Chapter

An Alternative Framework for Developing Material Models for Finite-Strain Elastoplasticity

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Abstract

Contemporary plasticity theories and their related material models for finite deformations are either based on additive decomposition of a strain-rate tensor or on multiplicative decomposition of a deformation gradient tensor into an elastic part and a plastic part. From the standpoint of the nonlinear continuum mechanics, the former theories, which are used to model hypoelastic-plastic materials, are rather incomplete theories, while the latter theories, which are used to model hyperelastic-plastic materials, are not even continuum-based theories, while none of their related material models are thermodynamically consistent. Recently, a nonlinear continuum theory for finite deformations of elastoplastic media was proposed, which allows for the development of objective and thermodynamically consistent material models. Therefore, the analysis results of the models are independent of the description and the particularities of their mathematical formulation. Here by the description we mean total or updated Lagrangian description and by the particularities of formulation, the ability to describe the model in various stress spaces using internal mechanical power conjugate stress measures and strain rates. In this chapter, an alternative framework for developing objective and thermodynamically consistent hypoelastic-plastic- and hyperelastic-plastic-based material models is presented using the first nonlinear continuum theory of finite deformations of elastoplastic media.

Keywords: nonlinear continuum theory for finite deformations of elastoplastic media, objective and thermodynamically consistent formulation, J_2 generalised plasticity with isotropic hardening, hypoelastic-plastic- and hyperelastic-plastic-based material models with internal damping

1. Introduction

There are two types of phenomenological flow plasticity theories and their related material models used at present to model plastic behaviour of deformable bodies within the framework of finite-strain elastoplasticity. The first type of theories are considered to be ad hoc extensions of small-strain flow plasticity theories into the area of finite deformations to describe materials, in which small elastic deformations are accompanied by finite inelastic deformations during the deformation process. They are based on additive decomposition of a strain-rate

tensor into an elastic part and a plastic part to describe the plastic flow in the material. As an example of such material is ductile metal, which at present is modelled mainly by a kind of a hypoelastic-plastic-based material model, whose constitutive equation does not have a form in terms of a finite-strain measure. Without a need for completeness, let us just mention a few comprehensive studies in technical literature, such as [1–4], where detailed descriptions of the most frequently used contemporary hypoelastic-plastic-based material models are presented.

The second type of flow plasticity theories, in which multiplicative decomposition of a deformation gradient tensor into an elastic part and a plastic part is used to describe the plastic flow in the material, is based on the theory of single-crystal plasticity [5–7]. The theories and their related material models, which are now considered as 'proper material models' to model plastic behaviour of the deformable body, assume that the intermediate configuration of the body is stress-free [1] or at least locally unstressed [4]. As a result, there cannot exist a deformation or strain tensor field that meets the conditions of compatibility [4]. Therefore these theories treat the kinematics of motion differently between the initial and current or intermediate configurations of the body. This means that the motion, the displacement field and the deformation gradient, all of which have an exact physical meaning in continuum mechanics, are considered in accordance with the continuum theory between the initial and current configurations of the body, but not between the configurations where one is an intermediate configuration. Here the motion and the displacement fields are disregarded, and as a result, the deformation gradient loses its physical meaning. Moreover, it should also be noted that the assumption of an unstressed intermediate configuration is not compatible with the theory of nonlinear continuum mechanics, as it violates proper stress transformations, resulting from the invariance of the internal mechanical power, when switching from one stress space to the other in any configuration of the body. As a result, contemporary multiplicative plasticity theories and their related material models in realty are not continuum-based.

