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Single Photon Eigen-Problem with Complex Internal Dynamics

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1. Introduction

Linearized single photon Hamiltonian is used for the analysis of its features in coordinate systems of various geometries. As it could have been expected, based on the general theory of relativity, it turned out that space geometry and physical features are closely interrelated. In Cartesian's coordinates single photons are spatial plane waves, while in cylindrical coordinates they are one-dimensional plane waves the amplitudes of which falls in planes normal to the direction of propagation. The most general information on single photon characteristics has been obtained by the analysis in spherical coordinates. The analysis in this system has shown that single photon spin essentially influences its behavior and that the wave functions of single photon can be normalized for zero orbital momentum, only.

A free photon Hamiltonian is linearized in the second part of this paper using Pauli's matrices. Based on the correspondence of Pauli's matrices kinematics and the kinematics of spin operators, it has been proved that a free photon integral of motion is a sum of orbital momentum and spin momentum for a half one spin. Linearized Hamiltonian represents a bilinear form of products of spin and momentum operators. Unitary transformation of this form results in an equivalent Hamiltonian, which has been analyzed by the method of Green's functions. The evaluated Green's function has given possibility for interpretation of photon reflection as a transformation of photon to anti-photon with energy change equal to double energy of photon and for spin change equal to Dirac's constant. Since photon is relativistic quantum object the exact determining of its characteristics is impossible. It is the reason for series of experimental works in which photon orbital momentum, which is not

integral of motion, was investigated. The exposed theory was compared to the mentioned experiments and in some elements the satisfactory agreement was found.

2. Eigen-problem of single photon Hamiltonian

In the first part of this work the eigen-problem of single photon Hamiltonian was formulated and solutions were proposed. Based on the general theory of relativity, it turned out that space geometry and physical features are closely interrelated. Because of that the analyses was provided in Cartesian's, cylindrical and spherical coordinate systems.

2.1 Introduction
Classical expression for free photon energy is:

$$E = c\sqrt{p_x^2 + p_y^2 + p_z^2} \, , \tag{1.1}$$

where c is the light velocity in vacuum and p_x, p_y and p_z are the components of photon momentum. If instead of classical momentum components we use quantum-mechanical operators $p_\nu \rightarrow \hat{p}_\nu = -i\hbar\dfrac{\partial}{\partial x_\nu}$; $\nu = (x,y,z)$, where $\hbar \equiv \dfrac{h}{2\pi} = 1{,}05456 \cdot 10^{-34}$ Js is Dirac's constant,

we obtain quantum-mechanical single photon Hamiltonian:

$$\hat{H} = \pm c\sqrt{\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2} \, . \tag{2.2}$$

This Hamiltonian is not a linear operator that contradicts the principle of superposition (Gottifried, 2003; Kadin, 2005). Klein and Gordon (Sapaznjikov, 1983) skirted this problem solving the eigen-problem of square of Hamiltonian (2.2):

$$\hat{H}^2\varphi = E^2\varphi \, , \tag{2.3}$$

since the square of Hamiltonian is a linear operator. This approach has given satisfactory description of single photon behaving. Up to now it is considered that this approach gives real picture of photon. Here will be demonstrated that Kline–Gordon picture of photon is incomplete.

Here we shall try to examine single photon behavior by means of linearized Hamiltonian (2.2). Linearization procedure is analogous to the procedure that was used by Dirac's in the analysis of relativistic electron Hamiltonian (Dirac, 1958). We shall take that

$$\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2 = \left(\hat{\alpha}\,\hat{p}_x + \hat{\beta}\,\hat{p}_y + \hat{\chi}\,\hat{p}_z\right)^2 , \tag{2.4}$$

i.e. we shall transform the sum of squares into the square of the sum using $\hat{\alpha}, \hat{\beta}$ and $\hat{\chi}$ matrices. In accordance with (2.4) these matrices fulfill the following relations:

$$\begin{aligned}
\hat{\alpha}^2 &= \hat{\beta}^2 = \hat{\chi}^2 = 1; \\
\hat{\alpha}\,\hat{\beta} + \hat{\beta}\,\hat{\alpha} &= \hat{\alpha}\,\hat{\chi} + \hat{\chi}\,\hat{\alpha} = \hat{\beta}\,\hat{\chi} + \hat{\chi}\,\hat{\beta} = 0.
\end{aligned} \tag{2.5}$$

It is easy to show (Tošić, et al., 2008; Delić, et al., 2008) that (2.5) conditions are fulfilled by Pauli's matrices

$$\hat{\alpha} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \qquad \hat{\beta} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \qquad \hat{\chi} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.6}$$

Combining (2.6), (2.4) and (2.2), we obtain linearized photon Hamiltonian which completely reproduces the quantum nature of light (Holbrow, et al., 2001; Torn, et al., 2004) in the form:

$$\hat{H} = \pm c \begin{pmatrix} \hat{p}_z & \hat{p}_x - i\hat{p}_y \\ \hat{p}_x + i\hat{p}_y & -\hat{p}_z \end{pmatrix} = \pm \frac{\hbar c}{i} \begin{pmatrix} \dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y} & -\dfrac{\partial}{\partial z} \end{pmatrix}. \tag{2.7}$$

Since linearized Hamiltonian is a 2×2 matrix, photon eigen-states must be columns and rows which two components. Operators of other physical quantities must be represented in the form of diagonal 2×2 matrices.

At the end of this presentation, it is important to underline the orbital momentum operator $\begin{pmatrix} \hat{\vec{L}} & 0 \\ 0 & \hat{\vec{L}} \end{pmatrix}$; $\hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}}$ does not commute with Hamiltonian (2.7). It means that it is not integral of motion as in Klein-Gordon theory (Davidov, 1963). It can be shown that integral of motion represents total momentum $\begin{pmatrix} \hat{\vec{J}} & 0 \\ 0 & \hat{\vec{J}} \end{pmatrix}$, where $\hat{\vec{J}}$ is the sum of orbital momentum $\hat{\vec{L}}$ and rotation momentum $\hat{\vec{S}}$ which corresponds to 1/2 spin.

In further the eigen-problem of linearized single photon Hamiltonian will be analyzed in Cartesian's, cylindrical and spherical coordinates.

2.2 Photons in Cartesian's picture

The eigen-problem of single photon Hamiltonian in Cartesian coordinates (we shall take it with plus sign) has the following form:

$$\frac{\hbar c}{i} \begin{pmatrix} \dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y} & -\dfrac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = E \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \tag{2.8}$$

wherefrom we obtain the following system of equations from:

$$\left(\frac{\partial}{\partial z} - ik \right) \Psi_1 + \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y} \right) \Psi_2 = 0 ; \tag{2.9a}$$

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \right) \Psi_1 - \left(\frac{\partial}{\partial z} + ik \right) \Psi_2 = 0 , \tag{2.9b}$$

where $k = \dfrac{E}{\hbar c}$. It follows from (2.9a) that:

$$\Psi_1 = \left(\frac{\partial}{\partial z} - ik \right)^{-1} \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y} \right) \Psi_2 . \tag{2.10}$$

Since the operators $\dfrac{\partial}{\partial z} \pm ik$ and $\dfrac{\partial}{\partial x} \pm i\dfrac{\partial}{\partial y}$ commute, through (2.10) we come to the following relation:

$$\left(\frac{\partial}{\partial z} + ik\right)\left(\frac{\partial}{\partial z} - ik\right)\Psi_1 + \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\Psi_1 = 0 \,. \tag{2.11}$$

