroth order and unknown argument  $x = m_L r$ , which includes the unknown phase velocity of wave.

The second equation can be written in the form

$$\Psi^*_{,rr} - (1/r)\Psi^*_{,r} - \{ m_T^2 + (1/r^2) \} \Psi^* = 0 \quad m_T = k \sqrt{\left( 1 - (v/v_T)^2 \right)}. \quad (15)$$



**Figure 1.**  
Plots of the first five Macdonald functions.

The corresponding solution under conditions (12) is expressed by the Macdonald function  $K_1\left(r\sqrt{k^2 - k_T^2}\right)$

$$\Psi^*(r) = A_\Psi K_1(m_T r) \quad (16)$$

of the first order and unknown argument  $x = m_T r$ , which includes the unknown wave velocity. The amplitude coefficient  $A_\Psi$  is assumed to be constant and arbitrary.

Note that the Macdonald functions have the property of attenuation with increasing arguments which is shown in **Figure 1**. Therefore, the propagation along the vertical coordinate  $z$  waves (15) and (16) can be considered as the waves with amplitudes  $\Phi^*(r)$ ,  $\Psi^*(r)$ , which attenuate with increasing the radial coordinate  $r$ .

This means that amplitudes can decrease essentially with increasing the distance from the surface of cylindrical cavity. In this sense, the waves (15) and (16) are the surface ones. This forms also the sense of conditions (12). The same conditions are used in the analysis of classical Rayleigh surface wave which propagates along the plane surface of isotropic elastic medium [5–9]. But the Rayleigh wave attenuates as an exponential function when being moved from the free surface, whereas the cylindrical surface Biot's wave attenuates as the Macdonald functions. At that, the arguments in exponential function and Macdonald functions are identical and depend on the wave velocity.

### 2.3 Boundary conditions: equations for unknown wave number

The boundary conditions correspond to the absence of stresses on surface  $r = r_0$

$$\sigma_{rr}(r = r_0, z, t) = 0, \quad \sigma_{rz}(r = r_0, z, t) = 0. \quad (17)$$

The stresses

$$\sigma_{rr} = 2\mu u_{r,r} + \lambda((u_r/r) + u_{r,r} + u_{z,z}), \quad \sigma_{rz} = \mu(u_{r,z} + u_{z,r}) \quad (18)$$

are written through the potentials

$$\sigma_{rr} = (\lambda + 2\mu)(\Phi_{,rr} - \Psi_{,rz}) + \lambda\{(1/r)(\Phi_{,r} - \Psi_{,z}) + \Phi_{,zz} + \Psi_{,rz} + (1/r)\Psi_{,z}\}, \quad (19)$$

$$\sigma_{rz} = \mu[(\Phi_{,rz} - \Psi_{,zz}) + \Phi_{,zr} + \Psi_{,rr} + (1/r)\Psi_{,r} - (1/r^2)\Psi]. \quad (20)$$

Then the boundary conditions (17) can be written in the form.

$$[2\mu(\Phi_{,rr} - \Psi_{,rz}) + \lambda\Delta\Phi]_{r=r_0} = 0, \mu[2(\Phi_{,rz} - \Psi_{,zz}) + \Delta\Psi - (1/r^2)\Psi]_{r=r_0} = 0 \quad (21)$$

In the work [1], Biot has used the expressions.

$$\Delta\Phi - (1/v_L)^2\Phi_{,tt} = 0, \Delta\Psi - (1/r^2)\Psi - (1/v_T)^2\Psi_{,tt} = 0 \text{ and rewrite Eq. (21) in such a way } [(\Phi_{,rr} - \Psi_{,rz}) + (\lambda/2\mu)(1/v_L)^2\Phi_{,tt}]_{r=r_0} = 0, \\ [2(\Phi_{,rz} - \Psi_{,zz}) + (1/v_T)^2\Psi_{,tt}]_{r=r_0} = 0.$$

Then the substitution of solutions (14) and (16) into the boundary conditions (21) gives two homogeneous algebraic equations relative to the unknown constant amplitude coefficients

$$\left[1 - (v/v_L)^2 - \frac{\lambda}{\mu}(v/v_L)^2 \frac{K_0(m_L r_0)}{K_0(m_L r_0) + K_2(m_L r_0)}\right] A_\Phi - \sqrt{(1 - (v/v_L)^2)} A_\Psi = 0, \quad (22)$$

$$2\sqrt{(1 - (v/v_L)^2)} A_\Phi + (2 - (v/v_T)^2) \frac{K_1(m_T r_0)}{K_1(m_L r_0)} A_\Psi = 0. \quad (23)$$

An analysis of these equations that describe the cylindrical surface wave is very similar to the analysis that has been carried out by Rayleigh for the classical wave propagating along the plane surface. Some novelty in analysis of systems (22) and (23) is consideration of the system relative to quantities  $K_1(m_L r_0)A_\Phi$  and  $K_1(m_S r_0)A_\Psi$

$$\left\{ \left(1 - (v/v_L)^2\right) \left[ \frac{K_0(m_L r_0)}{K_1(m_L r_0)} + \frac{1}{m_L r_0} \right] - \frac{\lambda}{2\mu}(v/v_L)^2 \frac{K_0(m_L r_0)}{K_1(m_L r_0)} \right\} K_1(m_L r_0) A_\Phi \\ + \sqrt{(1 - (v/v_S)^2)} \left[ \frac{K_0(m_T r_0)}{K_1(m_T r_0)} + \frac{1}{m_T r_0} \right] K_1(m_T r_0) A_\Psi = 0, \quad (24)$$

$$2\sqrt{(1 - (v/v_L)^2)} K_1(m_L r_0) A_\Phi + (2 - (v/v_T)^2) K_1(m_T r_0) A_\Psi = 0. \quad (25)$$

Solving of systems (24) and (25) gives two results. First, the solution is found accurate within one amplitude factor. Second, an equation for determination of phase velocity of cylindrical surface wave can be obtained in an explicit form.

The work of Biot (1952) has demonstrated some art in handling the Macdonald functions and has written Eq. (24) through only functions of the zeroth and first orders. For that, the known formulas

$$K'_0(x) = -K_1(x), \quad K'_1(x) = -K''_0(x), \\ K''_0(x) + (1/x)K'_0(x) = K_0(x), \quad K''_0(x) = (1/x)K_1(x) + K_0(x) \quad (26)$$

have been used [3]. As a result, the equation for determination of phase velocity of cylindrical wave has the form

$$(2 - (v/v_T)^2) \left\{ \left[ 2 - (v/v_T)^2 \right] \frac{K_0(m_L r_0)}{K_1(m_L r_0)} + \frac{(1 - (v/v_L)^2)}{m_L r_0} \right\} \\ - 4 \sqrt{(1 - (v/v_L)^2)} \sqrt{(1 - (v/v_T)^2)} \left[ \frac{K_0(m_T r_0)}{K_1(m_T r_0)} + \frac{1}{m_T r_0} \right] = 0. \quad (27)$$



Let us write the corresponding equation for the Rayleigh wave [5–9] as

$$4\sqrt{1 - (v/v_L)^2}\sqrt{1 - (v/v_S)^2} - [2 - (v/v_S)^2]^2 = 0. \quad (28)$$

Thus, a presence of Macdonald functions in Eq. (27) complicates essentially an analysis of this equation because according to relations  $m_L = k\sqrt{1 - (v/v_L)^2}$ ,

$m_S = k\sqrt{1 - (v/v_T)^2}$  these functions have the unknown velocity in argument.

