Chapter

Eight-by-Eight Spacetime Matrix Operator and Its Applications

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Abstract

A recent journal article by the authors introduced the eight-by-eight spacetime matrix operator $\hat{M}$ which played a key role in the formulation of Lorentz invariant matrix equations for both the classical electrodynamic Maxwell field equations and the quantum mechanical relativistic Dirac equation for free space. Those new equations we referred to as the Maxwell spacetime matrix and the Dirac spacetime matrix equations. These matrix equations will be briefly reviewed at the beginning of this chapter. Next we will show how the same matrix operator $\hat{M}$ plays a central role in the matrix formulation of other fundamental equations in both electromagnetic and quantum theories. These include the electromagnetic wave and charge continuity equations, the Lorentz conditions and electromagnetic potentials, the electromagnetic potential wave equations, and the quantum mechanical Klein-Gordon equation. In addition, a new generalized spacetime matrix equation, again employing the operator $\hat{M}$, will be described which is a generalization of the Maxwell and Dirac spacetime matrix equations. We will explore time-harmonic plane-wave solutions of this equation as well as the properties of these solutions.

Keywords: special theory of relativity, matrix operators, classical electrodynamics, relativistic quantum mechanics, matter waves, electromagnetic waves, optics, applied mathematics

1. Introduction

The eight-by-eight spacetime matrix operator $\hat{M}$ plays a key role in the matrix formulation of a number of well-known fundamental equations in both the fields of classical electrodynamics and relativistic quantum mechanics (see [1]). The spacetime matrix operator is defined by Eq. (1):

$$\hat{M} \equiv \begin{bmatrix} -\partial_4 & 0 & 0 & 0 & 0 & -\partial_3 & +\partial_2 & -\partial_1 \\ 0 & -\partial_4 & 0 & 0 & +\partial_3 & 0 & -\partial_1 & -\partial_2 \\ 0 & 0 & -\partial_4 & 0 & -\partial_2 & +\partial_1 & 0 & -\partial_3 \\ 0 & 0 & 0 & -\partial_4 & +\partial_1 & +\partial_2 & +\partial_3 & 0 \\ 0 & +\partial_3 & -\partial_2 & +\partial_1 & +\partial_4 & 0 & 0 & 0 \\ -\partial_3 & 0 & +\partial_1 & +\partial_2 & 0 & +\partial_4 & 0 & 0 \\ +\partial_2 & -\partial_1 & 0 & +\partial_3 & 0 & 0 & +\partial_4 & 0 \\ -\partial_1 & -\partial_2 & -\partial_3 & 0 & 0 & 0 & 0 & +\partial_4 \end{bmatrix}. \qquad (1)$$

Compact matrix equation	Compact matrix equation description
$\hat{M}\vert f\rangle = \vert o\rangle$	Maxwell spacetime matrix equation for free space
$\hat{M}\vert f\rangle = \vert j\rangle$	Maxwell matrix equation with charges and currents
$\hat{M}\hat{M}\vert f\rangle = \hat{M}\vert j\rangle$	Charge continuity and electromagnetic wave equations
$\hat{M}\vert a\rangle = \vert f\rangle$	Lorentz conditions and electromagnetic potentials
$\hat{M}\hat{M}\vert a\rangle = \vert j\rangle$	Electromagnetic potential wave equations
$\hat{M}\vert\phi\rangle + \kappa\vert\phi\rangle = \vert o\rangle$	Dirac spacetime matrix equation for free space
$\hat{M}\hat{M}\vert\phi\rangle - \kappa^2\vert\phi\rangle = \vert o\rangle$	Klein-Gordon spacetime matrix equation for free space
$\hat{M}\vert\psi\rangle + \kappa\vert\psi\rangle = \vert o\rangle$	Generalized spacetime matrix equation for free space

Table 1.
Compact matrix equations where the spacetime matrix operator $\hat{M}$ plays a central role.

The partial derivative symbols are defined by the following:

$$\partial_1 \equiv \frac{\partial}{\partial x} \qquad \partial_2 \equiv \frac{\partial}{\partial y} \qquad \partial_3 \equiv \frac{\partial}{\partial z} \qquad \partial_4 \equiv \frac{1}{ic}\frac{\partial}{\partial t}. \tag{2}$$

The imaginary quantity i represents the square root of minus one, and the physical quantity c corresponds to the speed of light in free space.

Eight compact matrix equations are listed in **Table 1**, each containing the spacetime matrix operator $\hat{M}$. Each of these equations, as well as the ket $|\ \rangle$ vector appearing in these equations, will be discussed in greater detail in the following sections of this chapter. An excellent introduction to bra $\langle\ |$ and ket $|\rangle$ vector notation may be found in [2]. The Gaussian system of units (see [3], p. 781) is employed throughout this chapter.

2. Eight-by-eight spacetime matrix operator properties

The spacetime matrix operator $\hat{M}$, defined in Eq. (1), may also be expressed by the following equation:

$$\hat{M} = M_1\partial_1 + M_2\partial_2 + M_3\partial_3 + M_4\partial_4. \tag{3}$$

The four eight-by-eight matrices M_μ, where $\mu = 1, 2, 3, 4$, are simply referred to as the spacetime matrices. These matrices have the following properties:

1. Each matrix M_μ is equal to its own multiplicative inverse

$$M_\mu = M_\mu^{-1}. \tag{4}$$

2. These matrices satisfy the anti-commutation relation

$$M_\mu M_\nu + M_\nu M_\mu = 2\delta_{\mu\nu} I. \tag{5}$$

3. Each matrix M_μ is Hermitian

$$M_\mu = M_\mu^\dagger. \tag{6}$$

4. In addition

$$\hat{M}\hat{M} = \hat{M}^2 = I\,\square^2. \tag{7}$$

The symbol $\delta_{\mu\nu}$ is the Kronecker delta, and I represents the eight-by-eight identity matrix. The d'Alembertian (see [4], p. 290) and the Laplacian (see [4], p. 15) operators are defined by

$$\square^2 \equiv \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \qquad \text{and} \qquad \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{8}$$

Some authors use the $\square$ symbol to represent the d'Alembertian operator.

3. Maxwell spacetime matrix equation

The Maxwell field equations play a fundamental role in both classical electrodynamics and physical optics. The propagations of electromagnetic waves through free space (see [4], pp. 514–522), nonconducting media (see [3], pp. 295–309), thin-film optical filters [5], and solid-state crystalline materials [6] are just a few examples where the Maxwell field equations play an important role.

3.1 Maxwell spacetime matrix equation for free space

An earlier eight-by-eight matrix representation of the Maxwell field equations was first introduced by the authors back in 1993 [7]. An improved updated version using the spacetime matrix operator $\hat{M}$ was published recently [1]. For free space, in the absence of charges and currents, this later version is given by

$$\begin{bmatrix} -\partial_4 & 0 & 0 & 0 & 0 & -\partial_3 & +\partial_2 & -\partial_1 \\ 0 & -\partial_4 & 0 & 0 & +\partial_3 & 0 & -\partial_1 & -\partial_2 \\ 0 & 0 & -\partial_4 & 0 & -\partial_2 & +\partial_1 & 0 & -\partial_3 \\ 0 & 0 & 0 & -\partial_4 & +\partial_1 & +\partial_2 & +\partial_3 & 0 \\ 0 & +\partial_3 & -\partial_2 & +\partial_1 & +\partial_4 & 0 & 0 & 0 \\ -\partial_3 & 0 & +\partial_1 & +\partial_2 & 0 & +\partial_4 & 0 & 0 \\ +\partial_2 & -\partial_1 & 0 & +\partial_3 & 0 & 0 & +\partial_4 & 0 \\ -\partial_1 & -\partial_2 & -\partial_3 & 0 & 0 & 0 & 0 & +\partial_4 \end{bmatrix} \begin{bmatrix} iE_1 \\ iE_2 \\ iE_3 \\ 0 \\ B_1 \\ B_2 \\ B_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{9}$$

The compact matrix form of Eq. (9) is given by

$$\hat{M}|f\rangle = |o\rangle. \tag{10}$$

The wave function $|f\rangle$ is an eight-by-one ket vector containing, in general, six nonzero scalar components associated with the electric field vector $\mathbf{E} = (E_1\, E_2\, E_3)$ and the magnetic induction vector $\mathbf{B} = (B_1\, B_2\, B_3)$. The elements (4,1) and (8,1) in $|f\rangle$ have purposely been set equal to zero. The case when these two elements are nonzero will be considered when the generalized spacetime matrix equation for free space is discussed. The ket vector $|o\rangle$ represents the eight-by-one null vector.

