Chapter

A Direct Construction of Intergroup Complementary Code Set for CDMA

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Abstract

A collection of mutually orthogonal complementary codes (CCs) is said to be complete complementary codes (CCCs) where the number of CCs are equal to the number of constituent sequences in each CC. Intergroup complementary (IGC) code set is a collection of multiple disjoint code groups with the following correlation properties: (1) inside the zero-correlation zone (ZCZ), the aperiodic autocorrelation function (AACF) of any IGC code is zero for all nonzero time shifts; (2) the aperiodic cross-correlation function (ACCF), of two distinct IGC codes, is zero for all time shifts inside the ZCZ when they are taken from the same code groups; and (3) the ACCF, for two IGC codes from two different code groups, is zero everywhere. IGC code set has a larger set size than CCC, and both can be applicable in multicarrier code-division multiple access (CDMA). In this chapter, we present a direct construction of IGC code set by using second-order generalized Boolean functions (GBFs), and our IGC code set can support interference-free code-division multiplexing. We also relate our construction with a graph where the ZCZ width depends on the number of isolated vertices present in a graph after the deletion of some vertices. Here, the construction that we propose can generate IGC code set with more flexible parameters.

Keywords: complementary code (CC), code-division multiple access (CDMA), generalized, Boolean function (GBF), intergroup complementary (IGC) code set, zero-correlation zone, (ZCZ) sequences

1. Introduction

Code-division multiple access (CDMA) [1] is an important communication technology where sequence signatures with good correlation properties are used to separate multiple users. In CDMA systems, multipath interference (MPI) and multiple access interference (MAI) degrade the performance where MPI and MAI occur due to the multipath propagation, non-ideal synchronization, and non-ideal correlation properties of spreading codes. Spreading code plays a significant role on the overall performance of a CDMA system. The interference-resist capability and system capacity are determined by the correlation properties and available number of spreading codes. Due to ideal auto- and cross-correlation properties, complete complementary codes (CCCs) have been applied to asynchronous multicarrier CDMA (MC-CDMA) [2] communications in order to provide zero interference performance.

Golay proposed a pair of sequences in Golay [3] known as Golay complementary pair (GCP) which is a set of two equal length sequences with the property that the sum of their aperiodic autocorrelation function (AACF) is zero everywhere except at the zero shift. Tseng and Liu [4] extended the idea of GCP to complementary set or complementary code (CC) which contains two or more than two sequences. Davis and Jedwab [5] proposed a direct construction of GCP called Golay-Davis-Jedwab (GDJ) pair by using second-order generalized Boolean functions (GBFs) to reduce peak-to-mean envelope power ratio (PMEPR) for OFDM system. As a generalization of GDJ pair, Paterson introduced a construction of CC Paterson [6] by associating each CC with a graph. Recently, a construction of CC has been reported in Sarkar et al. [7] which is a generalization of Paterson's CC construction. Later, Rathinakumar and Chaturvedi extended Paterson's construction to CCC Rathinakumar and Chaturvedi [8] which is a collection of mutually orthogonal CCs. Although CCs have ideal AACF and aperiodic cross-correlation function (ACCF), they are unable to support a maximum number of users as the set size cannot be larger than the flock size [9–11], where the flock size denotes the number of constituent sequences in each CC. The application of CCC has been extended for the enabling of interference-free MC-CDMA communication by designing a fractional-delay-resilient receiver in Liu et al. [12].

The binary Z-complementary sequences were first introduced by Fan et al. [13] and later extended to quadriphase Z-complementary sequences by Li et al. [14].

Recently, a construction of binary Z-complementary pairs has been reported in Adhikary et al. [15]. A direct construction of polyphase Z-complementary codes has been reported in Sarkar et al. [16], which is an extension of Rathinakumar's CCC construction. Due to favorable correlation properties of Z-complementary codes, it can be easily utilized for MC-CDMA system as spreading sequences to mitigate MPI and MAI efficiently [17]. The theoretical bound given in Liu et al. [18] shows that the Z-complementary codes have a much larger set size than CCCs.

IGC code set was first proposed by Li et al. [19] based on CCCs. Their code assignment algorithm shows that the CDMA systems employing the IGC codes (IGC-CDMA) outperform traditional CDMA with respect to bit error rate (BER). However, the ZCZ width of IGC codes [19] is fixed to the length of the elementary codes of the original CCCs, which limits the number of IGC codes. Another improved construction method of IGC codes is proposed in Feng et al. [20] based on the CCCs, interleaving operation, and orthogonal matrix which provides a flexible choice of the ZCZ width. However, there is no such construction which can directly produce IGC code set without having operation on CCCs; it motivates us to give a direct construction of IGC code set.

This chapter contains a direct method to construct IGC code set by applying second-order GBFs. This construction is capable of generating IGC code set with more flexible parameters such as ZCZ width and set size. We also relate our construction with a graph, and it has been shown that ZCZ width and set size of the IGC code set obtained by using our method depend on the number of isolated vertices present in a graph which is achieved by deleting some vertices from a graph.

2. Preliminary

2.1 Correlations of sequences

The ACCF between two sequences $\mathbf{a} = (a_0, a_1, ..., a_{L-1})$ and $\mathbf{b} = (b_0, b_1, ..., b_{L-1})$ is defined as follows:

$$\mathrm{C}(\mathbf{a},\mathbf{b})(\tau)=\begin{cases}\Sigma_{i=0}^{L-1-\tau}a_{i+\tau}b_i^*, & 0\le\tau<L,\\ \Sigma_{i=0}^{L+\tau-1}a_jb_{i-\tau}^* & -L<\tau<0,\\ 0, & \text{otherwise},\end{cases}\tag{1}$$

where τ is an integer. The above defined function in Eq. (1) is said to be AACF of $\mathbf{a}$ (or, $\mathbf{b}$) if $\mathbf{a}=\mathbf{b}$. The AACF of $\mathbf{a}$ at τ is denoted by $A(\mathbf{a})(\tau)$.

***Definition* 1** *An ordered set* $\{\mathbf{a}^0,\mathbf{a}^1,\ldots,\mathbf{a}^{P-1}\}$ *containing P sequences of equal length L is called* CC *if*

$$\sum_{i=0}^{P-1}A(\mathbf{a}^j)(\tau)=\begin{cases}LP, & \tau=0,\\ 0, & \text{otherwise}.\end{cases}\tag{2}$$

***Definition* 2** *Let* $\{\mathbf{C}^0,\mathbf{C}^1,\ldots,\mathbf{C}^{K-1}\}$ *be a set of K* CCs *where each of the* CC *contains* $P(K\le P)$ *constituent sequences of length L. The αth constituent sequence of* $\mathbf{C}^i$ *is* $\mathbf{C}_{i,\alpha}=(\mathrm{C}_{i,\alpha,0},\mathrm{C}_{i,\alpha,1}\ \ \mathrm{C}_{i,\alpha,L-1})$ *where* $\alpha=0,1,\ldots,\ P-1,\ \ i=0,1,\ldots,\ K-1$. *The ACCF of the* CCs *is given by*

$$\mathrm{C}(\mathbf{C}^i,\mathbf{C}^j)(\tau)=\sum_{\alpha=0}^{P-1}\mathrm{C}(\mathbf{C}_{i,\alpha}\mathbf{C}_{j,\alpha})(\tau)=0,\ \ \forall\tau,\ \ i\ne j.\tag{3}$$

The code set is said to be CCC when $K=P$.

***Definition* 3** *Given an IG*C *code set I* (*K,P,L,Z*) (Li et al. [19]), *K denotes a number of codes*, *P denotes the number of constituent sequences in each code*, *L denotes the length of each constituent sequence, and Z denotes ZCZ width, where* $K=PL/Z$. *The K codes can be divided into P code groups denoted by* $I^g(g=0,1,\ldots,P-1)$, *each group contains* $K/P=L/Z$ *codes. The code set* $I(K,P,L,Z)$ *has the following properties*:

$$\mathrm{C}(\mathbf{C}^i,\mathbf{C}^j)(\tau)=\begin{cases}PL, & i=j,\ \tau\ =0,\\ 0, & i=j,\ 0<|\tau|<Z,\\ 0, & i\ne j,\ \mathbf{C}^i,\mathbf{C}^j\in I^g,\ |\tau|<Z,\\ 0, & \mathbf{C}^i\in I^{g_1},\ \mathbf{C}^j\in I^{g_2},g_1\ne g_2,\ \ |\tau|<L,\\ \text{others}, & \text{otherwise}.\end{cases}\tag{4}$$

2.2 Generalized Boolean functions

Let $f:\{0,1\}^m\to\mathbb{Z}_q$ (q is average number, not less than 2) be a function of m variables $x_0, x_1, \ldots, x_{m-1}$. The product of k distinct variables $\chi_{i_0}\chi_{i_1}\ldots\chi_{i_{k-1}}(0\le i_0<i_1<\cdots<i_{k-1}\le m-1)$ is called a monomial of degree k. The monomials $1, x_0, \ldots, x_{m-1}, x_0x_1, \ldots, x_{m-2}x_{m-1}, \ldots, x_0x_1\ldots x_{m-1}$ are the list of 2^m monomials over the variables $x_0, x_1, \ldots, x_{m-1}$. A GBF f can uniquely be presented as a linear combination of these 2^m monomials, where the coefficient of each monomial belongs to $\mathbb{Z}_q$. We denote the complex valued sequence corresponding to the GBF f by $\psi(f)$ and define it as

$$\psi(f)=\left(\omega^{f0},\omega^{f_1},\ldots,\omega^{f_{2^m-1}}\right),\tag{5}$$

where $f_i=f(i_0,i_1,\ldots,i_{m-1})$, $\omega=\exp\left(2\pi\sqrt{-1}/q\right)$, and $(i_0,i_1,\ldots,i_{m-1})$ are the binary representation of the integer $i\left(i=\sum_{i=0}^{m-1}i_j2^j\right)$. Let $\mathbf{C}$ be an order set of P Boolean functions given by $\mathbf{C}=\{f_0,f_1,\ldots,f_{P-1}\}$. Then the complex valued code

corresponding to the set of Boolean function $\mathbf{C}$ is denoted by $\psi(\mathbf{C})$, given by $\psi(\mathbf{C}) = \{\psi(f_0), \psi(f_1), ..., \psi(f_{P-1})\}$. The code can also be viewed as a matrix where $\psi(f_{i-1})$ is the ith row of the matrix.