If the cavity radius is not small, then the Macdonald functions can be represented by the simple formula  $K_0(r) = K_1(r) = e^{-r}\sqrt{\pi/2r}$ , and Eq. (27) is reduced to the Rayleigh Eq. (28).

Strictly speaking, the analytical part of analysis is ended by obtaining Eq. (27). Further analysis can be continued with the aim of the numerical methods. Biot in [1] has shown some comments and conclusions based on resources of the 1950s.

A possibility of analytical approach is still saved in the problem on existence of the appropriate wave velocity. First of all, Eq. (27) depends on the elastic constants, and this dependence can be shown in the form of dependence on the ratio of known velocities  $(v_L/v_T)$ . If the notation  $(v^2/v_T^2) = z$  is used, then Eq. (27) can be written in the form

$$\begin{aligned} & (2 - z(v_L/v_T)^2) \left\{ (2 - z) \frac{K_0\left(r_0 k \sqrt{1 - z(v_L/v_T)^2}\right)}{K_1\left(r_0 k \sqrt{1 - z(v_L/v_T)^2}\right)} + \frac{(1 - z(v_L/v_T)^2)}{r_0 k \sqrt{1 - z(v_L/v_T)^2}} \right\} \\ & - 4 \sqrt{(1 - z)} \sqrt{1 - z(v_L/v_T)^2} \left[ \frac{K_0(r_0 k \sqrt{1 - z})}{K_1(r_0 k \sqrt{1 - z})} + \frac{1}{r_0 k \sqrt{1 - z}} \right] = 0. \end{aligned} \quad (29)$$

It seems appropriate to recall here the most known ways of proving the existence of velocity of the classical Rayleigh wave. An initial equation is always Eq. (28). Two different notations  $(v^2/v_T^2) = z$  and  $v = (1/\theta)$  are used, which generate two different representations of Eq. (28)

$$z\{z^3 - 8(z - 1)[z - 2(1 - (v_T^2/v_L^2))]\} = 0, \quad (30)$$

$$(2\theta^2 - (1/v_T^2))^2 - 4\theta^2 \sqrt{\theta^2 - (1/v_T^2)} \sqrt{\theta^2 - (1/v_L^2)} = 0. \quad (31)$$

Finding the real root of Eq. (30) is the key step in the analysis of the Rayleigh wave [5–9]. For more than 100 years of analysis of this wave, many methods of proving the existence of real velocity of wave were elaborated.

First of all, the sufficiently useful and exact empirical Viktorov's formula [5].

$$(v/v_T) = \sqrt{z} \approx \frac{0.87 + 1.12v}{1 + v} \quad (v \text{ the Poisson ratio}) \quad (32)$$

should be shown.

Let us show further briefly some phenomenological methods. Note that the restriction on the Rayleigh wave velocity is already obtained from a statement of the problem—it is less of the velocity of plane transverse wave. This restriction can be written in the form  $z < 1$  or  $\theta > (1/c_T)$ .

*Method 1* (graphical method [10, 11]). Eq. (30) is considered as a sum of two summands  $Z_1 + Z_2 = 0$ . The first summand  $Z_1 = z^3$  describes a cubic parabola; the lower branch of which lies in the first quadrant of the plane  $zOZ_1$ . The second summand describes a quadratic parabola  $Z_2 = -8(z - 1)\{z - 2[1 - (c_T^2/c_L^2)]\}$ , which is concave in the direction of coordinate axis  $OZ_2$ . Further the ratio  $(c_T^2/c_L^2) = (\mu/(\lambda + 2\mu))$  can be estimated from below and top  $0 \leq c_T^2/c_L^2 \leq 1/2$  with allowance for the shear modulus  $\mu$  that is positive. These parabolas are intersected on the interval  $(0; 1)$ . More exactly, one of the roots  $z = z_C$  of Eq. (30) can be estimated  $0.764 \leq (z = (c/c_T)^2) \leq 0.912$ . Here, the minimal value corresponds to the case when the parabola is tangent to the abscissa axis, and the maximal value corresponds to the case when the parabola is moved partially into the fourth quadrant. Thus, the velocity of Rayleigh wave is close to the velocity of plane transverse wave, but always less of its  $0.874 \leq (c/c_T) \leq 0.955$ .

*Method 2* (method of finding the interval, on ends of which the equation possesses the different by sign values [2, 11]). This method is based on the analysis of Eq. (30). The value of equation that corresponds to the point  $c_R = c_T$  is positive and equal to 1. The second point is chosen as  $c_R = \varepsilon c_T$ , where  $\varepsilon$  is assumed as the small quantity (this point is close to 0). When this value is substituted into Eq. (30), then expression  $-2\varepsilon^2[1 - (c_T^2/c_L^2)]$  is always negative. Hence, at least one root of equation lies in the interval  $(\varepsilon c_T, c_T)$ .

*Method 3* (another method of finding the interval, on ends of which the equation possesses the different by sign values [5]). This method is based on the analysis of Eq. (31). The right point is chosen as  $\theta = ((1/c_T))$  (similar to method 2). Then Eq. (31) possesses the positive value. The left point corresponds to  $\theta \rightarrow \infty$ . Further an expression (31) is expanded into the power series near the point at infinity. This series starts with the term  $-2\theta^2[(1/c_T^2) - (1/c_L^2)]$ , which is always negative. So this equation possesses in the chosen points the different sign values. Thus, at least one root of the equation lies in the interval  $((1/c_T), \infty)$ .

*Method 4* (method based on assumption relative to the Poisson ratio [7]). This assumption consists in the choice of value of Poisson ratio that is often used in the analysis of seismic waves in Earth's crust  $\nu = \lambda/[2(\lambda + \mu)] = (1/4) \rightarrow \lambda = \mu$ . Then cubic Eq. (31) (the zeroth root  $\theta_1 = 0$  is ignored from a physical considerations) can be solved exactly, and the roots possess the values  $\theta_2 = 4$ ,  $\theta_3 = 2 + (2/\sqrt{3})$ ,  $\theta_4 = 2 - (2/\sqrt{3})$ . Since the condition  $\theta < 1$  has been fulfilled, then the corresponding root is equal to  $\theta_4 = 0.8453$ .

The main conclusion from the shown above methods is that they really allow to establish an existence of real root of Rayleigh equation (the real value of velocity of harmonic Rayleigh wave). They give the positive answer on the question whether the Rayleigh wave exists. In the case of other surface waves including the cylindrical wave under consideration, the experience of the classical Rayleigh wave analysis can be quite useful.