In the same manner, from (2.9b) and (2.10), we come to the relation:

$$\left(\frac{\partial}{\partial z} - ik\right)\left(\frac{\partial}{\partial z} + ik\right)\Psi_2 + \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\Psi_2 = 0 \,. \tag{2.12}$$

The two last relations are of identical form and can be substituted by one unique relation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right)\Psi(x, y, z) = 0 \,. \tag{2.13}$$

If we take in (2.13) that $k^2 = k_x^2 + k_y^2 + k_z^2$ and $\Psi(x, y, z) = A(x)\,B(y)\,C(z)$, we come to the following equation:

$$\frac{1}{A}\frac{d^2 A}{dx^2} + k_x^2 + \frac{1}{B}\frac{d^2 B}{dy^2} + k_y^2 + \frac{1}{C}\frac{d^2 C}{dz^2} + k_z^2 = 0 \, , \tag{2.14}$$

which is fulfilled if:

$$\frac{d^2 A}{dx^2} + k_x^2 A = 0; \quad \frac{d^2 B}{dy^2} + k_y^2 B = 0; \quad \frac{d^2 C}{dz^2} + k_z^2 C = 0 \,. \tag{2.15}$$

Equations (2.15) can be easily solved and each of them has two linearly independent particular integrals:

$$\begin{aligned} A_1 &= a_1 e^{ixk_x}; \quad A_2 = a_2 e^{-ixk_x}; \\ B_1 &= b_1 e^{iyk_y}; \quad B_2 = b_2 e^{-iyk_y}; \\ C_1 &= c_1 e^{izk_z}; \quad C_2 = c_2 e^{-izk_z}. \end{aligned} \tag{2.16}$$

Based on these expressions, we conclude that eigen-vector of single photon $\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$ has the following form:

$$\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} D\,e^{i\vec{k}\vec{r}} \\ D\,e^{-i\vec{k}\vec{r}} \end{pmatrix}. \tag{2.17}$$

Since $\vec{k}$ is a continuous variable, the normalization of (2.17) mast be done to δ-function, wherefrom follows:

$$D^2 \int d^3\vec{r} \left(e^{-i\vec{k}'\vec{r}} e^{i\vec{k}'\vec{r}}\right) \begin{pmatrix} e^{i\vec{k}\vec{r}} \\ e^{-i\vec{k}\vec{r}} \end{pmatrix} = \delta(\vec{k} - \vec{k}') \,. \tag{2.18}$$

Solving these integrals, we come to: $2\,D^2\,(2\pi)^3 = 1$, wherefrom we get the normalized single photon eigen-vector as:

$$\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \frac{1}{4\sqrt{\pi^3}} \begin{pmatrix} e^{i\vec{k}\vec{r}} \\ e^{-i\vec{k}\vec{r}} \end{pmatrix}. \tag{2.19}$$

As it can be seen from (2.19), the components of single photon eigen-vector are progressive plane wave $\sim e^{i\vec{k}\vec{r}}$ and the regressive one $\sim e^{-i\vec{k}\vec{r}}$. Since we consider a free single photon, the obtained conclusion is physically acceptable.

2.3 Photons in cylindrical picture

In this section of first part of the paper we are going to analyze the same problem in cylindrical coordinates. Since solving of partial equation of $(\Delta + k^2)\Psi = 0$ type in cylindrical coordinates requires more general approach than that which was used in Cartesian's coordinates, it is necessary to examine single photon eigen-problem in cylindrical system.
In order to examine this problem, we shall start from the equation (2.13) in which Laplacian $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \equiv \Delta$ will be given in cylindrical coordinates (ρ,φ,z) where $\rho \ni [0,\infty]$, $\varphi \ni [0,2\pi]$
and $z \ni [-\infty,+\infty]$. The Laplacian in cylindrical coordinates has the following form:

$$\Delta = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2}$$

and therefore (2.13) with $\Psi(x,y,z) \Rightarrow \Phi(\rho,\varphi,z)$, reduces to:

$$\frac{\partial^2\Phi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\Phi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\varphi^2} + \frac{\partial^2\Phi}{\partial z^2} + k^2\Phi = 0 \cdot \tag{2.20}$$

The square of wave vector k will be separated into two parts $k_x^2 + k_y^2 + k_z^2 = q^2 + k_z^2$. On the basis of this the equation (2.20) can be written as follows:

$$\frac{\partial^2\Phi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\Phi}{\partial\rho} + q^2\Phi + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\varphi^2} = -\frac{\partial^2\Phi}{\partial z^2} - k_z^2\Phi \cdot \tag{2.21}$$

By the substitution:

$$\Phi(\rho,\varphi,z) = F(\rho,\varphi)G(z), \tag{2.22}$$

the equation (2.21) reduces to:

$$\frac{1}{F}\left(\frac{\partial^2 F}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial F}{\partial\rho} + q^2 F + \frac{1}{\rho^2}\frac{\partial^2 F}{\partial\varphi^2}\right) = -\frac{1}{G}\left(\frac{d^2 G}{\partial z^2} + k_z^2 G\right). \tag{2.23}$$

This equation is fulfilled if:

$$\frac{\partial^2 F}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial F}{\partial\rho} + q^2 F + \frac{1}{\rho^2}\frac{\partial^2 F}{\partial\varphi^2} = 0 ; \tag{2.24a}$$

$$\frac{d^2 G}{\partial z^2} + k_z^2 G = 0.$$ (2.24b)

Now we separate the variables by substitution:

$$F(\rho, \varphi) = X(\rho) S(\varphi),$$ (2.25)

after which, the (2.24a) goes over to:

$$\frac{1}{X}\left(\rho^2 \frac{\partial^2 X}{\partial \rho^2} + \rho \frac{\partial X}{\partial \rho} + q^2 \rho^2 X\right) = -\frac{1}{S}\frac{d^2 S}{\partial \varphi^2} \equiv m^2.$$ (2.26)

Introduction of the variables separation constant m^2 represents generalization with respect to approach used in previous section. Since the function $S(\varphi)$ must be single-sign $S(\varphi) = S(\varphi+2\pi)$ we must that m is integer, i.e. $m = 0, \pm 1, \pm 2,...$
Relation (2.26) is separated into two differential equations:

$$\frac{d^2 S}{d\varphi^2} + m^2 S = 0;$$ (2.27a)

$$\frac{d^2 X}{d\rho^2} + \frac{1}{\rho}\frac{dX}{d\rho} + (q^2 - \frac{m^2}{\rho^2})X = 0.$$ (2.27b)

The equation (2.24b) has two particular integrals:

$$G_1 = g_1 e^{izk_z}; \quad G_2 = g_2 e^{-izk_z},$$ (2.28)

while the solution of the equation (2.27a) is:

$$S_m(\varphi) = s_0 e^{im\varphi}.$$ (2.29)

By the substitution of argument $\rho = b\xi$, the equation (2.27b) reduces to

$$\frac{d^2 X}{d\xi^2} + \frac{1}{\xi}\frac{dX}{d\xi} + \left(q^2 b^2 - \frac{m^2}{\xi^2}\right)X = 0,$$ (2.30)

and taking that $b = \frac{1}{q}$, we translate (2.30) into Bessel's equation with integer index m:

$$\frac{d^2 X}{d\xi^2} + \frac{1}{\xi}\frac{dX}{d\xi} + \left(1 - \frac{m^2}{\xi^2}\right)X = 0.$$ (2.31)