For any given GBF f of m variables, the function $f(1-x_0, 1-x_1, ..., 1-x_{m-1})$ is denoted by $\tilde{f}$. For a $\mathbb{Z}_q$ valued vector $\mathbf{e} = (e_0, e_1, ..., e_{L-1})$, we denote the vector $\overline{\mathbf{e}}$ by $(\overline{e}_0, \overline{e}_1, ..., \overline{e}_{L-1})$ where $\overline{e}_i = \frac{q}{2} - e_i (i = 0, 1, \ ..., L-1)$. Now, we define the following notations $\overline{\mathbf{a}}$ and $\mathbf{a}^*$, where $\overline{\mathbf{a}}$ is derived from $\mathbf{a}$ by reversing it and $\mathbf{a}^*$ is the complex conjugate of $\mathbf{a}$.

2.3 Quadratic forms and graphs

In this context, we introduce some lemmas and new notations which will be used for our proposed construction.

***Definition* 4** *Let f be a GBF of variables $x_0, x_1, ..., \ x_{m-1}$ over $\mathbb{Z}_q$. Consider a list of $k (0 \le k < m)$ indices $0 \le j_0 < j_1 < \cdots j_k < m$, and write $\mathbf{x} = (x_{j_0}, x_{j_1}, ..., \ x_{j_{k-1}})$. Consider $\mathbf{c} = (c_0, c_1, ..., c_{k-1})$ to be a fixed binary vector. Then we define $\psi(f|_{\mathbf{x}=\mathbf{c}})$ as a complex valued vector with $\omega^{f(i_0 i_1 ... i_{m-1})}$ as a ith component if $i_{j_\alpha} = c_\alpha$ for each $0 \le \alpha < k$ and equal to zero otherwise. For $k = 0$, the complex valued vector $\psi(f|_{\mathbf{x}=\mathbf{c}})$ is nothing, but the vector $\psi(f)$ is defined before.*

Let $Q : \{0,1\}^m \to \mathbb{Z}_q$ be a quadratic form of m variables $x_0, x_1, ..., \ x_{m-1}$. A quadratic GBF is of the form Rathinakumar and Chaturvedi [8]

$$f = Q + \sum_{i=0}^{m-1} g_i x_i + g', \tag{6}$$

where $g', g_i \in \mathbb{Z}_q$ are arbitrary.

For a quadratic GBF, f, $G(f)$ denotes the graph of f. The $G(f)$ is obtained by joining the vertices x_i and x_j by an edge if there is a term $q_{i,j} x_i x_j (0 \le i < j \le m-1)$ in the GBF f with $q_{i,j} \ne 0 \left(q_{i,j} \in \mathbb{Z}_q\right)$. Consider a function $f|_{x_j=c}$, derived by fixing x_j at c in f. The graph of $f|_{x_j=c}$ is denoted by $G\left(f|_{x_j=c}\right)$ which is obtained by deleting the vertex x_j and all the edges which are connected to x_j from $G(f)$. Then $G(f|_{\mathbf{x}=\mathbf{c}})$ is obtained from $G(f)$ by deleting $x_{j_0}, x_{j_1}, ..., x_{j_{k-1}}$. $G(f|_{\mathbf{x}=\mathbf{c}})$ represent the same graph for all $\mathbf{c} \in \{0,1\}^k$. Therefore, for all $\mathbf{c}$ in $\{0,1\}^k$, $f\Big|_{\mathbf{x}=\mathbf{c}}$ have the same quadratic form. Note that the quadratic forms of f and $\tilde{f}$ are the same; thus, they have associated with the same graph.

***Lemma* 1** *Construction of* CCC [8].

Let f: $\{0, \ 1\}^m \to \mathbb{Z}_q$ be a GBF *and f its reversal. Assume that $G(f|_{\mathbf{x}=\mathbf{c}})$ is a path for each $\mathbf{c} \in \{0, \ 1\}^k$ and the edges in the path have the same weight $q/2$. Let $(b_0, b_1, ..., b_{k-1})$ be the binary representation of the integer t. Define the order sets of* GBFs $\mathbf{C}^t$ *to be*

$$\left\{ f + \frac{q}{2} \left(\sum_{\alpha=0}^{k-1} d_\alpha x_{j_\alpha} + \sum_{\alpha=0}^{k-1} b_\alpha x_{j_\alpha} + d x_\gamma \right) : d, d_\alpha \in \{0,1\} \right\}, \tag{7}$$

and the order set of GBFs $\tilde{\mathbf{C}}t$ *to be*

$$\left\{ \tilde{f} + \frac{q}{2} \left(\sum_{\alpha=0}^{k-1} d_\alpha \overline{x}_{j_\alpha} + \sum_{\alpha=0}^{k-1} b_\alpha \overline{x}_{j_\alpha} + \overline{d} x_\gamma \right) : \overline{d}, d_\alpha \in \{0,1\} \right\}, \tag{8}$$

where x_γ is one of the end vertices in the path. Then

$$\{\psi(\mathbf{C}^t) : 0 \le t < 2^k\} \cup \left\{\psi^*\left(\tilde{\mathbf{C}}^t\right) : 0 \le t < 2^k\right\} \tag{9}$$

generates a set of CCC, *where $\psi^*(\cdot)$ denotes the complex conjugate of $\psi(\cdot)$.*

3. Construction of IGC code set from GBFs

In this section, we propose a direct construction of IGC code set by using Boolean algebra and graph theory. Before proposing the main theorem of the construction, we define some sets and vectors and present some lemmas. First we define some notations which will be used throughout in our construction:

- $\mathbf{x} = \left(x_{j_0}, x_{j_1}, \ldots x_{j_{k-1}}\right) \in \mathbb{Z}_2^k$, $\mathbf{x}' = \left(x_{m-p'} x_{m-p+1}, \ldots, x_{m-1}\right) \in \mathbb{Z}_2^p$.
- $\mathbf{b} = (b_0, b_1, \ldots, b_{k-1})$, $\mathbf{b}_i = (b_{i,0}, b_{i,1}, \ldots, b_{i,k-1}) \in \mathbb{Z}_2^k (i = 1, 2, \ldots, 2^k)$.
- $\mathrm{d} = (d_0, d_1, \ldots, d_{k-1}) \in \mathbb{Z}_2^k$, $\mathbf{d}' = \left(d'_1, d'_2, \ldots, d'_p\right)$ and $\mathbf{d}'_j = \left(d'_{j,1}, d'_{j,2}, \ldots, d'_{j,p}\right) \in \mathbb{Z}_2^p (j = 1, 2, \ldots, 2^p)$.
- $\Gamma = \left(g_{m-p} g_{m-p+1'} \ldots g_{m-1}\right) \in \mathbb{Z}_q^p$.
- $\mathbf{a}\cdot\mathbf{b}$ denotes the dot products of any two vectors $\mathbf{a}$ and $\mathbf{b}$ which are of the same length.
- $\mathbf{A}\otimes\mathbf{B}$ denotes the Kronecker product of any two matrices of arbitrary size.
- $\left[(\cdot)_{i,j}\right]$, $i = 0, 1, \ldots, M-1$ and $j = 1, 2, \ldots, N$ denotes a matrix of order $M \times N$.