### 3. Cylindrical wave propagating along the surface of the cylindrical cavity in the direction of vertical axis: The case of transversal isotropy of medium

Let us return to the initial statement of problem and consider an infinite medium with cylindrical circular cavity that has the symmetry axis  $Oz$  and radius  $r_0$ . The medium is assumed to be the transversely isotropic elastic one. It is assumed further that the wave is harmonic in time, and attenuating deep into medium wave



propagates in the direction of axis  $Oz$  along the cavity surface. Such a problem can be considered as some generalization of Biot's [1] problem that is solved in the assumption of isotropy of medium on the case of transversal isotropy of medium. Therefore, it seems expedient to recall some facts from the theory of elasticity of transversally isotropic medium.

### 3.1 Some information on transversally isotropic medium

Let us consider the case when  $Ox_3$  is the axis of symmetry and  $Ox_1x_2$  is the plane of isotropy. This symmetry corresponds to the hexagonal crystalline system. The matrix of elastic properties is characterized by 5 independent elastic constants  $C_{11}$ ,  $C_{12}$ ,  $C_{13}$ ,  $C_{33}$ ,  $C_{44}$  and 12 non-zero components [11–13]:

$$C_{IK} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & (1/2)(C_{11} - C_{12}) \end{pmatrix}. \quad (33)$$

Then the constitutive relations  $\sigma \sim \varepsilon$  have the form [12, 14].

$$\begin{aligned} \sigma_{11} &= C_{11kl}\varepsilon_{kl} = C_{11}\varepsilon_{11} + C_{12}\varepsilon_{22} + C_{13}\varepsilon_{33}, \\ \sigma_{22} &= C_{22kl}\varepsilon_{kl} = C_{12}\varepsilon_{11} + C_{22}\varepsilon_{22} + C_{13}\varepsilon_{33}, \\ \sigma_{33} &= C_{33kl}\varepsilon_{kl} = C_{13}\varepsilon_{11} + C_{13}\varepsilon_{22} + C_{33}\varepsilon_{33}, \\ \sigma_{12} &= (C_{11} - C_{12})\varepsilon_{12}, \sigma_{13} = 2C_{44}\varepsilon_{13}, \sigma_{23} = 2C_{44}\varepsilon_{23}, \end{aligned} \quad (34)$$

or in notations  $\sigma \sim u$  [12, 14].

$$\begin{aligned} \sigma_{11} &= C_{11}u_{1,1} + C_{12}u_{2,2} + C_{13}u_{3,3}, \sigma_{22} = C_{12}u_{1,1} + C_{11}u_{2,2} + C_{13}u_{3,3}, \\ \sigma_{33} &= C_{13}u_{1,1} + C_{13}u_{2,2} + C_{11}u_{3,3}, \sigma_{12} = (1/2)(C_{11} - C_{12})(u_{1,2} + u_{2,1}), \\ \sigma_{13} &= C_{44}(u_{1,3} + u_{3,1}), \sigma_{23} = (1/2)C_{44}(u_{2,3} + u_{3,2}). \end{aligned} \quad (35)$$

Also, five independent elastic technical constants are often used.

$E_x = E_y$ ,  $E_x = E_y$ ,  $E_z$ ,  $G_{xy}$ ,  $G_{xz} = G_{yz}$ ,  $\nu_{xy}$ ,  $\nu_{xz} = \nu_{yz}$ ,  $G_{xy} = E_x / (1 + 2\nu_{xy})$ . They are evaluated through  $C_{NM}$  by the following formulas:

The longitudinal Young modulus that corresponds to tension along the symmetry axis  $Oz$

$$E_z = C_{33} - \left[ 2(C_{13})^2 / (C_{11} + C_{12}) \right]. \quad (36)$$

The transverse Young modulus that corresponds to tension in the isotropy plane  $Oxy$

$$E_x = (C_{11} - C_{12}) \left[ (C_{11} + C_{12})C_{33} - 2(C_{13})^2 \right] / \left[ C_{11}C_{33} + (C_{13})^2 \right]. \quad (37)$$

The shear modulus that corresponds to the shear along the isotropy plane  $Oxy$

$$G_{xy} = C_{66} = (1/2)(C_{11} - C_{12}). \quad (38)$$

The shear modulus that corresponds to the shear along the symmetry axis  $Oz$

$$G_{xz} = C_{44}. \quad (39)$$

The Poisson ratio that corresponds to the shear along the symmetry axis  $Oz$  under tension in the isotropy plane and characterizes the shortening in this plane

$$v_{xz} = C_{13}/(C_{11} + C_{12}). \quad (40)$$

Sometimes, the corresponding Lamé moduli are used.

$$\begin{aligned} \lambda_{xy} + 2\mu_{xy} &= C_{11}, \quad \lambda_{xy} = C_{12}, \quad \mu_{xy} = (1/2)(C_{11} - C_{12}) \\ \lambda_{xz} + 2\mu_{xz} &= C_{33}, \quad \lambda_{xz} = C_{13}, \quad \mu_{xz} = C_{44}. \end{aligned} \quad (41)$$

The Poisson ratio (40) is determined by the known formula of isotropic theory  $v_{xz} = \lambda_{xz}/2(\lambda_{xy} + \mu_{xy})$ .

The Poisson ratio  $v_{xy}$  that corresponds to the shear along the symmetry axis  $Oz$  under tension along the isotropy plane is determined also by the classical formula  $v_{xy} = \lambda_{xy}/(\lambda_{xy} + \mu_{xy})$ .

The constants  $C_{11}$ ,  $C_{12}$ ,  $C_{13}$ ,  $C_{33}$ ,  $C_{44}$  are represented through the technical constants  $E$ ,  $E'$ ,  $\nu$ ,  $\nu'$ ,  $G'$  by the formulas.

$$\begin{aligned} C_{11} &= \frac{1 - (\nu')^2(E/E')}{1 - \nu^2 + (1 + 2\nu)(\nu')^2(E/E')} E, \quad C_{12} = \frac{\nu - (\nu')^2(E/E')}{1 - \nu^2 + (1 + 2\nu)(\nu')^2(E/E')} E, \\ C_{13} &= \frac{\nu'(1 - \nu)}{1 - \nu^2 + (1 + 2\nu)(\nu')^2(E/E')} E, \\ C_{33} &= \frac{1 - \nu^2}{1 - \nu^2 + (1 + 2\nu)(\nu')^2(E/E')} E', \quad C_{44} = G'. \end{aligned} \quad (42)$$

Let us comment briefly some features of transversally isotropic materials. They can be divided on the natural and artificial ones. An example of the classical natural material is the rock. An example of the modern material is a family of fibers "Kevlar®." Kevlar® KM2 [15] is characterized by elastic constants  $E_x = 1.34 \text{ GPa}$ ,  $E_z = 84.62 \text{ GPa}$ ,  $G_{xz} = 24.40 \text{ GPa}$ ,  $v_{xy} = 0.24$ ,  $v_{xz} = 0.60$ .

