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Chapter

Distinctive Characteristics of Cosserat Plate Free Vibrations

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Abstract

In this chapter, we present the theoretical analysis of the distinctive characteristics of Cosserat plate vibrations. This analysis is based on the dynamic model of the Cosserat plates, which we developed as an extension of the Reissner plate theory. Primarily, we describe the validation of the model, which is based on the comparison with three-dimensional exact solutions. We present the results of the computer simulations, which allow us to identify different characteristics of the plate vibrations. Particularly, we illustrate and discuss the detection and the classification of the additional high resonance frequencies of a plate depending on the shape and orientation of microelements incorporated into the Cosserat plates.

Keywords: variational principle, Cosserat plate vibrations, frequencies of micro-vibrations

1. Introduction

The theory of asymmetric elasticity introduced in 1909 by the Cosserat brothers [1] gave rise to a variety of Cosserat plate theories. In 1960s, Green and Naghdi specialized their general theory of Cosserat surface to obtain the linear Cosserat plate [2], while independently Eringen proposed a complete theory of plates in the framework of Cosserat elasticity [3]. Numerous plate theories were formulated afterwards; for the review of the latest developments in the area of Cosserat plates we recommend to turn to [4].

The first theory of Cosserat plates based on the Reissner plate theory was developed in [5] and its finite element modeling is provided in [6]. The parametric theory of Cosserat plate, presented by the authors in [7], includes some additional assumptions leading to the introduction of the splitting parameter. This provided the highest level of approximation to the original three-dimensional problem. The theory provides the equilibrium equations and constitutive relations, and the optimal value of the minimization of the elastic energy of the Cosserat plate. The paper [7] also provides the analytical solutions of the presented plate theory and the three-dimensional Cosserat elasticity for simply supported rectangular plate. The comparison of these solutions showed that the precision of the developed Cosserat plate theory is similar to the precision of the classical plate theory developed by Reissner [8, 9].

The numerical modeling of bending of simply supported rectangular plates is given in [10]. We developed the Cosserat plate field equations and a rigorous formula for the optimal value of the splitting parameter. The solution of the

Cosserat plate was shown to converge to the Reissner plate as the elastic asymmetric parameters tend to zero. The Cosserat plate theory demonstrates the agreement with the size effect, confirming that the plates of smaller thickness are more rigid than is expected from the Reissner model. The modeling of Cosserat plates with simply supported rectangular holes is also provided. The finite element analysis of the perforated Cosserat plates is given in [11].

The extension of the static model of Cosserat elastic plates to the dynamic problems is presented in [12]. The computations predict a new kind of natural frequencies associated with the material microstructure and were shown to be compatible with the size effect principle reported in [10] for the Cosserat plate bending.

This chapter represents an extension of the paper [12] for different shapes and orientations of micro-elements incorporated into the Cosserat plates. It is based on the generalized variational principle for elastodynamics and includes a non-diagonal rotatory inertia tensor. The numerical computations of the plate free vibrations showed the existence of some additional high frequencies of micro-vibrations depending on the orientation of micro-elements. The comparison with three-dimensional Cosserat elastodynamics shows a high agreement with the exact values of the eigenvalue frequencies.

2. Cosserat linear elastodynamics

2.1 Fundamental equations

The Cosserat linear elasticity balance laws are

$$\sigma_{ji,j} = \frac{\partial p_i}{\partial t}, \tag{1}$$

$$\varepsilon_{ijk}\sigma_{jk} + \mu_{ji,j} = \frac{\partial q_i}{\partial t}, \tag{2}$$

where the σ_{ji} is the stress tensor, μ_{ji} the couple stress tensor, $p_i = \rho \frac{\partial u_i}{\partial t}$ and $q_i = J_{ji} \frac{\partial \phi_j}{\partial t}$ are the linear and angular momenta, ρ and J_{ji} are the material density and the rotatory inertia characteristics, ε_{ijk} is the Levi-Civita tensor.

We will also consider the constitutive equations as in [13]:

$$\sigma_{ji} = (\mu + \alpha)\gamma_{ji} + (\mu - \alpha)\gamma_{ij} + \lambda\gamma_{kk}\delta_{ij}, \tag{3}$$

$$\mu_{ji} = (\gamma + \varepsilon)\chi_{ji} + (\gamma - \varepsilon)\chi_{ij} + \beta\chi_{kk}\delta_{ij}, \tag{4}$$

and the kinematic relations in the form

$$\gamma_{ji} = u_{i,j} + \varepsilon_{ijk}\phi_k \text{ and } \chi_{ji} = \phi_{i,j}, \tag{5}$$

Here u_i and ϕ_i represent the displacement and microrotation vectors, γ_{ji} and χ_{ji} represent the strain and bend-twist tensors, μ, λ are the Lamé parameters and α, β, γ, ε are the Cosserat elasticity parameters.

The constitutive Eqs. (3)–(4) can be written in the reverse form [5].

$$\gamma_{ji} = (\mu' + \alpha')\sigma_{ji} + (\mu' - \alpha')\sigma_{ij} + \lambda'\sigma_{kk}, \tag{6}$$

$$\chi_{ji} = (\gamma' + \varepsilon')\mu_{ji} + (\gamma' - \varepsilon')\mu_{ij} + \beta'\mu_{kk}, \tag{7}$$

where $\mu' = \frac{1}{4\mu}$, $\alpha' = \frac{1}{4\alpha}$, $\gamma' = \frac{1}{4\gamma}$, $\varepsilon = \frac{1}{4\varepsilon}$, $\lambda' = \frac{-\lambda}{6\mu\left(\lambda+\frac{2\mu}{3}\right)}$ and $\beta' = \frac{-\beta}{6\mu\left(\beta+\frac{2\gamma}{3}\right)}$.

We will consider the boundary conditions given in [12].

$$u_i = u_i^0, \phi_i = \phi_i^0, \text{ on } \mathcal{G}_1^t = \partial B_0 \backslash \partial B_\sigma \times [t_0, t], \tag{8}$$

$$\sigma_{ji} n_i = \sigma_j^0, \mu_{ji} n_i = \mu_j^0 \text{ on } \mathcal{G}_2^t = \partial B_\sigma \times [t_0, t], \tag{9}$$

and initial conditions

$$u_i(x, 0) = U_i^0, \phi_i(x, 0) = \Phi_i^0, \text{ in } B_0, \tag{10}$$

$$\dot{u}_i(x, 0) = \dot{U}_i^0, \dot{\phi}_i(x, 0) = \dot{\Phi}_i^0, \text{ in } B_0, \tag{11}$$

where u_i^0 and ϕ_i^0 are prescribed on $\mathcal{G}_1$, σ_j^0 and μ_j^0 on $\mathcal{G}_2$, and n_i is the unit vector normal to the boundary ∂B_0 of the elastic body B_0.

2.2 Cosserat elastic energy

The strain stored energy U_C of the body B_0 is defined by the integral [13]:

$$U_C = \int_{B_0} \mathrm{W}\{\boldsymbol{\gamma}, \boldsymbol{\chi}\} dv, \tag{12}$$

where

$$\begin{aligned} \mathrm{W}\{\boldsymbol{\gamma}, \boldsymbol{\chi}\} = & \frac{\mu + \alpha}{2}\gamma_{ij}\gamma_{ij} + \frac{\mu - \alpha}{2}\gamma_{ij}\gamma_{ji} + \frac{\lambda}{2}\gamma_{kk}\gamma_{nn} \\ & + \frac{\gamma + \varepsilon}{2}\chi_{ij}\chi_{ij} + \frac{\gamma - \varepsilon}{2}\chi_{ij}\chi_{ji} + \frac{\beta}{2}\chi_{kk}\chi_{nn}, \end{aligned} \tag{13}$$

is non-negative. The relations Eqs. (3)–(4) can be written in the form [12]:

$$\boldsymbol{\sigma} = \nabla_{\boldsymbol{\gamma}} \mathrm{W} \text{ and } \boldsymbol{\mu} = \nabla_{\boldsymbol{\chi}} \mathrm{W}. \tag{14}$$

The stress energy is given as

$$U_K = \int_{B_0} \Phi\{\boldsymbol{\sigma}, \boldsymbol{\mu}\} dv, \tag{15}$$

where

$$\begin{aligned} \Phi\{\boldsymbol{\sigma}, \boldsymbol{\mu}\} = & \frac{\mu' + \alpha'}{2}\sigma_{ij}\sigma_{ij} + \frac{\mu' - \alpha'}{2}\sigma_{ij}\sigma_{ji} + \frac{\lambda'}{2}\sigma_{kk}\sigma_{nn} \\ & + \frac{\gamma' + \varepsilon'}{2}\mu_{ij}\mu_{ij} + \frac{\gamma' - \varepsilon'}{2}\mu_{ij}\mu_{ji} + \frac{\beta'}{2}\mu_{kk}\mu_{nn}, \end{aligned} \tag{16}$$

and the relations Eqs. (6)–(7) can be written as [12].

$$\boldsymbol{\gamma} = \frac{\partial \Phi}{\partial \boldsymbol{\sigma}}, \text{ and } \boldsymbol{\chi} = \frac{\partial \Phi}{\partial \boldsymbol{\mu}}. \tag{17}$$

We consider the work done by the stresses $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$ over the strains $\boldsymbol{\gamma}$ and $\boldsymbol{\chi}$ as in [13].

$$U = \int_{B_0} [\boldsymbol{\sigma} \cdot \boldsymbol{\gamma} + \boldsymbol{\mu} \cdot \boldsymbol{\chi}] dv \tag{18}$$

and

$$U = U_K = U_C \tag{19}$$

Here $\boldsymbol{\sigma} \cdot \boldsymbol{\gamma} = \sigma_{ji}\gamma_{ji}$ and $\boldsymbol{\mu} \cdot \boldsymbol{\chi} = \mu_{ji}\chi_{ji.}$
The stored kinetic energy T_C is defined as

$$T_C = \int_{B_0} \Upsilon_C dv = \frac{1}{2}\int_{B_0} \left(\rho \left(\frac{\partial \mathbf{u}}{\partial t} \right)^2 + \mathbf{J} \left(\frac{\partial \boldsymbol{\phi}}{\partial t} \right)^2 \right) dv, \tag{20}$$