Let f be a GBF of m variables $x_0, x_1, \ldots, x_{m-1}$ over $\mathbb{Z}_q$. For $\mathbf{b} \in \mathbb{Z}_2^k$, $\mathbf{d}' \in \mathbb{Z}_2^p$, we define order sets $S_{\mathbf{bd}'}$ and $\tilde{S}_{\mathbf{bd}'}$ corresponding to the GBF f as follows:

$$S_{\mathbf{bd}'}\left\{f + \frac{q}{2}\left(\sum_{\alpha=0}^{k} d_\alpha x_{j_\alpha} + \sum_{\alpha=0}^{-1} b_\alpha x_{j_\alpha} + \sum_{\alpha=1}^{p} d'_\alpha x_{m-p+\alpha-1} + d x_\gamma\right) : d, d_\alpha \in \{0,1\}\right\}, \tag{10}$$

or

$$S_{\mathbf{bd}'} = \left\{f + \frac{q}{2}\left((d + \mathbf{b})\cdot\mathbf{x} + \mathbf{d}'\cdot\mathbf{x}' + d x_\gamma\right) : d \in \mathbb{Z}_2, \mathbf{d} \in \mathbb{Z}_2^k\right\}, \tag{11}$$

and

$$\tilde{S}_{\mathbf{bd}'} = \left\{\tilde{f} + \frac{q}{2}\left((\mathrm{d} + \mathrm{b})\cdot\overline{\mathbf{x}} + \mathbf{d}'\cdot\overline{\mathbf{x}}' + \overline{d} x_\gamma\right) : d \in \mathbb{Z}_2, \mathbf{d} \in \mathbb{Z}_2^k\right\} \tag{12}$$

From the above expression, it is clear that each of the order sets $S_{\mathbf{bd}'}$ and $\tilde{S}_{\mathbf{bd}'}$ contains 2^{k+1} GBFs.

***Lemma* 2** *Let f be a* GBF *of m variables with the property that each $\mathbf{c} \in \{0,1\}^k$, $G(f|_{\mathbf{x}=\mathbf{c}})$ contains a path over $m-k-p (0 \le k < m,\ p \ge 0)$ vertices and p isolated vertices labeled $m-p,\ m-p+1, ...,\ m-1$ such that $0 \le k+p \le m-2 (m \ge 2)$. Further, assume that there was no edges between deleted vertices (as defined before, the restricted variables in a* GBF *are considered as the vertices to be deleted in the graph of the Boolean function) and isolated vertices before the deletion. Let x_γ be one of the end vertices of the path in $G(f|_{\mathbf{x}=\mathbf{c}})$ and the weight of each edge in the path be $q/2$. Let $\mathbf{a}'_1, \mathbf{a}'_2, ..., \mathbf{a}'_{2^{m-p}}$ be binary vector representations of $0, 1, 2^{m-p}-1$ of length $m-p$ and $\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_{2^p}$ be binary vector representations of $0, 1, 2^p-1$ of length p. Also let l be a positive integer such that $l = \Sigma_{i=0}^{k-1} d_i 2^i + d2^k$. Then for any choice of $g', g_j \in \mathbb{Z}_q$, the codes $\psi(S_{\mathbf{bd}'})$ and $\psi(\tilde{S}_{\mathbf{bd}'})$ can be expressed as*

$$\begin{aligned} \psi(S_{\mathbf{bd}'}) &= \left[\psi(F_{\mathbf{b}l})\omega^{\left(\Gamma+\frac{q}{2}\mathbf{d}'\right)\cdot\mathbf{r}_{j'}}\right], \quad l = 0, 1, ..., 2^{k+1}-1, j' = 1, 2, ..., 2^p, \\ \psi\left(\tilde{S}_{\mathbf{bd}'}\right) &= \left[\psi\left(F'_{\mathbf{b}l}\right)\omega^{\left(\Gamma+\frac{q}{2}\mathbf{d}'\right)\cdot\bar{\mathbf{r}}_{j'}}\right], \quad l = 0, 1, ..., 2^{k+1}-1, j' = 1, 2, ..., 2^p, \end{aligned} \tag{13}$$

where

$$\begin{aligned} \psi(F_{\mathbf{b}l}) &= \left(\omega^{F_{\mathbf{b}l}\left(\mathbf{a}'_1\right)}, , \omega^{F_{\mathbf{b}l}\left(\mathbf{a}'_2\right)}, ..., , \omega^{F_{\mathbf{b}l}\left(\mathbf{a}'_{2^{m-p}}\right)}\right), \\ \psi(F_{\mathrm{b}l'}) &= \left(\omega^{F'_{\mathbf{b}l}\left(\mathbf{a}'_1\right)}, , \omega^{F'_{\mathbf{b}l}\left(\mathbf{a}'_2\right)}, ..., , \omega^{F'_{\mathbf{b}l}\left(\mathbf{a}'_{2^{m-p}}\right)}\right), \\ F_{\mathbf{b}l} &= f' + \frac{q}{2}\left((\mathbf{d}+\mathbf{b})\cdot\mathbf{x} + dx_\gamma\right), \\ F'_{\mathbf{b}l} &= \tilde{f'} + \frac{q}{2}\left((\mathbf{d}+\mathbf{b})\cdot\bar{\mathbf{x}} + \bar{d}x_\gamma\right), \\ f' &= Q + \sum_{i=0}^{m-p-1} g_i x_i + g'. \end{aligned} \tag{14}$$

***Proof* 1** *Since there are no edges between the deleted and isolated vertices before the deletion of k vertices $x_{j_0}, x_{j_1}, ..., x_{j_{k-1}}$, the quadratic form Q presented in $G(f)$ can be expressed as*

$$Q = \frac{q}{2}\sum_{\alpha=0}^{m-k-p-1} x_{\pi(\alpha)}x_{\pi(\alpha+1)} + \sum_{0 \le \mu < v \le k-1} b'_{j_\mu, j_v} x_{j_\mu}x_{j_v} + \sum_{\alpha=0}^{m-k-p-1}\sum_{\sigma=0}^{k-1} c'_{\pi(\alpha)j_\sigma} x_{\pi(\alpha)}x_{j_\sigma}, \tag{15}$$

where π is a permutation over the set $\{0,\ 1,\ ..., m-1\} \setminus \{j_0, j_1,\ ..., j_{k-1}\} \cup \{m-p,\ m-p+1,\ ...,\ m-1\}$, $b'_{j_\mu, j_v} (\in \mathbb{Z}_q)$ which *denotes the weight between the vertices x_{j_μ} and x_{j_v} and $c'_{\pi(\alpha), j_\sigma} (\in \mathbb{Z}_q)$ denotes the weight between the vertices $x_{\pi(\alpha)}$ and $x_{j\sigma}$. Therefore, f' is a* GBF *of $m-p$ variables $x_0, x_1, ..., x_{m-p-1}$, and the* GBF *f of m variables can be expressed as.*

$$f = f' + \sum_{i=m-p}^{m-1} g_i x_i \tag{16}$$

or

$$f = f' + \Gamma\cdot\mathbf{x}'. \tag{17}$$

Now we define a GBF $F_l^{\mathbf{b}}$ *over m variables by*

$$\begin{aligned} F_l^{\mathbf{b}} &= f + \frac{q}{2}\left((\mathbf{d}+\mathbf{b})\cdot\mathbf{x} + \mathbf{d}'\cdot\mathbf{x}' + dx_\gamma\right) \\ &= F_{\mathbf{b}l} + \left(\Gamma + \frac{q}{2}\mathbf{d}'\right)\cdot\mathbf{x}'. \end{aligned} \tag{18}$$

Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{2^m}$ *be binary vector representations of* $0, 1, 2^m - 1$ *of length* m*, given in* **Table 1**. *The truth table given in Table 1 can also be expressed as the truth table given in* **Table 2**. **Table 3** *contains a truth table over* $m - p$ *variables*.

From **Tables 1–3**, *it is observed that the code* $\psi(S_{\mathbf{bd}'})$ *can be expressed as*

$$\psi(S_{\mathbf{bd}'}) = \left[\omega^{F_l^{\mathbf{b}}(\mathbf{a}_j)}\right],\ l = 0,1,\ 2^{k+1} - 1, j = 1, 2, \ldots, 2^m \tag{19}$$

$\mathbf{a}_1$
$\mathbf{a}_2$
⋮
$\mathbf{a}_{2^m}$

Table 1.
Truth table over m variables.

$\mathbf{a}_1'\mathbf{r}_1$
$\mathbf{a}_{2'}\mathbf{r}_1$
⋮
$\mathbf{a}'_{2m-p}\mathbf{r}_1$
$\mathbf{a}_1'\mathbf{r}_2$
$\mathbf{a}_{2'}\mathbf{r}_2$
⋮
$\mathbf{a}'_{2m-p}\mathbf{r}_2$
⋮
$\mathbf{a}_1'\mathbf{r}_{2p}$
$\mathbf{a}_2'\mathbf{r}_{2p}$
⋮
$\mathbf{a}'_{2^{m-p}}\mathbf{r}_{2p}$

Table 2.
Truth table over m variables.