An example of composite materials can be four fibrous composites of micro- and nanolevels, which are described in [15]. The corresponding elastic constants for some variants of these materials are as follows [15]:

10% of carbon microfibers

$$E_x = 3.59 \text{ GPa}, \quad E_z = 25.22 \text{ GPa}, \quad G_{xz} = 1.17 \text{ GPa}, \quad v_{xy} = 0.39, \quad v_{xz} = 0.58.$$

10% of graphite microwhiskers

$$E_x = 3.69 \text{ GPa}, \quad E_z = 102.4 \text{ GPa}, \quad G_{xz} = 1.14 \text{ GPa}, \quad v_{xy} = 0.39, \quad v_{xz} = 0.62.$$

10% of zig-zag carbon nanotubes

$$E_x = 3.70 \text{ GPa}, \quad E_z = 67.21 \text{ GPa}, \quad G_{xz} = 1.14 \text{ GPa}, \quad v_{xy} = 0.39, \quad v_{xz} = 0.62.$$

10% of chiral carbon nanotubes

$$E_x = 3.67 \text{ GPa}, \quad E_z = 126.4 \text{ GPa}, \quad G_{xz} = 1.14 \text{ GPa}, \quad v_{xy} = 0.39, \quad v_{xz} = 0.62.$$

The shown above values are typical for the transversally isotropic materials, and therefore they are briefly commented below.

*Comment 1.* The Young modulus in the direction along the symmetry axis  $E_z$  exceeds essentially the Young modulus in the isotropy plane  $E_x$  (from 6 to 34 times in examples above but can in some cases exceed 100 times).

*Comment 2.* The Lamé moduli  $\lambda_x$  and  $\lambda_z$  repeat the relations between  $E_x$  and  $E_z$ .

*Comment 3.* The Poisson ratio  $\nu_{xz}$  along the symmetry axis  $Oz$  exceeds the classical red line in 0.5 for values of this ratio.

*Comment 4.* The shear moduli  $G_{xy}$  and  $G_{xz}$  are differed quite moderately.

### 3.2 The basic formulas for elastic transversely isotropic medium with axial symmetry

Let us write the basic formulas for the case of symmetry axis  $Oz$ . Then displacements are characterized by two components  $u_r(r, z, t)$ ,  $u_z(r, z, t)$ . The motion equations in stresses have the form.

$$\sigma_{rr,r} + \sigma_{rz,z} + (1/r)(\sigma_{rr} - \sigma_{\varphi\varphi}) = 0, \sigma_{rz,r} + (1/r)\sigma_{\varphi z, \varphi} + \sigma_{zz,z} + (1/r)\sigma_{rz} = 0. \quad (43)$$

The substitution of constitutive equations.

$$\begin{aligned} \sigma_{rr} &= C_{11}u_{r,r} + C_{12}(1/r)u_r + C_{13}u_{z,z}, \sigma_{zz} = C_{13}u_{r,r} + C_{13}(1/r)u_r + C_{33}u_{z,z}, \\ \sigma_{rz} &= (1/2)C_{44}(u_{z,r} + u_{r,z}), \sigma_{\varphi\varphi} = \sigma_{r\varphi} = 0 \end{aligned} \quad (44)$$

into the motion Eqs. (43) gives the motion equations in displacements

$$C_{11}[u_{r,rr} + (1/r)u_{r,r} - (1/r^2)u_r] + C_{44}u_{r,zz} + [C_{13} + C_{44}]u_{z,rz} = \rho u_{r,tt}, \quad (45)$$

$$C_{44}(u_{z,rr} + (1/r)u_{z,r}) + C_{33}u_{z,zz} + [C_{13} + C_{44}](u_{r,rz} + (1/r)u_{r,z}) = \rho u_{z,tt}. \quad (46)$$

Note that Eqs. (45) and (46) include only four constants (the constant  $C_{12}$  is not represented in these equations). This means that displacements and strains are described by only four constants. But the stress state is already described by all five constants.

### 3.3 Three classical ways of introducing the potentials in transversely isotropic elasticity

The basic equations of the theory of transversely isotropic elasticity are frequently analyzed by the use of potentials. The potentials are introduced in theory of elasticity mainly for static problems. Transition to the dynamic problems is associated with complications that are sometimes impassable. Because the problem on waves is related to the dynamic ones, let us show further the possible complications with introducing the potentials.

*Way 1* [12]. It is proposed for the axisymmetric problems of equilibrium (not motion) and is based on introducing one only potential  $\varphi(r, z)$  as the function of stresses. The formulas for stresses include four unknown parameters  $a, b, c, d$ , which is characteristic for representations in the transversely isotropic elasticity.

$$\sigma_{rr} = -\{\varphi_{,rr} + b(1/r)\varphi_{,r} + a\varphi_{,zz}\}_{,z}, \sigma_{\theta\theta} = -\{b\varphi_{,rr} + (1/r)\varphi_{,r} + a\varphi_{,zz}\}_{,z}, \quad (47)$$

$$\sigma_{zz} = -\{c\varphi_{,rr} + c(1/r)\varphi_{,r} + d\varphi_{,zz}\}_{,z}, \sigma_{rz} = -\{\varphi_{,rr} + (1/r)\varphi_{,r} + a\varphi_{,zz}\}_{,r}. \quad (48)$$

The next step consists in substitution of formulas (47) and (48) into the first equation of equilibrium and the equations that are obtained from the Cauchy relations and formulas for the strain tensor. This permits to determine the unknown parameters through the elastic constants represented in the equilibrium equations. Further, the second equation of equilibrium gives the biharmonic equation for finding the potentials

$$\Delta_{s1}\Delta_{s2}\varphi = 0, \quad (49)$$

where  $\Delta_{sN}\varphi = \varphi_{,rr} + (1/r)\varphi_{,r} + \left(1/(s_N)^2\right)\varphi_{,zz}$   $N = 1; 2$  are some “complicated” copies of classical expressions  $\Delta\varphi = \varphi_{,rr} + (1/r)\varphi_{,r} + \varphi_{,zz}$  associated with the Laplace operator. Two constants  $s_N$  are determined from the algebraic equations

$$s^4 - [(a+c)/d]s^2 + (1/d) = 0, \quad (50)$$

$$s_{1,3} = \pm \sqrt{\frac{a+c + \sqrt{(a+c)^2 - 4d}}{2d}}, \quad s_{2,4} = \pm \sqrt{\frac{a+c - \sqrt{(a+c)^2 - 4d}}{2d}}.$$

Thus, a transition from the isotropic case to the transversally isotropic one complicates the procedure of solving the static problems. Here a necessity of solving the classical biharmonic equation is changed on necessity of solving some generalization of this equation in the form (49).

