Chapter

Loop-like Solitons

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Abstract

The physical phenomena that take place in nature generally have complicated nonlinear features. A variety of methods for examining the properties and solutions of nonlinear evolution equations are explored by using the Vakhnenko equation (VE) as an example. One remarkable feature of the VE is that it possesses loop-like soliton solutions. Loop-like solitons are a class of interesting wave phenomena, which have been involved in some nonlinear systems. The VE can be written in an alternative form, known as the Vakhnenko-Parkes equation (VPE). The VPE can be written in Hirota bilinear form. The Hirota method not only gives the N-soliton solution but enables one to find a way from the Bäcklund transformation through the conservation laws and associated eigenvalue problem to the inverse scattering transform (IST) method. This method is the most appropriate way of tackling the initial value problem (Cauchy problem). The standard procedure for IST method is expanded for the case of multiple poles, specifically, for the double poles with a single pole. In recent papers some physical phenomena in optics and magnetism are satisfactorily described by means of the VE. The question of physical interpretation of multivalued (loop-like) solutions is still an open question.

Keywords: nonlinear evolution equations, solutions, Vakhnenko equation, Hirota method, Bäcklund transformation, inverse scattering problem, N-soliton solution, spectral data
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1. The high-frequency perturbations in a relaxing medium

From the nonequilibrium thermodynamic standpoint, models of a relaxing medium are more general than equilibrium models. To develop physical models for wave propagation through media with complicated inner kinetics, notions based on the relaxational nature of a phenomenon are regarded to be promising. Thermodynamic equilibrium is disturbed owing to the propagation of fast perturbations. There are processes of the interaction that tend to return the equilibrium. The parameters characterizing this interaction are referred to as the inner variables unlike the macroparameters such as the pressure p, mass velocity u and density ρ. In essence, the change of macroparameters caused by the changes of inner parameters is a relaxation process.

We restrict our attention to barotropic media. An equilibrium state equation of a barotropic medium is a one-parameter equation. As a result of relaxation, an additional variable ξ (the inner parameter) appears in the state equation

$$p = p(\rho, \xi) \tag{1}$$

and defines the completeness of the relaxation process. There are two limiting cases with corresponding sound velocities:

i. Lack of relaxation (inner interaction processes are frozen) for which $\xi = 1$:

$$p = p(\rho, 1) \equiv p_f(\rho), \quad c_f^2 = dp_f/d\rho; \tag{2}$$

ii. Relaxation which is complete (there is local thermodynamic equilibrium) for which $\xi = 0$:

$$p = p(\rho, 0) \equiv p_e(\rho), \quad c_e^2 = dp_e/d\rho. \tag{3}$$

Slow and fast processes are compared by means of the relaxation time τ_p.

To analyse the wave motion, we use the following hydrodynamic equations in Lagrangian coordinates:

$$\frac{\partial V}{\partial t} - \frac{1}{\rho_0}\frac{\partial u}{\partial x} = 0, \qquad \frac{\partial u}{\partial t} + \frac{1}{\rho_0}\frac{\partial p}{\partial x} = 0. \tag{4}$$

The following dynamic state equation is applied to account for the relaxation effects:

$$\tau_p\left(\frac{dp}{dt} - c_f^2\frac{d\rho}{dt}\right) + (p - p_e) = 0. \tag{5}$$

Here $V \equiv \rho^{-1}$ is the specific volume and x is the Lagrangian space coordinate. Clearly, for the fast processes $(\omega\tau_p \gg 1)$, we have relation (2), and for the slow ones $(\omega\tau_p \ll 1)$, we have (3).

The closed system of equations consists of two motion equations (4) and dynamic state equation (5). The motion equations (4) are written in Lagrangian coordinates since the state equation (5) is related to the element of mass of the medium.

The substantiation of (5) within the framework of the thermodynamics of irreversible processes has been given in [1, 2]. We note that the mechanisms of the exchange processes are not defined concretely when deriving the dynamic state equation (5). In this equation the thermodynamic and kinetic parameters appear only as sound velocities c_e and c_f and relaxation time τ_p. These are very common characteristics and they can be found experimentally. Hence, it is not necessary to know the inner exchange mechanism in detail.

Combining the relationships (4) and (5), we obtain for low-frequency perturbations $(\tau_p\omega \ll 1)$ the Korteweg-de Vries-Burgers (KdVB) equation:

$$\begin{aligned} &\frac{\partial p}{\partial t} + c_e\frac{\partial p}{\partial x} + \alpha_e c_e^3 p\frac{\partial p}{\partial x} - \beta_e\frac{\partial^2 p}{\partial x^2} + \gamma_e\frac{\partial^3 p}{\partial x^3} = 0, \\ &\beta_e = \frac{c_e^2\tau_p}{2c_f^2}\left(c_f^2 - c_e^2\right), \quad \gamma_e = \frac{c_e^3\tau_p^2}{8c_f^4}\left(c_f^2 - c_e^2\right)\left(c_f^2 - 5c_e^2\right), \end{aligned} \tag{6}$$

whilst for high-frequency waves $(\tau_p\omega \gg 1)$, we have obtained the following equation:

$$\frac{\partial^2 p}{\partial x^2} - c_f^{-2}\frac{\partial^2 p}{\partial t^2} + \alpha_f c_f^2 \frac{\partial^2 p^2}{\partial x^2} + \beta_f \frac{\partial p}{\partial x} + \gamma_f p = 0,$$
$$\beta_f = \frac{c_f^2 - c_e^2}{\tau_p c_e^2 c_f}, \quad \gamma_f = \frac{c_f^4 - c_e^4}{2\tau_p^2 c_e^4 c_f^2}. \tag{7}$$

Equation (6) with ($\beta_e = 0$) is the well-known the Korteweg-de Vries (KdV) equation. The investigation of the KdV equation in conjunction with the nonlinear Schrodinger (NLS) and sine-Gordon equations gives rise to the theory of solitons [4–13].

We focus our main attention on (7). It has a dissipative term $\beta_f \partial p/\partial x$ and a dispersive term $\gamma_f p$. Without the nonlinear and dissipative terms, we have a linear Klein-Gordon equation.

Let us write down (7) in dimensionless form. In the moving coordinate system with velocity c_f, after factorization the equation has the form in the dimensionless variables $\tilde{x} = \sqrt{\frac{\gamma_f}{2}}(x - c_f t), \tilde{t} = \sqrt{\frac{\gamma_f}{2}} c_f t, \tilde{u} = \alpha_f c_f^2 p$ (tilde over variables $\tilde{x}, \tilde{t}$ and $\tilde{u}$ is omitted)

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)u + \alpha\frac{\partial u}{\partial x} + u = 0. \tag{8}$$

The constant $\alpha = \beta_f/\sqrt{2\gamma_f}$ is always positive. Equation (8) without the dissipative term has the form of the nonlinear equation [14, 15]:

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)u + u = 0. \tag{9}$$

Historically, (9) has been called the Vakhnenko equation (VE), and we will follow this name.

We note that (9) follows as a particular limit of the following generalized Korteweg-de Vries equation:

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \beta\frac{\partial^3 u}{\partial x^3}\right) = \gamma u \tag{10}$$

derived by Ostrovsky [16] to model small-amplitude long waves in a rotating fluid (γu is induced by the Coriolis force) of finite depth. Subsequently, (9) was known by different names in the literature, such as the Ostrovsky-Hunter equation, the short-wave equation, the reduced Ostrovsky equation, and the Ostrovsky-Vakhnenko equation depending on the physical context in which it is studied.

The consideration here of (9) has interest from the viewpoint of the investigation of the propagation of high-frequency perturbations.

2. Loop-like stationary solutions

The travelling wave solutions are solutions which are stationary with respect to a moving frame of reference. In this case, the evolution equation (a partial differential equation) becomes an ordinary differential equation (ODE) which is considerably easier to solve.

For the VE (9) it is convenient to introduce a new dependent variable z and new independent variables η and τ defined by

$$z = (u - v)/|v|, \quad \eta = (x - vt)/|v|^{1/2}, \quad \tau = t|v|^{1/2}, \tag{11}$$

where v is a nonzero constant [15]. Then the VE becomes begin equation:

$$z_{\eta\tau} + \left(zz_{\eta}\right)_{\eta} + z + c = 0, \tag{12}$$

where $c = \pm 1$ corresponding to $v \gtrless 0$. We now seek stationary solutions of (12) for which z is a function of η only so that $z_{\tau} = 0$ and z satisfies the ODE:

$$\left(zz_{\eta}\right)_{\eta} + z + c = 0. \tag{13}$$

After one integration (13) gives

$$\begin{aligned} \frac{1}{2}\left(zz_{\eta}\right)^2 &= f(z), \\ f(z) &= -\frac{1}{3}z^3 - \frac{1}{2}cz^2 + \frac{1}{6}A = -\frac{1}{3}(z - z_1)(z - z_2)(z - z_3). \end{aligned} \tag{14}$$

where A is a constant, and for periodic solutions z_1, z_2 and z_3 are real constants such that $z_1 \leq z_2 \leq z_3$. On using results 236.00 and 236.01 of [17], we may integrate (14) to obtain

$$\eta = \frac{\sqrt{6}z_1}{\sqrt{z_3 - z_1}}F(\varphi, m) + \sqrt{6(z_3 - z_1)}E(\varphi, m), \tag{15}$$

$$\sin\varphi = \frac{z_3 - z}{z_3 - z_2}, \quad m = \frac{z_3 - z_2}{z_3 - z_1}. \tag{16}$$

where $F(\varphi, m)$ and $E(\varphi, m)$ are incomplete elliptic integrals of the first and second kind, respectively. We have chosen the constant of integration in (15) to be zero so that $z = z_3$ at $\eta = 0$. The relations (15) give the required solution in parametric form, with z and η as functions of the parameter φ.