$\mathbf{a}_1'$
$\mathbf{a}_2'$
⋮
$\mathbf{a}'_{2^{m-p}}$

Table 3.
Truth table over $m - p$ *variables.*

or

$$\psi(S_{\mathbf{bd}'}) = \left[\psi(F_{\mathbf{b}l})\omega^{\left(\Gamma+\frac{q}{2}\mathbf{d}'\right)\cdot\mathbf{r}_{j'}}\right], \quad l = 0, 1, ..., 2^{k+1}-1, j' = 1, 2, ..., 2^p, \tag{20}$$

where

$$\psi(F_{\mathbf{b}l}) = \left(\omega^{F_{\mathbf{b}l}\left(\mathbf{a}'_1\right)}, \omega^{F_{\mathbf{b}l}\left(\mathbf{a}'_2\right)}, ..., \omega^{F_{\mathbf{b}l}\left(\mathbf{a}'_{2^m-0}\right)}\right), \quad l = 0, 1, \quad 2^{k+1}-1. \tag{21}$$

Similarly, we can show that

$$\psi\left(\tilde{S}_{\mathbf{bd}'}\right) = \left[\psi\left(F'_{\mathbf{b}l}\right)\omega^{\left(\Gamma+\frac{q}{2}\mathbf{d}'\right)\cdot\bar{\mathbf{r}}_{j'}}\right], \qquad l = 0, 1, \quad 2^{k+1}-1, j' = 1, 2, ..., 2^p. \tag{22}$$

***Example* 1** *Let f be a* GBF *of* four *variables over $\mathbb{Z}_4$, given by*

$$f(x_0, x_1, x_2, x_3) = 2x_1x_2 + 3x_0(x_1 + x_2) + x_0 + x_2 + x_3 + 1. \tag{23}$$

From the $G(f)$, given in **Figure 1**, *it is clear that after the deletion of the vertex* x_0, *the resultant graph contains a path over the vertices x_1, x_2 and an isolated vertex x_3. For this example $k = 1$ and $p = 1$. Therefore, the vectors* **b**, **d**, **d**′, **x**, Γ *and* **x**′ *are of length one and belong to $\mathbb{Z}_2$.*

Hence, $\mathbf{b} = \left(b_{j_0}\right) = (b_0) = b_0$, $\mathbf{d} = (d_0) = d_0$, $\mathbf{d}' = \left(d'_1\right) = d'_1$, $\mathbf{x} = \left(x_{j_0}\right) = (x_0) = x_0$, $\Gamma = \left(g_{m-p}\right) = \left(g_0\right) = 1$, *and* $\mathbf{x}' = \left(x_{m-p}\right) = (x_3) = x_3$. *The set of Boolean functions $S_{b_0d'_1}$ and $\tilde{S}_{b_0d'_1}$ are given below*:

$$S_{b_0d'_1} = \left\{f + \frac{q}{2}\left(d_0x_0 + b_0x_0 + d'_1x_3 + dx_2\right) \;:\; d, \;\; d_0 \in \{0, 1\}\right\}, \tag{24}$$

and

$$\tilde{s}_{b_0d'_1} = \left\{\tilde{f} + \frac{q}{2}\left(d_0\overline{x}_0 + b_0\overline{x}_0 + d'_1\overline{x}_3 + \overline{d}x_2\right) : d, \;\; d_0 \in \{0, 1\}\right\}. \tag{25}$$

The GBFs F_{b_0l} *and* F'_{b_0l} *are given by*

$$F_{b_0l} = 2x_1x_2 + 3x_0(x_1 + x_2) + x_0 + x_2 + 1 + \frac{q}{2}(d_0x_0 + b_0x_0 + dx_2) \tag{26}$$

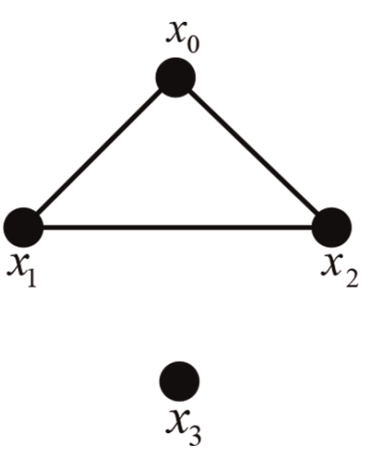

Figure 1.
The graph of the GBF $2x_1x_2 + 3x_0(x_1 + x_2) + x_0 + x_2 + x_3 + 1$.

and

$$F'_{b_0 l} = 2\overline{x}_1\overline{x}_2 + 3\overline{x}_0(\overline{x}_1 + \overline{x}_2) + \overline{x}_0 + \overline{x}_2 + 1 + \frac{q}{2}\left(d_0\overline{x}_0 + b_0\overline{x}_0 + \overline{d}x_2\right), \tag{27}$$

where $l = 0, 1, 2, 3$.
The codes corresponding to the sets of Boolean functions are listed below:

$$1)\quad \psi(S_{00}) = \begin{bmatrix} \omega^1\omega^2\omega^1\omega^1\omega^2\omega^2\omega^0\omega^3\omega^2\omega^3\omega^2\omega^2\omega^3\omega^3\omega^1\omega^0 \\ \omega^1\omega^0\omega^1\omega^3\omega^2\omega^0\omega^0\omega^1\omega^2\omega^1\omega^2\omega^0\omega^3\omega^1\omega^1\omega^2 \\ \omega^1\omega^2\omega^1\omega^1\omega^0\omega^0\omega^2\omega^1\omega^2\omega^3\omega^2\omega^2\omega^1\omega^1\omega^3\omega^2 \\ \omega^1\omega^0\omega^1\omega^3\omega^0\omega^2\omega^2\omega^3\omega^2\omega^1\omega^2\omega^0\omega^1\omega^3\omega^3\omega^0 \end{bmatrix} = \begin{bmatrix} \omega^0\psi(\mathbf{C}^0) & \omega^1\psi(\mathbf{C}^0) \end{bmatrix} \tag{28}$$

where

$$\psi(\mathbf{C}^0) = \begin{bmatrix} \omega^1\omega^2\omega^1\omega^1\omega^2\omega^2\omega^0\omega^3 \\ \omega^1\omega^0\omega^1\omega^3\omega^2\omega^0\omega^0\omega^1 \\ \omega^1\omega^2\omega^1\omega^1\omega^0\omega^0\omega^2\omega^1 \\ \omega^1\omega^0\omega^1\omega^3\omega^0\omega^2\omega^2\omega^3 \end{bmatrix}.$$

$$2)\quad \psi(\tilde{S}_{00}) = \begin{bmatrix} \omega^0\omega^1\omega^3\omega^3\omega^0\omega^0\omega^1\omega^0\omega^3\omega^0\omega^2\omega^2\omega^3\omega^3\omega^0\omega^3 \\ \omega^2\omega^1\omega^1\omega^3\omega^2\omega^0\omega^3\omega^0\omega^1\omega^0\omega^0\omega^2\omega^1\omega^3\omega^2\omega^3 \\ \omega^0\omega^1\omega^3\omega^3\omega^2\omega^2\omega^3\omega^2\omega^3\omega^0\omega^2\omega^2\omega^1\omega^1\omega^2\omega^1 \\ \omega^2\omega^1\omega^1\omega^3\omega^0\omega^2\omega^1\omega^2\omega^1\omega^0\omega^0\omega^2\omega^3\omega^1\omega^0\omega^1 \end{bmatrix} = \begin{bmatrix} \omega^1\psi\left(\tilde{\mathbf{C}}^0\right) & \omega^0\psi\left(\tilde{\mathbf{C}}^0\right) \end{bmatrix} \tag{29}$$

where

$$\psi\left(\tilde{\mathbf{C}}^0\right) = \begin{bmatrix} \omega^3\omega^0\omega^2\omega^2\omega^3\omega^3\omega^0\omega^3 \\ \omega^1\omega^0\omega^0\omega^2\omega^1\omega^3\omega^2\omega^3 \\ \omega^3\omega^0\omega^2\omega^2\omega^1\omega^1\omega^2\omega^1 \\ \omega^1\omega^0\omega^0\omega^2\omega^3\omega^1\omega^0\omega^1 \end{bmatrix}.$$

$$3)\quad \psi(S_{01}) = \begin{bmatrix} \omega^1\omega^2\omega^1\omega^1\omega^2\omega^2\omega^0\omega^3\omega^0\omega^1\omega^0\omega^0\omega^1\omega^1\omega^3\omega^2 \\ \omega^1\omega^0\omega^1\omega^3\omega^2\omega^0\omega^0\omega^1\omega^0\omega^3\omega^0\omega^2\omega^1\omega^3\omega^3\omega^0 \\ \omega^1\omega^2\omega^1\omega^1\omega^0\omega^0\omega^2\omega^1\omega^0\omega^1\omega^0\omega^0\omega^3\omega^3\omega^1\omega^0 \\ \omega^1\omega^0\omega^1\omega^3\omega^0\omega^2\omega^2\omega^3\omega^0\omega^3\omega^0\omega^2\omega^3\omega^1\omega^1\omega^2 \end{bmatrix} = \begin{bmatrix} \omega^0\psi(\mathbf{C}^0) & \omega^3\psi(\mathbf{C}^0) \end{bmatrix} \tag{30}$$