Hypoelastic-plastic- and hyperelastic-plastic-based material models have been the subject of study over recent decades, and there are a few issues to be concerned about when the models are used in numerical analyses. These include energy accumulation and residual stresses along a closed elastic strain path in the case of hypoelastic-plastic-based material models in which the Jaumann rate is used to calculate the Cauchy stress tensor [8], residual stress accumulation up to unacceptable values during multiple loading cycles along a closed elastic strain path using the Jaumann rate and a few other rates [9] or shear oscillation in finite shearing problems [10, 11]. The aforementioned problems however can be eliminated by replacing the Jaumann rate with the Green-Naghdi or Truesdell rate in the formulation of the models [4]. Simo and Pistner showed that employing a constant spatial elasticity tensor in objective stress integration is not compatible with elasticity and that such models in fact fail to define the elastic material [12]. Equivalent rate descriptions of hyperelastic-based models in terms of different strain measures have been thoroughly discussed by Perić, who also showed that the Jaumann rate- and the Green-Naghdi rate-based models provide different levels of approximation to problems governed by the logarithmic strain-based Hencky hyperelastic law [4, 13]. We will show herein that all of the above problems actually resulted from the fact that the related material models use thermodynamically inconsistent formulation.

The aim of this chapter is to present an alternative framework for developing objective and thermodynamically consistent hypoelastic-plastic- and hyperelastic-plastic-based material models using the first nonlinear continuum theory for finite deformations of elastoplastic media [14]. We will show that the strain-rate tensor

additive decomposition-based theories are in reality finite-strain theories, but they are constrained when the plastic flow in them is defined in terms of a Cauchy's stress tensor-based yield surface in the current configuration of the body, while contemporary deformation gradient multiplicative split-based theories are not even continuum-based theories. Moreover, none of their related material models is thermodynamically consistent. In addition to this, we will show that all flow plasticity theories are just variants of the nonlinear continuum theory for finite deformations of elastoplastic media presented in this chapter, using the additive decomposition of a Lagrangian displacement field into an elastic part and a plastic part. Eventually, we will demonstrate the theory in numerical experiments using simple hypoelastic-plastic- and hyperelastic-plastic-based material models with internal damping and then briefly discuss their analysis results.

2. Theory

The Lagrangian description is used to describe the kinematics of motion and constitutive and evolution equations of the material of a deformable body. Though a single form of the constitutive equation of a material is sufficient to describe the material, all forms of the constitutive equation of the material are needed in order to prove that its formulation is thermodynamically consistent.

2.1 A short overview of the nonlinear continuum mechanical theory for finite deformations of elastoplastic media

The nonlinear continuum theory for finite deformations of elastic media has been developed in an elegant manner in the past decades [15–19]. The theory is particularly suitable for modelling elastic materials, whose constitutive equations are defined either in terms of a finite-strain tensor, as in the case of the St-Venant-Kirchhoff material, or derived from an appropriate strain energy density function, as in the case of the hyperelastic materials [17, 18]. Developing material models for finite-strain elastoplasticity within the framework of thermodynamics with internal variables of state, however, requires a somewhat different approach [20]. The constitutive and evolution equations of these materials either exist in rate forms only, or contain rate equations, which at some point during the solution process have to be integrated. Moreover, in nonlinear continuum mechanics, no kinematics of motion can be described without the mathematical definitions of the motion, the Lagrangian and Eulerian displacement fields and the deformation gradient, respectively.

Starting with the definitions (see **Figure 1**), the motion $\mathbf{x} = \mathbf{\Phi}(\mathbf{X}, t)$ from the mathematical pint of view is a vector function or vector field, which maps each material point $^0P \in {}^0\Omega$ with a position vector $\mathbf{X}$ in the initial configuration of the body, into a spatial pint $P \in {}^t\Omega$ with a position vector $\mathbf{x}$ in the current configuration of the body. The function must exist whenever the body moves, and it determines the position vector of a material particle at each time instant $t \geq 0$ [16]:

$$\{\mathbf{x}|\mathbf{x} = \mathbf{\Phi}(\mathbf{X}, t), \text{ for } \mathbf{X} \in {}^0\Omega, \mathbf{x} \in {}^t\Omega \text{ and } t \geq 0\}. \tag{1}$$