For $c = 1$ (i.e., $v > 0$), there are periodic solutions for $0 < A < 1$ with $\lambda < 0$, $z_2 \in (-1, 0)$ and $z_3 \in (0, 0.5)$; an example of such a periodic wave is illustrated by curve 2 in **Figure 1**. Here we introduce a new independent variable ζ defined by

$$\frac{d\eta}{d\zeta} = z. \tag{17}$$

$A = 1$ gives the solitary wave limit:

$$u = \frac{3}{2}v\,\mathrm{sech}^2(\zeta/2), \quad \eta = -\zeta + 3\tanh(\zeta/2) \tag{18}$$

as illustrated by curve 1 in **Figure 1**. The periodic waves and the solitary wave have a loop-like structure as illustrated in **Figure 1**. For $c = -1$ (i.e., $v < 0$), there are periodic waves for $-1 < A < 0$ with $\lambda > 0$, $z_2 \in (0, 1)$ and $z_3 \in (1, 1.5)$; an example of such a periodic wave is illustrated by curve 2 in **Figure 2**. When $A = 0$ and $\lambda = 6$, then the periodic wave solution simplifies to

$$u(\eta)/|v| = -\frac{1}{6}\eta^2 + \frac{1}{2}, \qquad -3 \leq \eta \leq 3, \qquad u(\eta + 6) = u(\eta). \tag{19}$$

This is shown by curve 1 in **Figure 2**. For $A \simeq -1$ the solution has a sinusoidal form (curve 3 in **Figure 2**). Note that there are no solitary wave solutions.

A remarkable feature of the equation (9) is that it has a solitary wave (18) which has a loop-like form, i.e., it is a multivalued function (see **Figure 1**). Whilst loop

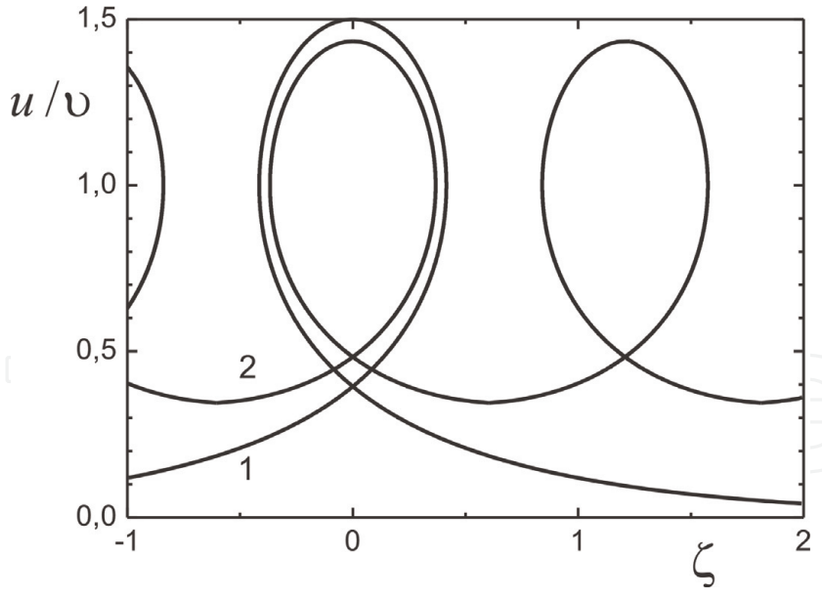

Figure 1.
Travelling wave solutions with $v > 0$.

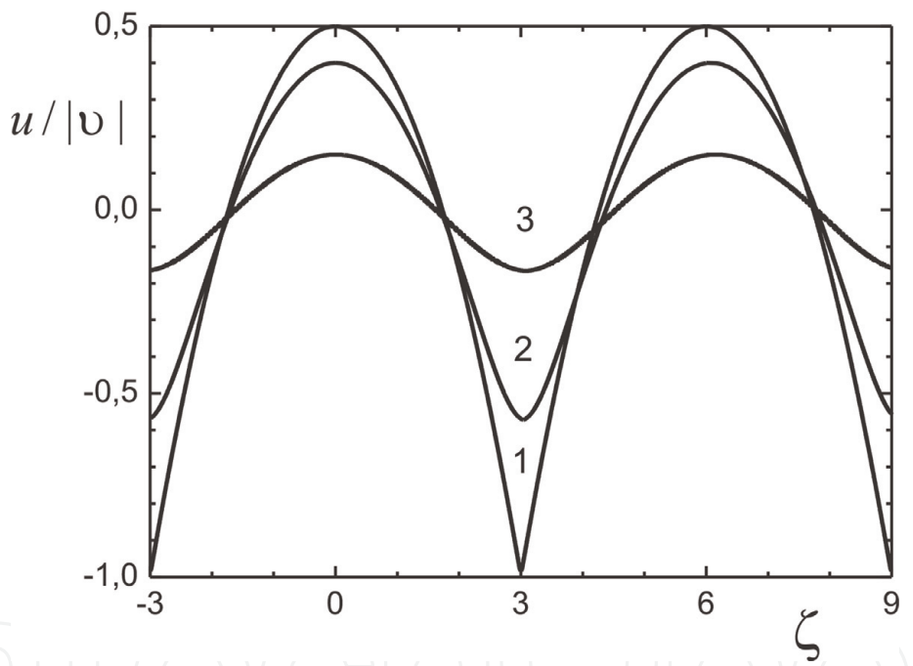

Figure 2.
Travelling wave solutions with $v < 0$.

solitary waves (18) are rather intriguing, it is the solution to the initial value problem that is of more interest in a physical context. An important question is the stability of the loop-like solutions. Although the analysis of stability does not link with the theory of solitons directly, the method applied in [15] is instructive, since it is successful in a nonlinear approximation. Stability of the loop-like solutions has been proved in [15]. From a physical viewpoint, the stability or otherwise of solutions is essential to their interpretation.

3. The Vakhnenko-Parkes equation

The multivalued solutions obtained in Section 2 obviously mean that the study of the VE (9) in the original coordinates (x, t) leads to certain difficulties.

These difficulties can be avoided by writing down the VE in new independent coordinates. We have succeeded in finding these coordinates. Historically, working separately, we (Vyacheslav Vakhnenko in Ukraine and John Parkes in the UK) independently suggested such independent coordinates in which the solutions become one-valued functions. It is instructive to present the two derivations here. In one derivation a physical approach, namely, a transformation between Euler and Lagrange coordinates, was used, whereas in the other derivation, a pure mathematical approach was used.

Let us define new independent variables (X, T) by the transformation

$$\varphi dT = dx - u\,dt, \qquad X = t. \tag{20}$$

The function φ is to be obtained. It is important that the functions $x = \theta(X, T)$ and $u = U(X, T)$ turn out to be single-valued. In terms of the coordinates (X, T), the solution of the VE (9) is given by single-valued parametric relations. The transformation into these coordinates is the key point in solving the problem of the interaction of solitons as well as explaining the multivalued solutions [3]. The transformation (20) is similar to the transformation between Eulerian coordinates (x, t) and Lagrangian coordinates (X, T). We require that $T = x$ if there is no perturbation, i.e., if $u(x, t) \equiv 0$. Hence $\varphi = 1$ when $u(x, t) \equiv 0$.

The function φ is the additional dependent variable in the equation system (22), (24) to which we reduce the original Eq. (9). We note that the transformation inverse to (20) is

$$dx = \varphi dT + U dX, \quad t = X, \quad U(X, T) \equiv u(x, t). \tag{21}$$

It follows that

$$\frac{\partial x}{\partial X} = U, \quad \frac{\partial x}{\partial T} = \varphi, \quad \frac{\partial t}{\partial X} = 1, \quad \frac{\partial t}{\partial T} = 0.$$

Hence

$$\frac{\partial \varphi}{\partial X} = \frac{\partial U}{\partial T} \tag{22}$$

and

$$\frac{\partial}{\partial X} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial T} = \varphi\frac{\partial}{\partial x}. \tag{23}$$

By using (23), we can write Eq. (9) in terms of $\varphi(X, T)$ and $U(X, T)$, namely,

$$U_{XT} + \varphi U = 0. \tag{24}$$

Equations (22) and (24) are the main system of equations. It can be reduced to a nonlinear equation (27) in one unknown W defined by

$$W_X = U. \tag{25}$$

From (22), (25) and the requirement that $\varphi = 1$ when $U \equiv 0$, we have

$$\varphi = 1 + W_T. \tag{26}$$

Then, by eliminating φ and U between (24), (25) and (26), we arrive at a transformed form of the VE (9), namely,

$$W_{XXT} + (1 + W_T)W_X = 0. \tag{27}$$

Alternatively, by eliminating φ between (22) and (24), we obtain

$$UU_{XXT} - U_X U_{XT} + U^2 U_T = 0. \tag{28}$$

Furthermore it follows from (21) that the original independent coordinates (x, t) are given by

$$x = \theta(X, T) = x_0 + T + W, \qquad t = X, \tag{29}$$

where x_0 is an arbitrary constant. Since the functions $\theta(X, T)$ and $U(X, T)$ are single-valued, the problem of multivalued solutions has been resolved from the mathematical point of view.

Alternatively, in a pure mathematical approach, we may start by introducing new independent variables X and T defined by

$$x = T + \int_{-\infty}^{X} U(X', T)dX' + x_1, \qquad t = X, \tag{30}$$

where x_1 is an arbitrary constant. From (30), we obtain (23) but with

$$\varphi(X, T) = 1 + \int_{-\infty}^{X} U_T dX'. \tag{31}$$

Now, on introducing (25), (30) and (31) may be identified with (29) and (26), respectively. The derivation of (27) and (28) proceeds as before.

The transformation into new coordinates, as has already been pointed out, was obtained by us independently of each other; nevertheless, we published the result together [18, 19]. Following the papers [20–23] hereafter, Eq. (27) (or in alternative form (28)) is referred to as the Vakhnenko-Parkes equation (VPE).

The travelling wave solution (15) and (16) for Equation (9) is also a travelling wave solution when written in terms of the transformed coordinates (X,T). In order to do this, we need to express the independent variable ζ, as introduced in (17), in terms of X and T.