The kinetic energy T_K is given as

$$T_K = \int_{B_0} \Upsilon_K \{\mathbf{p}, \mathbf{q}\} dv = \frac{1}{2}\int_{B_0} (\mathbf{p}^2 \rho^{-1} + \mathbf{q}^2 \mathbf{J}^{-1}) dv, \tag{21}$$

where

$$\mathbf{p} = \frac{\partial \Upsilon_C}{\partial \dot{\mathbf{u}}} = \rho \frac{\partial \mathbf{u}}{\partial t} \text{ and } \mathbf{q} = \frac{\partial \Upsilon_C}{\partial \dot{\boldsymbol{\phi}}} = \mathbf{J} \frac{\partial \boldsymbol{\phi}}{\partial t}, \tag{22}$$

and

$$\frac{\partial \mathbf{u}}{\partial t} = \frac{\partial \Upsilon_K}{\partial \mathbf{p}} = \mathbf{p}\rho^{-1} \text{ and } \frac{\partial \boldsymbol{\phi}}{\partial t} = \frac{\partial \Upsilon_K}{\partial \mathbf{q}} = \mathbf{q}\mathbf{J}^{-1}, \tag{23}$$

The work T_W done by the inertia forces over displacement and microrotation is given as in [12].

$$T_W = \int_{B_0} \Upsilon_W dv = \int_{B_0} \left(\frac{\partial \mathbf{p}}{\partial t} \cdot \mathbf{u} + \frac{\partial \mathbf{q}}{\partial t} \cdot \boldsymbol{\phi} \right) dv \tag{24}$$

Keeping in mind that the variation of $\mathbf{p}$ $\mathbf{u}$, $\mathbf{q}$, $\boldsymbol{\phi}$, $\delta\mathbf{u}$, and $\delta\boldsymbol{\phi}$ is zero at t_0 and t_k we can integrate by parts

$$\int_{t_0}^{t_k} T_K dt = \int_{t_0}^{t_k} T_I dt = \frac{1}{2}\int_{B_0} (\mathbf{p} \cdot \mathbf{u} + \mathbf{q} \cdot \boldsymbol{\phi}) dv \Big|_{t_0}^{t_k} - \int_{t_0}^{t_k} T_W dt \tag{25}$$

$$\delta \int_{t_0}^{t_k} T_K = -\delta \int_{t_0}^{t_k} T_W \tag{26}$$

or

$$\delta T_C = \delta T_K = -\delta T_W \tag{27}$$

and therefore

$$\int_{t_0}^{t_k} \int_{B_0} \left(\left(\mathbf{p} \cdot \delta \left(\frac{\partial \mathbf{u}}{\partial t} \right) + \mathbf{q} \cdot \delta \left(\frac{\partial \phi}{\partial t} \right) \right) \right) dv dt = -\int_{t_0}^{t_k} \int_{B_0} \left(\frac{\partial \mathbf{p}}{\partial t} \cdot \delta \mathbf{u} + \frac{\partial \mathbf{q}}{\partial t} \cdot \delta \boldsymbol{\phi} \right) dv dt \tag{28}$$

2.3 Variational principle for elastodynamics

We modify the HPR principle [14] for the case of Cosserat elastodynamics in the following way: for any set $\mathcal{A}$v of all admissible states $\mathfrak{s} = [\mathbf{u}, \boldsymbol{\phi}, \boldsymbol{\gamma}, \boldsymbol{\chi}, \boldsymbol{\sigma}, \boldsymbol{\mu}]$ that satisfy the strain-displacement and torsion-rotation relations Eq. (5), the zero variation

$$\delta\Theta(\mathfrak{s}) = 0$$

of the functional

$$\Theta(\mathfrak{s}) = \int_{t_0}^{t_k} \left[U_K + T_C - \int_{B_0} \left(\boldsymbol{\sigma} \cdot \boldsymbol{\gamma} + \boldsymbol{\mu} \cdot \boldsymbol{\chi} + \mathbf{p} \frac{\partial \mathbf{u}}{\partial t} + \mathbf{q} \frac{\partial \phi}{\partial t} \right) dv \right] dt$$

$$+ \int_{t_0}^{t} \int_{\mathcal{G}_1} [\boldsymbol{\sigma}_\mathbf{n} \cdot (\mathbf{u} - \mathbf{u_0}) + \boldsymbol{\mu}_\mathbf{n} (\boldsymbol{\phi} - \boldsymbol{\phi_0})] da dt + \int_{t_0}^{t} \int_{\mathcal{G}_2} [\boldsymbol{\sigma_0} \cdot \mathbf{u} + \boldsymbol{\mu_0} \cdot \boldsymbol{\phi}] da dt \tag{29}$$

at $\mathfrak{s} \in \mathcal{A}$ is equivalent of $\mathfrak{s}$ to be a solution of the system of equilibrium Eqs. (1)–(2), constitutive relations Eqs. (6)–(7), which satisfies the mixed boundary conditions Eqs. (8)–(9).

Proof of the variational principle for elastodynamics

Let us consider the variation of the functional $\Theta(\mathfrak{s})$:

$$\delta\Theta(\mathfrak{s}) = \int_{t_0}^{t_k} [\delta U_K + \delta T_C] dt$$

$$- \int_{t_0}^{t_k} \int_{B_0} \left[\delta\boldsymbol{\sigma} \cdot \boldsymbol{\gamma} + \boldsymbol{\sigma} \cdot \delta\boldsymbol{\gamma} + \delta\boldsymbol{\mu} \cdot \boldsymbol{\chi} + \boldsymbol{\mu} \cdot \delta\boldsymbol{\chi} + \frac{\partial \mathbf{u}}{\partial t} \delta\mathbf{p} + \mathbf{p} \cdot \delta\left(\frac{\partial \mathbf{u}}{\partial t} \right) + \delta\mathbf{q} \cdot \frac{\partial \phi}{\partial t} + \mathbf{q} \cdot \delta\left(\frac{\partial \phi}{\partial t} \right) \right] dv dt$$

$$+ \int_{t_0}^{t} \int_{\mathcal{G}_1} [\delta\boldsymbol{\sigma}_\boldsymbol{n} \cdot (\mathbf{u} - \mathbf{u_0}) + \boldsymbol{\sigma}_\boldsymbol{n} \delta\mathbf{u} + \delta\boldsymbol{\mu}_\boldsymbol{n} \cdot (\boldsymbol{\phi} - \boldsymbol{\phi_0}) + \boldsymbol{\mu}_\boldsymbol{n} \delta\boldsymbol{\phi}] da dt$$

$$+ \int_{t_0}^{t} \int_{\mathcal{G}_2} [\boldsymbol{\sigma_0} \cdot \delta\mathbf{u} + \boldsymbol{\mu_0} \cdot \delta\boldsymbol{\phi}] da dt$$

Taking into account Eq. (5) we can perform the integration by parts

$$\int_{B_0} \boldsymbol{\sigma} \cdot \delta\boldsymbol{\gamma} dv = \int_{\partial B_0} \boldsymbol{\sigma}_\boldsymbol{n} \cdot \delta\mathbf{u} da - \int_{B_0} \delta\mathbf{u} \cdot \mathbf{div}\boldsymbol{\sigma} dv + \int_{B_0} \boldsymbol{\varepsilon}\boldsymbol{\sigma} \cdot \delta\boldsymbol{\phi} dv$$

$$\int_{B_0} \boldsymbol{\mu} \cdot \delta\boldsymbol{\chi} dv = \int_{\partial B_0} \boldsymbol{\mu}_\boldsymbol{n} \cdot \delta\boldsymbol{\phi} da - \int_{B_0} \delta\boldsymbol{\phi} \cdot \mathbf{div}\boldsymbol{\mu} dv$$

and based on Eqs. (17)–(23)

$$\delta\Phi = \frac{\partial \Phi}{\partial \boldsymbol{\sigma}} \cdot \delta\boldsymbol{\sigma} + \frac{\partial \Phi}{\partial \boldsymbol{\mu}} \cdot \delta\boldsymbol{\mu}, \quad \delta\Upsilon_C = \frac{\partial \Upsilon_C}{\partial \left(\frac{\partial \mathbf{u}}{\partial t} \right)} \cdot \delta\left(\frac{\partial \mathbf{u}}{\partial t} \right) + \frac{\partial \Upsilon_C}{\partial \left(\frac{\partial \phi}{\partial t} \right)} \cdot \delta\left(\frac{\partial \phi}{\partial t} \right).$$

Then keeping in mind that $\delta T_K = -\delta T$ and Eq. (28) we can rewrite the expression for the variation of the functional $\delta\Theta(\mathfrak{s})$ in the following form

$$\delta\Theta(\mathfrak{s}) = \int_{t_0}^{t}\int_{B_0}\left[\left(\frac{\partial\Phi}{\partial\boldsymbol{\sigma}} - \boldsymbol{\gamma}\right)\cdot\delta\boldsymbol{\sigma}\right]dvdt + \int_{t_0}^{t}\int_{B_0}\left[\left(\frac{\partial\Phi}{\partial\boldsymbol{\mu}} - \boldsymbol{\chi}\right)\cdot\delta\boldsymbol{\mu}\right]dvdt$$
$$+ \int_{t_0}^{t}\int_{B_0}\left[\left(\rho\frac{\partial\mathbf{u}}{\partial t} - \mathbf{p}\right)\cdot\delta\left(\frac{\partial\mathbf{u}}{\partial t}\right)\right]dvdt + \int_{t_0}^{t}\int_{B_0}\left[\left(\mathbf{J}\frac{\partial\varphi}{\partial t} - \mathbf{q}\right)\cdot\delta\left(\frac{\partial\varphi}{\partial t}\right)\right]dvdt$$
$$+ \int_{t_0}^{t}\int_{B_0}\left[\left(\mathbf{div}\boldsymbol{\sigma} - \frac{\partial\mathbf{p}}{\partial t}\right)\cdot\delta\mathbf{u}\right]dvdt + \int_{t_0}^{t}\int_{B_0}\left[\left(\mathbf{div}\boldsymbol{\mu} + \boldsymbol{\varepsilon}\cdot\boldsymbol{\sigma} - \frac{\partial\mathbf{q}}{\partial t}\right)\cdot\delta\mathbf{u}\right]dvdt$$
$$+ \int_{t_0}^{t}\int_{\mathcal{G}_1}[(\mathbf{u}-\mathbf{u_0})\cdot\delta\boldsymbol{\sigma}_\mathbf{n}]dadt + \int_{t_0}^{t}\int_{\mathcal{G}_1}[(\boldsymbol{\phi}-\boldsymbol{\phi_0})\cdot\delta\boldsymbol{\mu}_\mathbf{n}]dadt$$
$$+ \int_{t_0}^{t}\int_{\mathcal{G}_2}[(\boldsymbol{\sigma}_\mathbf{n}-\boldsymbol{\sigma_0})\cdot\delta\mathbf{u}]dadt + \int_{t_0}^{t}\int_{\mathcal{G}_2}[(\boldsymbol{\mu}_\mathbf{n}-\boldsymbol{\mu_0})\cdot\delta\boldsymbol{\phi}]dadt$$

3. Dynamic Cosserat plate theory

In this section we review our stress, couple stress and kinematic assumptions of the Cosserat plate [7]. We consider the thin plate P, where h is the thickness of the plate and $x_3 = 0$ represents its middle plane. The sets T and B are the top and bottom surfaces contained in the planes $x_3 = h/2$, $x_3 = -h/2$ respectively and the curve Γ is the boundary of the middle plane of the plate.