The Maxwell spacetime matrix equation (9) when expanded is equivalent to two divergences and two curl equations, namely,

$$\nabla \cdot \mathbf{E} = 0 \qquad \text{and} \qquad \nabla \cdot \mathbf{B} = 0 \tag{11}$$

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{B} = 0 \qquad \text{and} \qquad \nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{E} = 0. \tag{12}$$

We recognize these four equations as the traditional Maxwell field equations (Gaussian units) for free space in the absence of charges, currents, and ordinary matter terms (see [8], pp. 362–368).

For electromagnetic waves, time-harmonic plane-wave solutions of the form

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E_o} \exp\{i(\mathbf{k}\cdot\mathbf{r} - \omega t)\} \qquad \text{and} \qquad \mathbf{B}(\mathbf{r}, t) = \mathbf{B_o} \exp\{i(\mathbf{k}\cdot\mathbf{r} - \omega t)\} \tag{13}$$

will next be substituted back into the previous four vector equations. This yields the following set of equations:

$$\mathbf{k}\cdot\mathbf{E_o} = 0 \qquad \text{and} \qquad \mathbf{k}\cdot\mathbf{B_o} = 0 \tag{14}$$

$$\mathbf{k}\times\mathbf{E_o} = +\frac{\omega}{c}\,\mathbf{B_o} \qquad \text{and} \qquad \mathbf{k}\times\mathbf{B_o} = -\frac{\omega}{c}\,\mathbf{E_o}. \tag{15}$$

The quantities $\mathbf{k}$ and ω correspond to the wave vector and the angular frequency associated with the electromagnetic wave; $\mathbf{r}$ and t represent the position vector and the instantaneous time. From the preceding equations, we find the vectors $\mathbf{E}_o$, $\mathbf{B}_o$, and $\mathbf{k}$ are mutually perpendicular. That is,

$$\mathbf{k}\perp\mathbf{E_o} \qquad \mathbf{E_o}\perp\mathbf{B_o} \qquad \mathbf{k}\perp\mathbf{B_o}. \tag{16}$$

These properties represent transverse electromagnetic waves. We also obtain the important results

$$E_o = B_o \tag{17}$$

and

$$\omega = kc \qquad \lambda f = c \qquad \text{where} \qquad \omega = 2\pi f \qquad k = 2\pi/\lambda. \tag{18}$$

The quantities k, f, and λ represent the wave number, the frequency, and the wavelength, respectively, associated with the electromagnetic wave. So for free space, the magnitudes of the electromagnetic field vectors $\mathbf{E_o}$ and $\mathbf{B_o}$ are equal, a well-known result in electromagnetic wave propagation. Recall we are using Gaussian units.

3.2 Maxwell spacetime matrix equation with charges and currents

The Maxwell spacetime matrix equation, with the addition of charge and current terms [1], is given by

$$\begin{bmatrix} -\partial_4 & 0 & 0 & 0 & 0 & -\partial_3 & +\partial_2 & -\partial_1 \\ 0 & -\partial_4 & 0 & 0 & +\partial_3 & 0 & -\partial_1 & -\partial_2 \\ 0 & 0 & -\partial_4 & 0 & -\partial_2 & +\partial_1 & 0 & -\partial_3 \\ 0 & 0 & 0 & -\partial_4 & +\partial_1 & +\partial_2 & +\partial_3 & 0 \\ 0 & +\partial_3 & -\partial_2 & +\partial_1 & +\partial_4 & 0 & 0 & 0 \\ -\partial_3 & 0 & +\partial_1 & +\partial_2 & 0 & +\partial_4 & 0 & 0 \\ +\partial_2 & -\partial_1 & 0 & +\partial_3 & 0 & 0 & +\partial_4 & 0 \\ -\partial_1 & -\partial_2 & -\partial_3 & 0 & 0 & 0 & 0 & +\partial_4 \end{bmatrix} \begin{bmatrix} iE_1 \\ iE_2 \\ iE_3 \\ 0 \\ B_1 \\ B_2 \\ B_3 \\ 0 \end{bmatrix} = \frac{4\pi}{c} \begin{bmatrix} J_{e1} \\ J_{e2} \\ J_{e3} \\ c\rho_m \\ iJ_{m1} \\ iJ_{m2} \\ iJ_{m3} \\ -ic\rho_e \end{bmatrix}. \tag{19}$$

The compact matrix form of the Maxwell spacetime matrix equation is given by

$$\hat{M}|f\rangle = |j\rangle. \tag{20}$$

Eq. (19), when expanded, is equivalent to two divergences and two curl equations. The resulting four vector equations are referred to as the microscopic Maxwell field equations (see [8], pp. 283–290). They are given by

$$\nabla \cdot \mathbf{E} = +4\pi\rho_e \qquad \text{and} \qquad \nabla \cdot \mathbf{B} = +4\pi\rho_m \tag{21}$$

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{B} = -\frac{4\pi}{c}\mathbf{J}_m \qquad \text{and} \qquad \nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{E} = +\frac{4\pi}{c}\mathbf{J}_e. \tag{22}$$

The various scalar and vector quantities appearing in the microscopic Maxwell vector equations are the electric field vector $\mathbf{E} = (E_1\, E_2\, E_3)$, the magnetic induction vector $\mathbf{B} = (B_1\, B_2\, B_3)$, the electric current density vector $\mathbf{J}_e = (J_{e1}\, J_{e2}\, J_{e3})$, the magnetic current density vector $\mathbf{J}_m = (J_{m1}\, J_{m2}\, J_{m3})$, the electric charge density ρ_e, the magnetic charge density ρ_m, and the speed of light c in free space. Both magnetic charge and magnetic current density (see [8], pp. 283–290) have been included in the Maxwell vector equations for purposes of completeness. They, of course, may be set equal to zero since hypothetical magnetic monopoles have not been discovered in nature. The ket vector $|f\rangle$ represents the eight-by-one column vector on the left-hand side of Eq. (19). The ket vector $|j\rangle$ corresponds to the eight-by-one column vector on the right-hand side of Eq. (19) multiplied by the factor $4\pi/\mathrm{c}$.

3.3 Charge continuity and electromagnetic wave equations

Charge continuity equations for electric (see [8], p. 15) and magnetic charges as well as the electromagnetic wave equations involving electric and magnetic charges and currents may be easily obtained by simply multiplying both sides of the Maxwell spacetime matrix equation in compact form (20) by the spacetime matrix operator $\hat{M}$. That is,

$$\hat{M}\hat{M}|f\rangle = \hat{M}|j\rangle. \tag{23}$$

Expanding this single matrix equation yields the charge continuity and electromagnetic wave equations:

$$\nabla \cdot \mathbf{J}_e + \frac{\partial}{\partial t}\rho_e = 0 \qquad \text{and} \qquad \nabla \cdot \mathbf{J}_m + \frac{\partial}{\partial t}\rho_m = 0 \tag{24}$$

$$\Box^2\mathbf{E} = \frac{4\pi}{c^2}\frac{\partial}{\partial t}\mathbf{J}_e + 4\pi\nabla\rho_e + \frac{4\pi}{c}\nabla \times \mathbf{J}_m \qquad \text{and} \qquad \Box^2\mathbf{B} = \frac{4\pi}{c^2}\frac{\partial}{\partial t}\mathbf{J}_m + 4\pi\nabla\rho_m - \frac{4\pi}{c}\nabla \times \mathbf{J}_e. \tag{25}$$

3.4 Lorentz conditions and electromagnetic potentials

By using the spacetime matrix operator $\hat{M}$, we can determine the relationship between electromagnetic fields and vector-scalar potentials as well as determine expressions for the Lorentz conditions (see [9], pp. 179–181) in a single matrix equation. The following matrix equation provides the desired relation:

$$\begin{bmatrix} -\partial_4 & 0 & 0 & 0 & 0 & -\partial_3 & +\partial_2 & -\partial_1 \\ 0 & -\partial_4 & 0 & 0 & +\partial_3 & 0 & -\partial_1 & -\partial_2 \\ 0 & 0 & -\partial_4 & 0 & -\partial_2 & +\partial_1 & 0 & -\partial_3 \\ 0 & 0 & 0 & -\partial_4 & +\partial_1 & +\partial_2 & +\partial_3 & 0 \\ 0 & +\partial_3 & -\partial_2 & +\partial_1 & +\partial_4 & 0 & 0 & 0 \\ -\partial_3 & 0 & +\partial_1 & +\partial_2 & 0 & +\partial_4 & 0 & 0 \\ +\partial_2 & -\partial_1 & 0 & +\partial_3 & 0 & 0 & +\partial_4 & 0 \\ -\partial_1 & -\partial_2 & -\partial_3 & 0 & 0 & 0 & 0 & +\partial_4 \end{bmatrix} \begin{bmatrix} -A_{e1} \\ -A_{e2} \\ -A_{e3} \\ -\phi_m \\ -iA_{m1} \\ -iA_{m2} \\ -iA_{m3} \\ i\phi_e \end{bmatrix} = \begin{bmatrix} iE_1 \\ iE_2 \\ iE_3 \\ 0 \\ B_1 \\ B_2 \\ B_3 \\ 0 \end{bmatrix}. \quad (26)$$