From the expressions for z in (11) and (17), we obtain

$$\frac{d\eta}{d\zeta} = \frac{U - v}{|v|} \tag{32}$$

so that

$$|v|\eta = \int U d\zeta - v\zeta. \tag{33}$$

From the definition of η in (17), and the expressions for x and t given by (29), we obtain

$$|v|\eta = |v|^{1/2}[W - v(X - VT)], \quad \text{where} \quad V := v^{-1}. \tag{34}$$

The expressions for $|v|\eta$ in (33) and (34) are equivalent if

$$\zeta = |v|^{1/2} Z, \quad \text{where} \quad Z := X - VT - X_0 \tag{35}$$

and X_0 is an arbitrary constant, so that

$$W = \int U dZ \text{ and } U = W_Z. \tag{36}$$

Hence, from (34), it follows that

$$W = \frac{\sqrt{|v|}}{p}[(z_1 + c)w + (z_3 - z_1)E(w|m)] + W_0, \tag{37}$$

where $w = p\sqrt{|v|}Z$ and W_0 is an arbitrary constant. Then

$$\frac{U}{|v|} = c + z_3 - (z_3 - z_2)\mathrm{sn}^2(w|m), \quad \text{where } w = p\sqrt{|v|}Z. \tag{38}$$

Eqs. (37) and (38) give the travelling wave solutions to the VPE in the forms (27) and (28), respectively. Eq. (38) is also the travelling wave solution of the VE (9) expressed in terms of the new coordinates (X,T). In the limiting case $m = 1$, (38) gives a solitary wave in the following two forms: For $v > 0$

$$U/v = \frac{3}{2}\mathrm{sech}^2\left(\frac{1}{2}\sqrt{v}Z\right) \tag{39}$$

and, for $v < 0$,

$$U/|v| = -1 + \frac{3}{2}\mathrm{sech}^2\left(\frac{1}{2}\sqrt{|v|}Z\right). \tag{40}$$

These two solutions are illustrated by curve 1 in **Figures 3** and **4**, respectively. The other curves illustrate examples of the solution given by (38) when $m \neq 1$. Curves 1 and 2 in **Figure 3** relate to curves 1 and 2, respectively, in **Figure 1**. Curves 1, 2 and 3 in **Figure 4** relate to curves 1, 2 and 3, respectively, in **Figure 2**.

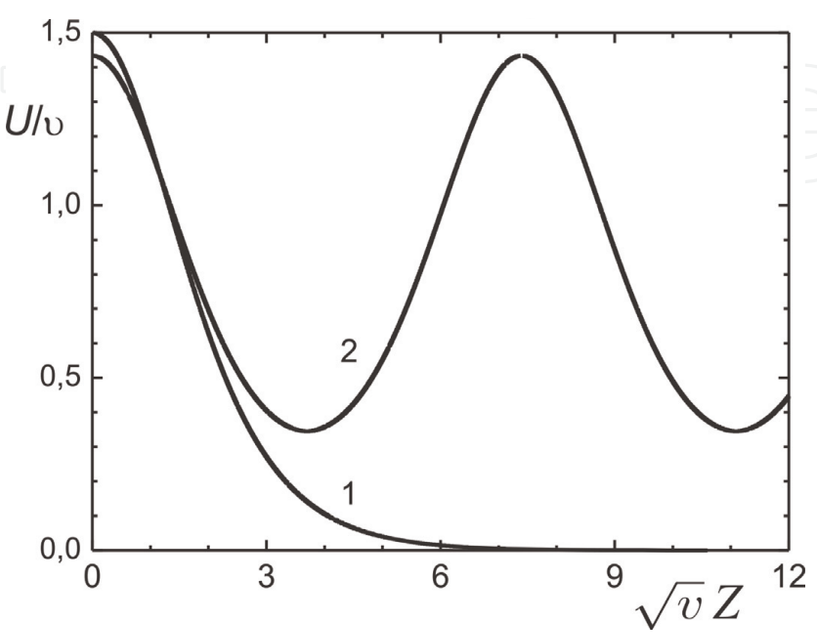

Figure 3.
Travelling wave solutions with $v > 0$ in coordinates (X, T).

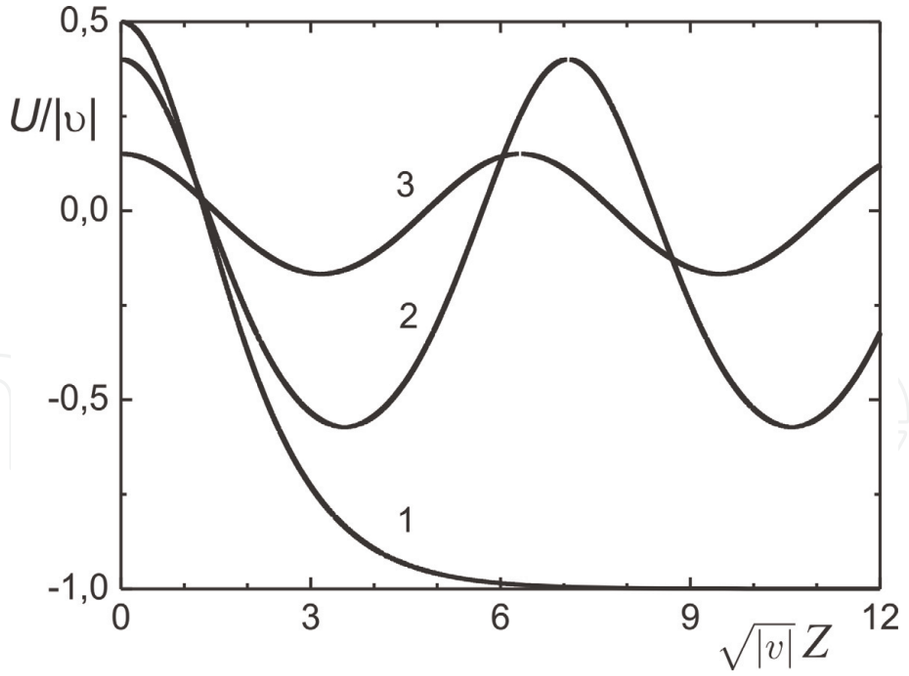

Figure 4.
Travelling wave solutions with $v < 0$ in coordinates (X, T).

There are two important observations to be made. Firstly, all the travelling wave solutions in terms of the new coordinates are single-valued. Secondly, the periodic solution shown by curve 1 in **Figure 2**, i.e., the solution consisting of parabolas, is not periodic in terms of the new coordinates. Hence, we reveal some accordance between curve 1 in **Figure 3** and curve 1 in **Figure 4**. These features are important for finding the solutions by the inverse scattering method [24–30].

4. From Hirota method to the inverse scattering method

The Hirota method gives the N-soliton solution as well as enables one to find a way from the Bäcklund transformation through the conservation laws and associated eigenvalue problem to the inverse scattering method [24]. Thus, the Hirota method allows us to formulate the inverse scattering method which is the most appropriate way of tackling the initial value problem (Cauchy problem).

In the Hirota method the equation, in our case the VPE (27), under investigation should be transformed into the Hirota bilinear form [9, 24]:

$$\left(D_T D_X^3 + D_X^2\right) f \cdot f = 0, \tag{41}$$

with

$$W = 6(\ln f)_X, \tag{42}$$

The Hirota bilinear D-operator is defined as (see Section 5.2 in [9])

$$D_T^n D_X^m a \cdot b = \left(\frac{\partial}{\partial T} - \frac{\partial}{\partial T'}\right)^n \left(\frac{\partial}{\partial X} - \frac{\partial}{\partial X'}\right)^m a(T, X) b(T', X') \Big|_{T=T', X=X'}. \tag{43}$$

Now we present a Bäcklund transformation for VPE (27) written in the bilinear form (41). This type of Bäcklund transformation was first introduced by Hirota [31]

and has the advantage that the transformation equations are linear with respect to each dependent variable. This Bäcklund transformation can be transformed to the ordinary one [24]:

$$\left(D_X^3 - \lambda\right) f' \cdot f = 0, \tag{44}$$

$$\left(3D_X D_T + 1 + \mu D_X\right) f' \cdot f = 0, \tag{45}$$

where $\lambda = \lambda(X)$ is an arbitrary function of X and $\mu = \mu(T)$ is an arbitrary function of T.

The inverse scattering transform (IST) method is arguably the most important discovery in the theory of solitons. The method enables one to solve the initial value problem for a nonlinear evolution equation. Moreover, it provides a proof of the complete integrability of the equation.

The essence of the application of the IST is as follows. The initial equation VPE (27) is written as the compatibility condition for two linear equations. These equations are presented in (47) and (48). Then $W(X, 0)$ is mapped into the scattering $S(0)$ for (47). It is important that since the variable $W(X, T)$ contained in the spectral equation (47) evolves according to (27), the spectrum λ always retains constant values. The time evolution of $S(T)$ is simple and linear. From a knowledge of $S(T)$, we reconstruct $W(X, T)$.

The use of the IST is the most appropriate way of tackling the initial value problem. In order to apply the IST method, one first has to formulate the associated eigenvalue problem. This can be achieved by finding a Bäcklund transformation associated with the VPE.

Now we will show that the IST problem for the VPE in the form (27) has a third-order eigenvalue problem that is similar to the one associated with a higher-order KdV equation [32, 33], a Boussinesq equation [33–37] and a model equation for shallow water waves [9, 38].

Introducing the function

$$\psi = f'/f, \tag{46}$$

and taking into account (42), we find that (44) and (45) reduce to

$$\psi_{XXX} + W_X \psi_X - \lambda\psi = 0, \tag{47}$$

$$3\psi_{XT} + (1 + W_T)\psi + \mu\psi_X = 0, \tag{48}$$

respectively, where we have used results similar to (X.1)–(X.3) in [9].