It means that the solution of (2.27b) is the m–order Bessel's function: J_m, i.e.

$$X(\rho) = a_0 J_m(q\rho).$$ (2.32)

Taking into account (2.28), (2.29) and (2.32), we obtain the components of single photon eigen-vector:

$$\Phi_1(\rho, \varphi, z) = D_1 J_m(q\rho) e^{izk_z} e^{im\varphi}; \quad \Phi_2(\rho, \varphi, z) = D_2 J_m(q\rho) e^{-izk_z} e^{im\varphi}.$$ (2.33a)

Since q and k_z are continuous variables, while m is a discrete one the normalization of eigen-vector must be done partially to δ–functions and partially to Kronecker's symbol. It means that normalization condition is the following:

$$\left(|D_1|^2 + |D_2|^2\right)\int_0^{2\pi} d\varphi\, \mathrm{e}^{\pm i(m-m')\varphi} \int_{-\infty}^{+\infty} dz\, \mathrm{e}^{\pm i(k_z - k_z')z_z} \int_0^{\infty} \rho\, d\rho\, J_m(q'\rho) J_m(q\rho) = q^{-1}\delta_{nm}\delta(k_z - k_z')\delta(q - q') \,.$$

Using formula for normalization of Bessel functions with integer index (Korn & Korn, 1961):

$$\int_0^{\infty} dx\, x\, J_m(k'x) J_m(kx) = \frac{1}{k}\delta(k - k') \,,$$

the normalization condition reduces into: $|D_1|^2 + |D_2|^2 = \dfrac{1}{4\pi^2}$. It means that normalized single photon eigen-vector in cylindrical coordinates is given by:

$$\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} D_1 J_m(q\rho)\mathrm{e}^{im\rho}\mathrm{e}^{izk_z} \\ D_2 J_m(q\rho)\mathrm{e}^{im\rho}\mathrm{e}^{-izk_z} \end{pmatrix}. \tag{2.34}$$

The first component Φ_1 corresponds to photon (velocity $+c$), while second component Φ_2 corresponds to anti-photon (velocity $-c$). From this formula we conclude that single photon eigen-vector components are progressive and regressive plane waves along z-axis. In the (x,y) planes components change periodically with polar angle φ and decrease by the rule $\rho^{-1/2}$ with distance between the axis and envelope of cylinder. The last is concluded on the basis of asymptotic behaving of Bessel's functions (Korn & Korn, 1961): $J_m(\rho) \approx \dfrac{\sin \rho}{\sqrt{\rho}}$, when $\rho \to \infty$. We have seen during the analysis of a photon in Cartesian's coordinates that only zero values of parameters of variables separation are physically imposed. In cylindrical coordinates, due to physical reasons again, one parameter of variable separation had zero value, while the other has to be a square of integer. The last is necessary since the solution must be single-sign function.

2.4 Photon in spherical picture

The analysis of single photon eigen-problem in spherical coordinates, as it well be shown later, requires introduction of two variable separation parameters. We start from the equation (2.13), where the Laplace's operator will be written down in spherical coordinates (r,θ,φ), where $r \in [0,\infty]$, $\theta \in [0,\pi]$ and $\varphi \in [0,2\pi]$. In these coordinates it is of the form:

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \varphi^2}. \tag{2.35}$$

It means that (2.13), with $\Psi(x,y,z) \to \Omega(r,\theta,\varphi)$, becomes:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \Omega}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \Omega}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \Omega}{\partial \varphi^2} + k^2\Omega = 0. \tag{2.36}$$

In the first stage of variables separation, we shall take that:

$$\Omega(r,\theta,\varphi) = R(r)Q(\theta,\varphi), \tag{2.37}$$

after which substitution into (2.36), it goes over to:

$$\frac{1}{R}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + k^2 r^2 R\right] = -\frac{1}{Q}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Q}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Q}{\partial\varphi^2}\right] = \Lambda^2, \tag{2.38}$$

where Λ^2 is the variable separation parameter. Double equality in (2.38) gives two equations:

$$\frac{d^2 R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \left(k^2 - \frac{\Lambda^2}{r^2}\right)R = 0; \tag{2.39a}$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Q}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Q}{\partial\varphi^2} + \Lambda^2 Q = 0. \tag{2.39b}$$

It should be noted that equation (2.39b) represents eigen-problem of $\frac{\hat{L}^2}{\hbar^2}$ operator. It means that Λ^2 determines orbital quantum numbers. In this equation we shall take that:

$$Q(\theta,\varphi) = T(\theta)S(\varphi), \tag{2.40}$$

after this substitution, which goes over to:

$$\frac{1}{B}\left[\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + T\Lambda^2\sin^2\theta\right] = -\frac{1}{S}\frac{\partial^2 S}{\partial\varphi^2} = m^2. \tag{2.41}$$

In this double equality the variable separation parameter m must be integer since the solution $S(\varphi)$ must be single-signed function. The same requirement appeared in the previous section where single photon vas analyzed in cylindrical coordinates. The equation (2.41) gives two second order differential equations:

$$\frac{d^2 S}{d\varphi^2} + m^2 S = 0; \tag{2.42a}$$

$$\frac{d^2 T}{d\theta^2} + \cot\theta\frac{dT}{d\theta} + \left(\Lambda^2 - \frac{m^2}{\sin^2\theta}\right)T = 0. \tag{2.42b}$$

When the solution (2.42a) is:

$$S_m(\varphi) = s_0 e^{im\varphi}; \quad m = 0,\pm 1,\pm 2, \ldots, \tag{2.43}$$

the equation (2.42b) is associated Legendre's equation (Gottifried, 2003; Davidov, 1963). The complete procedure of solving of this equation cannot be found in literature. Instead of the general solving procedure of the equation (2.42b) is solved for $m = 0$. Its solutions are Legendre's polynomials (Korn & Korn, 1961; Janke, et al., 1960). Differentiating these polynomials m-th times it was possible to conclude that solution (2.42b) can be expressed through m-th Legendre's polynomials derivations.

In order to avoid such an artificial solving of the equation (2.42b), we shall expose, briefly, its solving by means of potential series. This solving procedure may be comprehended as methodological contribution of this part of the paper. At the first stage, we translate the equation (2.42b) into algebraic form by means of substitution of argument $\cos\theta = \zeta$:

$$\left(1-\zeta^2\right)\frac{d^2B}{d\zeta^2} - 2\zeta\frac{dB}{d\zeta} + \left(\zeta^2 - \frac{m^2}{1-\zeta^2}\right)B = 0; \quad \zeta \in [-1,+1]. \tag{2.44}$$

The term $\frac{m^2}{1-\zeta^2}$ in (2.44) does not allow the solving of this equation by means of potential series. Consequently this term must be eliminated from the equation. The strategy of elimination is the following: by the substitution of $T = U \cdot V$, where U is an arbitrary function, the equation (2.44) reduces to the same form but with arbitrary constant in linear function with is multiplied by first derivative of V function. This arbitrary coefficient will be taken in the form $-2(2s+1)$ where s is arbitrary. Arbitrary constant s will be determined in a way which eliminates the term $\frac{m^2}{1-\zeta^2}$ from equation for V function. By the described strategy the (2.44) becomes:

$$\left(1-\zeta^2\right)\frac{d^2V}{d\zeta^2} - 2(2s+1)\zeta\frac{dV}{d\zeta} + \left(\Lambda^2 - 2s - 4s^2\right)V = 0. \tag{2.45}$$