The compact matrix form of Eq. (26) is given by

$$\hat{M}|a\rangle = |f\rangle. \quad (27)$$

The ket vector $|a\rangle$ corresponds to the eight-by-one column vector on the left-hand side of Eq. (26). Equation (26), when expanded, yields the Lorentz conditions and the relationship between electromagnetic fields and potentials:

$$\nabla \cdot \mathbf{A}_e + \frac{1}{c}\frac{\partial}{\partial t}\phi_e = 0 \qquad \text{and} \qquad \nabla \cdot \mathbf{A}_m + \frac{1}{c}\frac{\partial}{\partial t}\phi_m = 0 \quad (28)$$

$$\mathbf{E} = -\nabla\phi_e - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}_e - \nabla \times \mathbf{A}_m \qquad \text{and} \qquad \mathbf{B} = -\nabla\phi_m - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}_m + \nabla \times \mathbf{A}_e. \quad (29)$$

The new scalar and vector quantities appearing in the above equations are the electric vector potential $\mathbf{A}_e = (A_{e1}\ A_{e2}\ A_{e3})$, the magnetic vector potential $\mathbf{A}_m = (A_{m1}\ A_{m2}\ A_{m3})$, the electric scalar potential ϕ_e, and the magnetic scalar potential ϕ_m. So again we see how the eight-by-eight spacetime matrix operator $\hat{M}$ plays a central role in tying together important electromagnetic relations.

3.5 Electromagnetic potential wave equations

It is well-known that the electromagnetic vector and scalar potentials satisfy wave equations (see [9], pp. 179–181). This can be easily shown by multiplying both sides of Eq. (27) by the spacetime matrix operator $\hat{M}$. This gives

$$\hat{M}\hat{M}|a\rangle = \hat{M}|f\rangle. \quad (30)$$

Next replace the term $\hat{M}|f\rangle$ by the ket vector $|j\rangle$ using Eq. (20). This yields

$$\hat{M}\hat{M}|a\rangle = |j\rangle. \quad (31)$$

Expanding this single matrix equation yields eight partial differential equations which can be easily combined to form the following four potential wave equations:

$$\Box^2\phi_e = -4\pi\rho_e \qquad \text{and} \qquad \Box^2\phi_m = -4\pi\rho_m \quad (32)$$

$$\Box^2\mathbf{A}_e = -\frac{4\pi}{c}\mathbf{J}_e \qquad \text{and} \qquad \Box^2\mathbf{A}_m = -\frac{4\pi}{c}\mathbf{J}_m. \quad (33)$$

The single compact matrix (Eq. (31)) is therefore equivalent to these four potential wave equations.

4. Dirac spacetime matrix equation

The nonrelativistic Schrödinger wave equation (see [10], pp. 143–146) plays a fundamental role in quantum mechanical phenomena where the spin property of nonrelativistic particles may be ignored. This equation is usually first met in modern physics textbooks. However, when a particle with half-integer spin and/or moving at relativistic speeds is involved, the relativistic Dirac equation [11] comes into play.

4.1 Dirac spacetime matrix equation for free space

Using the spacetime matrix operator $\hat{M}$, the authors introduced in their most recent publication [1] a modified version of the traditional Dirac equation, referred to as the Dirac spacetime matrix equation. In the absence of electromagnetic potentials [11], the Dirac spacetime matrix equation for free space is given by

$$\begin{bmatrix} -\partial_4 & 0 & 0 & 0 & 0 & -\partial_3 & +\partial_2 & -\partial_1 \\ 0 & -\partial_4 & 0 & 0 & +\partial_3 & 0 & -\partial_1 & -\partial_2 \\ 0 & 0 & -\partial_4 & 0 & -\partial_2 & +\partial_1 & 0 & -\partial_3 \\ 0 & 0 & 0 & -\partial_4 & +\partial_1 & +\partial_2 & +\partial_3 & 0 \\ 0 & +\partial_3 & -\partial_2 & +\partial_1 & +\partial_4 & 0 & 0 & 0 \\ -\partial_3 & 0 & +\partial_1 & +\partial_2 & 0 & +\partial_4 & 0 & 0 \\ +\partial_2 & -\partial_1 & 0 & +\partial_3 & 0 & 0 & +\partial_4 & 0 \\ -\partial_1 & -\partial_2 & -\partial_3 & 0 & 0 & 0 & 0 & +\partial_4 \end{bmatrix} \begin{bmatrix} iU_1 \\ iU_2 \\ iU_3 \\ 0 \\ L_1 \\ L_2 \\ L_3 \\ 0 \end{bmatrix} + \kappa \begin{bmatrix} iU_1 \\ iU_2 \\ iU_3 \\ 0 \\ L_1 \\ L_2 \\ L_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (34)$$

The compact matrix form of Eq. (34) is given by

$$\hat{M}|\phi\rangle + \kappa|\phi\rangle = |o\rangle. \quad (35)$$

The wave function $|\phi\rangle$ is an eight-by-one ket vector containing, in general, six nonzero scalar components associated with two vector quantities $\mathbf{U} = (U_1\ U_2\ U_3)$ and $\mathbf{L} = (L_1\ L_2\ L_3)$. The elements (4,1) and (8,1) in $|\phi\rangle$ have purposely been set equal to zero. The case when these two elements are nonzero will also be considered when the generalized spacetime matrix equation for free space is discussed later in this chapter. The ket vector $|o\rangle$ represents the eight-by-one null vector. The constant κ is defined by

$$\kappa \equiv m_o c/\hbar. \quad (36)$$

Here m_o represents the rest mass of the matter-wave particle under consideration, c again is the speed of light in free space, and $\hbar$ is equal to the Planck constant h divided by 2π.

The Dirac spacetime matrix equation (34) when expanded is equivalent to eight partial differential equations. These eight equations can be rewritten as two divergence and two curl equations [1], namely,

$$\nabla \cdot \mathbf{U} = 0 \qquad \text{and} \qquad \nabla \cdot \mathbf{L} = 0 \quad (37)$$

$$\nabla \times \mathbf{U} = -\frac{1}{c}\frac{\partial}{\partial t}\mathbf{L} - i\kappa\mathbf{L} \qquad \text{and} \qquad \nabla \times \mathbf{L} = +\frac{1}{c}\frac{\partial}{\partial t}\mathbf{U} - i\kappa\mathbf{U}. \quad (38)$$

We refer to these equations as the Dirac spacetime vector equations for free space. It is noted that these equations resemble the four Maxwell field equations for free space in the absence of charge, current, and ordinary matter terms.

The simplest solutions of these vector equations are time-harmonic plane-wave solutions of the form

$$\mathbf{U}(\mathbf{r},t) = \mathbf{U_o} \exp\{i(\mathbf{p}\cdot\mathbf{r} - Et)/\hbar\} \qquad \text{and} \qquad \mathbf{L}(\mathbf{r},t) = \mathbf{L_o} \exp\{i(\mathbf{p}\cdot\mathbf{r} - Et)/\hbar\}. \tag{39}$$

The quantities $\mathbf{p}$ and E correspond to the linear momentum and the total energy of the associated matter-wave particle; $\mathbf{r}$ and t represent the position vector and the instantaneous time. For particles with nonzero rest mass m_o, the following special theory of relativity equations (see [10], pp. 21–25) may also be useful:

$$E = \gamma m_o c^2 \qquad p = \gamma m_o v \qquad \text{where} \qquad \gamma = 1/\sqrt{1-\beta^2} \qquad \beta = v/c. \tag{40}$$