Way 2 [12, 16]. This way is also proposed for the static problems. Here, two potentials are introduced which are linked immediately with displacements

$$u_r = \phi_{1,r} + \phi_{2,r}, \quad u_z = k_1\phi_{1,z} + k_2\phi_{2,z}. \quad (51)$$

A substitution of representations (51) into equations of equilibrium (45), (46)

$$C_{11}[u_{r,rr} + (1/r)u_{r,r} - (1/r^2)u_r] + C_{44}u_{r,zz} + [C_{13} + C_{44}]u_{z,rz} = 0,$$

$$C_{44}(u_{z,rr} + (1/r)u_{z,r}) + C_{33}u_{z,zz} + [C_{13} + C_{44}](u_{r,rz} + (1/r)u_{r,z}) = 0$$

allows to determine the unknown constants  $k_1, k_2$ . An idea consists in that both equations must be transformed in identical equations relative to the potentials by comparing some coefficients

$$\frac{k_{1(2)}(C_{13} + C_{44}) + C_{44}}{C_{11}} = \frac{kC_{33}}{k_{1(2)}C_{44} + (C_{13} + C_{44})} = V.$$

This expression gives the quadratic equation for  $k_{1(2)}$  and  $V$

$$V^2 + \frac{C_{13}(2C_{44} + C_{33}) - C_{11}C_{33}}{C_{11}C_{44}}V + \frac{C_{33}}{C_{11}} = 0. \quad (52)$$

Note that the simple link  $V_N = (1/s_N)$  exists between constants  $V_N$  and  $s_N$ , which makes the ways 1 and 2 very similar. Then the potentials fulfill the equations

$$\Delta_{rzn}\varphi_N = \varphi_{N,rr} + (1/r)\varphi_{N,r} + \left(1/(V_N)^2\right)\varphi_{N,zz}. \quad (53)$$

The stresses are expressed through new potentials in such a way

$$\sigma_{rr} = -(C_{11} - C_{12})(1/r)(\phi_{1,rr} + \phi_{2,rr}) - C_{44}((1+k_1)\phi_{1,zz} + (1+k_2)\phi_{2,zz}),$$

$$\sigma_{\theta\theta} = -(C_{11} - C_{12})(1/r)(\phi_{1,rr} + \phi_{2,rr}) - ((C_{13}k_1 - C_{12}V_1)\phi_{1,zz} + (C_{13}k_2 - C_{12}V_2)\phi_{2,zz}),$$

$$\sigma_{zz} = ((C_{33}k_1 - C_{13}V_1)\phi_{1,zz} + (C_{33}k_2 - C_{13}V_2)\phi_{2,zz}),$$

$$\sigma_{rz} = C_{44}((1+k_1)\phi_{1,rz} + (1+k_2)\phi_{2,rz}). \quad (54)$$

Way 3 [1, 16]. This way is proposed for equations of motion, but only for the isotropic theory of elasticity. It can be used for the static problems of transversely

isotropic theory of elasticity. The initial equations here are the equations of motion (43) without inertial summands

$$C_{11}[u_{r,rr} + (1/r)u_{r,r} - (1/r^2)u_r] + C_{44}u_{r,zz} + [C_{13} + C_{44}]u_{z,rz} = 0, \quad (55)$$

$$C_{44}(u_{z,rr} + (1/r)u_{z,r}) + C_{33}u_{z,zz} + [C_{13} + C_{44}](u_{r,rz} + (1/r)u_{r,z}) = 0. \quad (56)$$

The potentials are introduced like (51), but the representations are complicated by necessity of introducing two new unknown parameters:

$$u_r = \Phi_{,r} - \Psi_{,z}, \quad u_z = n\Phi_{,z} + m\Psi_{,r} + m(1/r)\Psi, \quad (57)$$

A substitution of representations (57) into equations of motion (45) and (46) gives equations relative to the potentials. Eq. (45) gives two equations:

$$\Phi_{,rr} + (1/r)\Phi_{,r} + \frac{C_{44} + n(C_{13} + C_{44})}{C_{11}}\Phi_{,zz} = 0, \quad (58)$$

$$\Psi_{,rr} + (1/r)\Psi_{,r} - (1/r^2)\Psi + \frac{C_{44}}{C_{11} - m(C_{13} + C_{44})}\Psi_{,zz} = 0, \quad (59)$$

whereas Eq. (46) gives three equations:

$$\Phi_{,rr} + (1/r)\Phi_{,r} + \frac{nC_{33}}{nC_{44} + (C_{13} + C_{44})}\Phi_{,zz} = 0, \quad (60)$$

$$\Psi_{,rrz} + (1/r)\Psi_{,rz} - (1/r)^2\Psi_{,z} + \frac{C_{33}m - (C_{13} + C_{44})}{C_{44}m}\Psi_{,zzz} = 0, \quad (61)$$

$$\Psi_{,rr} + (1/r)\Psi_{,r} - (1/r)^2\Psi + \frac{C_{33}m - (C_{13} + C_{44})}{C_{44}m}\Psi_{,zz} = 0. \quad (62)$$

The last two equations are identical. Eqs. (58) and (60) and (59) and (62) have to be identical. This means that the coefficients in these equations have to be identical. As a result, two equations can be obtained for the determination of unknown constants  $n, m$ .

$$\frac{C_{44} + n(C_{13} + C_{44})}{C_{11}} = \frac{nC_{33}}{nC_{44} + (C_{13} + C_{44})} \rightarrow \quad (63)$$

$$n^2 - n \frac{C_{11}C_{33} - (C_{44})^2 - (C_{13} + C_{44})^2}{C_{44}(C_{13} + C_{44})} + 1 = 0,$$

$$n_{1,2} = \frac{C_{11}C_{33} - (C_{44})^2 - (C_{13} + C_{44})^2}{2C_{44}(C_{13} + C_{44})} \quad (64)$$

$$\times \left( 1 \pm \sqrt{1 - 4 \left[ \frac{C_{44}(C_{13} + C_{44})}{C_{11}C_{33} - (C_{44})^2 - (C_{13} + C_{44})^2} \right]^2} \right),$$

$$\frac{C_{44}}{C_{11} - m(C_{13} + C_{44})} = \frac{C_{33}m - (C_{13} + C_{44})}{C_{44}m} \rightarrow m^2 \quad (65)$$

$$+ m \left[ \frac{(C_{44})^2 + C_{11}C_{33} + (C_{13} + C_{44})^2}{C_{33}(C_{13} + C_{44})} \right] + \frac{C_{11}}{C_{33}} = 0.$$



$$m_{1,2} = -\frac{(C_{44})^2 + C_{11}C_{33} + (C_{13} + C_{44})^2}{2C_{33}(C_{13} + C_{44})} \times \left\{ 1 \pm \sqrt{1 - 4\frac{C_{11}}{C_{33}} \left[ \frac{C_{33}(C_{13} + C_{44})}{(C_{44})^2 + C_{11}C_{33} + (C_{13} + C_{44})^2} \right]^2} \right\}. \quad (66)$$

The unknown potentials  $\Phi(r, z)$  and  $\Psi(r, z)$  have to be determined from the simple Eqs. (63) and (65) which are the classical Bessel equations of orders 0 and 1 and arguments depending on some rational combination of elastic constants.

Thus, three ways of introduction of potentials in the static problems of transversely isotropic theory of elasticity are shown. The different attempts to transfer these ways into the dynamic problems meet some troubles—the presence of inertial summands generates new additional conditions for the unknown constants in representations of potentials. Introducing the new constants does not help—the number of conditions is still more than the number of all constants.