This equation is suitable for solving by means of potential series. Arbitrary function U is given by $U = \left(1-\zeta^2\right)^s$, where $s = \pm m/2$. This means that function T has the form:

$$T = \left(1-\zeta^2\right)^s V. \tag{2.46}$$

Since $\zeta \in [-1,+1]$ the exponent s must not be negative since T would then have singularities in $\zeta = \pm 1$ not allowing the normalization. Fortunate circumstance is that the exponent of the function $1 - \zeta^2$ has $\pm$ sign. This means that for $m > 0$ can be taken $s = + m/2 = |m|/2$. If $m < 0$, we take $s = - m/2 = |m|/2$. Based on this reasoning the equation (2.45) becomes:

$$\left(1-\zeta^2\right)\frac{d^2V}{d\zeta^2} - 2(|m|+1)\zeta\frac{dV}{d\zeta} + \left[\Lambda^2 - |m|(|m|+1)\right]V = 0. \tag{2.47}$$

The solution of this equation will looked for in the form of potential series:

$$V = \sum_{n=0}^{\infty} v_n \zeta^n, \tag{2.48}$$

after which substitution in (2.47) we obtain recurrent formula for series coefficients:

$$v_{n+2} = -\frac{\Lambda^2 - (n+|m|)(n+|m|+1)}{(n+1)(n+1)} v_n; \quad n = 0,1,2,\dots \tag{2.49}$$

Here arises a dilemma whether to leave the whole series or to cut it and retain a polynomial instead of series. In order to solve this dilemma, we shall analyze a special case of formula (2.49) when $m = \Lambda = 0$. In this case formula (2.49) becomes:

$$v_{n+2} = \frac{n}{n+2} v_n ; \quad n = 1,3,5, \ldots, \tag{2.50}$$

wherefrom it turns out that $v_n = \dfrac{v_1}{2n+1}$, and this means that series solution (2.48) becomes:

$$V = \zeta + \frac{\zeta^3}{3} + \frac{\zeta^5}{5} \equiv \int \frac{d\zeta}{1-\zeta^2} = \frac{1}{2} \ln \frac{\zeta-1}{\zeta+1} . \tag{2.51}$$

From this formula is obvious that the series has singularities for $\zeta = \pm 1$. This resolves above mentioned dilemma: the series must be cut and the polynomial obtained in this way must be taken as solution. From the formula (2.49) it is clear that the series will be cut if:

$$\Lambda^2 = l\,(l+1) ; \quad l = 0,1,2, \ldots \tag{2.52}$$

Now is clear that the series is cut when $l = |m| + n$, wherefrom it follows that the degree of polynomial is $l = |m| - n$ and that quantum number m per module must not exceed l: $|m| \le l$. The obtained polynomials of $l - |m|$ degree are called the associated Legendre's polynomials (Korn & Korn, 1961; Janke, et al., 1960) and by means of them T function is expressed as:

$$T_{l,|m|}(\zeta) = (1-\zeta^2)^{|m|/2} L_{l-|m|}(\zeta) . \tag{2.53}$$

The product of functions (2.43) and (2.53) normalized per angles gives spherical harmonics (Gottifried, 2003; Davidov, 1963):

$$Y_{l,|m|} = \frac{(-1)^{l+|m|}}{2^l \, l!} \frac{e^{im\varphi}}{\sqrt{2\pi}} \sqrt{\frac{2l+1}{2} \frac{(l-|m|)!}{(l+|m|)!}} \sin^{|m|}\theta \frac{d^{l+|m|}}{d(\cos\theta)^{l+|m|}} (\sin\theta)^{2l} . \tag{2.54}$$

Finally we shall solve the equation (2.39a) in which Λ^2 is substituted by $l\,(l+1)$. It means that it goes over to:

$$\frac{d^2 R}{dr^2} + \frac{2}{r} \frac{dR}{dr} + \left(k^2 - \frac{l(l+1)}{r^2} \right) R = 0; \quad r \in [0,\infty) . \tag{2.55}$$

Substituting the function R with $r^{-1/2} J(r)$ and substituting the argument r by ρ/k, we translate last equation into Bessel's equation (Korn & Korn, 1961; Janke, et al., 1960) with $l+1/2$ index having two linearly independent particular solution $J_{l+1/2}(kr)$ and $J_{-l-1/2}(kr)$. Consequently the solutions of (2.39a) are:

$$R_1 = w_1 (kr)^{-1/2} J_{l+1/2}(kr) ; \quad R_2 = w_2 (kr)^{-1/2} J_{-l-1/2}(kr) . \tag{2.56}$$

It is necessary for further to quote behaving of Bessel's functions with half integer indices. It can be easily shown that:

$$J_{1/2}(kr) = \frac{\sin kr}{\sqrt{kr}} ; \quad J_{-1/2}(kr) = \frac{\cos kr}{\sqrt{kr}} , \tag{2.57}$$

As well as using recurrent formula for Bessel's functions (Janke, et al., 1960):

$$\frac{d}{dx}J_p = \frac{1}{2}J_{p-1} - \frac{1}{2}J_{p+1}, \tag{2.58}$$

and taking that $p = +1/2$ and $p = -1/2$, we obtain respectively:

$$J_{3/2}(x) = x^{-3/2}\sin x - x^{-1/2}\cos x\,;\, J_{-3/2}(x) = -x^{-3/2}\cos x - x^{-1/2}\sin x\,. \tag{2.59}$$

Due to the factor $x^{-3/2}$ functions $J_{\pm 3/2}$ have strong singularities in zero so that they cannot be normalized in the interval $0 \leq r < \infty$. Due to the same reasons neither $J_{\pm 5/2}$, $J_{\pm 7/2}$, etc. cannot be normalized. It can be concluded that only solutions for A which are proportional to $J_{\pm 1/2}$ have chances to be normalized. Those solutions are:

$$R_1 = \frac{W}{\sqrt{r}}\frac{\sin kr}{\sqrt{kr}}\,;\, R_2 = \frac{W}{\sqrt{r}}\frac{\cos kr}{\sqrt{kr}}\,. \tag{2.60}$$

The very important conclusion of this analysis is: only free photons with zero orbital momentum have chances to be normalized exist. For $l > 0$ photon eigen-vector cannot be normalized.

We shall now examine whether the components of photon eigen-vector proportional to R_1 and R_2 can be normalized. Those components are:

$$\Omega_1 = \frac{W}{\sqrt{r}}Y_{00}(\theta,\varphi)\frac{\sin kr}{\sqrt{kr}}\,;\, \Omega_2 = \frac{W}{\sqrt{r}}Y_{00}(\theta,\varphi)\frac{\cos kr}{\sqrt{kr}}\,. \tag{2.61}$$

The normalization condition is the following:

$$W^2 \int_0^{2\pi}d\varphi \int_0^{\pi}d\theta \sin\theta \left|Y_{0,0}(\theta,\varphi)\right|^2 \int_0^{\infty}dr\, r^2 \left[J_{1/2}(k'r)J_{1/2}(kr) + J_{-1/2}(k'r)J_{-1/2}(kr)\right] =$$
$$= \frac{W^2}{\sqrt{k'k}}\int_0^{\infty}dr\cos(k-k')r = \frac{1}{k^2}\delta(k-k')\,. \tag{2.62}$$

It is not difficult to show that: $\int_0^{\infty}dr\cos(k-k')r = 0$, so that the condition (2.62) becomes meaningless. This means that even for $l = 0$ photon eigen-vector cannot be normalized.