The set of points $P = \left(\Gamma \times \left[-\frac{h}{2}, \frac{h}{2}\right]\right) \cup T \cup B$ forms the entire surface of the plate and $\Gamma_u \times \left[-\frac{h}{2}, \frac{h}{2}\right]$ is the lateral part of the boundary where displacements and microrotations are prescribed. The notation $\Gamma_\sigma = \Gamma \backslash \Gamma_u$ of the remainder we use to describe the lateral part of the boundary edge $\Gamma_\sigma \times \left[-\frac{h}{2}, \frac{h}{2}\right]$ where stress and couple stress are prescribed. We also use notation P_0 for the middle plane internal domain of the plate.

In our case we consider the vertical load and pure twisting momentum boundary conditions at the top and bottom of the plate, which can be written in the form:

$$\sigma_{33}(x_1, x_2, h/2, t) = \sigma^t(x_1, x_2, t),\ \sigma_{33}(x_1, x_2, -h/2, t) = \sigma^b(x_1, x_2, t), \tag{30}$$
$$\sigma_{3\beta}(x_1, x_2, \pm h/2, t) = 0, \tag{31}$$
$$\mu_{33}(x_1, x_2, h/2, t) = \mu^t(x_1, x_2, t),\ \mu_{33}(x_1, x_2, -h/2, t) = \mu^b(x_1, x_2, t), \tag{32}$$
$$\mu_{3\beta}(x_1, x_2, \pm h/2, t) = 0, \tag{33}$$

where $(x_1, x_2) \in P_0$.

We will also consider the rotatory inertia $\mathbf{J}$ in the form

$$\mathbf{J} = \begin{pmatrix} J_{11} & J_{12} & 0 \\ J_{12} & J_{22} & 0 \\ 0 & 0 & J_{33} \end{pmatrix}$$

Let $\mathcal{A}$ denote the set of all admissible states that satisfy the Cosserat plate strain-displacement relation Eq. (5) and let Θ be a functional on $\mathcal{A}$ defined by

$$\Theta(s,\eta) = U_K^S + T_C^S - \int_{P_0}\left(\mathcal{S}\cdot\mathcal{E} + \mathbf{P}\cdot\frac{\partial\mathcal{U}}{\partial t} - \hat{P}\cdot\mathcal{W} + v\Omega_3^0\right)da + \int_{\Gamma_\sigma}\mathcal{S}_o\cdot(\mathcal{U}-\mathcal{U}_o)ds + \int_{\Gamma_u}\mathcal{S}_n\cdot\mathcal{U}ds, \tag{34}$$

for every $s = [\mathcal{U}, \mathcal{E}, \mathcal{S}] \in \mathcal{A}$. Here $\hat{P} = (\hat{p}_1, \hat{p}_2)$ and $\mathcal{W} = (W, W^*)$, $\hat{p}_1 = \eta p$ and $\hat{p}_2 = \frac{2}{3}(1-\eta)p$

Here the plate stress and kinetic energy density by the formulas

$$U_K^{\mathcal{S}} = \int_{P_0} \Phi(\mathcal{S}) da,\ T_K^{\mathcal{S}} = \int_{P_0} \Upsilon_C\left(\frac{\partial \mathcal{U}}{\partial t}\right) da \tag{35}$$

where P_0 is the internal domain of the middle plane of the plate.

$$\begin{aligned}
\Phi(\mathcal{S}) = &-\frac{3\lambda(M_{\alpha\alpha})(M_{\beta\beta})}{h^3\mu(3\lambda+2\mu)} \\
&+\frac{3(\alpha+\mu)M_{\alpha\beta}^2}{2h^3\alpha\mu}+\frac{3(\alpha+\mu)}{160h^3\alpha\mu}\left[8\hat{Q}_\alpha\hat{Q}_\alpha+15Q_\alpha\hat{Q}_\alpha+20\hat{Q}_\alpha Q_\alpha^*+8Q_\alpha^*Q_\alpha^*\right] \\
&+\frac{3(\alpha-\mu)M_{\alpha\beta}^2}{2h^3\alpha\mu}+\frac{\alpha-\mu}{280h^3\alpha\mu}\left[21Q_\alpha\left(5\hat{Q}_\alpha+4Q_\alpha^*\right)\right] \\
&-\frac{\gamma-\varepsilon}{160h\gamma\varepsilon}\left[24R_{\alpha\alpha}^2+45R_{\alpha\alpha}^*+60R_{\alpha\beta}R_{\alpha\beta}^*+48R_{12}R_{21}\right] \\
&+\frac{3(\gamma+\varepsilon)S_\alpha^*S_\alpha^*}{2h^3\gamma\varepsilon}+\frac{\gamma+\varepsilon}{160h^3\gamma\varepsilon}\left[8R_{\alpha\beta}^2+15R_{\alpha\beta}^*R_{\alpha\beta}^*+20R_{\alpha\beta}R_{\alpha\beta}^*\right] \\
&-\frac{3\beta}{80h\gamma(3\beta+2\gamma)}\left[-8(R_{\alpha\alpha})(R_{\beta\beta})-15\left(R_{\alpha\alpha}^*\right)\left(R_{\beta\beta}^*\right)-20(R_{\alpha\alpha})(R_{\alpha\alpha}^*)\right] \\
&-\frac{\beta}{4\gamma(3\beta+2\gamma)}\left[(2R_{\alpha\alpha}+3R_{\alpha\alpha}^*)t-h\left(V^2+T^2\right)\right] \\
&+\frac{\lambda}{560h\mu(3\lambda+2\mu)}\left[\frac{5+3\eta}{(1+\eta)}pM_{\alpha\alpha}\right] \\
&+\frac{(\lambda+\mu)h}{840\mu(3\lambda+2\mu)}\left(\frac{140+168\eta+51\eta^2}{4(1+\eta)^2}\right)p^2+\frac{(\lambda+\mu)h}{2\mu(3\lambda+2\mu)}\sigma_0^2+\frac{\varepsilon h}{12h\gamma\varepsilon}\left[(3T^2+V^2)\right]
\end{aligned} \tag{36}$$

and

$$\begin{aligned}
\Upsilon_C\left(\frac{\partial \mathcal{U}}{\partial t}\right) = &\frac{h\rho}{2}\left(\frac{\partial W}{\partial t}\right)^2+\frac{4h\rho}{15}\left(\frac{\partial W^*}{\partial t}\right)^2+\frac{2h\rho}{3}\left(\frac{\partial W}{\partial t}\frac{\partial W^*}{\partial t}\right)+\frac{h^3\rho}{24}\left(\frac{\partial \Psi_\alpha}{\partial t}\right)^2 \\
&+\frac{4hJ_{\alpha\beta}}{15}\left(\frac{\partial \Omega_\alpha^0}{\partial t}\frac{\partial \Omega_\beta^0}{\partial t}\right)+\frac{hJ_{\alpha\beta}}{2}\left(\frac{\partial \hat{\Omega}_\alpha^{\ 0}}{\partial t}\frac{\partial \hat{\Omega}_\beta^{\ 0}}{\partial t}\right)+\frac{2hJ_{\alpha\beta}}{3}\left(\frac{\partial \Omega_\alpha^0}{\partial t}\frac{\partial \hat{\Omega}_\beta^{\ 0}}{\partial t}\right)+\frac{hJ_{33}}{6}\left(\frac{\partial \Omega_3}{\partial t}\right)^2.
\end{aligned}$$

$\mathcal{S}$, $\mathcal{U}$ and $\mathcal{E}$ are the Cosserat plate stress, displacement and strain sets

$$\mathcal{S} = \left[M_{\alpha\beta}, Q_\alpha, Q_\alpha^*, \hat{Q}_\alpha, R_{\alpha\beta}, R_{\alpha\beta}^*, S_\beta^*\right], \tag{37}$$

$$\mathcal{S}_n = \left[\check{M}_\alpha, \check{Q}^*, \hat{\check{Q}}_\alpha, \check{R}_\alpha, \check{R}_\alpha^*, \check{S}^*\right], \tag{38}$$

$$\mathcal{S}_o = \left[\Pi_{o\alpha}, \Pi_{o3}, \Pi_{o3}^*, M_{o\alpha}, M_{o\alpha}^*, M_{o3}^*\right], \tag{39}$$

$$\mathcal{U} = \left[\Psi_\alpha, W, \Omega_3, \Omega_\alpha^0, W^*, \Omega_\alpha^0\right], \tag{40}$$

$$\mathcal{E} = \left[e_{\alpha\beta}, \omega_\beta, \omega_\alpha^*, \hat{\omega}_\alpha, \tau_{3\alpha}, \tau_{\alpha\beta}, \tau_{\alpha\beta}^*\right], \tag{41}$$

where $M_{\alpha\beta}n_\beta = \Pi_{o\alpha}$, $R_{\alpha\beta}n_\beta = M_{o\alpha}$, $Q_\alpha^* n_\alpha = \Pi_{o3}$, $S_\alpha^* n_\alpha = M_{o3}^*$, $\hat{Q}_\alpha n_\alpha = \Pi_{o3}^*$, $R_{\alpha\beta}^* n_\beta = M_{o\alpha}^*$, $\check{M}_\alpha = M_{\alpha\beta}n_\beta$, $\check{Q}^* = Q_\beta^* n_\beta$, $\check{R}_\alpha = R_{\alpha\beta}n_\beta$, $\check{S}^* = S_\beta^* n_\beta$, $\hat{\check{Q}} = \check{Q}_\beta n_\beta$, $\check{R}_\alpha^* = \check{R}_{\alpha\beta}^* n_\beta$. ($n_\beta$ is the outward unit normal vector to Γ_u).