$$4)\quad \psi(\tilde{S}_{01}) = \begin{bmatrix} \omega^2\omega^3\omega^1\omega^1\omega^2\omega^2\omega^3\omega^2\omega^3\omega^0\omega^2\omega^2\omega^3\omega^3\omega^0\omega^3 \\ \omega^0\omega^3\omega^3\omega^1\omega^0\omega^2\omega^1\omega^2\omega^1\omega^0\omega^0\omega^2\omega^1\omega^3\omega^2\omega^3 \\ \omega^2\omega^3\omega^1\omega^1\omega^0\omega^0\omega^1\omega^0\omega^3\omega^0\omega^2\omega^2\omega^1\omega^1\omega^2\omega^1 \\ \omega^0\omega^3\omega^3\omega^1\omega^2\omega^0\omega^3\omega^0\omega^1\omega^0\omega^0\omega^2\omega^3\omega^1\omega^0\omega^1 \end{bmatrix} = \begin{bmatrix} \omega^3\psi\left(\tilde{\mathbf{C}}^0\right) & \omega^0\psi\left(\tilde{\mathbf{C}}^0\right) \end{bmatrix} \tag{31}$$

$$5)\quad \psi(S_{10}) = \begin{bmatrix} \omega^1\omega^0\omega^1\omega^3\omega^2\omega^0\omega^0\omega^1\omega^2\omega^1\omega^2\omega^0\omega^3\omega^1\omega^1\omega^2 \\ \omega^1\omega^2\omega^1\omega^1\omega^2\omega^2\omega^0\omega^3\omega^2\omega^3\omega^2\omega^2\omega^3\omega^3\omega^1\omega^0 \\ \omega^1\omega^0\omega^1\omega^3\omega^0\omega^2\omega^2\omega^3\omega^2\omega^1\omega^2\omega^0\omega^1\omega^3\omega^3\omega^0 \\ \omega^1\omega^2\omega^1\omega^1\omega^0\omega^0\omega^2\omega^1\omega^2\omega^3\omega^2\omega^2\omega^1\omega^1\omega^3\omega^2 \end{bmatrix} \tag{32}$$
$$= \begin{bmatrix} \omega^0\psi(\mathbf{C}^1) & \omega^1\psi(\mathbf{C}^1) \end{bmatrix}$$

where

$$\psi(\mathbf{C}^1) = \begin{bmatrix} \omega^1\omega^0\omega^1\omega^3\omega^2\omega^0\omega^0\omega^1 \\ \omega^1\omega^2\omega^1\omega^1\omega^2\omega^2\omega^0\omega^3 \\ \omega^1\omega^0\omega^1\omega^3\omega^0\omega^2\omega^2\omega^3 \\ \omega^1\omega^2\omega^1\omega^1\omega^0\omega^0\omega^2\omega^1 \end{bmatrix}.$$

$$6)\quad \psi(\tilde{S}_{10}) = \begin{bmatrix} \omega^2\omega^1\omega^1\omega^3\omega^2\omega^0\omega^3\omega^0\omega^1\omega^0\omega^0\omega^2\omega^1\omega^3\omega^2\omega^3 \\ \omega^0\omega^1\omega^3\omega^3\omega^0\omega^0\omega^1\omega^0\omega^3\omega^0\omega^2\omega^2\omega^3\omega^3\omega^0\omega^3 \\ \omega^2\omega^1\omega^1\omega^3\omega^0\omega^2\omega^1\omega^2\omega^1\omega^0\omega^0\omega^2\omega^3\omega^1\omega^0\omega^1 \\ \omega^0\omega^1\omega^3\omega^3\omega^2\omega^2\omega^3\omega^2\omega^3\omega^0\omega^2\omega^2\omega^1\omega^1\omega^2\omega^1 \end{bmatrix} \tag{33}$$
$$= \begin{bmatrix} \omega^1\psi\left(\tilde{\mathbf{C}}^1\right) & \omega^0\psi\left(\tilde{\mathbf{C}}^1\right) \end{bmatrix}$$

where

$$\psi\left(\tilde{\mathbf{C}}^1\right) = \begin{bmatrix} \omega^1\omega^0\omega^0\omega^2\omega^1\omega^3\omega^2\omega^3 \\ \omega^3\omega^0\omega^2\omega^2\omega^3\omega^3\omega^0\omega^3 \\ \omega^1\omega^0\omega^0\omega^2\omega^3\omega^1\omega^0\omega^1 \\ \omega^3\omega^0\omega^2\omega^2\omega^1\omega^1\omega^2\omega^1 \end{bmatrix}.$$

$$7)\quad \psi(S_{11}) = \begin{bmatrix} \omega^1\omega^0\omega^1\omega^3\omega^2\omega^0\omega^0\omega^1\omega^0\omega^3\omega^0\omega^2\omega^1\omega^3\omega^3\omega^0 \\ \omega^1\omega^2\omega^1\omega^1\omega^2\omega^2\omega^0\omega^3\omega^0\omega^1\omega^0\omega^0\omega^1\omega^1\omega^3\omega^2 \\ \omega^1\omega^0\omega^1\omega^3\omega^0\omega^2\omega^2\omega^3\omega^0\omega^3\omega^0\omega^2\omega^3\omega^1\omega^1\omega^2 \\ \omega^1\omega^2\omega^1\omega^1\omega^0\omega^0\omega^2\omega^1\omega^0\omega^1\omega^0\omega^0\omega^3\omega^3\omega^1\omega^0 \end{bmatrix}. \tag{34}$$
$$= \begin{bmatrix} \omega^0\psi(\mathbf{C}^1) & \omega^3\psi(\mathbf{C}^1) \end{bmatrix}$$

$$8)\quad \psi(\tilde{S}_{11}) = \begin{bmatrix} \omega^0\omega^3\omega^3\omega^1\omega^0\omega^2\omega^1\omega^2\omega^1\omega^0\omega^0\omega^2\omega^1\omega^3\omega^2\omega^3 \\ \omega^2\omega^3\omega^1\omega^1\omega^2\omega^2\omega^3\omega^2\omega^3\omega^0\omega^2\omega^2\omega^3\omega^3\omega^0\omega^3 \\ \omega^0\omega^3\omega^3\omega^1\omega^2\omega^0\omega^3\omega^0\omega^1\omega^0\omega^0\omega^2\omega^3\omega^1\omega^0\omega^1 \\ \omega^2\omega^3\omega^1\omega^1\omega^0\omega^0\omega^1\omega^0\omega^3\omega^0\omega^2\omega^2\omega^1\omega^1\omega^2\omega^1 \end{bmatrix}. \tag{35}$$
$$= \begin{bmatrix} \omega^3\psi\left(\tilde{\mathbf{C}}^1\right) & \omega^0\psi\left(\tilde{\mathbf{C}}^1\right) \end{bmatrix}$$

***Theorem* 1** *Let f be a* GBF *over m variables as defined in Lemma* 1 *and Lemma* 2. *Suppose $I^0, I^1, \ldots, I^{2^{k+1}-1}$ are a list of $2^k + 1$ code groups defined by*

$$I^t = \left\{ I_{\mathbf{s}}^t : 0 \le s < 2^p \right\} = \left\{ \psi(S_{\mathbf{bd}'}) : \mathbf{d}' \in \{0,1\}^p \right\} \tag{36}$$

and

$$I^{2^k+t} = \left\{ I_{\mathrm{s}}^{2^k+t} : 0 \le s < 2^p \right\} = \left\{ \psi^* \left(\tilde{S}_{\mathbf{bd}'} \right) : \mathbf{d}' \in \{0,1\}^p \right\}, \tag{37}$$

which forms an IGC code set $I\left(2^{k+p+1}, 2^{k+1}, 2^m, 2^{m-p}\right)$.

***Proof* 2** *Let* $\psi\left(S_{\mathbf{bd}_1'}\right)$ *and* $\psi\left(S_{\mathbf{bd}_2'}\right)$ *be any two codes from a code group* I^t. *Then the ACCF of* $\psi\left(S_{\mathbf{bd}_1'}\right)$ *and* $\psi\left(S_{\mathbf{bd}_2'}\right)$ *at the time shift* $\eta 2^{m-p} + \tau$ (*where* $0 \le \eta < 2^p$, $\eta \in \mathbb{Z}$, $0 \le \tau < 2^{m-p}$, $\tau \in \mathbb{Z}$) *is*