The last possibility for normalization free photons eigen-vector is so called box quantization method. In this method the sphere is substituted by cube enveloping it and cyclic boundary conditions are required: $e^{ikr} = e^{ik(r+L)}$, wherefrom it follows that wave vector is quantized:

$$k = \frac{2\pi}{L}n\,;\quad n = 1,2,3,\dots \tag{2.63}$$

Since $k = 2\pi/\lambda$, it gives that:

$$L = n\lambda\,;\quad n = 1,2,3,\dots \tag{2.64}$$

It is seen that the first harmonic of electromagnetic waves has the wave length equal to the cube edge.

Photon energy is determined in the standard way:

$$E = \hbar ck = \frac{h}{2\pi}c\frac{2\pi}{L}n = h\nu_0 n\,;\quad \nu_0 = \frac{c}{L}\,. \tag{2.65}$$

This expression for energy is in full accordance with Plank's hypothesis (Planck, 1901). In the normalization condition (2.62) the following translations has to be used:

$$\delta(k-k') \to \delta_{nn} \xrightarrow[n=m]{} 1 \; ; \quad \int\limits_0^\infty dr \to \int\limits_0^L dr = L \; ; \quad \frac{\cos(k-k')r}{\sqrt{kk'}} \to \frac{\cos\dfrac{2\pi}{L}(n-m)r}{\dfrac{2\pi}{L}\sqrt{nm}} \xrightarrow[n=m]{} \frac{L}{2\pi n} \; .$$

Combining this and (2.62) we obtain that the normalization constant is $W = \dfrac{1}{\sqrt{2\pi n}}$. On the basis of this the normalized photon eigen-vector is given by:

$$\begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix} = \frac{1}{\sqrt{2\pi n}} \begin{pmatrix} Y_{00}(\theta,\varphi) r^{-1/2} J_{1/2}(\dfrac{2\pi}{L}nr) \\ Y_{00}(\theta,\varphi) r^{-1/2} J_{-1/2}(\dfrac{2\pi}{L}nr) \end{pmatrix} = \frac{n}{2} \frac{1}{(2\pi n)^{3/2}} \begin{pmatrix} \dfrac{\sin\dfrac{2\pi}{L}nr}{\sqrt{\dfrac{2\pi}{L}nr}} \\ \dfrac{\cos\dfrac{2\pi}{L}nr}{\sqrt{\dfrac{2\pi}{L}nr}} \end{pmatrix} \; ; \quad n = 1,2,3,\dots \quad (2.66)$$

As it can be seen the analysis of single photon eigen-problem in spherical coordinates has shown it orbital momentum of photon is equal to zero and that the spin $S = 1/2$ is its unique rotational characteristics (Yao, et al., 2005). Physically it is fully understandable that orbital momentum of a free photon is equal to zero since it moves along the straight line. On straight line photon radius-vector $\vec{r}$ and its momentum $\vec{p} = m_f \dot{\vec{r}}$ are parallel and this gives that $\vec{l} = \vec{r} \times \vec{p} = 0$.

3. Free photon as a system with complex internal dynamics

In the second part of this work the free photon Hamiltonian will be linearized using Pauli's matrices. Based on the correspondence of Pauli matrices kinematics and the kinematics of spin operators, the unitary transformation of this form (equivalent Hamiltonian), will be analyzed by the method of Green's functions. Since photon is relativistic quantum object the exact determining of its characteristics is impossible. It is the reason for series of experimental works in which photon orbital momentum, which is not integral of motion, will be theoretically investigated.

3.1 Introduction

The fact that photon Hamiltonian is not a linear operator has a set of consequences that have not been studied sufficiently so far. The main reason is that photon characteristics have been mainly examined by means of Klein-Gordon's equation (Gottifried, 2003; Davidov, 1963; Messiah, 1970; Davydov, 1976), which represents eigen-problem of photon Hamiltonian square. In this part of our paper we shall linearized photon Hamiltonian and examine some of photon characteristics witch follow from linearized Hamiltonian. The analogy with Dirac's approach to the problem of electrons will be used (Gottifried, 2003; Dirac, 1958).

Firstly will be examined integrals of motion of free photon and will be shown that the photon integral of motion is not orbital momentum. It will be shown that the integral of motion is total momentum being the sun of orbital one and spin momentum.

The evaluated Green's function has given possibility for interpretation of photon reflection as a transformation of photon to anti-photon with energy change equal to double energy of photon and for spin change equal to Dirac's constant (Dirac, 1958; Messiah, 1970). Since photon is relativistic quantum object the exact determining of its characteristics is impossible.

The discussion of obtained results and their comparison to the experimental data will be done at the last part.

3.2 Linearized photon Hamiltonian

We shall not deal with this eigen-problem in further of this paper. Instead of this we shall look for integrals of motion, i.e. those operators that commute with free-photon Hamiltonian (2.7). It is obvious that any function depending on momentum components represents an integral of motion, but this fact is not of physical interest.

It is of particular importance whether orbital momentum:

$$\hat{\vec{L}} = \begin{pmatrix} \hat{\vec{L}} & 0 \\ 0 & \hat{\vec{L}} \end{pmatrix}; \quad \hat{\vec{L}} = \vec{r} \times \hat{\vec{p}} \tag{3.1}$$

is photon integral of motion, since in non-relativistic quantum mechanics operator $\hat{\vec{L}}$ is integral of motion of electron (Messiah, 1970; Davydov, 1976). The components of orbital momentum are given as follows:

$$\hat{L}_x = y\hat{p}_z - z\hat{p}_y ; \quad \hat{L}_y = z\hat{p}_x - x\hat{p}_z ; \quad \hat{L}_z = x\hat{p}_y - y\hat{p}_x . \tag{3.2}$$

If we use commutation relations for components of radius vector and the components of momentum: $[x_i, p_j] = i\hbar \, \delta_{ij}, \ i,j \in (x,y,z)$ and look for commutators of (3.2) with Hamiltonian (2.7), we come to the following relations:

$$\left[\hat{L}_x, \hat{H}\right] = \pm i\hbar c \left(\hat{p}_z \hat{\beta} - \hat{p}_y \hat{\chi}\right); \quad \left[\hat{L}_y, \hat{H}\right] = \pm i\hbar c \left(\hat{p}_x \hat{\chi} - \hat{p}_z \hat{\alpha}\right); \quad \left[\hat{L}_z, \hat{H}\right] = \pm i\hbar c \left(\hat{p}_y \hat{\alpha} - \hat{p}_x \hat{\beta}\right), \tag{3.3}$$

based on which it follows that orbital momentum is not a free photon integral of motion.

It should be pointed out that signs in (3.3) are obtained on the basis of obvious symmetry properties $\hat{H}(-\vec{r}) = \hat{H}(\vec{r})$ and $\vec{L}(-\vec{r}) = \vec{L}(\vec{r})$, where $\vec{r}$ is radius-vector.