### 3.4 Solving the problem on the propagation in the direction of vertical axis surface cylindrical wave for the case of transversal isotropy of medium

Consider now equations of motion (45) and (46) and introduce the potentials by the formula (57). A substitution of formula (57) into equations of motion gives five equations relative to the potentials. Eq. (57) gives two equations:

$$\Phi_{,rr} + (1/r)\Phi_{,r} + \frac{C_{44} + n(C_{13} + C_{44})}{C_{11}}\Phi_{,zz} = \frac{\rho}{C_{11}}\Phi_{,tt}, \quad (67)$$

$$\Psi_{,rr} + (1/r)\Psi_{,r} - (1/r^2)\Psi + \frac{C_{44}}{C_{11} - m(C_{13} + C_{44})}\Psi_{,zz} = \frac{\rho}{C_{11} - m(C_{13} + C_{44})}\Psi_{,tt}. \quad (68)$$

Eq. (46) gives three equations:

$$\Phi_{,rr} + (1/r)\Phi_{,r} + \frac{nC_{33}}{nC_{44} + (C_{13} + C_{44})}\Phi_{,zz} = \frac{n\rho}{nC_{44} + (C_{13} + C_{44})}\Phi_{,tt}, \quad (69)$$

$$\Psi_{,rrz} + (1/r)\Psi_{,rz} - (1/r)^2\Psi_{,z} + \frac{C_{33}m - (C_{13} + C_{44})}{C_{44}m}\Psi_{,zzz} = \frac{\rho}{C_{44}}\Psi_{,ztt}, \quad (70)$$

$$\Psi_{,rr} + (1/r)\Psi_{,r} - (1/r)^2\Psi + \frac{C_{33}m - (C_{13} + C_{44})}{C_{44}m}\Psi_{,zz} = \frac{\rho}{C_{44}}\Psi_{,tt}. \quad (71)$$

Two last equations are identical. Also the equations for potential  $\Phi$  must be identical as well as the equations for potential  $\Psi$  must be identical. Let us assume additionally that the problem in hand considering the solution in the form of harmonic in time cylindrical wave with unknown wave number  $k$  and known frequency  $\omega$ :

$$\Phi(r, z, t) = \widehat{\Phi}(r)e^{i(kz - \omega t)}, \quad \Psi(r, z, t) = \widehat{\Psi}(r)e^{i(kz - \omega t)}. \quad (72)$$

Note that characterization of an attenuation of wave depth down functions  $\widehat{\Phi}(r)$ ,  $\widehat{\Psi}(r)$  is unknown. They must be found from equations, which are obtained by substitution of representations (72) into Eqs. (67) and (71):

$$\widehat{\Phi}_{,rr} + (1/r)\widehat{\Phi}_{,r} - \left[ \frac{C_{44} + n(C_{13} + C_{44})}{C_{11}} k^2 - k_{L(11)}^2 \right] \widehat{\Phi} = 0, \quad (73)$$

$$k_{L(11)} = (\omega/v_{L(11)}), v_{L(11)} = \sqrt{C_{11}/\rho},$$

$$\widehat{\Phi}_{,rr} + (1/r)\widehat{\Phi}_{,r} - \frac{n}{nC_{44} + (C_{13} + C_{44})} (C_{33}k^2 - C_{11}k_{L(11)}^2) \widehat{\Phi} = 0, \quad (74)$$

$$\widehat{\Psi}_{,rr} + (1/r)\widehat{\Psi}_{,r} - (1/r^2) \widehat{\Psi} - \frac{C_{44}}{C_{11} - m(C_{13} + C_{44})} (k^2 - k_{T(44)}^2) \widehat{\Psi} = 0, \quad (75)$$

$$k_{T(44)} = (\omega/v_{L(44)}), v_{L(44)} = \sqrt{C_{44}/\rho},$$

$$\widehat{\Psi}_{,rr} + (1/r)\widehat{\Psi}_{,r} - (1/r)^2 \widehat{\Psi} - \left[ \frac{C_{33}m - (C_{13} + C_{44})}{C_{44}m} k^2 - k_{T(44)}^2 \right] \widehat{\Psi} = 0. \quad (76)$$

As a result, two equations can be obtained that permit to determine the constants  $n, m$

$$n^2 - 2N_1n + N_2 = 0, m^2 + 2M_1m + M_2 = 0, \quad (77)$$

$$N_{\pm}(M_{\pm}) = N_1(M_1) \pm \sqrt{[N_1(M_1)]^2 - N_2(M_2)}, \quad (78)$$

$$2N_1 = \frac{[C_{11}C_{33} - (C_{13} + C_{44})^2]k^2 - C_{11}[C_{11} - C_{44}]k_{L(11)}^2 - (C_{44})^2}{C_{44}(C_{13} + C_{44})k^2}, \quad (79)$$

$$N_2 = \frac{C_{44} - C_{11}k_{L(11)}^2}{C_{44}k^2} = 0$$

$$2M_1 = \frac{[(C_{44})^2 - C_{11}C_{33} - (C_{13} + C_{44})^2]k^2 - [(C_{44})^2 - C_{11}C_{44}]k_{T(44)}^2}{(C_{13} + C_{44})(C_{33}k^2 - C_{44}k_{T(44)}^2)}, \quad (80)$$

$$M_2 = \frac{C_{11}}{(C_{33}k^2 - C_{44}k_{T(44)}^2)} k^2.$$

Note that restriction on the kind of solution (it has to be a wave) allows to unite two different conditions into one—conditions for equaling coefficients in summands with the second derivative by time  $t$  and vertical coordinate  $z$ . In this case, the number of unknown constants coincides with the number of conditions which are necessary for the determination of potentials. As a result, the wave attenuation-transformed potentials can be determined from the equations of Bessel type:

$$\widehat{\Phi}_{,rr} + (1/r)\widehat{\Phi}_{,r} - M_{L(11)}^2 \widehat{\Phi} = 0, \quad M_{L(11)} = \sqrt{\frac{C_{44} + n(C_{13} + C_{44})}{C_{11}} k^2 - k_{L(11)}^2}, \quad (81)$$

$$\widehat{\Psi}_{,rr} + (1/r)\widehat{\Psi}_{,r} - \left[ (1/r^2) + M_{T(44)}^2 \right] \widehat{\Psi} = 0, \quad (82)$$

$$M_{T(44)} = \sqrt{\frac{C_{44}}{C_{11} - m(C_{13} + C_{44})} (k^2 - k_{T(44)}^2)},$$

A success in the determination of transformed potentials is accompanied by a complication of conditions which provide the wave attenuation. They have the form.

$$\frac{C_{44} + n(C_{13} + C_{44})}{C_{11}}k^2 - k_{L(11)}^2 > 0, \quad \frac{C_{33}m - (C_{13} + C_{44})}{C_{44}m}k^2 - k_{T(44)}^2 > 0. \quad (83)$$

Let us recall that the similar conditions for the case of isotropic medium  $k^2 - k_L^2 > 0$ ,  $k^2 - k_T^2 > 0$  are slightly simpler and coincide with the corresponding conditions of classical Rayleigh surface wave [5–9, 17]. A complexity of conditions (83) is increased by the complex form of dependence of constants  $n$ ,  $m$  on the wave number  $k$ .