$$\begin{aligned}
&\mathrm{C}\left(\psi\left(S_{\mathbf{bd}_1'}\right), \psi\left(S_{\mathbf{bd}_2'}\right)\right)(\eta 2^{m-p} + \tau) \\
&= \sum_{l=0}^{2^{k+1}-1} \mathrm{C}(\psi(F_{\mathbf{b}l}), \psi(F_{\mathbf{b}l})(\tau) \sum_{i=1}^{2^p-\eta} \omega^{\left(\Gamma+\frac{q}{2}\mathbf{d}_1'\right)\cdot\mathbf{r}_{i+\eta} - \left(\Gamma+\frac{q}{2}\mathbf{d}_2'\right)\cdot\mathbf{r}_i} \\
&\quad + \sum_{l=0}^{2^{k+1}-1} \mathrm{C}(\psi(F_{\mathbf{b}l}), \psi(F_{\mathbf{b}l})(\tau - 2^{m-p}) \sum_{i=1}^{2^p-\eta-1} \omega^{\left(\Gamma+\frac{q}{2}\mathbf{d}_1'\right)\cdot\mathbf{r}_{i+\eta+1} - \left(\Gamma+\frac{q}{2}\mathbf{d}_2'\right)\cdot\mathbf{r}_i} \\
&= \mathrm{C}(\psi(\mathbf{C}^t), \psi(\mathbf{C}^t))(\tau) \sum_{i=1}^{2^p-\eta} \omega^{\left(\Gamma+\frac{q}{2}\mathbf{d}_1'\right)\cdot\mathbf{r}_{i+\eta} - \left(\Gamma+\frac{q}{2}\mathbf{d}_2'\right)\cdot\mathbf{r}_i} \\
&\quad + \mathrm{C}(\psi(\mathbf{C}^t), \psi(\mathbf{C}^t))(\tau - 2^{m-p}) \sum_{i=1}^{2^p-\eta-1} \omega^{\left(\Gamma+\frac{q}{2}\mathbf{d}_1'\right)\cdot\mathbf{r}_{i+\eta+1} - \left(\Gamma+\frac{q}{2}\mathbf{d}_2'\right)\cdot\mathbf{r}_i} \\
&= A(\psi(\mathbf{C}^t))(\tau) \sum_{i=1}^{2^p-\eta} \omega^{\left(\mathrm{r}+\frac{q}{2}\mathbf{d}_1'\right)\cdot\mathbf{r}_{i+\eta} - \left(\Gamma+\frac{q}{2}\mathbf{d}_2'\right)\cdot\mathbf{r}_i} \\
&\quad + A(\psi(\mathbf{C}^t))(\tau - 2^{m-p}) \sum_{i=1}^{2^p-\eta-1} \omega^{\left(\Gamma+\frac{q}{2}\mathbf{d}_1'\right)\cdot\mathbf{r}_{i+\eta+1} - \left(\Gamma+\frac{q}{2}\mathbf{d}_2'\right)\cdot\mathbf{r}_i}.
\end{aligned} \tag{38}$$

For $\mathbf{d}_1' = \mathbf{d}_2' = \mathbf{d}'$, *the ACCF given in* Eq. (38) *reduced to AACF as follows*:

$$A(\psi(S_{\mathbf{bd}'}))(\eta 2^{m-p} + \tau) = \begin{cases} 2^{m+k-p+1} & \\ \times \sum_{i=1}^{2^p-\eta} \omega^{\left(\Gamma+\frac{q}{2}\mathbf{d}_1'\right)\cdot\mathbf{r}_{i+\eta} - \left(\Gamma+\frac{q}{2}\mathbf{d}_2'\right)\cdot\mathbf{r}_i}, & \tau = 0,\ 0 \le \eta < 2^p, \\ 0, & 0 < |\tau| < 2^{m-p}, \\ & 0 \le \eta < 2^p. \end{cases} \tag{39}$$

For $\mathbf{d}_1' \ne \mathbf{d}_2'$, *the ACCF given in* Eq. (38) *can be expressed as*

$$\mathrm{C}\left(\psi\left(S_{\mathbf{bd}_1'}\right), \psi\left(S_{\mathbf{bd}_2'}\right)\right)(\eta 2^{m-p} + \tau) = \begin{cases} 2^{m+k-p+1} & \\ \times \sum_{i=1}^{2^p-\eta} \omega^{\left(\Gamma+\frac{q}{2}\mathbf{d}_1'\right)\cdot\mathbf{r}_{i+\eta} - \left(\Gamma+\frac{q}{2}\mathbf{d}_2'\right)\mathbf{r}_i}, & \tau = 0,\ 0 < \eta < 2^p, \\ 0, & \tau = 0, \eta = 0, \\ 0, & 0 < |\tau| < 2^{m-p}, \\ & 0 \le \eta < 2^p. \end{cases} \tag{40}$$

The terms in Eqs. (39) and (40) *are derived from the autocorrelation properties of the* $\mathrm{CC}\psi(\mathbf{C}^t)$. *It is also observed that the codes from the same code group* I^t *have ideal auto- and cross-correlation properties inside the ZCZ width* 2^{m-p}. *Similarly, we can show that*

$$A\left(\psi^*\left(\tilde{S}_{\mathbf{bd}'}\right)\right)(\eta 2^{m-p}+\tau)=\begin{cases}2^{m+k-p+1} & \tau=0,\ 0\le\eta<2^p,\\ \times\Sigma_{i=1}^{2^p-\eta}\omega^{\left(\Gamma+\frac{q}{2}\mathbf{d}_2'\right)\cdot\bar{\mathbf{r}}_i-\left(\Gamma+\frac{q}{2}\mathbf{d}_1'\right)\cdot\bar{\mathbf{r}}_{i+\eta}}, & \\ 0, & 0<|\tau|<2^{m-p},\\ & 0\le\eta<2^p,\end{cases}\tag{41}$$

and for $\mathbf{d}_1'\neq\mathbf{d}_2'$,

$$\mathrm{C}\left(\psi^*\left(\tilde{S}_{\mathbf{bd}_1'}\right),\psi^*\left(\tilde{S}_{\mathbf{bd}_2'}\right)\right)(\eta 2^{m-p}+\tau)=\begin{cases}2^{m+k-p+1} & \tau=0,\ 0<\eta<2^p,\\ \times\Sigma_{i=1}^{2^p-\eta}\omega^{\left(\Gamma+\frac{q}{2}\mathbf{d}_2'\right)\cdot\bar{\mathbf{r}}_i-\left(\Gamma+\frac{q}{2}\mathbf{d}_1'\right)\cdot\bar{\mathbf{r}}_{i+\eta}}, & \\ 0, & \tau=0,\ \eta=0,\\ 0, & 0<|\tau|<2^{m-p},\\ & 0\le\eta<2^p.\end{cases}\tag{42}$$

From Eqs. (41) and (42), *we get that the codes from the same code group* I^{2^k+t} *have ideal auto- and cross-correlation properties inside the ZCZ width* 2^{m-p}.

Now we show that the ACCFs between any two codes of any two different code groups I^{t_1} *and* $I^{t_2}\left(0\le t_1,t_2<2^k\right)$ *are zeros everywhere. Let* $\psi\left(S_{\mathbf{b}_1\mathbf{d}_1'}\right)\in I^{t_1}$, $\psi\left(S_{\mathbf{b}_2\mathbf{d}_2'}\right)\in I^{t_2}$ *where* $\mathbf{b}_1,\mathbf{b}_2$ *are binary vector representations of* t_1,t_2, *and* $\mathbf{d}_1',\mathbf{d}_2'$ *are any two binary vectors in* $\mathbb{Z}_2^p$. *Then*

$$\begin{aligned}&\mathrm{C}\left(\psi\left(S_{\mathbf{b}_1\mathbf{d}_1'}\right),\psi\left(S_{\mathbf{b}_2\mathbf{d}_2'}\right)\right)(\eta 2^{m-p}+\tau)\\ &=\sum_{l=0}^{2^{k+1}-1}\mathrm{C}(\psi(F_{\mathbf{b}_1 l}),\psi(F_{\mathbf{b}_2 l})(\tau)\sum_{i=1}^{2^p-\eta}\omega^{\left(\Gamma+\frac{q}{2}\mathbf{d}_1'\right)\cdot\mathbf{r}_{i+\eta}-\left(\Gamma+\frac{q}{2}\mathbf{d}_2'\right)\cdot\mathbf{r}_i}\\ &\quad+\sum_{l=0}^{2^{k+1}-1}\mathrm{C}(\psi(F_{\mathbf{b}_1 l}),\psi(F_{\mathbf{b}_2 l})(\tau-2^{m-p})\sum_{i=1}^{2^p-\eta-1}\omega^{\left(\Gamma+\frac{q}{2}\mathbf{d}_1'\right)\cdot\mathbf{r}_{i+\eta+1}-\left(\Gamma+\frac{q}{2}\mathbf{d}_2'\right)\cdot\mathbf{r}_i}\\ &=\mathrm{C}(\psi(\mathbf{C}^t),\psi(\mathbf{C}^t))(\tau)\sum_{i=1}^{2^p-\eta}\omega^{\left(\Gamma+\frac{q}{2}\mathbf{d}_1'\right)\cdot\mathbf{r}_{i+\eta}-\left(\Gamma+\frac{q}{2}\mathbf{d}_2'\right)\cdot\mathbf{r}_i}\\ &\quad+\mathrm{C}(\psi(\mathbf{C}^t),\psi(\mathbf{C}^t))(\tau-2^{m-p})\sum_{i=1}^{2^p-\eta-1}\omega^{\left(\Gamma+\frac{q}{2}\mathbf{d}_1'\right)\cdot\mathbf{r}_{i+\eta+1}-\left(\Gamma+\frac{q}{2}\mathbf{d}_2'\right)\cdot\mathbf{r}_i}\\ &=0\quad\forall\tau,\eta.\end{aligned}\tag{43}$$