The plate characteristics provide the approximation of the components of the three-dimensional tensors σ_{ji} and μ_{ji}

$$\sigma_{\alpha\beta} = \frac{6}{h^2}\zeta M_{\alpha\beta}(x_1, x_2, t), \tag{42}$$

$$\sigma_{3\beta} = \frac{3}{2h}\left(1-\zeta^2\right)Q_\beta(x_1, x_2, t), \tag{43}$$

$$\sigma_{\beta 3} = \frac{3}{2h}\left(1-\zeta^2\right)Q_\beta^*(x_1, x_2, t) + \frac{3}{2h}\hat{Q}_\beta(x_1, x_2, t), \tag{44}$$

$$\sigma_{33} = -\frac{3}{4}\left(\frac{1}{3}\zeta^3 - \zeta\right)p_1(x_1, x_2, t) + \zeta p_2(x_1, x_2, t) + \sigma_0(x_1, x_2, t), \tag{45}$$

$$\mu_{\alpha\beta} = \frac{3}{2h}\left(1-\zeta^2\right)R_{\alpha\beta}(x_1, x_2, t) + \frac{3}{2h}R_{\alpha\beta}^*(x_1, x_2, t), \tag{46}$$

$$\mu_{\beta 3} = \frac{6}{h^2}\zeta S_\beta^*(x_1, x_2, t), \tag{47}$$

$$\mu_{3\beta} = 0, \tag{48}$$

$$\mu_{33} = \zeta V(x_1, x_2, t) + T(x_1, x_2, t), \tag{49}$$

where

$$p(x_1, x_2, t) = \sigma^t(x_1, x_2, t) - \sigma^b(x_1, x_2, t), \tag{50}$$

$$\sigma_0(x_1, x_2, t) = \frac{1}{2}\left(\sigma^t(x_1, x_2, t) + \sigma^b(x_1, x_2, t)\right), \tag{51}$$

$$V(x_1, x_2, t) = \frac{1}{2}\left(\mu^t(x_1, x_2, t) - \mu^b(x_1, x_2, t)\right), \tag{52}$$

$$T(x_1, x_2, t) = \frac{1}{2}\left(\mu^t(x_1, x_2, t) + \mu^b(x_1, x_2, t)\right). \tag{53}$$

The pressures p_1 and p_2 are chosen in the form

$$p_1(x_1, x_2, t) = \eta p(x_1, x_2, t), \tag{54}$$

$$p_2(x_1, x_2, t) = \frac{(1-\eta)}{2}p(x_1, x_2, t). \tag{55}$$

and $\eta \in \mathbb{R}$ is called *splitting parameter*.

The three-dimensional displacements u_i and microrotations ϕ_i

$$u_\alpha = \frac{h}{2}\zeta\Psi_\alpha(x_1, x_2, t), \tag{56}$$

$$u_3 = W(x_1, x_2, t) + \left(1-\zeta^2\right)W^*(x_1, x_2, t), \tag{57}$$

$$\phi_\alpha = \Omega_\alpha^0(x_1, x_2, t)\left(1-\zeta^2\right) + \hat{\Omega}_\alpha(x_1, x_2, t), \tag{58}$$

$$\phi_3 = \zeta\Omega_3(x_1, x_2, t), \tag{59}$$

and the three-dimensional strain and torsion tensors γ_{ji} and χ_{ji}

$$\gamma_{\alpha\beta} = \frac{6}{h^2}\zeta e_{\alpha\beta}(x_1, x_2, t), \tag{60}$$

$$\gamma_{3\beta} = \frac{3}{2h}\left(1-\zeta^2\right)\omega_\beta(x_1, x_2, t), \tag{61}$$

$$\gamma_{\beta 3} = \frac{3}{2h}\left(1-\zeta^2\right)\omega^*_\beta(x_1, x_2, t) + \frac{3}{2h}\hat{\omega}_\beta(x_1, x_2, t), \tag{62}$$

$$\chi_{\alpha\beta} = \frac{3}{2h}\left(1-\zeta^2\right)\tau_{\alpha\beta}(x_1, x_2, t) + \frac{3}{2h}\tau^*_{\alpha\beta}(x_1, x_2, t), \tag{63}$$

$$\chi_{3\beta} = \frac{6}{h^2}\zeta\tau^*_\beta(x_1, x_2, t), \tag{64}$$

where $\zeta = \frac{2x_3}{h}$.

Then zero variation of the functional

$$\delta\Theta(s, \eta) = 0$$

is equivalent to the plate bending system of equations (A) and constitutive formulas (B) mixed problems.

A. The bending equilibrium system of equations:

$$M_{\alpha\beta,\,\alpha} - Q_\beta = I_1\frac{\partial^2\Psi_\beta}{\partial t^2}, \tag{65}$$

$$Q^*_{\alpha,\,\alpha} + \hat{p}_1 = I_2\frac{\partial^2 W^*}{\partial t^2}, \tag{66}$$

$$R_{\alpha\beta,\,\alpha} + \varepsilon_{3\beta\gamma}\left(Q^*_\gamma - Q_\gamma\right) = I_{\alpha\beta}\frac{\partial^2\Omega^0_\alpha}{\partial t^2}, \tag{67}$$

$$\varepsilon_{3\beta\gamma}M_{\beta\gamma} + S^*_{\alpha,\,\alpha} = I_3\frac{\partial^2\Omega_3}{\partial t^2}, \tag{68}$$

$$\hat{Q}_{\alpha,\,\alpha} + \hat{p}_2 = I_2\frac{\partial^2 W}{\partial t^2}, \tag{69}$$

$$R^*_{\alpha\beta,\,\alpha} + \varepsilon_{3\beta\gamma}\hat{Q}_\gamma = I^0_{\alpha\beta}\frac{\partial^2\hat{\Omega}^0_\alpha}{\partial t^2}, \tag{70}$$

where $I_1 = \frac{h^3}{12}\rho$, $I_2 = \frac{2h}{3}\rho$, $I_{\alpha\beta} = \frac{5h}{6}J_{\alpha\beta}$, $I_3 = \frac{h^2}{6}J_{33}$, $I^0_{\alpha\beta} = \frac{2h}{3}J_{\alpha\beta}$, $\hat{p}_1 = \eta_{opt}p$, and $\hat{p}_2 = \frac{2}{3}\left(1-\eta_{opt}\right)p$, with the resultant traction boundary conditions:

$$M_{\alpha\beta}n_\beta = \Pi_{o\alpha}, \quad R_{\alpha\beta}n_\beta = M_{o\alpha}, \tag{71}$$

$$Q^*_\alpha n_\alpha = \Pi_{o3}, \quad S^*_\alpha n_\alpha = \Upsilon_{o3}, \tag{72}$$

at the part Γ_σ and the resultant displacement boundary conditions

$$\Psi_\alpha = \Psi_{o\alpha},\ W = W_o,\ \Omega^0_\alpha = \Omega^0_{o\alpha},\ \Omega_3 = \Omega_{o3}, \tag{73}$$

at the part Γ_u.

B. Constitutive formulas in the reverse form:[1]

$$M_{\alpha\alpha} = \frac{\mu(\lambda+\mu)h^3}{3(\lambda+2\mu)}\Psi_{\alpha,\alpha} + \frac{\lambda\mu h^3}{6(\lambda+2\mu)}\Psi_{\beta,\beta} + \frac{(3p_1+5p_2)\lambda h^2}{30(\lambda+2\mu)}, \tag{74}$$

$$M_{\beta\alpha} = \frac{(\mu-\alpha)h^3}{12}\Psi_{\alpha,\beta} + \frac{(\mu+\alpha)h^3}{12}\Psi_{\beta,\alpha} + (-1)^{\alpha'}\frac{\alpha h^3}{6}\Omega_3, \tag{75}$$

$$R_{\beta\alpha} = \frac{5(\gamma-\varepsilon)h}{6}\Omega^0_{\beta,\alpha} + \frac{5(\gamma+\varepsilon)h}{6}\Omega^0_{\alpha,\beta}, \tag{76}$$

$$R_{\alpha\alpha} = \frac{10h\gamma(\beta+\gamma)}{3(\beta+2\gamma)}\Omega^0_{\alpha,\alpha} + \frac{5h\beta\gamma}{3(\beta+2\gamma)}\Omega^0_{\beta,\beta}, \tag{77}$$

$$R^*_{\beta\alpha} = \frac{2(\gamma-\varepsilon)h}{3}\hat{\Omega}_{\beta,\alpha} + \frac{2(\gamma+\varepsilon)h}{3}\hat{\Omega}_{\alpha,\beta}, \tag{78}$$

$$R^*_{\alpha\alpha} = \frac{8\gamma(\gamma+\beta)h}{3(\beta+2\gamma)}\hat{\Omega}_{\alpha,\alpha} + \frac{4\gamma\beta h}{3(\beta+2\gamma)}\hat{\Omega}_{\beta,\beta}, \tag{79}$$

$$Q_\alpha = \frac{5(\mu+\alpha)h}{6}\Psi_\alpha + \frac{5(\mu-\alpha)h}{6}W_{,\alpha} + \frac{2(\mu-\alpha)h}{3}W^*_{,\alpha} + (-1)^\beta\frac{5h\alpha}{3}\Omega^0_\beta + (-1)^\beta\frac{5h\alpha}{3}\hat{\Omega}_\beta, \tag{80}$$