In order to find some rotation characteristics that commute with a free photon Hamiltonian, we shall first show that commutation relations for matrices $\hat{\alpha}, \hat{\beta}$ and $\hat{\chi}$, given in section 2.1 by expression (2.6), are:

$$\left[\hat{\alpha}, \hat{\beta}\right] = 2i\hat{\chi}; \quad \left[\hat{\chi}, \hat{\alpha}\right] = 2i\hat{\beta}; \quad \left[\hat{\beta}, \hat{\chi}\right] = 2i\hat{\alpha}, \tag{3.4}$$

while commutation relations for spin components (Dirac, 1958; Messiah, 1970):

$$\left[\hat{S}_x, \hat{S}_y\right] = i\hbar\hat{S}_z ; \quad \left[\hat{S}_z, \hat{S}_x\right] = i\hbar\hat{S}_y ; \quad \left[\hat{S}_y, \hat{S}_z\right] = i\hbar\hat{S}_x , \tag{3.5}$$

are very similar to (3.4). Comparing (3.4) to (3.5) we can establish the correspondence between spin operator components and matrices $\hat{\alpha}, \hat{\beta}$ and $\hat{\chi}$:

$$\hat{S}_x = \frac{\hbar}{2}\hat{\alpha} \; ; \quad \hat{S}_y = \frac{\hbar}{2}\hat{\beta} \; ; \quad \hat{S}_z = \frac{\hbar}{2}\hat{\chi} \; . \tag{3.6}$$

Commutators of matrices $\hat{\alpha}, \hat{\beta}$ and $\hat{\chi}$ with Hamiltonian are given by:

$$\left[\hat{\alpha}, \hat{H}\right] = \mp 2ic(\hat{p}_z\hat{\beta} - \hat{p}_y\hat{\chi}) \; ; \quad \left[\hat{\beta}, \hat{H}\right] = \mp 2ic(\hat{p}_x\hat{\chi} - \hat{p}_z\hat{\alpha}) \; ; \quad \left[\hat{\chi}, \hat{H}\right] = \mp 2ic(\hat{p}_y\hat{\alpha} - \hat{p}_x\hat{\beta}) \; . \tag{3.7}$$

We shall now look for a commutator of component $\hat{J}_x$ of total momentum, with photon Hamiltonian i.e. with $\hat{H}(\vec{r})$. Using upper signs in formulas (3.3) and (3.7) we obtain:

$$\left[\hat{J}_x, \hat{H}(\vec{r})\right] = \left[(\hat{L}_x + \hat{S}_x), \hat{H}(\vec{r})\right] = \left[(\hat{L}_x + \frac{\hbar}{2}\hat{\alpha}), \hat{H}(\vec{r})\right] = \left[\hat{L}_x, \hat{H}(\vec{r})\right] + \frac{\hbar}{2}\left[\hat{\alpha}, \hat{H}(\vec{r})\right] =$$
$$= i\hbar c\left(\hat{p}_z\hat{\beta} - \hat{p}_y\hat{\chi}\right) + \frac{\hbar}{2}(-2ic)\left(\hat{p}_z\hat{\beta} - \hat{p}_y\hat{\chi}\right) = 0 \; . \tag{3.8}$$

For lower signs in formulas (3.3) and (3.7) [1], we have:

$$\left[\hat{J}_x, \hat{H}(-\vec{r})\right] = \left[(\hat{L}_x + \hat{S}_x), \hat{H}(-\vec{r})\right] = \left[(\hat{L}_x + \frac{\hbar}{2}\hat{\alpha}), \hat{H}(-\vec{r})\right] = \left[\hat{L}_x, \hat{H}(-\vec{r})\right] + \frac{\hbar}{2}\left[\hat{\alpha}, \hat{H}(-\vec{r})\right] =$$
$$= i\hbar c\left(\hat{p}_z\hat{\beta} - \hat{p}_y\hat{\chi}\right) + \frac{\hbar}{2}(2ic)\left(\hat{p}_z\hat{\beta} - \hat{p}_y\hat{\chi}\right) = 0 \; . \tag{3.9}$$

It can be proved, in the same manner, that both y and z components of total momentum $\vec{J} = \vec{L} + \vec{S}$ commute with photon Hamiltonian (the expression (2.7) with sign +, i.e. $\hat{H}(\vec{r})$ will be called photon Hamiltonian). The expression (2.7) with sign –, i.e. $\hat{H}(-\vec{r})$ will be called anti-photon Hamiltonian. In the same manner can be proved that y and z components of total momentum $\vec{J} = \vec{L} - \vec{S}$ commute with anti-photon Hamiltonian.
The final conclusion is the following: total momentum $\vec{L} + \vec{S}$ is integral of motion for photon, while total momentum $\vec{L} - \vec{S}$ is integral of motion for anti-photon. Up to now we have the proof that total momentum $\vec{L} + \vec{S}$ is free photon integral of motion, but we do not know what magnitude of photon spin is.
If spin is $S = 1/2$, then the following relation is valid:

$$\left(\hat{S}_x - i\hat{S}_y\right)^2 = 0 \; . \tag{3.10}$$

For spin $S > 1/2$ the exponent in (3.10) is higher than 2, i.e. it must be 3,4, ... etc. In (3.10) we shall go over to matrices $\hat{\alpha}$ and $\hat{\beta}$ through relation (3.6). So we obtain:

$$\left(\hat{S}_x - i\hat{S}_y\right)^2 = \frac{\hbar^2}{4}(\hat{\alpha} - i\hat{\beta})^2 = \frac{\hbar^2}{4}[\hat{\alpha}^2 - \hat{\beta}^2 + 2i(\hat{\alpha}\hat{\beta} + \hat{\beta}\hat{\alpha})] = 0$$

---

[1] this corresponds to negative photon energies, i.e. corresponds to $\hat{H}(-\vec{r})$

(in the last stage of the upper proof the relations (3.5) from section 2.1 were used). Consequently, we can conclude that free photon integral of motion represents a total momentum which is the sum of orbital momentum and spin momentum which corresponds to the case when $S = 1/2$.

In the same way can be concluded that anti-photon integral of motion is the sum of orbital momentum and spin momentum which corresponds to spin $S = -1/2$. It should be noticed that negative spin is rather senseless concept so that $\pm S$ really means $\pm S_z$, where $S_z = \hbar/2$.

In nonrelativistic quantum mechanics (Gottifried, 2003; Davidov, 1963) the conclusion that $\hat{J}$ is integral of motion would mean that energy and total momentum of the quantum object can be measured simultaneously and exactly. Since photon is relativistic object (Berestetskii, et al., 1982) the maximal exactness of measuring of photon momentum is given by $\Delta p \, \Delta t \sim \hbar/c$, and consequently energy and total momentum can be determined with an error of the order $\Delta E \, \Delta t \sim \hbar$. The orbital momentum $\vec{L}$, as it follows from (3.3), is not integral of motion, but for relativistic object this fact is not essential, since for relativistic objects absolutely exact determining of physical characteristics is in possible.

Considering the correspondence (3.6), photon Hamiltonian which is given by $\hat{H} = c\left(\hat{\alpha}\hat{p}_x + \hat{\beta}\hat{p}_y + \hat{\chi}\hat{p}_z\right)$ can be expressed by means of spin operators in the following form:

$$\hat{H} = \frac{2c}{\hbar}\left(\hat{S}_x\hat{p}_x + \hat{S}_y\hat{p}_y + \hat{S}_z\hat{p}_z\right). \tag{3.11}$$

The obtained form of photon Hamiltonian, which includes operators of translation moment $\hat{\vec{P}}$ and spin $\hat{\vec{S}}$ suggest that a free photon has wealthy internal dynamics that consists of mutual action of its translation and spin characteristics. This "internal life" will be examined further in the paper.