Similarly, we can also show that the ACCFs between any two codes of any two different code groups $I^{2^k+t_1}$ *and* $I^{2^k+t_2}\left(0\le t_1,t_2<2^k\right)$ *are zeros everywhere, i.e.*,

$$\begin{aligned}&\mathrm{C}\left(\psi\left(\tilde{S}_{\mathbf{b}_1\mathbf{d}_1'}\right),\psi*\left(\tilde{S}_{\mathbf{b}_2\mathbf{d}_2'}\right)\right)(\eta 2^{m-p}+\tau)\\ &=\sum_{l=0}^{2^{k+1}-1}\mathrm{C}(\psi*\left(F'_{\mathbf{b}_1 l}\right),\psi*\left(F'_{\mathbf{b}_2 l}\right)(\tau)\sum_{i=1}^{2^p-\eta}\omega^{\left(\Gamma+\frac{q}{2}\mathbf{d}_2'\right)\cdot\mathbf{r}_i-\left(\Gamma+\frac{q}{2}\mathbf{d}_1'\right)\cdot\mathbf{r}_{i+\eta}}\\ &\quad+\sum_{l=0}^{2^{k+1}-1}\mathrm{C}(\psi*\left(F'_{\mathbf{b}_1 l}\right),\psi*\left(F'_{\mathbf{b}_2 l}\right)(\tau-2^{m-p})\sum_{i=1}^{2^p-\eta-1}\omega^{\left(\Gamma+\frac{q}{2}\mathbf{d}_2'\right)\cdot\mathbf{r}_i-\left(\Gamma+\frac{q}{2}\mathbf{d}_1'\right)\cdot\mathbf{r}_{i+\eta+1}}\\ &=\mathrm{C}\left(\psi*\left(\tilde{\mathbf{C}}^{t_1}\right),\psi*\left(\tilde{\mathbf{C}}^{t_2}\right)\right)(\tau-2^{m-p})\sum_{i=1}^{2^p-\eta-1}\omega^{\left(\Gamma+\frac{q}{2}\mathbf{d}_2'\right)\cdot\mathbf{r}_i-\left(\Gamma+\frac{q}{2}\mathbf{d}_1'\right)\cdot\mathbf{r}_{i+\eta}}\\ &\quad+\mathrm{C}\left(\psi*\left(\tilde{\mathbf{C}}^{t_1}\right),\psi*\left(\tilde{\mathbf{C}}^{t_2}\right)\right)(\tau-2^{m-p})\sum_{i=1}^{2^p-\eta-1}\omega^{\left(\Gamma+\frac{q}{2}\mathbf{d}_2'\right)\cdot\mathbf{r}_i-\left(\Gamma+\frac{q}{2}\mathbf{d}_1'\right)\cdot\mathbf{r}_{i+\eta+1}}\\ &=0\quad\forall\tau,\eta.\end{aligned}\tag{44}$$

The results in Eqs. (43) and (44) *are obtained by using the ideal cross-correlation properties of CCCs. To complete the proof, now we only need to show that the ACCFs of any code from* I^{t_u} *and* $I^{2^k+t_v}\left(u,\ v\in \mathrm{Z},\ 1\le u,\ v\le 2^k\right)$ *are zeros everywhere. In this case,* t_u *and* t_v *are any two integers in* $\left[0,2^k\right)$ and *may or may not be equal. Let* $\psi(S_{\mathrm{b}_u \mathrm{d}_1{}'}\in I^{t_u},\ \ \psi^*\left(\tilde{S}_{\mathrm{b}_v \mathrm{d}_2{}'}\right)\in I^{2^k+t_v}$ *where* b_u, b_v *are binary vector representations of* t_u, t_v, *respectively. Then*

$$\begin{aligned}
&\mathrm{C}\left(\psi\left(S_{\mathbf{b}_u\mathbf{d}_1'}\right),\psi*\left(\tilde{S}_{\mathbf{b}_v\mathbf{d}_2'}\right)\right)(\eta 2^{m-p}+\tau)\\
&=\sum_{l=0}^{2^{k+1}-1}\mathrm{C}(\psi*\left(F'_{\mathbf{b}_u l}\right),\psi*\left(F'_{\mathbf{b}_v l}\right)(\tau)\sum_{i=1}^{2^p-\eta}\omega^{\left(\Gamma+\frac{q}{2}\mathbf{d}_1'\right)\cdot\mathbf{r}_{i+\eta}-\left(\Gamma+\frac{q}{2}\mathbf{d}_2'\right)\cdot\mathbf{r}_i}\\
&\quad+\sum_{l=0}^{2^{k+1}-1}\mathrm{C}(\psi*\left(F'_{\mathbf{b}_u l}\right),\psi*(F_{\mathbf{b}vl})(\tau)\sum_{i=1}^{2^p-\eta}\omega^{\left(\Gamma+\frac{q}{2}\mathbf{d}_1'\right)\cdot\mathbf{r}_{i+\eta}-\left(\Gamma+\frac{q}{2}\mathbf{d}_2'\right)\cdot\mathbf{r}_i}\\
&=\mathrm{C}(\psi*(\mathbf{C}^{t_u}),\psi*(\mathbf{C}^{t_v}))(\tau-2^{m-p})\sum_{i=1}^{2^p-\eta-1}\omega^{\left(\Gamma+\frac{q}{2}\mathbf{d}_1'\right)\cdot\mathbf{r}_{i+\eta}-\left(\Gamma+\frac{q}{2}\mathbf{d}_2'\right)\cdot\mathbf{r}_i}\\
&\quad+\mathrm{C}\left(\psi*(\mathbf{C}^{t_u}),\psi*\left(\tilde{\mathbf{C}}^{t_v}\right)\right)(\tau-2^{m-p})\sum_{i=1}^{2^p-\eta-1}\omega^{\left(\Gamma+\frac{q}{2}\mathbf{d}_1'\right)\cdot\mathbf{r}_{i+\eta+1}-\left(\Gamma+\frac{q}{2}\mathbf{d}_2'\right)\cdot\mathbf{r}_i}\\
&=0\quad\forall\tau,\eta.
\end{aligned}\tag{45}$$

The above result is also obtained by using the ideal cross-correlation properties of CCCs. From Eqs. (39)-(45), *we observed that the AACFs and ACCFs of the codes of the same group are zeros inside the ZCZ width* 2^{m-p} *and the ACCFs of the codes from different code groups are zeros everywhere. Hence, we can conclude that* $I^0, I^1, \ldots, I^{2^{k+1}-1}$ *form an IGC code set* $I\left(2^{k+p+1},2^{k+1},2^m,2^{m-p}\right)$.

***Example* 2** *Let* f *be a GBF of* four *variables as given in Example* 1. *Then the obtained IGC code set* $I(8,4,16,8)$ *corresponding to the* GBF f *is given below*:

Code group 1:

$$\begin{aligned}
I_0^0&=\psi(S_{00})\\
&=\begin{bmatrix}
\omega^1\omega^2\omega^1\omega^1\omega^2\omega^2\omega^0\omega^3\omega^2\omega^3\omega^2\omega^2\omega^3\omega^3\omega^1\omega^0\\
\omega^1\omega^0\omega^1\omega^3\omega^2\omega^0\omega^0\omega^1\omega^2\omega^1\omega^2\omega^0\omega^3\omega^1\omega^1\omega^2\\
\omega^1\omega^2\omega^1\omega^1\omega^0\omega^0\omega^2\omega^1\omega^2\omega^3\omega^2\omega^2\omega^1\omega^1\omega^3\omega^2\\
\omega^1\omega^0\omega^1\omega^3\omega^0\omega^2\omega^2\omega^3\omega^2\omega^1\omega^2\omega^0\omega^1\omega^3\omega^3\omega^0
\end{bmatrix}\\
I_1^0&=\psi(S_{01})\\
&=\begin{bmatrix}
\omega^1\omega^2\omega^1\omega^1\omega^2\omega^2\omega^0\omega^3\omega^0\omega^1\omega^0\omega^0\omega^1\omega^1\omega^3\omega^2\\
\omega^1\omega^0\omega^1\omega^3\omega^2\omega^0\omega^0\omega^1\omega^0\omega^3\omega^0\omega^2\omega^1\omega^3\omega^3\omega^0\\
\omega^1\omega^2\omega^1\omega^1\omega^0\omega^0\omega^2\omega^1\omega^0\omega^1\omega^0\omega^0\omega^3\omega^3\omega^1\omega^0\\
\omega^1\omega^0\omega^1\omega^3\omega^0\omega^2\omega^2\omega^3\omega^0\omega^3\omega^0\omega^2\omega^3\omega^1\omega^1\omega^2
\end{bmatrix}.
\end{aligned}\tag{46}$$