$$Q^*_\alpha = \frac{5(\mu-\alpha)h}{6}\Psi_\alpha + \frac{5(\mu-\alpha)^2h}{6(\mu+\alpha)}W_{,\alpha} + \frac{2(\mu+\alpha)h}{3}W^*_{,\alpha} + (-1)^\alpha\frac{5h\alpha}{3}\left(\Omega^0_\beta + \frac{(\mu-\alpha)}{(\mu+\alpha)}\hat{\Omega}_\beta\right), \tag{81}$$

$$\hat{Q}_\alpha = \frac{8\alpha\mu h}{3(\mu+\alpha)}W_{,\alpha} + (-1)^\alpha\frac{8\alpha\mu h}{3(\mu+\alpha)}\hat{\Omega}_\beta, \tag{82}$$

$$S^*_\alpha = \frac{5\gamma\varepsilon h^3}{3(\gamma+\varepsilon)}\Omega_{3,\alpha}, \tag{83}$$

and the optimal value η_{opt} of the splitting parameter is given as in [10]

$$\eta_{opt} = \frac{2\mathcal{W}^{(00)} - \mathcal{W}^{(10)} - \mathcal{W}^{(01)}}{2\left(\mathcal{W}^{(11)} + \mathcal{W}^{(00)} - \mathcal{W}^{(10)} - \mathcal{W}^{(01)}\right)}. \tag{84}$$

where

$$\mathcal{W}^{(ij)} = \mathcal{S}|_{\eta=i}\cdot\mathcal{E}|_{\eta=j}.$$

We also assume that the initial condition can be presented in the form

$$\mathcal{U}(x_1,x_2,0) = \mathcal{U}^0(x_1,x_2),\ \frac{\partial\mathcal{U}}{\partial t}(x_1,x_2,0) = \mathcal{V}^0(x_1,x_2)$$

4. Cosserat plate dynamic field equations

The Cosserat plate field equations are obtained by substituting the relations Eqs. (74)–(83) into the system of Eqs. (65)–(70) similar to [10]:

[1] In the following formulas a subindex $\beta = 1$ if $\alpha = 2$ and $\beta = 2$ if $\alpha = 1$.

$$L\mathcal{U} = K\frac{\partial^2 \mathcal{U}}{\partial t^2} + \mathcal{F}(\eta), \tag{85}$$

where

$$L = \begin{bmatrix} L_{11} & L_{12} & L_{13} & L_{14} & 0 & L_{16} & kL_{13} & 0 & L_{16} \\ L_{12} & L_{22} & L_{23} & L_{24} & L_{16} & 0 & kL_{23} & L_{16} & 0 \\ -L_{13} & -L_{23} & L_{33} & 0 & L_{35} & L_{36} & L_{77} & L_{38} & L_{39} \\ L_{41} & L_{42} & 0 & L_{44} & 0 & 0 & 0 & 0 & 0 \\ 0 & -L_{16} & -L_{38} & 0 & L_{55} & L_{56} & -kL_{35} & L_{58} & 0 \\ L_{16} & 0 & -L_{39} & 0 & L_{56} & L_{66} & -kL_{36} & 0 & L_{58} \\ -L_{13} & -L_{14} & L_{73} & 0 & L_{35} & L_{36} & L_{77} & L_{78} & L_{79} \\ 0 & -L_{16} & -L_{78} & 0 & L_{85} & L_{56} & -kL_{35} & L_{88} & kL_{56} \\ L_{16} & 0 & -L_{79} & 0 & L_{56} & L_{55} & -kL_{36} & kL_{56} & L_{99} \end{bmatrix},$$

$$K = \begin{bmatrix} \frac{h^3}{12}\rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{h^3}{12}\rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2h}{3}\rho & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{h^2}{6}J_{33} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{5h}{6}J_{11} & \frac{5h}{6}J_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{5h}{6}J_{12} & \frac{5h}{6}J_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{2h}{3}\rho & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2h}{3}J_{11} & \frac{2h}{3}J_{12} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2h}{3}J_{12} & \frac{2h}{3}J_{22} \end{bmatrix},$$

$$\mathcal{U} = \left[\Psi_1,\ \Psi_2,\ W,\ \Omega_3,\ \Omega_1^0,\ \Omega_2^0,\ W^*,\ \Omega_1^0,\ \Omega_2^0\right]^T,$$

$$\mathcal{F}(\eta) = \left[-\frac{3h^2\lambda\left(3p_{1,1}+5p_{2,1}\right)}{30(\lambda+2\mu)},\ -\frac{3h^2\lambda\left(3p_{1,2}+5p_{2,2}\right)}{30(\lambda+2\mu)},\ -p_1,\ 0,\ 0,\ 0,\ \frac{h^2\left(3p_1+4p_2\right)}{24},\ 0,\ 0\right]^T,$$

$$p_1 = \eta p, \quad p_2 = \frac{(1-\eta)}{2}p$$

The operators L_{ij} are given as follows

$$L_{11} = c_1 \frac{\partial^2}{\partial x_1^2} + c_2 \frac{\partial^2}{\partial x_2^2} - c_3, \quad L_{12} = (c_1 - c_2) \frac{\partial^2}{\partial x_1 x_2}, \quad L_{13} = c_{11} \frac{\partial}{\partial x_1},$$
$$L_{14} = c_{12} \frac{\partial}{\partial x_2}, \quad L_{16} = c_{13}, \quad L_{17} = k_1 c_{11} \frac{\partial}{\partial x_1},$$
$$L_{22} = c_2 \frac{\partial^2}{\partial x_1^2} + c_1 \frac{\partial^2}{\partial x_2^2} - c_3, \quad L_{23} = c_{11} \frac{\partial}{\partial x_2}, \quad L_{24} = -c_{12} \frac{\partial}{\partial x_1},$$
$$L_{33} = c_3 \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right), \quad L_{35} = -c_{13} \frac{\partial}{\partial x_2}, \quad L_{36} = c_{13} \frac{\partial}{\partial x_1},$$
$$L_{38} = -c_{10} \frac{\partial}{\partial x_2}, \quad L_{39} = c_{10} \frac{\partial}{\partial x_1}, \quad L_{41} = -c_{12} \frac{\partial}{\partial x_2},$$
$$L_{42} = c_{12} \frac{\partial}{\partial x_1}, \quad L_{44} = c_6 \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) - 2c_{12}, \quad L_{55} = c_7 \frac{\partial^2}{\partial x_1^2} + c_8 \frac{\partial^2}{\partial x_2^2} - 2c_{13},$$
$$L_{56} = (c_7 - c_8) \frac{\partial^2}{\partial x_1 x_2}, \quad L_{58} = -c_9, \quad L_{66} = c_8 \frac{\partial^2}{\partial x_1^2} + c_7 \frac{\partial^2}{\partial x_2^2} - 2c_{13},$$
$$L_{73} = c_5 \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right), \quad L_{77} = c_4 \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right), \quad L_{78} = -c_{14} \frac{\partial}{\partial x_2},$$
$$L_{79} = c_{14} \frac{\partial}{\partial x_1}, \quad L_{85} = c_7 \frac{\partial^2}{\partial x_1^2} + c_8 \frac{\partial^2}{\partial x_2^2} - 2c_{13}, \quad L_{88} = c_7 \frac{\partial^2}{\partial x_1^2} + c_8 \frac{\partial^2}{\partial x_2^2} - c_{15},$$
$$L_{99} = c_8 \frac{\partial^2}{\partial x_1^2} + c_7 \frac{\partial^2}{\partial x_2^2} - c_{15}.$$

The coefficients c_i are given as

$$c_1 = \frac{h^3 \mu(\lambda + \mu)}{3(\lambda + 2\mu)}, \quad c_2 = \frac{h^3(\alpha + \mu)}{12}, \quad c_3 = \frac{5h(\alpha + \mu)}{6}, \quad c_4 = \frac{5h(\alpha - \mu)^2}{6(\alpha + \mu)},$$
$$c_5 = \frac{h(5\alpha^2 + 6\alpha\mu + 5\mu^2)}{6(\alpha + \mu)}, \quad c_6 = \frac{h^3 \gamma \varepsilon}{3(\gamma + \varepsilon)}, \quad c_7 = \frac{10h\gamma(\beta + \gamma)}{3(\beta + 2\gamma)}, \quad c_8 = \frac{5h(\gamma + \varepsilon)}{6},$$
$$c_9 = \frac{10h\alpha^2}{3(\alpha + \mu)}, \quad c_{10} = \frac{5h\alpha(\alpha - \mu)}{3(\alpha + \mu)}, \quad c_{11} = \frac{5h(\alpha - \mu)}{6}, \quad c_{12} = \frac{h^3 \alpha}{6},$$
$$c_{13} = \frac{5h\alpha}{3}, \quad c_{14} = \frac{h\alpha(5\alpha + 3\mu)}{3(\alpha + \mu)}, \quad c_{15} = \frac{2h\alpha(5\alpha + 4\mu)}{3(\alpha + \mu)}.$$

5. Numerical validation

For the validation purposes we provide the algorithm and computation results for the three-dimensional Cosserat elastodynamics. We also present the analysis of the numerical results based on the plate theory for the microelements of different shapes and orientations incorporated into the Cosserat plate.

5.1 Analysis of Cosserat plate vibrations based on the three-dimensional theory

In our computations we consider the plates made of polyurethane foam—a material reported in the literature to behave Cosserat like—and the values of the technical elastic parameters presented in [15]: $E = 299.5\,\text{MPa}$, $\nu = 0.44$, $l_t = 0.62\,\text{mm}$, $l_b = 0.327\,\text{mm}$, $N^2 = 0.04$. Taking into account that the ratio β/γ is equal to 1 for bending [15], these values of the technical constants correspond

to the following values of Lamé and Cosserat parameters: $\lambda = 762.616\,\text{MPa}$, $\mu = 103.993\,\text{MPa}$, $\alpha = 4.333\,\text{MPa}$, $\beta = 39.975\,\text{MPa}$, $\gamma = 39.975\,\text{MPa}$, $\varepsilon = 4.505\,\text{MPa}$. We consider a low-density rigid foam usually characterized by the densities of 24–50 kg/m^3 [16]. In all further numerical computations we used the density value $\rho = 34$ kg/m^3 and different values the rotatory inertia $\mathbf{J}$.