From (47) and (48), it can be shown that

$$3\lambda\psi_T + (1 + W_T)\psi_{XX} - W_{XT}\psi_X + [W_{XXT} + (1 + W_T)W_X + \mu\lambda]\psi = 0 \tag{49}$$

and

$$[W_{XXT} + (1 + W_T)W_X]_X \psi + (3\psi_T + \mu\psi)\lambda_X = 0. \tag{50}$$

In view of (27), (49) becomes

$$3\lambda\psi_T + (1 + W_T)\psi_{XX} - W_{XT}\psi_X + \lambda\mu\psi = 0, \tag{51}$$

and (50) implies that $\lambda_X = 0$ so the spectrum λ of (47) remains constant. Constant λ is what is required in the IST problem. Equation (50) yields the equation $W_{XXT} + (1 + W_T)W_X = h(T)$, where $h(T)$ is an arbitrary function of T.

Now, according to (62) and (72), the inverse scattering method restricts the solutions to those that vanish as $|X| \to \infty$, so $h(T)$ is to be identically zero. Thus, the pair of equations (47) and (48) or (47) and (50) can be considered as the Lax pair for the VPE (27).

Since (47) and (48) are alternative forms of Eqs. (44) and (45), respectively, it follows that the pair of equations (47) and (48) is associated with the VPE (27) considered here. Thus, the IST problem is directly related to a spectral equation of third order, namely, (47). The inverse problem for certain third-order spectral equations has been considered by Kaup [33] and Caudrey [34, 35]. As expected, (47) and (48) are similar to, but cannot be transformed into, the corresponding equations for the Hirota-Satsuma equation (HSE) (see Eq. (A8a) and (A8b) in [39]). Clarkson and Mansfield [40] note that the scattering problem for the HSE is similar to that for the Boussinesq equation which has been studied comprehensively by Deift et al. [37].

5. The inverse scattering method for a third-order equation

5.1 Example of the use of the IST method to find the one-soliton solution

Consider the one-soliton solution of the VPE by application of the IST method. Let the initial perturbation be

$$W(X,0) = 6k(1 + \tanh(\eta)), \quad \eta = kX + \alpha. \tag{52}$$

For convenience we introduce new notation ξ_1 and β_1 instead of parameters k and α by

$$k = \frac{\sqrt{3}}{2}\xi_1, \quad \alpha = \frac{1}{2}\ln\left(\beta_1/2\sqrt{3}\xi_1\right) \tag{53}$$

then

$$W(X,0) = 6\sqrt{3}\xi_1 \frac{\partial}{\partial X}\ln\left[1 + \frac{\beta_1}{2\sqrt{3}\xi_1}\exp\left(\sqrt{3}\xi_1 X\right)\right] \tag{54}$$

is the initial condition for the VPE.

The first step in the IST method is to solve the spectral equation (47) with spectral parameter λ for the given initial condition $W(X,0)$. In our example it is (54). The solution is studied over the complex ζ-plane, where $\zeta^3 = \lambda$. One can verify by direct substitution of (55) in (47) that the solution $\psi(X,0;\zeta)$ of the linear ODE (47), normalized so that $\psi(X,0;\zeta)\exp(-\zeta X) \to 1$ at $X \to -\infty$, is given by

$$\psi(X,0;\zeta)\exp(-\zeta X) = 1 - \frac{\beta_1 \exp\left(\sqrt{3}\xi_1 X\right)}{1 + \beta_1 \frac{\exp\left(\sqrt{3}\xi_1 X\right)}{2\sqrt{3}\xi_1}}\left[\frac{\omega_2}{\mathrm{i}\omega_2\xi_1 - \zeta} + \frac{\omega_3}{-\mathrm{i}\omega_3\xi_1 - \zeta}\right], \tag{55}$$

where $\omega_j = e^{\mathrm{i}2\pi(j-1)/3}$ are the cube roots of 1 ($j = 1, 2, 3$). The constants β_1 and ξ_1, as we will show, are associated with the local spectral data.

The second step in the IST method is to obtain the evolution of β_1 and ξ_1. The time dependence of the solution $\psi(X,T)$ is described by Eq. (48). Analysing Eq. (48), we may assume that

$$\xi_1(T) = \xi_1(0) = \text{const.},$$
$$\beta_1(T) = \beta_1(0)\exp\left(-\frac{1}{\sqrt{3}\xi_1}T\right). \tag{56}$$

Below, the assumption of these relationships will be justified. Indeed, we know that the spectrum λ in (47) remains constant if $W(X,T)$ evolves according to Eq. (27). Therefore, as will be proved, the spectrum data evolve as in (70). In notations (77) and (78), from (70) we obtain the relations (56).

The final step in IST method is to select the solution $W(X,T)$ from (55) with $\xi_1(T)$, $\beta_1(T)$ as in (56). According to Eq. (2.7) in [33], we expand $\psi(X,T;\zeta)$ as an asymptotic series in ζ^{-1} to obtain

$$\psi(X,0;\zeta)\exp(-\zeta X) = 1 - \frac{1}{3\zeta}[W(X) - W(-\infty)] + O\left(\zeta^{-2}\right), \tag{57}$$

i.e., $W(X) - W(-\infty) = \lim_{\zeta\to\infty}[3\zeta(1 - \psi\exp(-\zeta X))]$. Taking into account the functional dependence (56), we find the required one-soliton solution of the VPE in form

$$W(X,T) = 6\sqrt{3}\xi_1\frac{\partial}{\partial X}\ln\left[1 + \frac{\beta_1}{2\sqrt{3}}\exp\left(\sqrt{3}\xi_1 X - \frac{1}{\sqrt{3}\xi_1}T\right)\right] + \text{const.} \tag{58}$$

Thus, for the example of the one-soliton solution, we have demonstrated the IST method.

5.2 The direct spectral problem

Let us consider the principal aspects of the inverse scattering transform problem for a third-order equation. The inverse problem for certain third-order spectral equations has been considered by Kaup [33] and Caudrey [34, 35]. The time evolution of ψ is determined from (48) or (51).

Following the method described by Caudrey [34], the spectral equation (47) can be rewritten

$$\frac{\partial}{\partial X}\ \psi = [\mathbf{A}(\zeta) + \mathbf{B}(X,\zeta)]\cdot\psi \tag{59}$$

with

$$\psi = \begin{pmatrix}\psi\\ \psi_X\\ \psi_{XX}\end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ \lambda & 0 & 0\end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & -W_X & 0\end{pmatrix}. \tag{60}$$

The matrix $\mathbf{A}$ has eigenvalues $\lambda_j(\zeta)$ and left and right eigenvectors $\tilde{\boldsymbol{v}}_j(\zeta)$ and $\boldsymbol{v}_j(\zeta)$, respectively. These quantities are defined through a spectral parameter λ as

$$\lambda_j(\zeta) = \omega_j\zeta, \quad \lambda_j^3(\zeta) = \lambda,$$
$$\boldsymbol{v}_j(\zeta) = \begin{pmatrix}1\\ \lambda_j(\zeta)\\ \lambda_j^2(\zeta)\end{pmatrix}, \qquad \tilde{\boldsymbol{v}}_j(\zeta) = \left(\lambda_j^2(\zeta)\ \ \lambda_j(\zeta)\ \ 1\right), \tag{61}$$

where, as previously, $\omega_j = e^{2\pi i(j-1)/3}$ are the cube roots of 1 ($j = 1, 2, 3$). Obviously the $\lambda_j(\zeta)$ are distinct, and they and $\tilde{\boldsymbol{v}}_j(\zeta)$ and $\boldsymbol{v}_j(\zeta)$ are analytic throughout the complex ζ-plane.

The solution of the linear equation (47) (or equivalently (59)) has been obtained by Caudrey [34] in terms of Jost functions $\boldsymbol{\phi}_j(X, \zeta)$ which have the asymptotic behaviour:

$$\boldsymbol{\Phi}_j(X, \zeta) \coloneqq \exp\left\{-\lambda_j(\zeta)X\right\}\boldsymbol{\phi}_j(X, \zeta) \to v_j(\zeta) \quad \text{as } X \to -\infty. \tag{62}$$

Caudrey [34] showed how Eq. (59) can be solved by expressing it as a Fredholm integral equation.

The complex ζ-plane is to be divided into regions such that, in the interior of each region, the order of the numbers $\mathrm{Re}(\lambda_i(\zeta))$ is fixed. As we pass from one region to another, this order changes, and hence, on a boundary between two regions, $\mathrm{Re}(\lambda_i(\zeta)) = \mathrm{Re}\big(\lambda_j(\zeta)\big)$ for at least one pair $i \neq j$. The Jost function $\boldsymbol{\phi}_j$ is regular throughout the complex ζ-plane apart from poles and finite singularities on the boundaries between the regions. At any point in the interior of any region of the complex ζ-plane, the solution of Eq. (59) is obtained by the relation (2.12) from [34]. It is the direct spectral problem.

5.3 The spectral data

The information about the singularities of the Jost functions $\boldsymbol{\phi}_j(X, \zeta)$ reside in the spectral data. First let us consider the poles. It is assumed that a pole $\zeta_i^{(k)}$ in $\boldsymbol{\phi}_i(X, \zeta)$ is simple, does not coincide with a pole of $\boldsymbol{\phi}_j(X, \zeta)$ and $j \neq i$ and does not lie on a boundary between two regions. Then, as proven in [34], the residue is

$$\mathrm{Res}\ \ \boldsymbol{\phi}_i\left(X, \zeta_i^{(k)}\right) = \sum_{\substack{j=1 \\ j \neq i}}^{n} \gamma_{ij}^{(k)}\ \ \boldsymbol{\phi}_j\left(X, \zeta_i^{(k)}\right) \tag{63}$$

and it can be found because we know the solution (47) in any regular regions from solving the direct problem (see Section 5.2). Note that, for $\boldsymbol{\phi}_j\left(X, \zeta_i^{(k)}\right)$, the point $\zeta_i^{(k)}$ lies in the interior of a regular region. The quantities $\zeta_i^{(k)}$ and $\gamma_{ij}^{(k)}$ constitute the discrete part of the spectral data.