The quantities γ and β are known as the Lorentz factor and the speed parameter, respectively. The symbol v represents the relativistic speed of the matter-wave particle. Substitution of the preceding time-harmonic plane-wave solutions back into the Dirac spacetime vector equations yield the following set of vector equations for matter waves:

$$\mathbf{p}c\cdot\mathbf{U_o} = 0 \qquad \text{and} \qquad \mathbf{p}c\cdot\mathbf{L_o} = 0 \tag{41}$$

$$\mathbf{p}c\times\mathbf{U_o} = +E\,\frac{(\gamma-1)}{\gamma}\,\mathbf{L_o} \qquad \text{and} \qquad \mathbf{p}c\times\mathbf{L_o} = -E\,\frac{(\gamma+1)}{\gamma}\mathbf{U_o}. \tag{42}$$

From the previous equations we find the three vectors $\mathbf{U}_o$, $\mathbf{L}_o$, and $\mathbf{p}c$ are mutually perpendicular. That is,

$$\mathbf{p}c\perp\mathbf{U_o} \qquad \mathbf{U_o}\perp\mathbf{L_o} \qquad \mathbf{p}c\perp\mathbf{L_o} \tag{43}$$

These properties represent transverse waves. In addition, we also obtain the important result:

$$(\gamma+1)\,U_o^2 = (\gamma-1)\,L_o^2. \tag{44}$$

The magnitudes of the vectors $\mathbf{U}_o$ and $\mathbf{L}_o$ are related through the Lorentz factor γ, which depends on the speed parameter β, which ultimately depends on the speed v of the nonzero rest-mass particle. Note, for γ much greater than unity, characteristic of a relativistic particle, the magnitudes of the vectors $\mathbf{U}_o$ and $\mathbf{L}_o$ are nearly equal. On the other hand, for γ close to unity, characteristic of a nonrelativistic particle, the magnitude of the vector $\mathbf{L}_o$ is much greater than the magnitude of the vector $\mathbf{U}_o$. One other important result is

$$E^2 = p^2c^2 + m_o^2c^4 \qquad \text{which implies} \qquad E = \pm\sqrt{p^2c^2 + m_o^2c^4}. \tag{45}$$

The $\pm$ sign is associated with the quantum mechanical energy E of a matter-wave particle, like a half-integer spin electron. This was first interpreted by Paul A. M. Dirac. He recognized the negative energy levels predicted by his relativistic equation could not be ignored. This led to his concept of a hole theory of positrons. For a detailed discussion on negative energy states (see [11]).

4.2 Klein-Gordon spacetime matrix equation

The Klein-Gordon equation (see [12], pp. 118–129) is yet another quantum mechanical relativistic equation which is the field equation of the quanta associated

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with spin-less (spin-0) particles. An example of a spin-less particle is the recently discovered Higgs boson.

A version of the Klein-Gordon equation can be easily derived by simply starting with the compact matrix form of the Dirac spacetime matrix equation for free space, namely, Eq. (35). Multiply both sides by the spacetime matrix operator $\hat{M}$. This gives

$$\hat{M}\hat{M}|\phi\rangle + \kappa\hat{M}|\phi\rangle = |o\rangle. \tag{46}$$

Next replace the term $\hat{M}|\phi\rangle$ with $-\kappa|\phi\rangle$ using Eq. (35). We obtain

$$\hat{M}\hat{M}|\phi\rangle - \kappa^2|\phi\rangle = |o\rangle. \tag{47}$$

We refer to this equation as the Klein-Gordon spacetime matrix equation for free space. Using the fourth property of the spacetime matrix operator $\hat{M}$, it can be easily shown that Eq. (47) is equivalent to the following two equations involving the vectors **U** and **L**:

$$\square^2\mathbf{U} - \kappa^2\mathbf{U} = 0 \qquad \text{and} \qquad \square^2\mathbf{L} - \kappa^2\mathbf{L} = 0. \tag{48}$$

Therefore, the vectors **U** and **L** also satisfy Klein-Gordon type equations.

5. Generalized spacetime matrix equation

In this section, we will introduce for the first time a new matrix equation where again the spacetime operator $\hat{M}$ plays a central role. We will refer to this equation as the generalized spacetime matrix equation for free space.

5.1 Big unanswered questions and mysteries in physics and astronomy

The number of unanswered questions and mysteries regarding the universe from the smallest to the largest, in the fields of physics and astronomy, is unimaginable. There are many references, too numerous to list here, which address this topic. However, an excellent comprehensive list of unsolved problems in physics appears in [13] for various broad areas of physics. These areas include general physics, quantum physics, cosmology, general relativity, quantum gravity, high-energy physics, particle physics, astronomy, astrophysics, nuclear physics, atomic physics, molecular physics, optical physics, classical mechanics, condensed matter physics, plasma physics, and biophysics. The following is a partial list of some of the most important questions and mysteries being addressed today by physicists and astronomers around the globe:

How did the universe begin and what is the ultimate fate of the universe?
Is the universe infinite or just very big?
Why is there more matter than antimatter in the universe?
What came before the big bang?
Why are the galaxies distributed in clumps and filaments?
Are there additional dimensions?
Is spacetime fundamentally continuous or discrete?
How can we create a quantum theory of gravity?
What is dark energy and dark matter?
Do dark gravity, dark charge, and dark antimatter exist?

What happens inside a black hole and do naked singularities exist?
Why does time seem to flow only in one direction?
Is time travel really possible?
Is string theory or M-theory a viable theory of everything?
What kind of physics underlies the standard model?
Are there really just three generations of leptons and quarks?
Do gravitons exist?
Are protons unstable?
Do magnetic monopoles exist?
What are the masses of neutrinos?
Do the quarks or leptons have any substructure?
Do tachyons exist and can information travel faster than light?
Why do the particles have the precise masses they do?
Do fundamental physical constants vary over time?
Why are the strengths of the fundamental forces what they are?
Do parallel universes exist and is there a multiverse?
Was our spatially 3-D universe formed out of a vacuum by a 2-D hologram?
Was the hologram formed by a flow of information? If so, what form?
Does pair production formed, spontaneously, out of a vacuum?
Are they likewise formed out of a flow of information?
Do life processes, such as ion flows through cell membranes, form likewise as flows of information?

As we can see, even with all of the discoveries made over the past several hundred years, there is so much we do not understand and so much yet to be discovered about our universe and possibly beyond.

So far we have described the first seven compact matrix equations listed in **Table 1** where the spacetime matrix operator $\hat{M}$ plays a fundamental role. We found that each of these seven equations correspond to a variety of fundamental equations, in both classical electrodynamics and relativistic quantum mechanics. In the next subsection, we will discuss in detail the eighth compact matrix equation listed in **Table 1**. This eighth equation is associated with a new matrix equation which we will refer to as the generalized spacetime matrix equation for free space. As we will see, there are several theoretical implications resulting from our study of the generalized spacetime matrix equation which perhaps may be added as unanswered questions or mysteries to the preceding list.

5.2 Generalized spacetime matrix equation for free space

We define the generalized spacetime matrix equation for free space by the following equation:

$$\begin{bmatrix} -\partial_4 & 0 & 0 & 0 & 0 & -\partial_3 & +\partial_2 & -\partial_1 \\ 0 & -\partial_4 & 0 & 0 & +\partial_3 & 0 & -\partial_1 & -\partial_2 \\ 0 & 0 & -\partial_4 & 0 & -\partial_2 & +\partial_1 & 0 & -\partial_3 \\ 0 & 0 & 0 & -\partial_4 & +\partial_1 & +\partial_2 & +\partial_3 & 0 \\ 0 & +\partial_3 & -\partial_2 & +\partial_1 & +\partial_4 & 0 & 0 & 0 \\ -\partial_3 & 0 & +\partial_1 & +\partial_2 & 0 & +\partial_4 & 0 & 0 \\ +\partial_2 & -\partial_1 & 0 & +\partial_3 & 0 & 0 & +\partial_4 & 0 \\ -\partial_1 & -\partial_2 & -\partial_3 & 0 & 0 & 0 & 0 & +\partial_4 \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \\ \Omega_1 \\ \Omega_2 \\ \Omega_3 \\ \Omega_4 \end{bmatrix} + \kappa \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \\ \Omega_1 \\ \Omega_2 \\ \Omega_3 \\ \Omega_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (49)$$

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The compact matrix form of Eq. (49) is given by

$$\hat{M}|\psi\rangle + \kappa|\psi\rangle = |o\rangle. \tag{50}$$

This is the eighth compact matrix equation in **Table 1**. Note the similarity between the generalized spacetime matrix equation for free space and the Dirac spacetime matrix equation for free space (34) when $\kappa = m_o c/\hbar$ and the Maxwell spacetime matrix equation for free space (9) when $\kappa = 0$. In those equations we purposely set the (4,1) and (8,1) elements in the ket vectors identically equal to zero. Doing so allowed us to convert those matrix equations to vector equations (involving three-dimensional vectors only) which are described in greater detail in [1].