Let us consider the plate B_0 being a rectangular cuboid $[0,a] \times 0, a] \times \left[-\frac{h}{2}, \frac{h}{2}\right]$. Let the sets T and B be the top and the bottom surfaces contained in the planes $x_3 = \frac{h}{2}$ and $x_3 = -\frac{h}{2}$ respectively, and the curve $\Gamma = \Gamma_1 \cup \Gamma_2$ be the lateral part of the boundary:

$$\Gamma_1 = \left\{(x_1, x_2, x_3) : x_1 \in \{0, a\}, x_2 \in [0, a], x_3 \in \left[-\frac{h}{2}, \frac{h}{2}\right]\right\},$$
$$\Gamma_2 = \left\{(x_1, x_2, x_3) : x_1 \in [0, a], x_2 \in \{0, a\}, x_3 \in \left[-\frac{h}{2}, \frac{h}{2}\right]\right\},$$

We solve the three-dimensional Cosserat equilibrium Eqs. (1)–(2) accompanied by the constitutive Eqs. (3)–(4) and strain-displacement and torsion-rotation relations Eq. (5) complemented by the following boundary conditions:

$$\Gamma_1 : u_2 = 0, u_3 = 0, \varphi_1 = 0, \sigma_{11} = 0, \mu_{12} = 0, \mu_{13} = 0; \tag{86}$$
$$\Gamma_2 : u_1 = 0, u_3 = 0, \varphi_2 = 0, \sigma_{22} = 0, \mu_{21} = 0, \mu_{23} = 0; \tag{87}$$
$$T : \sigma_{33} = p(x_1, x_2), \mu_{33} = 0; \tag{88}$$
$$B : \sigma_{33} = 0, \mu_{33} = 0. \tag{89}$$

where the initial distribution of the pressure is given as $p = \sin\left(\frac{\pi x_1}{a}\right) \sin\left(\frac{\pi x_2}{a}\right) \sin \omega t$ and the rotatory inertia tensor $\mathbf{J}$ is assumed to have a diagonal form

$$\mathbf{J} = \begin{bmatrix} J_x & 0 & 0 \\ 0 & J_y & 0 \\ 0 & 0 & J_z \end{bmatrix}. \tag{90}$$

Using the method of separation of variables and taking into account the boundary conditions Eqs. (86)–(87), we express the kinematic variables in the form:

$$u_1 = \cos\left(\frac{\pi x_1}{a}\right) \sin\left(\frac{\pi x_2}{a}\right) z_1(x_3) \sin \omega t, \tag{91}$$
$$u_2 = \sin\left(\frac{\pi x_1}{a}\right) \cos\left(\frac{\pi x_2}{a}\right) z_2(x_3) \sin \omega t, \tag{92}$$
$$u_3 = \sin\left(\frac{\pi x_1}{a}\right) \sin\left(\frac{\pi x_2}{a}\right) z_3(x_3) \sin \omega t, \tag{93}$$
$$\phi_1 = \sin\left(\frac{\pi x_1}{a}\right) \cos\left(\frac{\pi x_2}{a}\right) z_4(x_3) \sin \omega t, \tag{94}$$
$$\phi_2 = \cos\left(\frac{\pi x_1}{a}\right) \sin\left(\frac{\pi x_2}{a}\right) z_5(x_3) \sin \omega t, \tag{95}$$
$$\phi_3 = \cos\left(\frac{\pi x_1}{a}\right) \cos\left(\frac{\pi x_2}{a}\right) z_6(x_3) \sin \omega t, \tag{96}$$

where the functions $z_i(x_3)$ represent the transverse variations of the kinematic variables.

If we substitute the expressions Eqs. (91)–(96) into Eqs. (3)–(4) and then into Eqs. (1)–(2), we will obtain the following eigenvalue problem

$$\mathbf{B}z = \omega^2 \mathbf{A}z \tag{97}$$

where

$$\mathbf{B} = \begin{bmatrix} b_1L_2 + b_2L_0 & b_3L_0 & b_4L_1 & 0 & -b_5L_1 & -b_6L_0 \\ b_3L_0 & b_1L_2 + b_2L_0 & b_4L_1 & b_5L_1 & 0 & b_6L_0 \\ -b_4L_1 & b_4L_1 & b_7L_2 & b_6L_0 & -b_6L_0 & 0 \\ 0 & -b_5L_1 & b_6L_0 & b_9L_2 + b_{10}L_0 & b_{11}L_0 & b_{12}L_1 \\ b_5L_1 & 0 & -b_6L_0 & b_{11}L_0 & b_9L_2 + b_{10}L_0 & b_{12}L_1 \\ -b_6L_0 & b_6L_0 & 0 & -b_{12}L_1 & -b_{12}L_1 & b_{13}L_2 + b_2L_{14} \end{bmatrix}, \tag{98}$$

$$\mathbf{A} = \begin{bmatrix} -a^2\rho & 0 & 0 & 0 & 0 & 0 \\ 0 & -a^2\rho & 0 & 0 & 0 & 0 \\ 0 & 0 & -a^2\rho & 0 & 0 & 0 \\ 0 & 0 & 0 & -a^2J_x & 0 & 0 \\ 0 & 0 & 0 & 0 & -a^2J_y & 0 \\ 0 & 0 & 0 & 0 & 0 & -a^2J_z \end{bmatrix}, \tag{99}$$

$$z = [z_1, \quad z_2, \quad z_3, \quad z_4, \quad z_5, \quad z_6]^T, \tag{100}$$

and the differential operators L_i are defined as

$$L_0 = I, \quad L_1 = \frac{d}{dx_3}, \quad L_2 = \frac{d^2}{dx_3^2}$$

and the coefficients b_i are defined as

$$\begin{array}{lll} b_1 = a^2(\mu + \alpha), & b_2 = -\pi^2(\alpha + \lambda + 3\mu), & b_3 = -\pi^2(\lambda + \mu - \alpha), \\ b_4 = a\pi(\lambda + \mu - \alpha), & b_5 = 2a^2\alpha, & b_6 = 2a\pi\alpha, \\ b_7 = a^2(2\mu + \lambda), & b_8 = -2\pi^2(\alpha + \mu), & b_9 = a^2(\gamma + \varepsilon), \\ b_{10} = -\pi^2(\beta + \varepsilon + 3\gamma), & b_{11} = -\pi^2(\beta + \gamma - \varepsilon), & b_{12} = -a\pi(\beta + \gamma - \varepsilon), \\ b_{13} = a^2(\beta + 2\gamma), & b_{14} = -2\pi^2(\gamma + \varepsilon) - 4a^2\alpha & \end{array}$$

The system of differential Eq. (97) is complemented by the following boundary conditions $\mathbf{D}z = D_0$ for $x_3 = \frac{h}{2}$ and $\mathbf{D}z = 0$ for $x_3 = -\frac{h}{2}$.

$$\mathbf{D} = \begin{bmatrix} d_1L_1 & 0 & d_2L_0 & 0 & -d_3L_0 & 0 \\ 0 & d_1L_1 & d_2L_0 & d_3L_0 & 0 & 0 \\ d_4L_0 & d_4L_0 & d_5L_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_6L_1 & 0 & d_7L_0 \\ 0 & 0 & 0 & 0 & d_6L_1 & d_7L_0 \\ 0 & 0 & 0 & d_8L_0 & d_8L_0 & d_9L_1 \end{bmatrix}, \tag{101}$$

$$D_0 = [0, \quad 0, \quad a, \quad 0, \quad 0, \quad 0]^T, \tag{102}$$

and the coefficients d_i are defined as

$$\begin{array}{lll} d_1 = a(\mu + \alpha), & d_2 = -\pi(\mu - \alpha), & d_3 = 2a\alpha, \\ d_4 = a(\lambda + 2\mu), & d_5 = -\pi\lambda, & d_6 = a(\gamma + \varepsilon), \\ d_7 = a(\gamma - \varepsilon), & d_8 = \pi\beta, & d_9 = a(\beta + 2\gamma). \end{array}$$

The idea for the solution of the eigenvalue problem Eq. (97) is based on the following algorithm:

Step 1. *Fix certain frequency value*.

We fix certain value of the frequency ω and force the Cosserat body to vibrate at this frequency.

Step 2. *Solve the three-dimensional Cosserat system of equations*.

Mathematically, fixing certain value of ω implies that three-dimensional system of Eq. (97) has a constant right-hand side and therefore can be solved for the kinematic variables as a static system of equations. We solve the system Eq. (97) using the high-precision Runge-Kutta method incorporated in Mathematica software similar to how it was done in [7].

Step 3. *Find large amplitudes of the kinematic variables*.

We run ω through an interval of positive real values and take note where the solution changes its sign and the amplitude of the solutions starts to grow indefinitely. This corresponds to the oscillation of the Cosserat body at its resonant frequency. Thus, when the frequency ω coincides with the natural frequency of the plate the resonance will occur and the large amplitude linear vibrations can be observed (**Figure 1**).

The comparison of the eigenfrequencies of the Cosserat plate with the eigenfrequencies of the three-dimensional Cosserat elasticity is given in the **Table 1**. The rotatory inertia principle moments used are $J_x = 0.001, J_y = 0.001, J_z = 0.001$, which represent a ball-shaped microelement (**Figure 2**). The relative error of the natural macro frequencies associated with the rotation of the middle plane and the flexural motion is less than 1%.