If the conditions (83) are fulfilled, then the solution of wave equations for potentials can be written in the form.

$$\widehat{\Phi}(r) = \widehat{A}_\Phi K_0(M_{L(11)}r), \quad \widehat{\Psi}(r) = \widehat{A}_\Psi K_1(M_{T(44)}r). \quad (84)$$

With allowance for formulas (84), the representations of potentials becomes more definite

$$\Phi(r, z, t) = \widehat{A}_\Phi K_0(M_{L(11)}r) e^{i(kz - \omega t)}, \quad \Psi(r, z, t) = \widehat{A}_\Psi K_1(M_{T(44)}r) e^{i(kz - \omega t)}. \quad (85)$$

The formula (85) completes the first analytical part of solving the problem on cylindrical surface wave.

### 3.5 Boundary conditions: equations for the unknown wave number

This part of analysis can be treated as the second analytical part. The boundary conditions have the form identical for all kinds of symmetry of properties. That is, they have the form (17) or (21). The formulas for stresses depend already on the symmetry of medium. The expressions for stresses through the potential reflect the features of introducing the potentials. In this case, they have the form

$$\sigma_{rr} = (\lambda + 2\mu)(\Phi_{,rr} - \Psi_{,rz}) + \lambda \left\{ (1/r)(\Phi_{,r} - \Psi_{,z}) + n\Phi_{,zz} + m\Psi_{,rz} + m(1/r)\Psi_{,z} \right\}, \quad (86)$$

$$\sigma_{rz} = \mu [(\Phi_{,rz} - \Psi_{,zz}) + n\Phi_{,zr} + m\Psi_{,rr} + m(1/r)\Psi_{,r} - m(1/r^2)\Psi]. \quad (87)$$

Further, the representations (86) and (87) should be substituted into the boundary conditions, and the formulas on differentiation of Macdonald functions [3] should be taken into account:

$$[dK_0(M_{L(11)}rx)/dr] = -M_{L(11)}K_1(M_{L(11)}rx),$$

$$[d^2K_0(M_{L(11)}r)/dr^2] = M_{L(11)}(1/r)K_1(M_{L(11)}r) + (M_{L(11)})^2K_0(M_{L(11)}r),$$

$$[dK_1(M_{T(44)}r)/dr] = -(1/r)K_1(M_{T(44)}r) - M_{T(44)}K_0(M_{T(44)}r).$$

Then the boundary conditions are transformed into the algebraic equations relative to quantities  $K_1(M_{L(11)}r_o)\widehat{A}_\Phi$ ,  $K_1(M_{T(44)}r_o)\widehat{A}_\Psi$

$$\left[ M_{L(11)} \frac{1}{r_o} + \frac{v_L^2}{v_T^2} \left( \frac{(M_{L(11)})^2 - v_L^2 - v_T^2}{v_L^2} n k^2 \right) \frac{K_0(M_{L(11)} r_o)}{K_1(M_{L(11)} r_o)} \right] \widehat{A}_\Phi K_1(M_{L(11)} r_o) - i k \frac{v_L^2 - v_T^2}{v_T^2} \times \left[ \left( 2(1-m) + \frac{v_T^2}{v_L^2 - v_T^2} \right) \frac{1}{r_o} + \left( (1-m) + \frac{v_T^2}{v_L^2 - v_T^2} \right) M_{T(44)} \frac{K_0(M_{T(44)} r_o)}{K_1(M_{T(44)} r_o)} \right] \widehat{A}_\Psi K_1(M_{T(44)} r_o) = 0, \quad (88)$$

$$(1+n) i k \frac{K_0(M_{L(11)} r_o)}{K_1(M_{L(11)} r_o)} K_1(M_{L(11)} r_o) \widehat{A}_\Phi + \left[ m(M_{T(44)})^2 + k^2 \right] K_1(M_{T(44)} r_o) \widehat{A}_\Psi = 0. \quad (89)$$

When the determinant of linear homogeneous system of Eqs. (88) and (89) is equaled to zero, then the equations for the unknown wave number can be obtained:

$$(1+n) k^2 \frac{v_L^2 - v_T^2}{v_T^2} \frac{K_0(M_{L(11)} r_o)}{K_1(M_{L(11)} r_o)} \left[ \left( 2(1-m) + \frac{v_T^2}{v_L^2 - v_T^2} \right) (1/r_o) + \left( (1-m) + \frac{v_T^2}{v_L^2 - v_T^2} \right) M_{T(44)} \frac{K_0(M_{T(44)} r_o)}{K_1(M_{T(44)} r_o)} \right] - \left[ m(M_{T(44)})^2 + k^2 \right] \left[ M_{L(11)} \frac{1}{r_o} + \frac{v_L^2}{v_T^2} \left( \frac{(M_{L(11)})^2 - v_L^2 - v_T^2}{v_L^2} n k^2 \right) \frac{K_0(M_{L(11)} r_o)}{K_1(M_{L(11)} r_o)} \right] = 0. \quad (90)$$

Note that the sufficiently complex expression relative to the wave number is hidden coefficients  $M_{L(11)}$ ,  $M_{T(44)}$  of Macdonald's functions  $\frac{K_0(M_{L(11)} r_o)}{K_1(M_{L(11)} r_o)}$ ,  $\frac{K_0(M_{T(44)} r_o)}{K_1(M_{T(44)} r_o)}$ . Therefore, the analytical part of analysis is finished on these formulas. Further, the numerical approaches have to be utilized.

Note also that the simple and convenient condition from analysis of classical surface Rayleigh wave [6–10, 17], when the wave number depends only on ratio  $(v_L^2/v_T^2)$ , does not exist in the analysis of cylindrical surface wave. Here, the parameters  $M_{L(11)}$ ,  $M_{T(44)}$  depend on the complicated form on all elastic constants. Of course, the Macdonald functions can be represented approximately through their arguments. But only the numerical methods can give the final result—the value of wave number or phase velocity.

#### 4. Solving the problem on propagating in the direction of symmetry axis surface wave within the framework of linearized theory of elasticity with allowance for initial stresses

Note that analysis of cylindrical surface wave in isotropic medium was first carried out by Biot [1] in 1952 and the transversally isotropic medium with initial stresses was first carried out by Guz et al. in 1974 [18].