Now we consider the singularities on the boundaries between regions. However, in order to simplify matters, we first make some observations. The solution of the spectral problem can be facilitated by using various symmetry properties. In view of (47), we need only consider the first elements of

$$\boldsymbol{\phi}_i(X, \zeta) = \begin{pmatrix} \boldsymbol{\phi}_i(X, \zeta) \\ \boldsymbol{\phi}_i(X, \zeta)_X \\ \boldsymbol{\phi}_i(X, \zeta)_{XX} \end{pmatrix}, \tag{64}$$

whilst the symmetry

$$\boldsymbol{\phi}_1(X, \zeta/\omega_1) = \boldsymbol{\phi}_2(X, \zeta/\omega_2) = \boldsymbol{\phi}_3(X, \zeta/\omega_3) \tag{65}$$

means we need only to consider $\boldsymbol{\phi}_1(X, \zeta)$. In our case, for $\boldsymbol{\phi}_1(X, \zeta)$, the complex ζ-plane is divided into four regions by two lines (see **Figure 5**) given by

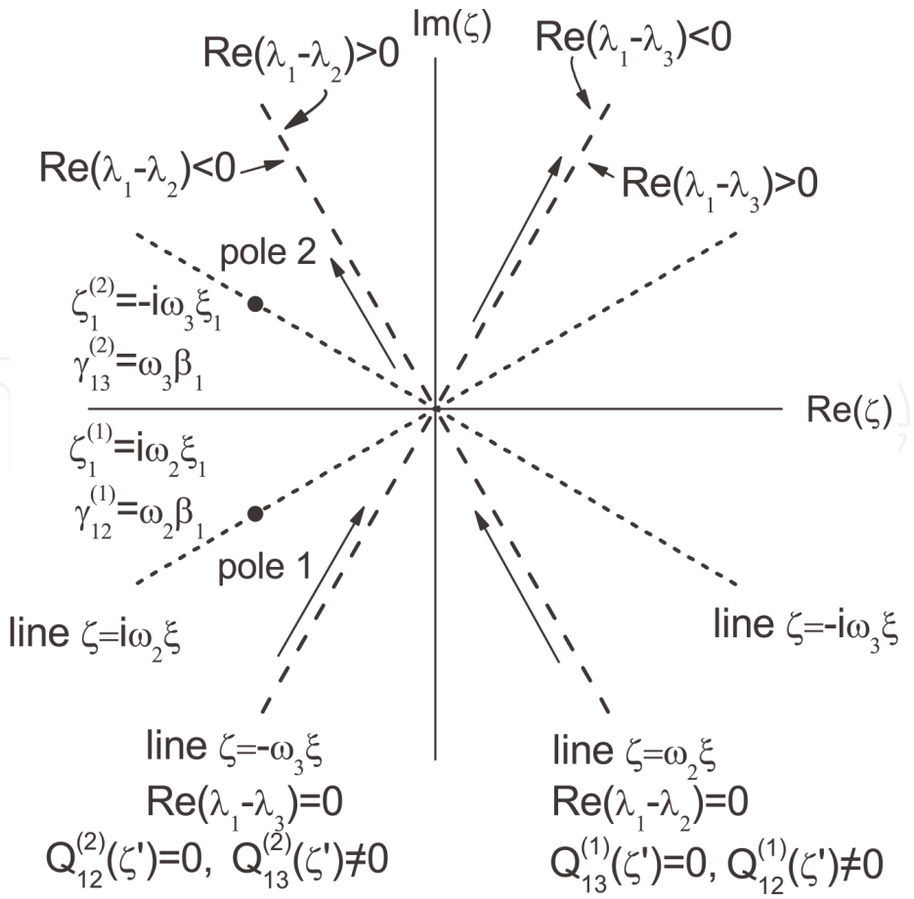

Figure 5.
The regular regions for Jost functions $\phi_1(X, \zeta)$ in the complex ζ-plane. The dashed lines determine the boundaries between regular regions. These lines are lines where the singularity functions $Q_{1j}(\zeta')$ are given. The dotted lines are the lines where the poles appear.

$$\begin{aligned} &\text{(i)} \quad \zeta' = \omega_2 \xi, \quad \text{where} \quad \mathrm{Re}(\lambda_1(\zeta)) = \mathrm{Re}(\lambda_2(\zeta)), \\ &\text{(ii)} \quad \zeta' = -\omega_3 \xi, \quad \text{where} \quad \mathrm{Re}(\lambda_1(\zeta)) = \mathrm{Re}(\lambda_3(\zeta)), \end{aligned} \tag{66}$$

where ξ is real (see **Figure 5**). The singularity of $\phi_1(X, \zeta)$ can appear only on these boundaries between the regular regions on the ζ-plane, and it is characterized by functions $Q_{1j}(\zeta')$ at each fixed $j \neq 1$. We denote the limit of a quantity, as the boundary is approached, by the superfix $\pm$ in according to the sign of $\mathrm{Re}\big(\lambda_1(\zeta) - \lambda_j(\zeta)\big)$ (see **Figure 5**).

In [34] (see Eq. (3.14) there) the jump of $\phi_1(X, \zeta)$ on the boundaries is calculated as

$$\phi_1^+(X, \zeta) - \phi_1^-(X, \zeta) = \sum_{j=2}^{3} Q_{1j}(\zeta) \phi_j^-(X, \zeta), \tag{67}$$

where, from (66), the sum is over the lines $\zeta' = \omega_2 \xi$ and $\zeta' = -\omega_3 \xi$ given by

$$\begin{aligned} &\text{(i)} \quad \zeta' = \omega_2 \xi, \quad \text{with} \quad Q_{12}^{(1)}(\zeta') \neq 0, \quad Q_{13}^{(1)}(\zeta') \equiv 0, \\ &\text{(ii)} \quad \zeta' = -\omega_3 \xi, \quad \text{with} \quad Q_{12}^{(2)}(\zeta') \equiv 0, \quad Q_{13}^{(2)}(\zeta') \neq 0. \end{aligned}$$

The singularity functions $Q_{1j}(\zeta')$ are determined by $W(X, 0)$ through the matrix $\mathbf{B}(X, \zeta)$ (60) (see Eq. (3.13) in [34])

$$Q_{1j}(\zeta) = \frac{1}{\tilde{\boldsymbol{v}}_j(\zeta) \cdot \quad v_j(\zeta)} \quad \tilde{\mathbf{v}}_j(\zeta) \cdot \int_{-\infty}^{\infty} \exp\left[\left(\lambda_1(\zeta) - \lambda_j(\zeta)\right)z\right] \mathbf{B}(z,\zeta) \cdot \phi_1^-(X,\zeta)dz. \tag{68}$$

The quantities $Q_{1j}(\zeta')$ along all the boundaries constitute the continuum part of the spectral data.

Thus, the spectral data are

$$S = \left\{\zeta_1^{(k)}, \gamma_{1j}^{(k)}, Q_{1j}(\zeta'); j = 2,3, k = 1,2,...,m\right\}. \tag{69}$$

One of the important features which is to be noted for the IST method is as follows. After the spectral data have been found from $\mathbf{B}(X,0;\zeta)$, i.e., at initial time, we need to seek the time evolution of the spectral data from Eq. (48). Analysing (48) at $X \to \infty$ together with (62)

$$\phi_i(X,T,\zeta) = \exp\left[-(3\lambda_i(\zeta))^{-1}T\right]\phi_i(X,0,\zeta),$$

the T-dependence is revealed as

$$\begin{aligned}
&\zeta_j^{(k)}(T) = \zeta_j^{(k)}(0),\\
&\gamma_{1j}^{(k)}(T) = \gamma_{1j}^{(k)}(0)\exp\left\{\left[-\left(3\lambda_j\left(\zeta_1^{(k)}\right)\right)^{-1} + \left(3\lambda_1\left(\zeta_1^{(k)}\right)\right)^{-1}\right]T\right\},\\
&Q_{1j}(T;\zeta') = Q_{1j}(0;\zeta')\exp\left\{\left[-\left(3\lambda_j(\zeta')\right)^{-1} + \left(3\lambda_1(\zeta')\right)^{-1}\right]T\right\}.
\end{aligned} \tag{70}$$

The final step in the application of the IST method is to reconstruct $B(X,T;\zeta)$ from the evaluated spectral data. In the next section, we show how to do this.

5.4 The inverse spectral problem

The final procedure in IST method is that of the reconstruction of the matrix $B(X,T;\zeta)$ and $W(X,T)$ from the spectral data S.

The spectral data define $\Phi_1(X,\zeta)$ uniquely in the form (see Eq. (6.20) in [34]))

$$\begin{aligned}
\Phi_1(X,T;\zeta) = 1 &- \sum_{k=1}^{K}\sum_{j=2}^{3}\gamma_{1j}^{(k)}(T)\frac{\exp\left\{\left[\lambda_j\left(\zeta_1^{(k)}\right) - \lambda_1\left(\zeta_1^{(k)}\right)\right]X\right\}}{\lambda_1\left(\zeta_1^{(k)}\right) - \lambda_1(\zeta)}\Phi_1\left(X,T;\omega_j\zeta_1^{(k)}\right)\\
&+ \frac{1}{2\pi\mathrm{i}}\int\sum_{j=2}^{3}Q_{1j}(T;\zeta')\frac{\exp\left\{\left[\lambda_j(\zeta') - \lambda_1(\zeta')\right]X\right\}}{\zeta' - \zeta}\Phi_1^-\left(X,T;\omega_j\zeta'\right)d\zeta'.
\end{aligned} \tag{71}$$

Eq. (71) contains the spectral data, namely, K poles with the quantities $\gamma_{1j}^{(k)}$ for the bound state spectrum as well as the functions $Q_{1j}(\zeta')$ given along all the boundaries of regular regions for the continuous spectrum. The integral in (71) is along all the boundaries (see the dashed lines in **Figure 5**). The direction of integration is taken so that the side chosen to be $\mathrm{Re}\left(\lambda_1(\zeta) - \lambda_j(\zeta)\right) < 0$ is shown by the arrows in **Figure 5** (for the lines (66), ξ sweeps from $-\infty$ to $+\infty$).