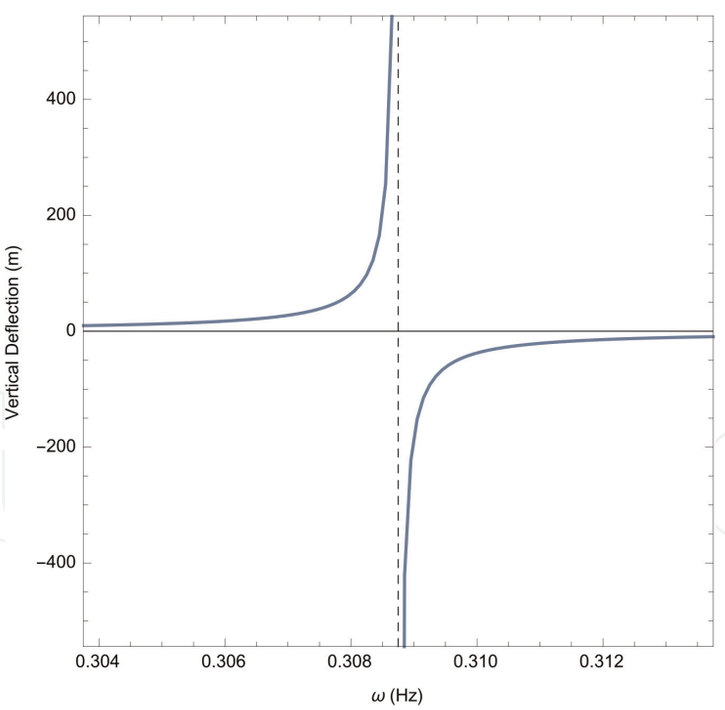

Figure 1.
Large amplitude linear vibrations of the Cosserat body forced to vibrate close to its natural frequency ω_1.

	ω_1, ω_2	ω_3, ω_7	ω_4	ω_5, ω_8	ω_6, ω_9
Plate theory	0.310	17.881	501.13	205.62	338.95
D Cosserat elasticity	0.309	17.763	530.82	211.98	317.87

Table 1.
Comparison of the eigenfrequencies ω_i (Hz) with the exact values of the 3D Cosserat elasticity.

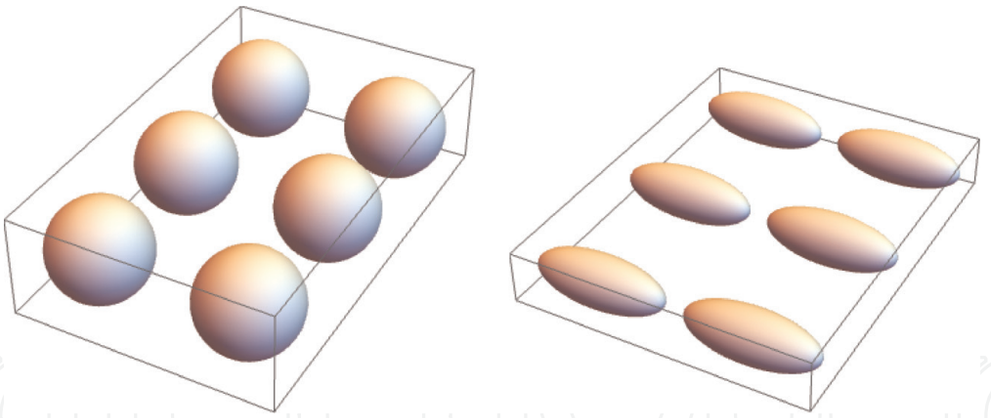

Figure 2.
Ball-shaped micro-elements: $J_x = 0.001$, $J_y = 0.001$, $J_z = 0.001$ (left) and horizontally stretched ellipsoid micro-elements: $J_x = 0.002$, $J_y = 0.001$, $J_z = 0.0001$(right).

5.2 Analysis of Cosserat plate vibrations based on the plate theory

We consider a plate $a \times a$ of thickness h with the boundary $G = G_1 \cup G_2$

$$G_1 = \{(x_1, x_2) : x_1 \in \{0, a\}, x_2 \in [0, a]\}$$
$$G_2 = \{(x_1, x_2) : x_2 \in \{0, a\}, x_1 \in [0, a]\}$$

and the following hard simply supported boundary conditions [7]:

$$G_1 : W = 0, W^* = 0, \Psi_2 = 0, \Omega_1^0 = 0, \hat{\Omega}_1^0 = 0, \Omega_3 = 0, \frac{\partial \Psi_1}{\partial n} = 0, \frac{\partial \Omega_2^0}{\partial n} = 0, \frac{\partial \hat{\Omega}_2^0}{\partial n} = 0;$$

$$G_2 : W = 0, W^* = 0, \Psi_1 = 0, \Omega_2^0 = 0, \hat{\Omega}_2^0 = 0, \Omega_3 = 0, \frac{\partial \Psi_2}{\partial n} = 0, \frac{\partial \Omega_1^0}{\partial n} = 0, \frac{\partial \hat{\Omega}_1^0}{\partial n} = 0.$$

Similar to [12] we apply the method of separation of variables for the eigenvalue problem Eq. (85) to solve for the kinematic variables Ψ_α, W, Ω_3, Ω_α^0, W^* and Ω_α^0. The kinematic variables can be further expressed in the following form

$$\Psi_1^{nm} = A_1 \cos\left(\frac{n\pi x_1}{a}\right) \sin\left(\frac{m\pi x_2}{a}\right) \sin(\omega t) + B_1 \sin\left(\frac{n\pi x_1}{a}\right) \cos\left(\frac{m\pi x_2}{a}\right) \sin(\omega t),$$
$$\Psi_2^{nm} = A_2 \sin\left(\frac{n\pi x_1}{a}\right) \cos\left(\frac{m\pi x_2}{a}\right) \sin(\omega t) + B_2 \cos\left(\frac{n\pi x_1}{a}\right) \sin\left(\frac{m\pi x_2}{a}\right) \sin(\omega t),$$
$$W^{nm} = A_3 \sin\left(\frac{n\pi x_1}{a}\right) \sin\left(\frac{m\pi x_2}{a}\right) \sin(\omega t) + B_3 \cos\left(\frac{n\pi x_1}{a}\right) \cos\left(\frac{m\pi x_2}{a}\right) \sin(\omega t),$$
$$\Omega_3^{nm} = A_4 \cos\left(\frac{n\pi x_1}{a}\right) \cos\left(\frac{m\pi x_2}{a}\right) \sin(\omega t) + B_4 \sin\left(\frac{n\pi x_1}{a}\right) \sin\left(\frac{m\pi x_2}{a}\right) \sin(\omega t),$$
$$\Omega_1^{0,nm} = A_5 \sin\left(\frac{n\pi x_1}{a}\right) \cos\left(\frac{m\pi x_2}{a}\right) \sin(\omega t) + B_5 \cos\left(\frac{n\pi x_1}{a}\right) \sin\left(\frac{m\pi x_2}{a}\right) \sin(\omega t),$$
$$\Omega_2^{0,nm} = A_6 \cos\left(\frac{n\pi x_1}{a}\right) \sin\left(\frac{m\pi x_2}{a}\right) \sin(\omega t) + B_6 \sin\left(\frac{n\pi x_1}{a}\right) \cos\left(\frac{m\pi x_2}{a}\right) \sin(\omega t),$$
$$W^{*,nm} = A_7 \sin\left(\frac{n\pi x_1}{a}\right) \sin\left(\frac{m\pi x_2}{a}\right) \sin(\omega t) + B_7 \cos\left(\frac{n\pi x_1}{a}\right) \cos\left(\frac{m\pi x_2}{a}\right) \sin(\omega t),$$
$$\hat{\Omega}_1^{0,nm} = A_8 \sin\left(\frac{n\pi x_1}{a}\right) \cos\left(\frac{m\pi x_2}{a}\right) \sin(\omega t) + B_8 \cos\left(\frac{n\pi x_1}{a}\right) \sin\left(\frac{m\pi x_2}{a}\right) \sin(\omega t),$$
$$\hat{\Omega}_2^{0,nm} = A_9 \cos\left(\frac{n\pi x_1}{a}\right) \sin\left(\frac{m\pi x_2}{a}\right) \sin(\omega t) + B_9 \sin\left(\frac{n\pi x_1}{a}\right) \cos\left(\frac{m\pi x_2}{a}\right) \sin(\omega t),$$

where A_i and B_i are constants.

	ω_1, ω_2	ω_3, ω_7	ω_4	ω_5, ω_8	ω_6, ω_9			
Shape	J_x	J_y	J_z	ω_1, ω_2	ω_3, ω_7	ω_4	ω_5, ω_8	ω_6, ω_9
Ball	0.001	0.001	0.001	17.88	0.31	501.13	205.62	338.95
Vertical ellipsoid	0.001	0.001	0.0001	17.88	0.31	501.13	650.22	338.95
Horizontal ellipsoid	0.0001	0.001	0.001	17.88	0.31	1363.01	205.62	394.08

Table 2.
Eigenfrequencies ω_i^{11} (Hz) for different shapes of micro-elements.

We solve an eigenvalue problem by substituting these expressions into the system of Eq. (85). The obtained nine sequences of positive eigenfrequencies ω_i^{nm} are associated with the rotation of the middle plane (ω_1^{nm} and ω_2^{nm}), flexural motion and its transverse variation (ω_3^{nm} and ω_7^{nm}), micro rotatory inertia (ω_4^{nm}, ω_5^{nm} and ω_6^{nm}) and its transverse variation (ω_8^{nm} and ω_9^{nm}) [12].

We perform all our numerical simulations for $a = 3.0$ m and $h = 0.1$ m. We consider different forms of micro elements: ball-shaped elements, horizontally and vertically stretched ellipsoids (see **Figure 2**). For simplicity we will use the notation ω_i for the first elements ω_i^{11} of the sequences ω_i^{nm}. The results of the computations are given in the **Table 2**. The shape of the micro-elements does not effect the natural macro frequencies ω_1 and ω_2 associated with the rotation of the middle plane and ω_3 and ω_7 associated with the flexural motion and its transverse variation. The ellipsoid elements have higher micro frequencies associated with the micro rotatory inertia (ω_4, ω_5 and ω_6) and its transverse variation (ω_8 and ω_9), than the ball-shaped elements.