Let us show below an analysis of the problem in hand that is carried out in Subchapter “Longitudinal Waves” of Chapter 4 “Waves in Cylindrical Media” of volume 2 of edition [19]. Here, the cylinder of circular cross-section is considered, and the longitudinal wave is defined as the wave propagating in the direction of cylinder axis  $Oy_3$ . The problem is assumed to be axisymmetric and is described within the framework of linearized theory of elasticity for bodies with initial stresses. The cylindrical coordinates  $(r', \theta, y_3)$  are introduced, and displacements are taken in the form

$$u_{r'} = u_{r'}(r', y_3, t), \quad u_\theta = 0, \quad u_{y_3} = u_{y_3}(r', y_3, t) \quad (91)$$

The medium is assumed isotropic or transversally isotropic. The main relations for transversal isotropy are described by independent constant

$$\omega_{1111}, \omega_{1122}, \omega_{1133}, \omega_{1221}, \omega_{1313}, \omega_{1331}, \omega_{3113}, \omega_{3333}. \quad (92)$$

Note that as shown in (92), eight constants are necessary in the linearized theory, but in the framework of linear theory, they have the form (33), and their number is five.

Further, the general solutions of basic equations in displacements are utilized. These equations have the form (3.174) [19]

$$\omega_{lm\alpha\beta}(\partial^2 u_\alpha / \partial x_k \partial x_\beta) = \rho \delta_{m\alpha}(\partial^2 u_\alpha / \partial \tau^2) \quad (93)$$

where only eight independent constants (92) must be taken into account.

The corresponding equations of linear theory of elasticity for the case of transversally isotropic medium without of initial stresses are written above as Eqs. (45) and (46).

The general solutions for the case of axial symmetry are expressed through one potential in the form (4.13) [19]

$$u_{r'} = -(\partial^2 / \partial r' \partial y_3) X', \quad (94)$$

$$u_3 = (\omega'_{1111} + \omega'_{1313})^{-1} [\omega'_{1111} \Delta'_1 + \omega'_{3113} (\partial^2 / \partial y_3^2) - \rho' (\partial^2 / \partial \tau^2)] X',$$

$$\Delta' = (\partial^2 / \partial r'^2) + (1/r')(\partial / \partial r').$$

Note that in Section 3 of this chapter, two potentials  $\Phi, \Psi$  are introduced by formula (57), which corresponds and generalizes the procedure used in Biot's analysis [1].

The longitudinal harmonic wave is described analytically through the potential in the form (101) [19]

$$X'(r', y_3, \tau) = X'_{(1)}(r') e^{i(ky_3 - \omega\tau)}, \quad (95)$$

where the unknown amplitude  $X'_{(1)}(r')$  has to be determined by substitution of solution (4.13) [19] into the second Eq. (3.362) [19] (for potential  $X'$ ). This gives Eq. (4.16) [19]:

$$\left\{ (\omega'_{1111} \omega'_{1331}) (\Delta'_1 - k^2 \xi_2^2) (\Delta'_1 - k^2 \xi_3^2) - k^2 \rho' C_{cp}^2 [(\omega'_{1111} + \omega'_{1331}) \Delta'_1 - k^2 (\omega'_{1111} + \omega'_{3113})] + \rho'^2 C_{cp}^2 \right\} X'_{(1)} = 0, \quad (96)$$



$$C_{cp} = \omega/k, \xi'_{2,3} = c' \pm \sqrt{c'^2 - (\omega'_{3333}\omega'_{3113}/\omega'_{1111}\omega'_{1331})},$$

$$c' = (1/2) \left[ \begin{array}{l} (\omega'_{3333}/\omega'_{1331}) + (\omega'_{3113}/\omega'_{1111}) \\ - \left( (\omega'_{1111} + \omega'_{1331})^2 / \omega'_{1111}\omega'_{1331} \right) \end{array} \right],$$

which further is written in the form

$$(\Delta'_1 - \zeta'^2_2)(\Delta'_1 - \zeta'^2_3) = 0 \quad (97)$$

The unknown quantities  $\zeta'_{2,3}$  must be found from the linear algebraic equation of the fourth degree (4.20) [19].

$$\begin{aligned} & \omega'_{1111}\omega'_{1331}(\zeta')^4 + k^4(\rho'C_{cp}^2 - \omega'_{3333})(\rho'C_{cp}^2 - \omega'_{3113}) \\ & + k^2 \left[ \begin{array}{l} \omega'_{1111}(\rho'C_{cp}^2 - \omega'_{3333}) + \omega'_{1331}(\rho'C_{cp}^2 - \omega'_{3113}) \\ + (\omega'_{1111} + \omega'_{3113})^2 \end{array} \right] (\zeta')^2 = 0, \end{aligned} \quad (98)$$

The solution (95) describes the surface wave, if amplitude  $X'_{(1)}(r')$  attenuates with increasing the radius. This is provided by the condition that quantities  $\zeta'_{2,3}$  is unequal and pure imaginary. Then the potential gains the form (4.22) [19].

$$X'_{(1)}(r') = B_{10}J_0(|\zeta'_2|r') + B_{20}K_0(|\zeta'_2|r') + B_{30}J_0(|\zeta'_3|r') + B_{40}K_0(|\zeta'_3|r'), \quad (99)$$

The shown part of analysis from introducing the potential by formula (94) to representation of solution by formula (99) inclusive can be compared with analogous part of analysis from Section 3 of this chapter (from introducing the potentials by formula (57) to the solution in the form of (85)). It is easy to see a difference in representations (99) and (85): formula (99) uses the Bessel functions and in particular the Macdonald function of zero index, whereas formula (85) uses (like the Biot's solution (14)) the Macdonald functions (16) of the zero and first indexes.

The next part of analysis of cylindrical wave consists in substitution of solution into boundary conditions of the form (99) [19]

$$Q'_{r'r'} = 0, \quad Q'_{r'3} = 0 \text{ when } r' = R'_1, R'_2. \quad (100)$$

The case of oscillatory behavior of wave in the direction of radius is considered with pointing that the case of surface wave is the same type. A substitution of solution (99) into conditions (4.79) [19] gives the dependence of velocity of surface wave or its wave number on frequency—a dispersion equation in the form of determinant of the fourth order in the form (4.26) [19].

$$\det|\alpha_{ij}| \equiv \Delta(\omega, k) = 0; \quad i, j = 1, 2, 3, 4. \quad (101)$$

This finishes the analytical part of analysis shown in [19]. It corresponds to the part of Section 3.5 of this chapter, where the explicit form of dispersive equations is proposed in the form (90) that includes the Macdonald functions of the zero and first orders which represent some generalization of dispersion Eq. (27) obtained by Biot.

## 5. Conclusions

This chapter proposes three fragments of analytical analysis of the cylindrical surface wave propagating in the vertical direction of circular cylindrical cavity. The


first fragment shows the analytical part of pioneer work of Biot. It represents the classicism of mathematical procedures and physical comments of Biot. Properly speaking, the clear and understandable Rayleigh's scheme is saved, but it is complemented by some findings reflecting the features of cylindrical waves. Two next fragments show the more late development of the Biot's problem. They are different by influence of the Biot's procedure. The approach shown in Section 3 is more close to the Biot's analytical scheme, whereas Section 4 proposes as an independent scheme that is more close to the Rayleigh scheme. Nevertheless, all fragments testify the mathematical complexity in solving the problem on the cylindrical surface waves. Thus, revisiting the old Biot's problem shows that it still generates new scientific and practical problems.

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