3.3 Unitary transformation of photon Hamiltonian

Photon Hamiltonian (3.11) represents bilinear form in which photon momentum operators are multiplied by spin operators. Since momentum characterizes translation photon motion, and spin characterizes rotation, it is obvious that the internal dynamic structure of a photon is determined by both its translation and rotation characteristics, and that their interaction – considering the form of Hamiltonian (3.11), leads to hybridization of excitations (Agranovich, 2009). Spin operators in (3.11) correspond to spin $S = 1/2$ and its can then is represented by Pauli's operators in the following manner (Tyablikov, 1967):

$$\hat{S}_x - i\hat{S}_y = \hbar P^+ ; \quad \hat{S}_x + i\hat{S}_y = \hbar P ; \quad \frac{1}{2} - \hat{S}_z = \hbar P^+ P . \tag{3.12}$$

Pauli's operators fulfill commutation relations:

$$\left[P_i, P_j^+\right] = \left[1 - 2P_i^+P_j\right]\delta_{ij} ; \quad \left[P_i, P_j\right] = \left[P_i^+, P_j^+\right] = 0 ; \quad P_i^2 = P_i^{+2} = 0 ; \quad \left(P^+P\right)_{e.v} = \begin{cases} 0 ; \\ 1 . \end{cases} \tag{3.13}$$

After substitution of (3.12) in (3.11) (in this formula sign + is retained), we obtain the following form of Hamiltonian:

$$\hat{H} = c\hat{p}_z + c\left[(\hat{p}_x - i\hat{p}_y)P + (\hat{p}_x + i\hat{p}_y)P^+ - 2\hat{p}_z P^+ P\right] .$$ (3.14)

This conversion to Pauli operators has been made because the physical picture of processes is clearer through operator's creation and annihilation of excitation.

Operators of moments are linear in operators of creation and annihilation of photon: $P \sim A + A^+$, so it can easily be concluded that mean value of photon Hamiltonian over states $\frac{1}{n!}(A^+)^n P^+|0\rangle$ is equal to zero. This means that the method of theory of perturbation would be inappropriate for Hamiltonian (3.14) analysis. This is why we would make unitary transformation of photon Hamiltonian with the goal to bring it into the form more suitable for calculation than the form (3.14), i.e. we shall go to equivalent Hamiltonian given by:

$$\hat{H}_{eq} = e^{\hat{W}} \hat{H} e^{-\hat{W}} ,$$ (3.15)

where:

$$\hat{W} = i\vec{k}\vec{r} + \rho(P - P^+) + i\lambda P^+ P ,$$ (3.16)

and ρ and λ are real parameters.

Equivalent Hamiltonian is found using Weil's identity (Tošić, 1978):

$$e^{\hat{W}} \hat{D} e^{-\hat{W}} = \sum_{n=0}^{\infty} \frac{(1)^n}{n!} \underbrace{\left[\hat{W},\left[\hat{W},...\left[\hat{W},...\left[\hat{W},\hat{D}\right]\right]...\right]\right]}_{n-times}.$$ (3.17)

It has included the terms of the following type: $P + P^+$, $P - P^+$ and P^+P. Undetermined parameter λ has been determined so that the member proportional to $P - P^+$ disappear from equivalent Hamiltonian. The final result of the described procedure is as follows:

$$\hat{H}_{eq} = E_0 + \hat{H} + \hat{H}_S ,$$ (3.18)

where $\hat{H}$ is starting Hamiltonian, and

$$E_0 = \hbar c(k_x \sin 2\rho + k_z \cos 2\rho) ;$$ (3.19a)

$$\hat{H}_S = -g(P + P^+) + 2aP^+ P ,$$ (3.19b)

where are:

$$g = \hbar c\sqrt{k_y^2 + k_x^2 \cos^2 2\rho + k_z^2 \sin^2 2\rho - k_x k_z \sin 4\rho} ; \quad a = \hbar c(k_x \sin 2\rho + k_z \cos 2\rho) \cdot$$

We shall further analyze free photon behavior using method of Green's functions (Tyablikov, 1967; Tošić, 1978; Rickayzen, 1980; Mahan, 1990; Šetrajčić, et al., 2008). Hamiltonian E_0 is irrelevant in Green function techniques. Starting Hamiltonian $\hat{H}$, as we have already concluded earlier, has zero mean value over states $\frac{1}{n!}(A^+)^n P^+|0\rangle$. This is why we shall exclude it from calculations. The analysis of photon internal processes will be made with Hamiltonian $\hat{H}_S$.

3.4 Green's function of free photons

Since Pauli operators figure in $\hat{H}_s$ Hamiltonian without various configuration indices, the analysis of spin processes in a free photon will be made by means of anticommutator Pauli Green function:

$$\Gamma(t) = \left\langle\left\langle P(t) \middle| P^+(0) \right\rangle\right\rangle = \Theta(t) \left\langle P(t) P^+(0) + P^+(0) P(t) \right\rangle, \tag{3.20}$$

where $\Theta(t)$ is Heaviside's step function (Tyablikov, 1967; Tošić, 1978; Rickayzen, 1980). Correlator of anticommutator Pauli's Green's function contains mean value of anticommutator of Pauli's operator of the same configuration index, and according to (3.13) it is equal to one. This fact simplifies evaluation of mean values by means of spectral intensity of Green function.

Differentiating $\Gamma(t)$ per time and using equation of motion for operator P, we come to the following equation:

$$i\hbar \frac{d\Gamma(t)}{dt} = i\hbar\,\delta(t) + 2a\,\Gamma(t) + 2g\,\Delta(t) \,. \tag{3.21}$$

The Green's function of type: $\left\langle\left\langle \mathrm{const} \middle| P^+ \right\rangle\right\rangle$ are equal to zero. The function Δt is given by:

$$\Delta(t) = \left\langle\left\langle P^+(t) P(t) \middle| P^+(0) \right\rangle\right\rangle. \tag{3.22}$$

Using the same procedure, for defining function $\Delta(t)$ we obtain the following equation:

$$i\hbar \frac{d\Delta(t)}{dt} = g\,\Gamma(t) - g\,F(t), \tag{3.23}$$

where:

$$F(t) = \left\langle\left\langle P^+(t) \middle| P^+(0) \right\rangle\right\rangle, \tag{3.24}$$

with defining following equation:

$$i\hbar \frac{dF(t)}{dt} = -2g\,\Delta(t) - 2a\,F(t) \,. \tag{3.25}$$

In differential equations (3.21), (3.23) and (3.25), Furrier's transformations time-frequency are then made:

$$f(t) = \int_{-\infty}^{+\infty} dt\, \mathrm{e}^{-i\omega t} f(\omega); \quad f \equiv (\Gamma, \Delta, F); \quad \delta(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt\, \mathrm{e}^{-i\omega t} \,, \tag{3.26}$$

so we obtain the system of algebraic equations:

$$\begin{aligned}
(E - 2a)\Gamma(\omega) - 2g\,\Delta(\omega) &= \frac{i\hbar}{2\pi}; \\
\Delta(\omega) &= g\left[\Gamma(\omega) - F(\omega)\right]; \\
E\,F(\omega) &= -2\left[g\,\Delta(\omega) + a\,F(\omega)\right].
\end{aligned} \tag{3.27}$$