In Eq. (49), we no longer restrict elements (4,1) and (8,1) to be equal to zero. The wave function $|\psi\rangle$ can be thought of as being composed of two four-dimensional vectors $\Delta = (\Delta_1\ \Delta_2\ \Delta_3\ \Delta_4)$ and $\Omega = (\Omega_1\ \Omega_2\ \Omega_3\ \Omega_4)$. The implications by avoiding the earlier restrictions on elements (4,1) and (8,1) will be investigated shortly. We will find some new predictions and surprises by removing these restrictions.

5.3 Eigenvalue spacetime matrix equations

Our primary goal now is to determine the properties of time-harmonic plane-wave solutions satisfying the generalized spacetime matrix (Eq. (49)) for free space. The approach we will take is to cast Eq. (49) into an eigenvalue equation and use the methods of linear algebra to determine the set of orthonormal eigenvectors and corresponding eigenvalues satisfying this eigenvalue equation. (For an excellent book on linear algebra and the solution of eigenvalue equations; see [14], pp. 189–190.) For now let $\kappa = m_o c/\hbar$, the same constant in the Dirac spacetime matrix equation. Later on we will look at the special case when $\kappa = 0$.

We first multiply Eq. (49) by the factor $\hbar c M_4$. The matrix M_4 is the fourth of the spacetime matrices first introduced in Eq. (3). After doing so, with minor algebraic manipulation, we obtain the following matrix equation:

$$\hbar c \begin{bmatrix} -\kappa & 0 & 0 & 0 & 0 & +\partial_3 & -\partial_2 & +\partial_1 \\ 0 & -\kappa & 0 & 0 & -\partial_3 & 0 & +\partial_1 & +\partial_2 \\ 0 & 0 & -\kappa & 0 & +\partial_2 & -\partial_1 & 0 & +\partial_3 \\ 0 & 0 & 0 & -\kappa & -\partial_1 & -\partial_2 & -\partial_3 & 0 \\ 0 & +\partial_3 & -\partial_2 & +\partial_1 & +\kappa & 0 & 0 & 0 \\ -\partial_3 & 0 & +\partial_1 & +\partial_2 & 0 & +\kappa & 0 & 0 \\ +\partial_2 & -\partial_1 & 0 & +\partial_3 & 0 & 0 & +\kappa & 0 \\ -\partial_1 & -\partial_2 & -\partial_3 & 0 & 0 & 0 & 0 & +\kappa \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \\ \Omega_1 \\ \Omega_2 \\ \Omega_3 \\ \Omega_4 \end{bmatrix} = i\hbar \frac{\partial}{\partial t} \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \\ \Omega_1 \\ \Omega_2 \\ \Omega_3 \\ \Omega_4 \end{bmatrix}. \tag{51}$$

The compact matrix form of this equation is given by

$$\hat{H}|\psi\rangle = i\hbar \frac{\partial}{\partial t}|\psi\rangle. \tag{52}$$

This equation has the same identical form as the nonrelativistic Schrödinger equation (see [12], pp. 118–129). However, the Hamiltonian matrix operator $\hat{H}$ is entirely different. This equation represents the canonical form of the generalized spacetime matrix (Eq. (49)).

For time-harmonic plane-wave solutions, the ket vector $|\psi\rangle$ may be expressed as

$$|\psi\rangle = |\psi_o\rangle \exp\left[+i(\mathbf{p}\cdot\mathbf{r} - Et)/\hbar\right]. \tag{53}$$

Again the quantities $\mathbf{p}$ and E correspond to the linear momentum vector and the total energy; $\mathbf{r}$ and t represent the position vector and the instantaneous time. After substituting the eight-by-one ket vector $|\psi\rangle$ back into Eq. (51), we obtain the following eigenvalue equation:

$$pc\begin{bmatrix} -\mu & 0 & 0 & 0 & 0 & +i\alpha_3 & -i\alpha_2 & +i\alpha_1 \\ 0 & -\mu & 0 & 0 & -i\alpha_3 & 0 & +i\alpha_1 & +i\alpha_2 \\ 0 & 0 & -\mu & 0 & +i\alpha_2 & -i\alpha_1 & 0 & +i\alpha_3 \\ 0 & 0 & 0 & -\mu & -i\alpha_1 & -i\alpha_2 & -i\alpha_3 & 0 \\ 0 & +i\alpha_3 & -i\alpha_2 & +i\alpha_1 & +\mu & 0 & 0 & 0 \\ -i\alpha_3 & 0 & +i\alpha_1 & +i\alpha_2 & 0 & +\mu & 0 & 0 \\ +i\alpha_2 & -i\alpha_1 & 0 & +i\alpha_3 & 0 & 0 & +\mu & 0 \\ -i\alpha_1 & -i\alpha_2 & -i\alpha_3 & 0 & 0 & 0 & 0 & +\mu \end{bmatrix}\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \\ \Omega_1 \\ \Omega_2 \\ \Omega_3 \\ \Omega_4 \end{bmatrix} = E\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \\ \Omega_1 \\ \Omega_2 \\ \Omega_3 \\ \Omega_4 \end{bmatrix}. \tag{54}$$

We will refer to Eq. (54) as the eigenvalue spacetime matrix equation. The compact matrix form of Eq. (54) is represented by

$$H|\psi\rangle = E|\psi\rangle. \tag{55}$$

The eight-by-eight matrix H is Hermitian which implies the eigenvalues E are real (see [14], p. 222). The following equations define various quantities appearing in Eq. (54):

$$\mu \equiv m_o c^2/pc \qquad \text{and} \qquad \mathbf{p} \equiv p\ (\alpha_1\ \alpha_2\ \alpha_3). \tag{56}$$

The quantity p is the magnitude of the linear momentum vector $\mathbf{p}$, and α_1, α_2, α_3 represent the direction cosines, associated with the direction of the linear momentum vector $\mathbf{p}$.

5.4 Wave propagation along the +z direction for $\kappa = m_o c/\hbar$

Without loss of generality, let us consider matter-wave propagation along the +z direction, that is,

$$\mathbf{p} = p\ (0\ 0\ 1). \tag{57}$$

Eq. (54) reduces to the following simplified form:

$$\begin{bmatrix} -E_o & 0 & 0 & 0 & 0 & +ipc & 0 & 0 \\ 0 & -E_o & 0 & 0 & -ipc & 0 & 0 & 0 \\ 0 & 0 & -E_o & 0 & 0 & 0 & 0 & +ipc \\ 0 & 0 & 0 & -E_o & 0 & 0 & -ipc & 0 \\ 0 & +ipc & 0 & 0 & +E_o & 0 & 0 & 0 \\ -ipc & 0 & 0 & 0 & 0 & +E_o & 0 & 0 \\ 0 & 0 & 0 & +ipc & 0 & 0 & +E_o & 0 \\ 0 & 0 & -ipc & 0 & 0 & 0 & 0 & +E_o \end{bmatrix}\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \\ \Omega_1 \\ \Omega_2 \\ \Omega_3 \\ \Omega_4 \end{bmatrix} = E\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \\ \Omega_1 \\ \Omega_2 \\ \Omega_3 \\ \Omega_4 \end{bmatrix} \tag{58}$$

where

$$E_o \equiv m_o c^2. \tag{59}$$

The matrix in Eq. (58) is an eight-by-eight square matrix. A compact matrix version of Eq. (58) may be expressed as follows:

$$H|\psi_n\rangle = E_n|\psi_n\rangle \qquad n = 1, 2, 3, \ldots 8. \tag{60}$$

At this point we are now in a position to determine eight eigenvectors $|\psi_n\rangle$ and the corresponding eigenvalues E_n satisfying the eigenvalue (Eq. (58)). We chose to use the matrix software program MATLAB [15] for determining the eigenvalues and eigenvectors. As it turns out, there are only two unique eigenvalues given by

$$E_+ = +E \quad \text{and} \quad E_- = -E \qquad \text{where} \qquad E = \sqrt{E_o^2 + p^2c^2}. \tag{61}$$

From the special theory of relativity (see [10], pp. 21–25), the following relations may also be of use:

$$E = \gamma E_o \qquad p = \gamma m_o v \qquad pc = \gamma\beta E_o \qquad \gamma = 1/\sqrt{1-\beta^2} \qquad \beta = v/c. \tag{62}$$

As before, γ and β are referred to as the Lorentz factor and speed parameter, respectively. For each of the two eigenvalues, there are four unique eigenvectors. The eight eigenvectors $|\psi_n\rangle$ form an orthonormal set, that is,

$$\langle\psi_m|\psi_n\rangle = \delta_{mn}. \tag{63}$$

The symbol δ_{mn} represents the Kronecker delta. In **Table 2** is a summary of the eigenvalues and orthonormal eigenvectors satisfying the eigenvalue spacetime matrix (Eq. (58)).