It is necessary to note that we should carry out the integration along the lines $\omega_2(\xi + i\varepsilon)$ and $-\omega_3(\xi + i\varepsilon)$ with $\varepsilon > 0$. In this case condition (62) is satisfied. Passing to the limit $\varepsilon \to 0$, we can obtain the solution which does not satisfy condition (62). However, for any finite $\varepsilon > 0$, the restricted region on X can be determined where the solution associated with a finite $\varepsilon > 0$ (for which the condition (62) is valid) and the solution associated with $\varepsilon = 0$ are sufficiently close to each other. In this sense, taking the integration at $\varepsilon = 0$, we remain within the inverse scattering theory [34], and so condition (62) can be omitted. The solution obtained at $\varepsilon = 0$ can be extended to sufficiently large finite X. Thus, we will interpret the solution obtained at $\varepsilon = 0$ as the solution of the VPE (27) which is valid for arbitrary but finite X.

By choosing appropriate values for ζ, the left-hand side in (71) can be $\Phi_1\left(X, T; \omega_j \zeta_1^{(k)}\right)$, or by allowing ζ to approach the boundaries from the appropriate sides, the left-hand side can be $\Phi_1^-\left(X, T; \omega_j \zeta'\right)$. We acquire a set of linear matrix/Fredholm equations in the unknowns $\Phi_1\left(X, T; \omega_j \zeta_1^{(k)}\right)$ and $\Phi_1^-\left(X, T; \omega_j \zeta'\right)$. The solution of this equation system enables one to define $\Phi_1(X, T; \zeta)$ from (71).

By knowing $\Phi_1(X, T; \zeta)$, we can take extra information into account, namely, that the expansion of $\Phi_1(X, T; \zeta)$ as an asymptotic series in $\lambda_1^{-1}(\zeta)$ connects with $W(X, T)$ as follows (cf. Eq. (2.7) in [33]):

$$\Phi_1(X, T; \zeta) = 1 - \frac{1}{3\lambda_1(\zeta)}[W(X, T) - W(-\infty)] + O\left(\lambda_1^{-2}(\zeta)\right). \tag{72}$$

Consequently, the solution $W(X, T)$ and the matrix $B(X, T; \zeta)$ can be reconstructed from the spectral data.

6. The interaction of the loop-like solitons

We will discuss the exact N-soliton solution of the VPE via the inverse scattering method [24]. To do this we consider (71) with $Q_{1j}(\zeta) \equiv 0$. Then there is only the bound state spectrum which is associated with the soliton solutions.

Let the bound state spectrum be defined by K poles. The relation (71) is reduced to the form

$$\Phi_1(X, T; \zeta) = 1 - \sum_{k=1}^{K} \sum_{j=2}^{3} \gamma_{1j}^{(k)}(T) \frac{\exp\left\{\left[\lambda_j\left(\zeta_1^{(k)}\right) - \lambda_1\left(\zeta_1^{(k)}\right)\right]X\right\}}{\lambda_1\left(\zeta_1^{(k)}\right) - \lambda_1(\zeta)} \Phi_1\left(X, T; \omega_j \zeta_1^{(k)}\right). \tag{73}$$

Eq. (73) involves the spectral data, namely, the poles $\zeta_1^{(k)}$ and the quantities $\gamma_{1j}^{(k)}$. First we will prove that $\mathrm{Re}\,\lambda = 0$ for compact support. From Eq. (47) we have

$$(\psi_X)_{XXX} + (U\psi_X)_X - \lambda\psi_X = 0, \tag{74}$$

and together with Eq. (47), this enables us to write

$$\frac{\partial}{\partial X}\left(\frac{\partial^2}{\partial X^2}\psi_X\psi^* - 3\psi_{XX}\psi_X^* + U\psi_X\psi^*\right) - 2\mathrm{Re}\,\lambda\psi_X\psi^* = 0. \tag{75}$$

Integrating Eq. (75) over all values of X, we obtain that, for compact support, $\mathrm{Re}\,\lambda = 0$ since, in the general case, $\int_{-\infty}^{\infty} \psi_X\psi^* dX \neq 0$.

As follows from Eqs. (2.12), (2.13), (2.36) and (2.37) of [33], $\psi_X(\zeta)$ is related to the adjoint states $\psi_X^A(-\zeta)$. In the usual manner, using the adjoint states and Eq. (14) from [35] and Eq. (2.37) from [33], one can obtain

$$\phi_{1X}(X,\zeta) = \frac{\mathrm{i}}{\sqrt{3}}[\phi_{1X}(X,-\omega_2\zeta)\phi_1(X,-\omega_3\zeta) - \phi_{1X}(X,-\omega_3\zeta)\phi_1(X,-\omega_2\zeta)]. \quad (76)$$

It is easily seen that if $\zeta_1^{(1)}$ is a pole of $\phi_1(X,\zeta)$, then there is a pole either at $\zeta_1^{(2)} = -\omega_2\zeta_1^{(1)}$ (if $\phi_1(X,-\omega_2\zeta)$ has a pole) or at $\zeta_1^{(2)} = -\omega_3\zeta_1^{(1)}$ (if $\phi_1(X,-\omega_3\zeta)$ has a pole). For definiteness let $\zeta_1^{(2)} = -\omega_2\zeta_1^{(1)}$. Then, as follows from (76), $-\omega_3\zeta_1^{(2)}$ should be a pole. However, this pole coincides with pole $\zeta_1^{(1)}$, since $-\omega_3\zeta_1^{(2)} = -\omega_3(-\omega_2)\zeta_1^{(1)} = \zeta_1^{(1)}$. Hence, the poles appear in pairs, $\zeta_1^{(2n-1)}$ and $\zeta_1^{(2n)}$, under the condition $\zeta_1^{(2n)}/\zeta_1^{(2n-1)} = -\omega_2$, where n is the pair number.

Let us consider N pairs of poles, i.e., in all there are $K = 2N$ poles over which the sum is taken in (76). For the pair n $(n = 1, 2, ..., N)$ we have the properties

$$\text{(i)} \quad \zeta_1^{(2n-1)} = \mathrm{i}\omega_2\xi_n, \qquad \text{(ii)} \quad \zeta_1^{(2n)} = -\mathrm{i}\omega_3\xi_n. \quad (77)$$

Since U is real and λ is imaginary, ξ_k is real. The relationships (77) are in line with the condition (2.33) from [33]. These relationships are also similar to Eqs. (6.24) and (6.25) in [34], whilst $\gamma_{1j}^{(k)}$ turns out to be different from $\tilde{\gamma}_{1j}^{(k)}$ for the Boussinesq equation (see Eqs. (6.24) and (6.25) in [34]). Indeed, by considering (76) in the vicinity of the first pole $\zeta_1^{(2n-1)}$ of the pair n and using the relation (73), one can obtain a relation between $\gamma_{12}^{(k)}$ and $\gamma_{13}^{(k)}$. In this case the functions $\phi_{1,X}(X,\zeta)$, $\phi_1(X,-\omega_2\zeta)$ and $\phi_{1,X}(X,-\omega_2\zeta)$ also have poles here, whilst the functions $\phi_1(X,-\omega_3\zeta)$ and $\phi_{1,X}(X,-\omega_3\zeta)$ do not have poles here. Substituting $\phi_1(X,\zeta)$ in the form (73) into Eq. (76) and letting $X \to -\infty$, we have the ratio $\gamma_{13}^{(2n)}/\gamma_{12}^{(2n-1)} = \omega_2$ and $\gamma_{12}^{(2n)} = \gamma_{13}^{(2n-1)} = 0$. Therefore, the properties of $\gamma_{ij}^{(k)}$ should be defined by the relationships

$$\begin{aligned} &\text{(i)} \quad \gamma_{12}^{(2n-1)} = \omega_2\beta_k, \quad \gamma_{13}^{(2n-1)} = 0, \\ &\text{(ii)} \quad \gamma_{12}^{(2n)} = 0, \qquad \gamma_{13}^{(2n)} = \omega_3\beta_k, \end{aligned} \quad (78)$$

where, as it will be proved below, β_k is real when $U = W_X$ is real.
By defining

$$\Psi_k(X,T) = \sum_{j=2}^{3}\gamma_{1j}^{(k)}(T)\exp\left\{\lambda_j\left(\zeta_1^{(k)}\right)X\right\}\Phi_1\left(X,T;\omega_j\zeta_1^{(k)}\right), \quad (79)$$

we may rewrite the relationship (73) as (see, for instance, Eqs. (6.33) and (6.34) in [34])

$$\Phi_1(X,T;\zeta) = 1 - \sum_{k=1}^{2N}\frac{\exp\left\{-\lambda_1\left(\zeta_1^{(k)}\right)X\right\}}{\lambda_1\left(\zeta_1^{(k)}\right) - \lambda_1(\zeta)}\Psi_k(X,T). \quad (80)$$

From (72) and (80), it may be shown that (cf. Eq. (6.38) in [34])

$$W(X,T) - W(-\infty) = -3\sum_{k=1}^{2N}\exp\left\{-\lambda_1\left(\zeta_1^{(k)}\right)X\right\}\Psi_k(X,T) = 3\frac{\partial}{\partial X}\ln\left(\det M(X,T)\right). \quad (81)$$

The $2N \times 2N$ matrix $M(X,T)$ is defined as in relationship (6.36) in [34] by

$$M_{kl}(X,T) = \delta_{kl} - \sum_{j=2}^{3} \gamma_{1j}^{(k)}(0) \frac{\exp\left\{\left[-\left(3\lambda_j\left(\zeta_1^{(k)}\right)\right)^{-1} + \left(3\lambda_1\left(\zeta_1^{(k)}\right)\right)^{-1}\right]T + \left(\lambda_j\left(\zeta_1^{(k)}\right) - \lambda_1\left(\zeta_1^{(l)}\right)\right)X\right\}}{\lambda_j\left(\zeta_1^{(k)}\right) - \lambda_1\left(\zeta_1^{(l)}\right)}, \tag{82}$$

and

$$n = 1,\ 2, \ldots, N,$$

$$\lambda_1\left(\zeta_1^{(2n-1)}\right) = i\omega_2\xi_n, \qquad \lambda_2\left(\zeta_1^{(2n-1)}\right) = i\omega_3\xi_n, \qquad \gamma_{12}^{(2n-1)} = \omega_2\beta_n, \qquad \gamma_{13}^{(2n-1)} = 0,$$

$$\lambda_1\left(\zeta_1^{(2n)}\right) = -i\omega_3\xi_n, \qquad \lambda_3\left(\zeta_1^{(2n)}\right) = -i\omega_2\xi_n, \qquad \gamma_{12}^{(2n)} = 0, \qquad \gamma_{13}^{(2n)} = \omega_3\beta_n.$$

For the N-soliton solution, there are N arbitrary constants ξ_n and N arbitrary constants β_n.