Solving this system of equations, we find that:

$$\Gamma(\omega) = \frac{i\hbar}{2\pi} \frac{E^2 + 2aE - 2g^2}{(E^2 - E_0^2)^2}, \tag{3.28}$$

where:

$$E_0 = 2\sqrt{a^2 + g^2} = 2\hbar ck. \tag{3.29}$$

In order to determine spectral intensity of function Γ, it is necessary to break down the right side of the formula (3.20) into common fractions. So, we obtain the following:

$$\Gamma(\omega) = \frac{i}{2\pi} \left[\frac{2g^2}{E_0^2} \frac{1}{\omega} + \left(\frac{1}{2} - \frac{g^2}{E_0^2} + \frac{a}{E_0} \right) \frac{1}{\omega - \omega_0} + \left(\frac{1}{2} - \frac{g^2}{E_0^2} - \frac{a}{E_0} \right) \frac{1}{\omega + \omega_0} \right], \tag{3.30}$$

where: $\omega = E/\hbar$ and $\omega_0 = E_0/\hbar$. Since function Γ is anticommutator function, its spectral intensity is given by the formula (Tyablikov, 1967; Tošić, 1978; Rickayzen, 1980):

$$I_\Gamma(\omega) = \frac{\Gamma(\omega + i\delta) + \Gamma(\omega - i\delta)}{e^{\frac{\hbar\omega}{k_B T}} + 1}; \quad \delta \to +0, \tag{3.31}$$

and using Dirac's formula:

$$\frac{1}{\omega - \omega_k \pm i\delta} = \text{P.V.} \left\{ \frac{1}{\omega - \omega_k} \right\} \mp i\pi\,\delta(\omega - \omega_k), \tag{3.32}$$

where P.V. denotes principal value of integral, we find the explicit expression for spectral intensity:

$$I_\Gamma(\omega) = \frac{2g^2}{E_0^2} \frac{\delta(\omega)}{e^{\frac{\hbar\omega}{k_B T}} + 1} + \left(\frac{1}{2} - \frac{g^2}{E_0^2} + \frac{a}{E_0} \right) \frac{\delta(\omega - \omega_0)}{e^{\frac{\hbar\omega}{k_B T}} + 1} + \left(\frac{1}{2} - \frac{g^2}{E_0^2} - \frac{a}{E_0} \right) \frac{\delta(\omega + \omega_0)}{e^{\frac{\hbar\omega}{k_B T}} + 1}. \tag{3.33}$$

Now we can defined the expression for correlation function of a free photon as:

$$\langle P^+(0)P(t) \rangle \equiv \int_{-\infty}^{+\infty} d\omega\, e^{-i\omega t} I_\Gamma(\omega) = \frac{2g^2}{E_0^2} \frac{1}{2} + \left(\frac{1}{2} - \frac{g^2}{E_0^2} + \frac{a}{E_0} \right) \frac{e^{-i\omega_0 t}}{e^{\frac{\hbar\omega_0}{k_B T}} + 1} + \left(\frac{1}{2} - \frac{g^2}{E_0^2} - \frac{a}{E_0} \right) \frac{e^{i\omega_0 t}}{e^{\frac{\hbar\omega_0}{k_B T}} + 1}. \tag{3.34}$$

Next, we can calculate expression for concentration of spin excitations of a free photon. It is obtained from (3.34), if we take in it that $t = 0$, i.e.

$$\langle P^+ P \rangle = \frac{1}{2} - \frac{a}{E_0} \tanh \frac{\hbar ck}{k_B T}. \tag{3.35}$$

Combining formulae for a over formula (3.19b), and E_0 from (3.29), and converting to sphere coordinate system, we find that:

$$\frac{a}{E_0} = \frac{1}{2} \left(\sin 2\rho \sin\theta \cos\varphi + \cos 2\rho \cos\theta \right).$$

In accordance with this and formula (3.35), we get the following expression for ordering parameter of spin subsystem in a free photon:

$$\sigma = 1 - 2\langle P^{+}P \rangle = (\sin 2\rho \sin \theta \cos \varphi + \cos 2\rho \cos \theta) \tanh \frac{\hbar ck}{k_{\mathrm{B}}T}. \qquad (3.36)$$

The set of results of this section requires some explanations. The most interesting results is that energy for spin translation from $\hbar/2$ to $-\hbar/2$ is $2\hbar ck$. This can be explained on the basis of measuring process in which incident photon bean is reflected by measuring devices. The momentum of incident phonon is $\hbar k$ while the momentum reflected phonon is $-\hbar k$. So we obtain the change of photon momentum $\Delta p = \hbar k - (-\hbar k) = 2\hbar k$, and consequently the energy change $\Delta E = 2\hbar ck$. The energy $-\hbar ck$ corresponds to anti-photon, so that we can consider the described process as a transformation of photon to anti-photon. In this process the spin change takes place, also (the Green's function $\Gamma(t) = \langle\langle P(t)|P^{+}(0)\rangle\rangle$ was calculated). Since photon and anti-photon spins have opposite signs the change the spin is $\Delta S = \hbar/2 - (-\hbar/2) = \hbar$. The value of ΔS is equal the value $\hbar$ and this is eigen-value of spin $s = 1$. This is the reason for behaving of photon as particle with spin $s = 1$.

The polar and azimuthally dependences of ordering parameter comes from the fact that incident bean must not be always orthogonal to the plane of measuring device.

4. Conclusions

1. The analysis of single photon behaving in coordinate systems of various geometries has shown the following:
    *   In Cartesian coordinate system the components of single photon eigen-vector are progressive and regressive plane waves. In other words, the photon behaving is characterized by coordinates and momentum, only.
    *   In cylindrical coordinate system the components of single photon eigen-vector of are progressive and regressive plane waves, but only along z axis. In comparison to the results of analysis in Cartesian coordinates the fact that in x y planes photon oscillations are attenuated according to $\rho^{-1/2}$ law represents a generalization. This conclusion has been based on asymptotic behaving of Bessel functions.
    *   The most authentic result is obtained by the analysis in spherical coordinates. Photon states do not depend on angles, and they are damped according to r^{-1} law, where r is radius of the sphere. It has also turned out that orbital momentum of a free photon is equal to zero (this is understandable considering the fact that it moves along a straight line). The important element determining the behavior of a free photon is spin. One components of photon eigen-vector corresponds to spin $S_z = 1/2$ projection, while the other component corresponds to spin $S_z = -1/2$ projection. This was concluded on the basis of the fact that its eigen-vector components are Bessel functions having indices $+1/2$ and $-1/2$.
        The last result shows that linearization of photon Hamiltonian gives more complete picture of single photon than Kline-Gordon's approach.
2. Concluding the exposed analysis we shall try to connect the results obtained in series of experimental investigation of photon orbital momentum (Beth, 1936; Leach, et al., 2002; Allen, et al., 1992; Allen, 1966; He, et al., 1995; Friese, et al., 1996; Markoski, et al., 2008; van Enk & Nienhuis, 2007; Santamato, et al., 1988; O'Neil, et al., 2002; Volke-Sepulveda, et al., 2002). We shall not describe all quoted experiments. Instead of it we shall describe the essential idea: the orbital momentum of photon was determines from the changes of

torque of rotating particles. These changes where lied in some interval, so that the values of orbital momentum have had determined dispersion. As it vas said at the end of first section, such result is expectable for relativistic objects, in this case for photons. The azimuthally dependence of measured results is also predicted by the theory exposed in last Section.

Ending this analysis it should by noticed out that on the bases of given analysis the photon reflection can be considered as a transformation of photons to anti-photons.

5. Acknowledgements

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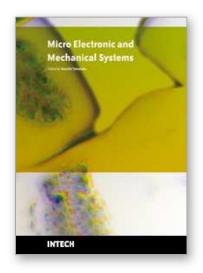
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