The constants a and b appearing in **Table 2** are defined by

$$a \equiv \frac{\sqrt{2}}{2}\sqrt{\frac{\gamma+1}{\gamma}} \qquad a^2 + b^2 = 1 \qquad b \equiv \frac{\sqrt{2}}{2}\sqrt{\frac{\gamma-1}{\gamma}}. \tag{64}$$

Inspection of the contents of **Table 2** reveals the following important results:

1. $|\psi_1\rangle$ and $|\psi_2\rangle$ represent transverse waves with positive energy $+\gamma E_o$.

2. $|\psi_3\rangle$ and $|\psi_4\rangle$ represent transverse waves with negative energy $-\gamma E_o$.

3. $|\psi_5\rangle$ and $|\psi_6\rangle$ represent non-transverse waves with positive energy $+\gamma E_o$.

4. $|\psi_7\rangle$ and $|\psi_8\rangle$ represent non-transverse waves with negative energy $-\gamma E_o$.

For wave propagation in the $+z$ direction, the transverse waves have eigenvector solutions $|\psi\rangle$ where elements (3,1), (4,1), (7,1), and (8,1) are identically equal to zero. In other words, $\Delta = (\Delta_1\ \Delta_2\ \ 0\ \ 0)$ and $\Omega = (\Omega_1\ \Omega_2\ \ 0\ \ 0)$. For this case, Δ_1, Δ_2 and Ω_1, Ω_2 correspond to the x and y components. Thus, for wave propagation in the $+z$ direction, the transverse wave solutions only have x and y vector components characteristic of a transverse wave in three dimensions.

E_n	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8
E	$+\gamma E_o$	$+\gamma E_o$	$-\gamma E_o$	$-\gamma E_o$	$+\gamma E_o$	$+\gamma E_o$	$-\gamma E_o$	$-\gamma E_o$
$\vert\psi_n\rangle$	$\vert\psi_1\rangle$	$\vert\psi_2\rangle$	$\vert\psi_3\rangle$	$\vert\psi_4\rangle$	$\vert\psi_5\rangle$	$\vert\psi_6\rangle$	$\vert\psi_7\rangle$	$\vert\psi_8\rangle$
Δ_1	0	$+b$	0	$+a$	0	0	0	0
Δ_2	$+ib$	0	$+ia$	0	0	0	0	0
Δ_3	0	0	0	0	$+b$	0	$+a$	0
Δ_4	0	0	0	0	0	$+ib$	0	$+ia$
Ω_1	$-a$	0	$+b$	0	0	0	0	0
Ω_2	0	$-ia$	0	$+ib$	0	0	0	0
Ω_3	0	0	0	0	0	$-a$	0	$+b$
Ω_4	0	0	0	0	$-ia$	0	$+ib$	0

Table 2.
Eigenvalues and orthonormal eigenvectors associated with the generalized spacetime matrix equation for wave propagation in the +z direction when $\kappa = m_o c/\hbar$.

On the other hand, for wave propagation in the $+z$ direction, the non-transverse waves have eigenvector solutions $|\psi\rangle$ where elements (1,1), (2,1), (5,1), and (6,1) are identically equal to zero. That is to say, $\Delta = (0\ \ 0\ \ \Delta_3\ \Delta_4)$ and $\Omega = (0\ \ 0\ \ \Omega_3\ \Omega_4)$. This implies, Δ_3 and Ω_3 represent z-components. Δ_4 and Ω_4 represent the fourth components (unknown origin) in a four-dimensional space. Thus, for wave propagation in the $+z$ direction, the non-transverse wave solutions have a z vector component (longitudinal in nature) and a fourth vector component (neither transverse nor longitudinal in nature, perhaps a "time" component) of a non-transverse wave in four dimensions.

5.5 Traditional Dirac equation

The authors, in their most recent publication [1], indicated solutions of their Dirac spacetime matrix equation for free space could be mapped into solutions satisfying the traditional Dirac matrix equation. We wish to explore this in greater detail. The traditional Dirac equation, in the absence of electromagnetic potential terms, is given by

$$\begin{bmatrix} +\partial_4 & 0 & -i\partial_3 & -\partial_2 - i\partial_1 \\ 0 & +\partial_4 & +\partial_2 - i\partial_1 & +i\partial_3 \\ +i\partial_3 & +\partial_2 + i\partial_1 & -\partial_4 & 0 \\ -\partial_2 + i\partial_1 & -i\partial_3 & 0 & -\partial_4 \end{bmatrix} \begin{bmatrix} \Sigma_1 \\ \Sigma_2 \\ \Sigma_3 \\ \Sigma_4 \end{bmatrix} + \kappa \begin{bmatrix} \Sigma_1 \\ \Sigma_2 \\ \Sigma_3 \\ \Sigma_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{65}$$

This equation corresponds to the special case employing the Dirac representation (see [12], pp. 694–706) for details. The compact matrix form of Eq. (65) is given by

$$\hat{D}|\sigma\rangle + \kappa|\sigma\rangle = |o\rangle. \tag{66}$$

The Dirac matrix operator $\hat{D}$ represents the four-by-four matrix operator on the left-hand side of Eq. (65), $|\sigma\rangle$ is the four-by-one ket vector appearing twice on the left-hand side, and $|o\rangle$ is the four-by-one null ket vector appearing on the right-hand side. For time-harmonic plane-wave solutions, the ket vector $|\sigma\rangle$ may be expressed as

$$|\sigma\rangle = |\sigma_o\rangle \exp\left[+i(\mathbf{p}\cdot\mathbf{r} - Et)/\hbar\right]. \tag{67}$$

Substituting this time-harmonic plane-wave solution back into the traditional Dirac equation (65) ultimately leads to the corresponding eigenvalue equation:

$$pc\begin{bmatrix} +\mu & 0 & +\alpha_3 & -i\alpha_2+\alpha_1 \\ 0 & +\mu & +i\alpha_2+\alpha_1 & -\alpha_3 \\ +\alpha_3 & -i\alpha_2+\alpha_1 & -\mu & 0 \\ +i\alpha_2+\alpha_1 & -\alpha_3 & 0 & -\mu \end{bmatrix}\begin{bmatrix}\Sigma_1\\ \Sigma_2\\ \Sigma_3\\ \Sigma_4\end{bmatrix} = E\begin{bmatrix}\Sigma_1\\ \Sigma_2\\ \Sigma_3\\ \Sigma_4\end{bmatrix}. \tag{68}$$

For the special case of wave propagation in the $+z$ direction, the preceding eigenvalue equation reduces to the following simplified form:

$$\begin{bmatrix} +E_o & 0 & +pc & 0 \\ 0 & +E_o & 0 & -pc \\ +pc & 0 & -E_o & 0 \\ 0 & -pc & 0 & -E_o \end{bmatrix}\begin{bmatrix}\Sigma_1\\ \Sigma_2\\ \Sigma_3\\ \Sigma_4\end{bmatrix} = E\begin{bmatrix}\Sigma_1\\ \Sigma_2\\ \Sigma_3\\ \Sigma_4\end{bmatrix}. \tag{69}$$

Again using the matrix software MATLAB, the four orthonormal eigenvectors and corresponding eigenvalues satisfying Eq. (69) are listed in the **Table 3**.

The quantities a and b appearing in **Table 3** are defined by

$$a \equiv \frac{\sqrt{2}}{2}\sqrt{\frac{\gamma+1}{\gamma}} \qquad a^2+b^2=1 \qquad b \equiv \frac{\sqrt{2}}{2}\sqrt{\frac{\gamma-1}{\gamma}}. \tag{70}$$

Note, the quantities a and b appearing in the traditional Dirac equation eigenvectors listed in **Table 3** are the same a and b quantities appearing in the generalized spacetime matrix equation eigenvectors listed in **Table 2** for $\kappa = m_o c/\hbar$.