Code group 2:

$$
\begin{aligned}
I_0^1 &= \psi(S_{10}) \\
&= \begin{bmatrix}
\omega^1\omega^0\omega^1\omega^3\omega^2\omega^0\omega^0\omega^1\omega^2\omega^1\omega^2\omega^0\omega^3\omega^1\omega^1\omega^2 \\
\omega^1\omega^2\omega^1\omega^1\omega^2\omega^2\omega^0\omega^3\omega^2\omega^3\omega^2\omega^2\omega^3\omega^3\omega^1\omega^0 \\
\omega^1\omega^0\omega^1\omega^3\omega^0\omega^2\omega^2\omega^3\omega^2\omega^1\omega^2\omega^0\omega^1\omega^3\omega^3\omega^0 \\
\omega^1\omega^2\omega^1\omega^1\omega^0\omega^0\omega^2\omega^1\omega^2\omega^3\omega^2\omega^2\omega^1\omega^1\omega^3\omega^2
\end{bmatrix} \\
I_1^1 &= \psi(S_{11}) \\
&= \begin{bmatrix}
\omega^1\omega^0\omega^1\omega^3\omega^2\omega^0\omega^0\omega^1\omega^0\omega^3\omega^0\omega^2\omega^1\omega^3\omega^3\omega^0 \\
\omega^1\omega^2\omega^1\omega^1\omega^2\omega^2\omega^0\omega^3\omega^0\omega^1\omega^0\omega^0\omega^1\omega^1\omega^3\omega^2 \\
\omega^1\omega^0\omega^1\omega^3\omega^0\omega^2\omega^2\omega^3\omega^0\omega^3\omega^0\omega^2\omega^3\omega^1\omega^1\omega^2 \\
\omega^1\omega^2\omega^1\omega^1\omega^0\omega^0\omega^2\omega^1\omega^0\omega^1\omega^0\omega^0\omega^3\omega^3\omega^1\omega^0
\end{bmatrix}.
\end{aligned}
\tag{47}
$$

Code group 3:

$$
\begin{aligned}
I_0^2 &= \psi^*\left(\tilde{S}_{00}\right) \\
&= \begin{bmatrix}
\omega^0\omega^3\omega^1\omega^1\omega^0\omega^0\omega^3\omega^0\omega^1\omega^0\omega^2\omega^2\omega^1\omega^1\omega^0\omega^1 \\
\omega^2\omega^3\omega^3\omega^1\omega^2\omega^0\omega^1\omega^0\omega^3\omega^0\omega^0\omega^2\omega^3\omega^1\omega^2\omega^1 \\
\omega^0\omega^3\omega^1\omega^1\omega^2\omega^2\omega^1\omega^2\omega^1\omega^0\omega^2\omega^2\omega^3\omega^3\omega^2\omega^3 \\
\omega^2\omega^3\omega^3\omega^1\omega^0\omega^2\omega^3\omega^2\omega^3\omega^0\omega^0\omega^2\omega^1\omega^3\omega^0\omega^3
\end{bmatrix} \\
I_1^2 &= \psi^*\left(\tilde{S}_{01}\right) \\
&= \begin{bmatrix}
\omega^2\omega^1\omega^3\omega^3\omega^2\omega^2\omega^1\omega^2\omega^1\omega^0\omega^2\omega^2\omega^1\omega^1\omega^0\omega^1 \\
\omega^0\omega^1\omega^1\omega^3\omega^0\omega^2\omega^3\omega^2\omega^3\omega^0\omega^0\omega^2\omega^3\omega^1\omega^2\omega^1 \\
\omega^2\omega^1\omega^3\omega^3\omega^0\omega^0\omega^3\omega^0\omega^1\omega^0\omega^2\omega^2\omega^3\omega^3\omega^2\omega^3 \\
\omega^0\omega^1\omega^1\omega^3\omega^2\omega^0\omega^1\omega^0\omega^3\omega^0\omega^0\omega^2\omega^1\omega^3\omega^0\omega^3
\end{bmatrix}.
\end{aligned}
\tag{48}
$$

Code group 4:

$$
\begin{aligned}
I_0^3 &= \psi^*\left(\tilde{S}_{10}\right) \\
&= \begin{bmatrix}
\omega^2\omega^3\omega^3\omega^1\omega^2\omega^0\omega^1\omega^0\omega^3\omega^0\omega^0\omega^2\omega^3\omega^1\omega^2\omega^1 \\
\omega^0\omega^3\omega^1\omega^1\omega^0\omega^0\omega^3\omega^0\omega^1\omega^0\omega^2\omega^2\omega^1\omega^1\omega^0\omega^1 \\
\omega^2\omega^3\omega^3\omega^1\omega^0\omega^2\omega^3\omega^2\omega^3\omega^0\omega^0\omega^2\omega^1\omega^3\omega^0\omega^3 \\
\omega^0\omega^3\omega^1\omega^1\omega^2\omega^2\omega^1\omega^2\omega^1\omega^0\omega^2\omega^2\omega^3\omega^3\omega^2\omega^3
\end{bmatrix} \\
I_1^3 &= \psi^*\left(\tilde{S}_{11}\right) \\
&= \begin{bmatrix}
\omega^0\omega^1\omega^1\omega^3\omega^0\omega^2\omega^3\omega^2\omega^3\omega^0\omega^0\omega^2\omega^3\omega^1\omega^2\omega^1 \\
\omega^2\omega^1\omega^3\omega^3\omega^2\omega^2\omega^1\omega^2\omega^1\omega^0\omega^2\omega^2\omega^1\omega^1\omega^0\omega^1 \\
\omega^0\omega^1\omega^1\omega^3\omega^2\omega^0\omega^1\omega^0\omega^3\omega^0\omega^0\omega^2\omega^1\omega^3\omega^0\omega^3 \\
\omega^2\omega^1\omega^3\omega^3\omega^0\omega^0\omega^3\omega^0\omega^1\omega^0\omega^2\omega^2\omega^3\omega^3\omega^2\omega^3
\end{bmatrix}.
\end{aligned}
\tag{49}
$$

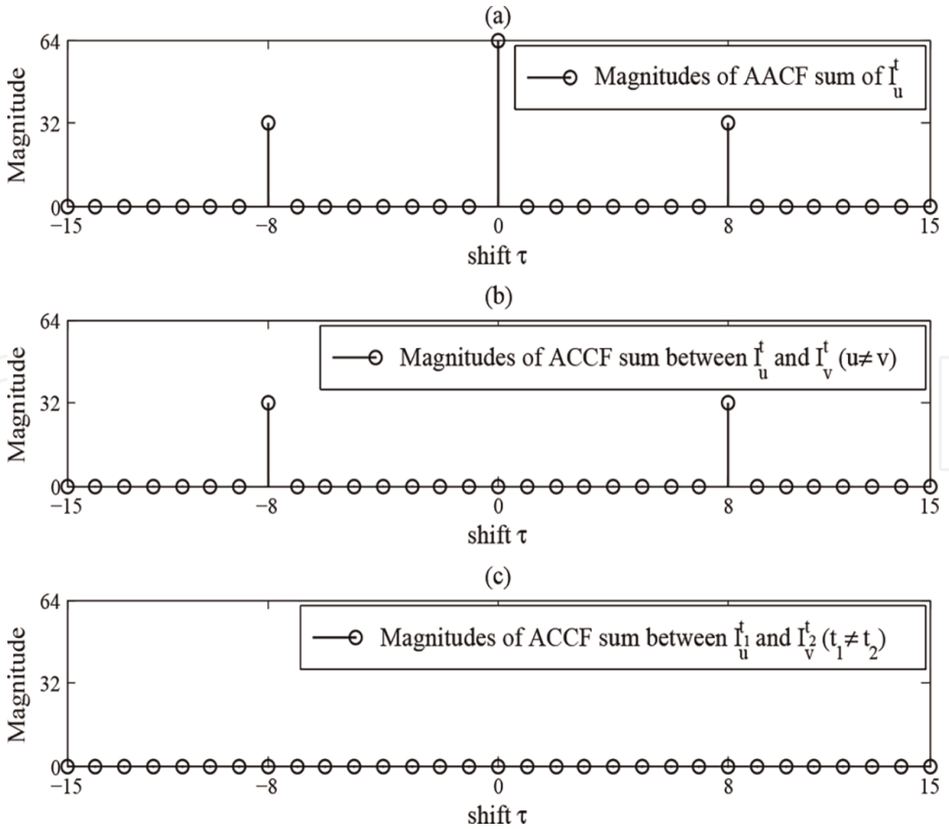

Figure 2.
Correlation plots of $I(8, 4, 16, 8)$.

The correlation properties of I (8, 4, 16, 8) are described in **Figure 2** where **Figure 2a** presents the absolute value of AACF sum of each code in $I(8, 4, 16, 8)$, **Figure 2b** shows absolute value of ACCF sum between any two distinct codes from the same code group, and **Figure 2c** presents the absolute value of ACCF sum between any two distinct codes from different code groups.

4. Summary

In this chapter, we have presented a direct construction of IGC code set by using second-order GBFs. The AACF sidelobes of the codes of constructed IGC code set are zeros within ZCZ width, and the ACCFs of any two different codes of the same code group are zeros inside the ZCZ width, whereas the ACCFs of any two different codes from two different code groups are zeros everywhere. We have shown that there is a relation between our proposed construction and graph. The ZCZ width of the proposed IGC code set depends on the number of isolated vertices present in a graph after the deletion of some vertices. We also have shown that the ZCZ width of the proposed IGC code set by our construction is flexible and it can extend their applications. It is observed that most of the constructions given in literature are based on CCCs, whereas our construction can produce IGC code set directly.

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