Let J_x, J_y and J_z be the principal moments of inertia of the microelements corresponding to the principal axes of their rotation. We assume that the quantities J_x, J_y and J_z are constant throughout the plate B_0. If the microelements are rotated around the z-axis by the angle θ the rotatory inertia tensor $\mathbf{J}$ can be expressed as

$$\mathbf{J} = \begin{pmatrix} J_x \cos^2\theta + J_y \sin^2\theta & \left(J_x - J_y\right) \sin 2\theta & 0 \\ \left(J_x - J_y\right) \sin 2\theta & J_x \sin^2\theta + J_y \cos^2\theta & 0 \\ 0 & 0 & J_z \end{pmatrix} \tag{103}$$

Angle θ	ω_1	ω_2	ω_3	ω_7	ω_4	ω_5	ω_8	ω_6	ω_9
0°	17.88	17.88	0.31	0.31	650.221	265.37	265.37	450.61	450.61
10°	17.88	17.88	0.31	0.31	650.221	255.59	279.40	429.89	469.93
20°	17.88	17.88	0.31	0.31	650.221	247.75	295.33	406.70	484.79
30°	17.88	17.88	0.31	0.31	650.221	242.57	313.65	382.94	495.14
40°	17.88	17.88	0.31	0.31	650.221	239.99	333.10	360.57	500.46
45°	17.88	17.88	0.31	0.31	650.221	239.68	338.95	354.35	501.13
50°	17.88	17.88	0.31	0.31	650.221	239.99	333.10	360.57	500.46
60°	17.88	17.88	0.31	0.31	650.221	242.57	313.65	382.94	495.14
70°	17.88	17.88	0.31	0.31	650.221	247.75	295.33	406.70	484.79
80°	17.88	17.88	0.31	0.31	650.221	255.59	279.40	429.89	469.93
90°	17.88	17.88	0.31	0.31	650.221	265.37	265.37	450.61	450.61

Table 3.
Eigenfrequencies ω_i^{11} (Hz) for different angles of rotation of horizontal ellipsoid micro-elements.

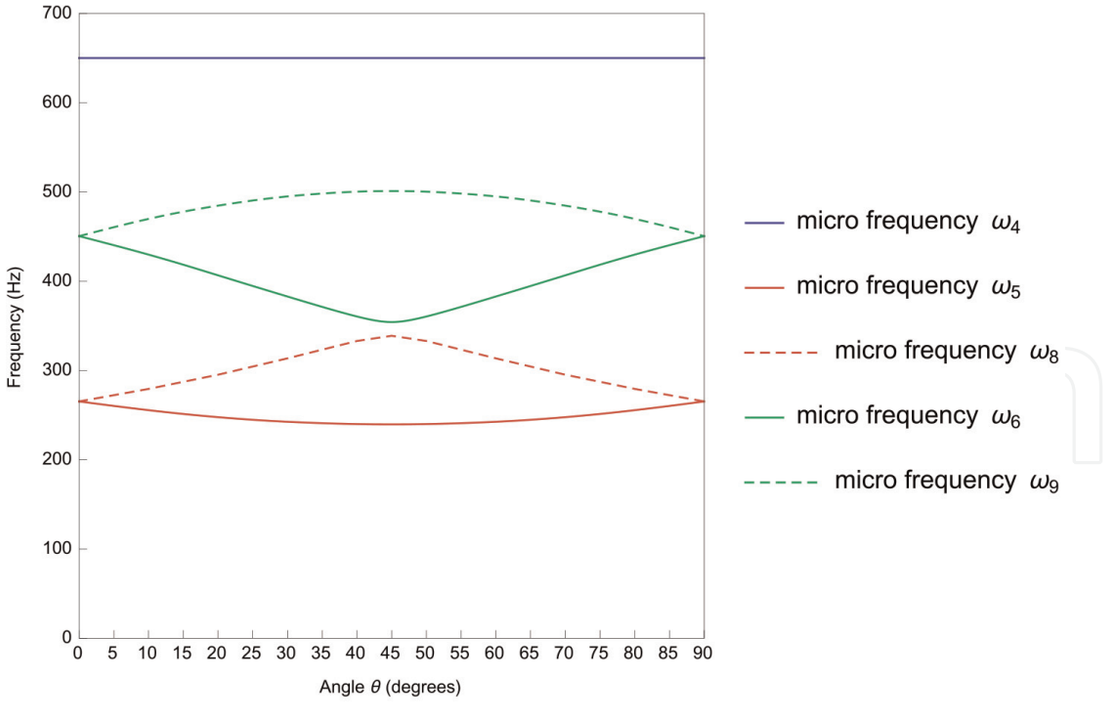

Figure 3.
Micro frequencies ω_4, ω_5, ω_8, ω_6 and ω_9.

The eigenfrequencies for different angles of microrotation of the microelements are given in the **Table 3** and the **Figure 3**. The rotatory inertia principle moments used are $J_x = 0.002, J_y = 0.001, J_z = 0.0001$, which represent a horizontally stretched ellipsoid microelement. The case when the microelements are not aligned with the edges of the plate the model predicts some additional natural frequencies related with the microstructure of the material.

6. Conclusions

In this chapter, we presented a mathematical model of Cosserat plate vibrations. The dynamic model of the plates has been developed as a dynamic extension of the Reissner plate theory. The equations has been presented in both tensorial and the matrix forms. We also described the validation of the model, which is based on the comparison with the three-dimensional Cosserat elastodynamics exact solutions. Based on the presented results of the computer simulations we were able to detect and classify the additional high resonance frequencies of a plate. We have shown that the frequencies depend on the shape and orientation of microelements (ball-shaped elements, horizontally and vertically stretched ellipsoids) incorporated into the Cosserat plates. We also have been able to identify that micro frequencies associated with the micro rotatory inertia and its transverse variation of the ellipsoid elements have higher micro frequencies than the ball-shaped elements. We also showed the dependence of the eigenfrequencies on the angles of rotation of the horizontal ellipsoid micro-elements. These results can be used to identify the characteristics of the plate micro-elements.

Appendix A: conventions and notations

A.1 Conventions

We use the following notation convention:

1. the values of the Latin subindex i take values in the set $\{1, 2, 3\}$
2. the values of the Greek indices α and β take values in the set $\{1, 2\}$
3. the Einstein summation notation is used throughout the chapter

A.2 Notations

$\{x_i\}$	artesian coordinates
P	Cosserat thin plate
h	plate thickness
μ, λ	Lamé parameters
α, β, γ, ε	Cosserat elasticity parameters
ρ	material density
J_{ji} or $\mathbf{J}$	rotatory inertia
σ_{ji} or $\boldsymbol{\sigma}$	the stress tensor
μ_{ji} or $\boldsymbol{\mu}$	the couple stress tensor
γ_{ji} or $\boldsymbol{\gamma}$	strain tensor
χ_{ji} or $\boldsymbol{\chi}$	bend-twist tensor
u_i or $\mathbf{u}$	displacement vector
ϕ_i or $\boldsymbol{\phi}$	microrotation vector
p_i or $\mathbf{p}$	linear momentum
q_i or $\mathbf{q}$	angular momentum
ε_{ijk}	Levi-Civita tensor
U_C	strain stored energy
U_K	stress energy
T_C	stored kinetic energy
T_W	work of inertia forces
$\mathcal{S}$	Cosserat plate stress set
$\mathcal{U}$	Cosserat plate displacement set
$\mathcal{E}$	Cosserat plate strain set
η	splitting parameter
p	pressure
ω	natural frequency of plate vibration
θ	angle of microelement orientation
$M_{\alpha\beta}$	bending and twisting moments
Q_α	shear forces
Q_α^*, $\hat{Q}_\alpha$	transverse shear forces
$R_{\alpha\beta}$	micropolar bending moments
$R_{\alpha\beta}^*$	micropolar twisting moments
S_α^*	micropolar couple moments

Ψ_α	rotations of the middle plane around x_α axis
W, W^*	vertical deflections of the middle plate
Ω_α^0	microrotations in the middle plate around x_α axis
Ω_3	rate of change of the microrotation

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References

[1] Cosserat E, Cosserat F. Theorie des corps deformables [Theory of Deformable Bodies]. Paris: A. Hermann et fils; 1909

[2] Green A, Naghdi P. The linear theory of an elastic Cosserat plate. Mathematical Proceedings of the Cambridge Philosophical Society. 1966; **63**:537-550

[3] Eringen A. Theory of micropolar plates. Journal of Applied Mathematics and Physics. 1967;**18**:12-31

[4] Altenbach H, Eremeyev V. On the theories of plates based on the Cosserat approach. Advances in Applied Mathematics and Mechanics. 2010;**21**: 73-92

[5] Steinberg L. Deformation of micropolar plates of moderate thickness. International Journal of Applied Mathematics and Mechanics. 2010; **6**(17):1-24

[6] Kvasov R, Steinberg L. Numerical modeling of bending of Cosserat elastic plates. In: Proceedings of the 5th Computing Alliance of Hispanic-Serving Institutions; 2011. pp. 67-70

[7] Steinberg L, Kvasov R. Enhanced mathematical model for Cosserat plate bending. Thin-Walled Structures. 2013; **63**:51-62

[8] Reissner E. On the theory of elastic plates. Journal of Mathematics and Physics. 1944;**23**:184-191

[9] Reissner E. The effect of transverse shear deformation on the bending of elastic plates. Journal of Applied Mechanics. 1945:69-77

[10] Kvasov R, Steinberg L. Numerical modeling of bending of micropolar plates. Thin-Walled Structures. 2013;**69**: 67-78

[11] Kvasov R, Steinberg L. Modeling of size effects in bending of perforated Cosserat plates. Modelling and Simulation in Engineering. 2017;**2017**:1-19

[12] Steinberg L, Kvasov R. Analytical modeling of vibration of micropolar plates. Applied Mathematics. 2015;**6**: 817-836

[13] Nowacki W. Theory of Asymmetric Elasticity. Oxford, New York: Pergamon Press; 1986

[14] Gurtin M. The linear theory of elasticity. In: Truesdell C, editor. Handbuch der Physik. Vol. VIa/2. Berlin: Springer-Verlag; 1972. pp. 1-296

[15] Lakes R. Experimental methods for study of Cosserat elastic solids and other generalized elastic continua. In: Mühlhaus H, editor. Continuum Models for Materials with Microstructures. New York: Wiley J; 1995. pp. 1-22

[16] Singh S. Blowing agents for polyurethane foams. Rapra Review Report. 2002;**12**(10)