E_n	E_1	E_2	E_3	E_4
E	$+\gamma E_o$	$+\gamma E_o$	$-\gamma E_o$	$-\gamma E_o$
$\vert\sigma_n\rangle$	$\vert\sigma_1\rangle$	$\vert\sigma_2\rangle$	$\vert\sigma_3\rangle$	$\vert\sigma_4\rangle$
Σ_1	$-\frac{\sqrt{2}}{2}a$	$-\frac{\sqrt{2}}{2}a$	$+\frac{\sqrt{2}}{2}b$	$+\frac{\sqrt{2}}{2}b$
Σ_2	$-\frac{\sqrt{2}}{2}a$	$+\frac{\sqrt{2}}{2}a$	$+\frac{\sqrt{2}}{2}b$	$-\frac{\sqrt{2}}{2}b$
Σ_3	$-\frac{\sqrt{2}}{2}b$	$-\frac{\sqrt{2}}{2}b$	$-\frac{\sqrt{2}}{2}a$	$-\frac{\sqrt{2}}{2}a$
Σ_4	$+\frac{\sqrt{2}}{2}b$	$-\frac{\sqrt{2}}{2}b$	$+\frac{\sqrt{2}}{2}a$	$-\frac{\sqrt{2}}{2}a$

Table 3.
Eigenvalues and orthonormal eigenvectors associated with the traditional Dirac equation for wave propagation in the $+z$ direction when $\kappa = m_o c/\hbar$.

5.6 Linear transformation equation

For the special case of a matter wave traveling through free space in the + z direction, we found the orthonormal set of eigenvectors and corresponding eigenvalues, for both the generalized spacetime matrix (Eq. (49)) and the traditional Dirac equation (65), when $\kappa = m_o c/\hbar$. These two sets of orthonormal eigenvectors are related [1] through the following linear transformation matrix equation:

$$\begin{bmatrix} \Sigma_1 \\ \Sigma_2 \\ \Sigma_3 \\ \Sigma_4 \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} 0 & 0 & 0 & 0 & +1 & -i & +1 & -i \\ 0 & 0 & 0 & 0 & +1 & +i & -1 & -i \\ -1 & +i & -1 & +i & 0 & 0 & 0 & 0 \\ -1 & -i & +1 & +i & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \\ \Omega_1 \\ \Omega_2 \\ \Omega_3 \\ \Omega_4 \end{bmatrix}. \tag{71}$$

The compact matrix form of Eq. (71) is given by

$$|\sigma\rangle = Z|\psi\rangle. \tag{72}$$

When we substitute each the eight eigenvectors $|\psi_n\rangle$ from **Table 2** back into Eq. (71), we obtain the following results:

1. The four transverse eigenvectors in **Table 2** map into the four eigenvectors in **Table 3**:

$$|\sigma_1\rangle = Z|\psi_1\rangle \qquad |\sigma_2\rangle = Z|\psi_2\rangle \qquad |\sigma_3\rangle = Z|\psi_3\rangle \qquad |\sigma_4\rangle = Z|\psi_4\rangle. \tag{73}$$

2. The four non-transverse eigenvectors in **Table 2** map into the same four eigenvectors in **Table 3**:

$$|\sigma_1\rangle = Z|\psi_5\rangle \qquad |\sigma_2\rangle = Z|\psi_6\rangle \qquad |\sigma_3\rangle = Z|\psi_7\rangle \qquad |\sigma_4\rangle = Z|\psi_8\rangle. \tag{74}$$

Therefore, whether we use the four transverse eigenvector solutions or the four non-transverse eigenvector solutions satisfying the generalized spacetime matrix (Eq. (49)), the same four eigenvector solutions satisfying the traditional Dirac equation (65) are obtained using Eq. (71). It is noted the four transverse eigenvector solutions could have been obtained from the four Dirac vector equations (37) and (38).

5.7 Wave propagation along the +z direction for $\kappa = 0$

For the special case of wave propagation in the +z direction, when $\kappa = 0$, time-harmonic plane-wave solutions satisfying the generalized spacetime matrix equation for free space (49) yield the set of eigenvectors and eigenvalues presented in **Table 4**. The eight eigenvectors $|\psi_n\rangle$ also form an orthonormal set, that is,

$$\langle\psi_m|\psi_n\rangle = \delta_{mn}. \tag{75}$$

E_n	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8
$p_n c$	$+pc$	$+pc$	$-pc$	$-pc$	$+pc$	$+pc$	$-pc$	$-pc$
$\lvert\psi_n\rangle$	$\lvert\psi_1\rangle$	$\lvert\psi_2\rangle$	$\lvert\psi_3\rangle$	$\lvert\psi_4\rangle$	$\lvert\psi_5\rangle$	$\lvert\psi_6\rangle$	$\lvert\psi_7\rangle$	$\lvert\psi_8\rangle$
Δ_1	0	$+b$	0	$+a$	0	0	0	0
Δ_2	$+ib$	0	$+ia$	0	0	0	0	0
Δ_3	0	0	0	0	$+b$	0	$+a$	0
Δ_4	0	0	0	0	0	$+ib$	0	$+ia$
Ω_1	$-a$	0	$+b$	0	0	0	0	0
Ω_2	0	$-ia$	0	$+ib$	0	0	0	0
Ω_3	0	0	0	0	0	$-a$	0	$+b$
Ω_4	0	0	0	0	$-ia$	0	$+ib$	0
v_n	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
	$+c$	$+c$	$-c$	$-c$	$+c$	$+c$	$-c$	$-c$

Table 4.
Eigenvalues and orthonormal eigenvectors associated with the generalized spacetime matrix equation for wave propagation in the +z direction when $\kappa = 0$.

The constants a and b appearing in **Table 4** are now defined by

$$a \equiv \frac{\sqrt{2}}{2} \qquad a^2 + b^2 = 1 \qquad b \equiv \frac{\sqrt{2}}{2} \tag{76}$$

Inspection of the contents of **Table 4** reveals the following important results:

1. $|\psi_1\rangle$ and $|\psi_2\rangle$ represent transverse waves moving with speed $+$c.

2. $|\psi_3\rangle$ and $|\psi_4\rangle$ represent transverse waves moving with speed $-$c.

3. $|\psi_5\rangle$ and $|\psi_6\rangle$ represent non-transverse waves moving with speed $+$c.

4. $|\psi_7\rangle$ and $|\psi_8\rangle$ represent non-transverse waves moving with speed $-$c.

For wave propagation in the +*z* direction, the transverse waves have eigenvector solutions $|\psi\rangle$ where elements (3,1), (4,1), (7,1), and (8,1) are identically equal to zero. In other words, $\Delta = (\Delta_1\ \Delta_2\ \ 0\ \ 0)$ and $\Omega = (\Omega_1\ \Omega_2\ \ 0\ \ 0)$. For this case, Δ_1, Δ_2 and Ω_1, Ω_2 correspond to the x and y components. Thus, for wave propagation in the +*z* direction, the transverse wave solutions only have x and y vector components, characteristic of a transverse wave in three dimensions. Only those waves propagating at a speed in free space of +*c* represent real electromagnetic waves.

On the other hand, for wave propagation in the +*z* direction, the non-transverse waves have eigenvector solutions $|\psi\rangle$ where elements (1,1), (2,1), (5,1), and (6,1) are identically equal to zero. That is to say, $\Delta = (0\ \ 0\ \ \Delta_3\ \Delta_4)$ and $\Omega = (0\ \ 0\ \ \Omega_3\ \Omega_4)$. This implies, Δ_3 and Ω_3 represent z-components. Δ_4 and Ω_4 represent the fourth components in a four-dimensional space. Thus, for wave propagation in the +*z* direction, the non-transverse wave solutions have a z vector component (longitudinal in nature) and a fourth vector component (neither transverse nor longitudinal in nature) of a non-transverse wave in four dimensions. Perhaps there is new physics regarding these additional solutions.

5.8 Unresolved issues regarding the generalized spacetime matrix equation

The eigenvectors and eigenvalues associated with the generalized spacetime matrix equation, for the special case of a time-harmonic plane-wave propagating in free space in the *+z* direction, have been determined for both $\kappa = m_o c/\hbar$ and $\kappa = 0$. The following are the key points found in this analysis:

1. For the case when $\kappa = m_o c/\hbar$, we found there were four orthonormal eigenvectors (two having positive energy eigenvalues $+\gamma E_o$ and two having negative energy eigenvalues $-\gamma E_o$) describing waves having transverse properties. From **Table 2**, each of these four eigenvectors have components $\Delta_3 = \Delta_4 = \Omega_3 = \Omega_4 = 0$. Using the linear transformation equation (71), these eigenvectors map nicely into four orthonormal eigenvectors satisfying the traditional Dirac equation.

2. For the case when $\kappa = m_o c/\hbar$, we found there were also four orthonormal eigenvectors (again two having positive energy eigenvalues $+\gamma E_o$ and two having negative energy eigenvalues $-\gamma E_o$) describing waves having non-transverse properties. From **Table 2**, each of these four eigenvectors have components $\Delta_1 = \Delta_2 = \Omega_1 = \Omega_2 = 0$. Again, using the linear transformation equation (71), these four eigenvectors map nicely into the same four orthonormal eigenvectors satisfying the traditional Dirac equation as mentioned in Case 1.