The final result for the N-soliton solution of the VPE is defined by relationship (81) with (82).

6.1 Examples of one- and two-soliton solutions of the VPE

In order to obtain the one-soliton solution of the VPE (27)

$$W_{XXT} + (1 + W_T)W_X = 0,$$

we need first to calculate the 2×2 matrix $M(X,T)$ according to (82) with $N = 1$. We find that the matrix is

$$\begin{pmatrix} 1 - \dfrac{\omega_2\beta_1}{\sqrt{3}\xi_1}\exp\left[\sqrt{3}\xi_1 X - \left(\sqrt{3}\xi_1\right)^{-1}T\right] & \dfrac{i\omega_3\beta_1}{2\xi_1}\exp\left[2i\omega_3\xi_1 X - \left(\sqrt{3}\xi_1\right)^{-1}T\right] \\ \dfrac{-i\omega_2\beta_1}{2\xi_1}\exp\left[-2i\omega_2\xi_1 X - \left(\sqrt{3}\xi_1\right)^{-1}T\right] & 1 - \dfrac{\omega_3\beta_1}{\sqrt{3}\xi_1}\exp\left[\sqrt{3}\xi_1 X - \left(\sqrt{3}\xi_1\right)^{-1}T\right] \end{pmatrix} \tag{83}$$

and its determinant is

$$\det M(X,T) = \left\{1 + \frac{\beta_1}{2\sqrt{3}\xi_1}\exp\left[\sqrt{3}\xi_1\left(X - \frac{T}{3\xi_1^2}\right)\right]\right\}^2. \tag{84}$$

Consequently, from Eq. (81) we have the one-soliton solution of the VPE

$$U(X,T) = W_X(X,T) = \frac{9}{2}\xi_1^2 \operatorname{sech}^2\left[\frac{\sqrt{3}}{2}\xi_1\left(X - \frac{T}{3\xi_1^2}\right) + \alpha_1\right], \tag{85}$$

where $\alpha_1 = \frac{1}{2}\ln\left(\beta_1/2\sqrt{3}\xi_1\right)$ is an arbitrary constant. Since U is real, it follows from (85) that β_1 is real. Note that with $\beta_1/\xi_1 < 0$ we have the real solution in the form of the singular soliton [41].

$$U(X,T) = \frac{9}{2}\xi_1^2 \sinh^{-2}\eta, \quad \eta = \frac{\sqrt{3}}{2}\xi_1\left(X - \frac{T}{3\xi_1^2}\right) + \alpha_1. \tag{86}$$

Let us now consider the two-soliton solution of the VPE. In this case $M(X,T)$ is a 4×4 matrix. We will not give the explicit form here, but we find that

$$\det M(X,T) = \left(1 + q_1^2 + q_2^2 + b^2 q_1^2 q_2^2\right)^2, \tag{87}$$

where

$$q_i = \exp\left[\frac{\sqrt{3}}{2}\xi_i\left(X - \frac{T}{3\xi_i^2}\right) + \alpha_i\right], \quad b^2 = \left(\frac{\xi_2 - \xi_1}{\xi_2 + \xi_1}\right)^2 \frac{\xi_1^2 + \xi_2^2 - \xi_1\xi_2}{\xi_1^2 + \xi_2^2 + \xi_1\xi_2}, \tag{88}$$

and $\alpha_i = \frac{1}{2}\ln\left(\beta_i / 2\sqrt{3}\xi_i\right)$ are arbitrary constants. The two-soliton solution to the VPE as found by the IST method is given by (81) together with (87).

Finally we note that comparison of (81) with $W = 6(\ln f)_X$ from (42) shows that

$$\ln\left(\det M(X,T)\right) = 2\ln(f). \tag{89}$$

so that $\det M(X,T)$ is a perfect square for arbitrary N.

6.2 The two-loop-like solitons of the VE

We discuss the two-loop soliton solution of the VE in more detail. Let us consider what happens in x-t space. The relations (20), (25) and (29) determine the solutions in x-t throughout the solutions in X-T. In these coordinates x-t, we have the loop-like solitons.

The shifts, δ_i, of the two-loop solitons u_1 and u_2 in the positive x-direction due to the interaction may be computed as follows. The larger loop soliton is always shifted forwards by the interaction. However, for smaller u_2 with $r = \xi_1/\xi_2$, there is a value $r_c = 0.88867$ in that we have a different form of the phase shift:

a. For $r_c < r < 1$, $\delta_1 < 0$ so the smaller loop soliton is shifted backwards.

b. For $r = r_c$, where $r_c = 0.88867$ is the root of $\ln b + 3/r = 0$, $\delta_1 = 0$, so the smaller loop soliton is not shifted by the interaction.

c. For $0 < r < r_c$, $\delta_1 > 0$ so the smaller loop soliton is shifted forwards.

At first sight it might seem that the behaviour in (b) and (c) contradicts conservation of 'momentum'. That this is not so is justified as follows. By integrating (9) with respect to x, we find that $\int_{-\infty}^{\infty} u\,dx = 0$; also, by multiplying (9) by x and integrating with respect to x, we obtain $\int_{-\infty}^{\infty} xu\,dx = 0$. Thus, in x-t space, the 'mass' of each soliton is zero, and 'momentum' is conserved whatever δ_1 and δ_2 may be. In particular δ_1 and δ_2 may have the same sign as in (c), or one of them may be zero as in (b).

Cases (a), (b) and (c) are illustrated in **Figures 6–8**, respectively; in these figures u is plotted against x for various values of t. For convenience in the figures, the interactions of solitons are shown in coordinates moving with speed $(v_1 + v_2)/2$.

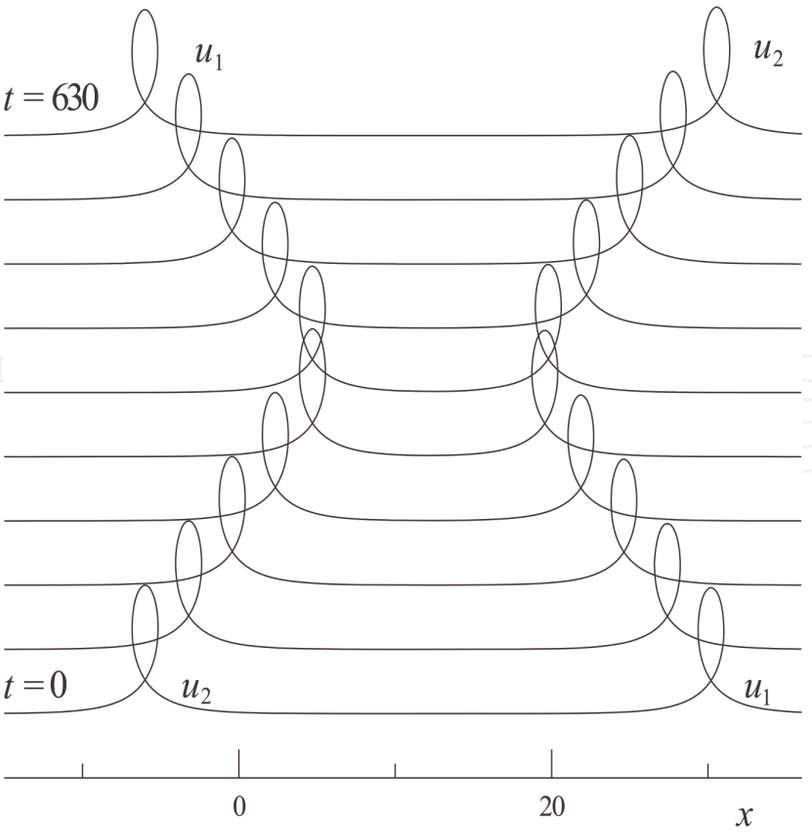

Figure 6.
The interaction process for two-loop solitons with $\xi_1 = 0.99$ and $\xi_2 = 1$ so that $r = 0.99$ and $\delta_1 < 0$.

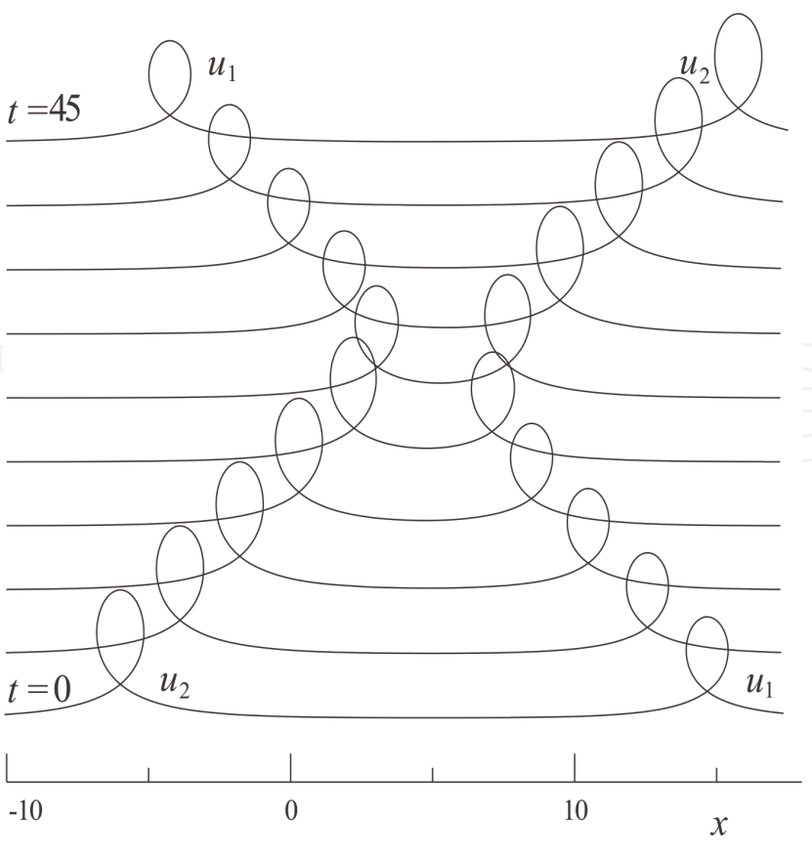

Figure 7.
The interaction process for two-loop solitons with $\xi_1 = 0.88867$ and $\xi_2 = 1$ so that $r = 0.88867$ and $\delta_1 = 0$.

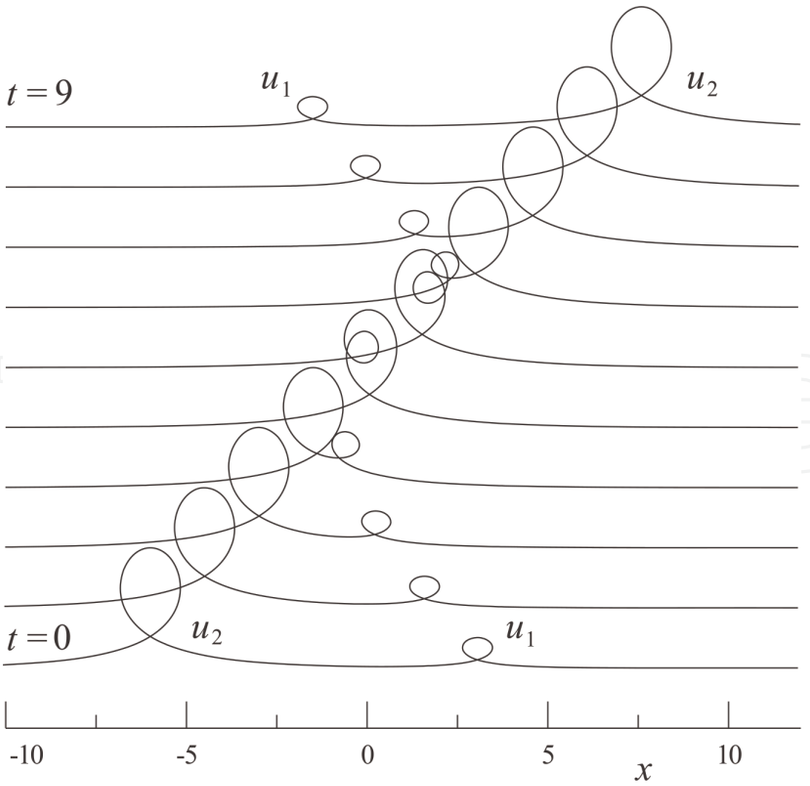

Figure 8.
The interaction process for two-loop solitons with $\xi_1 = 0.5$ and $\xi_2 = 1$ so that $r = 0.5$ and $\delta_1 > 0$.

7. Discussion on the loop-like solutions

We have already mentioned the important question on stability of loop-like solutions (Section 2).

7.1 Remarks on the existence and uniqueness theorem

In [42], the existence and uniqueness theorem is formulated for system (one) differential equations. The loop-like solutions take place on travelling waves. In this case, the initial equation is reduced to an ordinary differential equation (ODE) (see Section 2). It has been this equation which we are exploring. Now we note some important remarks. In particular, in order to investigate the ODE (the solution on travelling waves), it is still necessary to reconcile this solution with the initial problem, which is described by the differential equation in partial derivatives (evolution equation). Consequently, the ambiguous solutions for the ODE during their reconstruction into the initial coordinates should be checked by means of some restrictive conditions (see 7.2).

It is necessary to note that if the conditions of the existence and uniqueness theorem break down, then nevertheless, this does not restrict the existence of solutions. Hence, the solutions can exist, for example, the multivalued solutions. Here we point out an example: the exact solutions for the Camassa-Holm equation (CHE) and the Degasperis-Procesi equation (DPE) can be constructed as the component solutions, through separate parts (branches) of solutions (see [43]).

The selection of possible multivalued solutions will be discussed in 7.2.

7.2 Selection for the loop-like solutions

Solutions must satisfy the following conditions:

1. At the point $\eta = 0$, the solution must pass over the ellipse $(z^2 + 2v\eta^2)_{\eta\eta} = 0$ (see Eq. (4.3) in [3]);

2. According to the conservation law $\int_{\infty}^{-\infty} u(x,t)dx = \text{const} \equiv 0$ for $t > 0$. The 'mass' of individual soliton equals to zero. This condition will be satisfied if point 1 takes place.

3. As you know [3], taking into account dissipation in the physical process allows one to select a solution from an array of possible solutions that are inherent to the equation without dissipation. This condition also selects a solution as in a point 1 if $\alpha \to 0$.

4. During the interaction of the solitons [24, 29], you must take into account all the parts of loop-like soliton (see end of Section 7.3). The soliton has a form satisfying point 2.

Thus, we cannot arbitrarily combine the solutions at $\eta = 0$. The solutions, in particular, solitons should be specific loop-like form.

7.3 Physical interpretation of the multivalued solutions

From the mathematical point of view, an ambiguous solution does not present difficulties, whereas the physical interpretation of ambiguity always presents some difficulties. In this connection the problem of ambiguous solutions is regarded as important. The problem consists in whether the ambiguity has a physical nature or is related to the incompleteness of the mathematical model, in particular to the lack of dissipation.

We will consider the problem related to the singular points when dissipation takes place. At these points the dissipative term $\alpha \frac{\partial u}{\partial x}$ tends to infinity. The question arises: Are there solutions of Eq. (8) in a loop-like form? That the dissipation is likely to destroy the loop-like solutions can be associated with the following well-known fact [5]. For the simplest nonlinear equation without dispersion and without dissipation, namely,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \tag{90}$$

any initial smooth solution with boundary conditions

$$u|_{x\to+\infty} = 0, \qquad u|_{x\to-\infty} = u_0 = \quad \text{const.} > 0$$

becomes ambiguous in the final analysis. When dissipation is considered, we have the Burgers equation [47]:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \mu\frac{\partial^2 u}{\partial x^2} = 0.$$

The dissipative term in this equation and in Eq. (6) for low frequency is coincident. The inclusion of the dissipative term transforms the solutions so that they cannot be ambiguous as a result of evolution. The wave parameters are always

unambiguous. What happens in our case for high frequency when the dissipative term has the form αu (see Eq. (18) in [29])? Will the inclusion of dissipation give rise to unambiguous solutions?

By direct integration of Eq. (8) (written in terms of the variables (11)) within the neighbourhood of singular points $z = 0$ where $z_\eta \to \pm\infty$ and $z_\tau \ll z_\eta$, it can be derived (see [3]) that the dissipative term, with dissipation parameter less than some limit value α^*, does not destroy the loop-like solutions. Now we give a physical interpretation to ambiguous solutions.

Since the solution to the VE has a parametric form (15) and (16), there is a space of variables in which the solution is a single-valued function. Hence, we can solve the problem of the ambiguous solution. A number of states with their thermodynamic parameters can occupy one microvolume. It is assumed that the interaction between the separated states occupying one microvolume can be neglected in comparison with the interaction between the particles of one thermodynamic state. Even if we take into account the interaction between the separated states in accordance with the dynamic state equation (5), for high frequencies, a dissipative term arises which is similar to the corresponding term in Eq. (7) but with the other relaxation time. In this sense the separated terms are distributed in space, but describing the wave process, we consider them as interpenetratable. A similar situation, when several components with different hydrodynamic parameters occupy one microvolume, has been assumed in mixture theory (see, for instance, [48]). Such a fundamental assumption in the theory of mixtures is physically impossible (see [48], p.7), but it is appropriate in the sense that separated components are multi-velocity interpenetratable continua.

Consequently, the following three observations show that, in the framework of the approach considered here, there are multivalued solutions when we model high-frequency wave processes: (1) All parts of loop-like solution are stable to perturbations. (2) Dissipation does not destroy the loop-like solutions. (3) The investigation regarding the interaction of the solitons has shown that it is necessary to take into account the whole ambiguous solution and not just the separate parts.

7.4 Conclusion

Loop-like solitons are a class of interesting wave phenomena, which take place in some nonlinear systems. This interest consisted not only in the interpretation of the solutions obtained but also in the explanation of the experimental results. The ambiguous structure of the loop-like solutions is similar to the loop soliton solution to an equation that models a stretched rope [44]. Loop-like solitons on a vortex filament were investigated by Hasimoto [45] and Lamb, Jr. [46]. The loop-like solutions appear in description of physical phenomena, in particular, electromagnetic terahertz pulses in asymmetric molecules [49], high-frequency perturbations in a relaxation medium [3, 50, 51] and soliton in ferrites [52]. As a typical multivalued structure, loop soliton has been discussed in some possible physical fields including particle physics [53] and quantum field theory [54].

It must be admitted that we are a long way still from complete awareness of physical processes which can be described by loop-like solutions. However, the approach, considered here, will hopefully be interesting and useful in understanding the birth and death process for particles, since the mass and momentum of individual loop-like soliton are zero. Furthermore, the investigations in optics, magnetism and hydrodynamics clearly indicate the acceptability of the approach on loop-like solitons. Indeed, the phase shifts observed at interaction of solitons can be explained by means of loop-like solutions.

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