3. Therefore, for the case when $\kappa = m_o c/\hbar$, the generalized spacetime matrix equation (49) for free space provides eight orthonormal eigenvector solutions (both transverse and non-transverse) which map into four orthonormal eigenvector solutions satisfying the traditional Dirac equation (65).

4. For the case when $\kappa = 0$, we found there were four orthonormal eigenvectors (two associated with waves propagating in free space with speed *+c* and two associated with waves propagating in free space with speed *-c*) describing waves having transverse properties. From **Table 4**, each of these four eigenvectors have components $\Delta_3 = \Delta_4 = \Omega_3 = \Omega_4 = 0$. For the case of transverse waves propagating with *+c*, these eigenvectors are associated with real electromagnetic waves predicted by the traditional Maxwell equations.

5. For the case when $\kappa = 0$, we found there were also four orthonormal eigenvectors (two associated with waves propagating in free space with speed *+c* and two associated with waves propagating in free space with speed *-c*) describing waves having non-transverse properties. From **Table 4**, each of these four eigenvectors has components $\Delta_1 = \Delta_2 = \Omega_1 = \Omega_2 = 0$.

6. The generalized spacetime matrix equation for $\kappa = 0$ when $\Delta_4 \equiv 0$ and $\Omega_4 \equiv 0$ is simply the Maxwell spacetime matrix equation for free space. The generalized spacetime matrix equation for $\kappa = m_o c/\hbar$ when $\Delta_4 \equiv 0$ and $\Omega_4 \equiv 0$ is simply the Dirac spacetime matrix equation for free space. In addition, the Dirac spacetime matrix equation for free space is equivalent to the four Dirac spacetime vector equations (37) and (38) for free space resembling the four Maxwell vector equations (11) and (12) for free space.

In the de Broglie-Bohm picture of quantum mechanics, Hardy [16] and Bell [17] suggest empty waves represented by wave functions propagating in spacetime, but not carrying energy or momentum, can exist. This same concept was called ghost waves or ghost fields by Albert Einstein (see [18]). The controversy as to whether matter waves correspond to real waves or ghost waves has been and is still a subject of debate and controversy.

In Section 5.1, we mentioned that the number of unanswered questions and mysteries regarding the universe from the smallest to the largest, in the fields of physics and astronomy, is unimaginable. Allowing the elements Δ_4 and Ω_4 to have nonzero values in the generalized spacetime matrix equation certainly raises a number of unanswered questions. The following is the author's list of 12 unanswered questions and mysteries regarding our analysis of the generalized spacetime matrix equation for free space:

For relativistic quantum mechanics—matter waves:
What class of particles do the transverse eigenvectors represent?
Do the transverse eigenvectors represent real or ghost waves?
What class of particles do the non-transverse eigenvectors represent?
Do non-transverse eigenvectors represent real or ghost waves?
Are the transverse and non-transverse eigenvectors equivalent in some way?

For classical electrodynamics—electromagnetic waves:
What can be said about those waves propagating with speed *-c*?
Do these represent a new type of electromagnetic wave?
What can be said about those waves having a longitudinal component?
What can be said about those waves having a fourth component?
Could these be associated with undiscovered electromagnetic waves?

And two last questions:
Why do the *Dirac* and *Maxwell* vector equations resemble each other?
Does the spacetime matrix operator $\hat{M}$ have more surprises in store?

6. Conclusions

1. The four classical electromagnetic microscopic Maxwell field equations have been rewritten as a single matrix equation, referred to as the Maxwell spacetime matrix equation, using the spacetime matrix operator $\hat{M}$. The Maxwell spacetime matrix equation is relativistic invariant under a Lorenz transformation.

2. The square eight-by-eight matrix operator $\hat{M}$ has several benefits as summarized next. Other fundamental equations of electromagnetic theory have also been expressed as single matrix equations using the spacetime matrix operator $\hat{M}$, namely, the electromagnetic wave and charge continuity equations, the Lorentz conditions and electromagnetic potentials, and the electromagnetic potential wave equations.

3. The traditional relativistic Dirac equation for free space has been expressed as a new matrix equation, referred to as the Dirac spacetime matrix equation for free space, using the same spacetime matrix operator $\hat{M}$. The Dirac spacetime matrix equation is also relativistic invariant under a Lorenz transformation.

4. Solutions of the new Dirac spacetime matrix equation can be easily transformed into solutions satisfying the traditional relativistic Dirac equation using the linear transformation matrix Z.

5. The Dirac spacetime matrix equation is equivalent to four new relativistic quantum mechanical vector equations. We referred to these equations as the Dirac spacetime vector equations. In the absence of electromagnetic potentials, these vector equations resemble the four classical electromagnetic microscopic Maxwell field vector equations in the absence of charge and current densities.

6. Multiplication of the Dirac spacetime matrix equation by the spacetime matrix operator $\hat{M}$ leads to the relativistic Klein-Gordon spacetime matrix equation.

7. Four transverse orthonormal eigenvectors as well as the four non-transverse orthonormal eigenvectors satisfying the Dirac spacetime matrix equation map, via the linear transformation matrix Z, into the same set of four orthonormal eigenvectors satisfying the traditional Dirac equation.

8. A new generalized spacetime matrix equation employing the operator $\hat{M}$ was introduced. This equation is a generalization of the Maxwell and Dirac spacetime matrix equations for free space. We explored time-harmonic plane-wave solutions of this equation as well as their properties. Some of results obtained may suggest new physics.

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References

[1] Bocker R, Frieden B. A new matrix formulation of the Maxwell and Dirac equations. Heliyon. 2018;**4**(12):e01033. DOI: 10.1016/j.heliyon.2018.e01033

[2] Messiah A. Quantum Mechanics. New York: Dover; 2014. pp. 245-250. ISBN: 13:9780486784557

[3] Jackson J. Classical Electrodynamics. 3rd ed. New York: Wiley; 1999. DOI: 10.1119/1.19136

[4] Lorrain P, Corson D, Lorrain F. Electromagnetic Fields and Waves. 3rd ed. New York: Freeman; 1988. ISBN: 10: 0716718235

[5] Macleod H. Thin-Film Optical Filters. 2nd ed. New York: McGraw-Hill; 1989. pp. 1-312. ISBN: 0-07-044694-6

[6] Fowles G. Introduction to Modern Optics. New York: Holt, Rinehart and Winston; 1968. pp. 168-183. DOI: 10.1119/1.1975142

[7] Bocker R, Frieden B. Solution of the Maxwell field equations in vacuum for arbitrary charge and current distributions using the methods of matrix algebra. IEEE Transactions on Education. 1993;**36**:350-356. DOI: 10.1109/13.241610

[8] Ohanian H. Classical Electrodynamics. Boston: Allyn and Bacon; 1988. ISBN-10:0205105289

[9] Jackson J. Classical Electrodynamics. New York: Wiley; 1962. Library of Congress, Catalog Card Number 62-8774

[10] Serway R, Moses C, Moyer C. Modern Physics. Philadelphia: Saunders; 1989. ISBN: 0-03-004844-3

[11] Schiff L. Quantum Mechanics. 3rd ed. New York: McGraw-Hill; 1968. pp. 472-488. ISBN: 13: 978-0070552876

[12] Roman P. Advanced Quantum Theory. Palo Alto: Addison-Wesley; 1965. ISBN-13: 9780201064957

[13] Wikipedia. List of Unsolved Problems in Physics [Internet]. Available from: https://en.wikipedia.org/wiki/List_of_unsolved_problems_in_physics [Accessed: 29 April 2019]

[14] Strang G. Linear Algebra and its Applications. 2nd ed. New York: Academic Press; 1976. ISBN: 0-12-673660-X

[15] Gilat A. Matlab: An Introduction with Applications. 5th ed. New York: Wiley; 2014. ISBN: 9781118629864

[16] Hardy L. On the existence of empty waves in quantum theory. Physics Letters A. 1992;**167**:11-16. DOI: 10.1016/0375-9601(92)90618-V

[17] Bell J. Six possible worlds of quantum mechanics. Foundations of Physics. 1992;**22**:1201-1215. DOI: 10.1007/BF01889711

[18] Selleri F, Van der Merwe A. Quantum Paradoxes and Physical Reality. Dordrecht: Kluwer; 1990. ISBN: 978-94-009-1862-7