In Eq. (1) $^0\Omega$ is the domain of the function, which stands for the volume of the body in its initial configuration; $^t\Omega$ is the range of the function, which stands for the volume of the body in its current configuration; and t is time. The vector field that connects the position vectors of the material particle is the displacement field. It
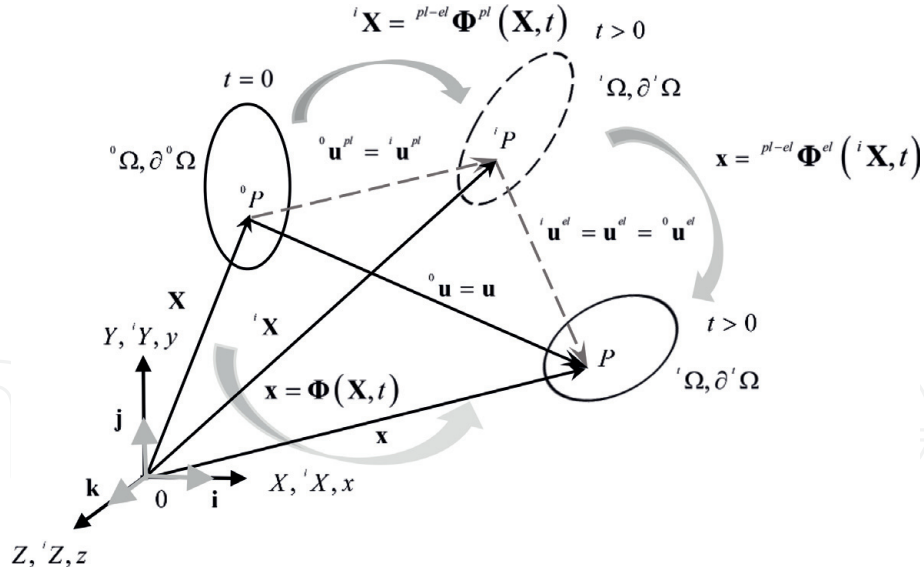
Figure 1.
The proper kinematics of motion of elastoplastic media.

must have both, Lagrangian $^0\mathbf{u} = {}^0\mathbf{u}(\mathbf{X},t)$ and Eulerian $\mathbf{u} = \mathbf{u}(\mathbf{x},t)$ forms; otherwise a physical phenomenon expressed in Lagrangian form cannot be re-expressed in Eulerian form and vice versa. The Lagrangian and Eulerian displacement fields are then defined as [16].

$$^0\mathbf{u} = {}^0\mathbf{u}(\mathbf{X},t) = \mathbf{x} - \mathbf{X} = \boldsymbol{\Phi}(\mathbf{X},t) - \mathbf{X}, \text{ for } t \geq 0 \text{ and } \mathbf{X} \in {}^0\Omega, \tag{2}$$

$$\mathbf{u} = \mathbf{u}(\mathbf{x},t) = \mathbf{x} - \mathbf{X} = \mathbf{x} - \boldsymbol{\Phi}^{-1}(\mathbf{x},t), \text{ for } t \geq 0 \text{ and } \mathbf{x} \in {}^t\Omega. \tag{3}$$

The deformation gradient $\mathbf{F}$ then becomes the derivative of the position vector of the material point after motion with respect to the position vector of the material point before motion $\mathbf{F} = \partial\mathbf{x}/\partial\mathbf{X}$ or simply the gradient of the vector function describing the motion.

When the motion is decomposed into several parts, so that the body moves from its initial configuration into its current configuration through several intermediate configurations, the above definitions apply between any two configurations of the body. Let us now consider a deformation process during which the body first undergoes plastic deformations and then elastic deformations at its each constituent. Then the Lagrangian plastic motion $^{pl-el}\boldsymbol{\Phi}^{pl} = {}^{pl-el}\boldsymbol{\Phi}^{pl}(\mathbf{X},t)$ and the Lagrangian plastic displacement field $^0\mathbf{u}^{pl} = {}^0\mathbf{u}^{pl}(\mathbf{X},t)$, defined over the initial volume of the body, take the following forms:

$$\left\{ {}^i\mathbf{X} \middle| {}^i\mathbf{X} = {}^{pl-el}\boldsymbol{\Phi}^{pl}(\mathbf{X},t), \text{ for } \mathbf{X} \in {}^0\Omega, {}^i\mathbf{X} \in {}^i\Omega \text{ and } t \geq 0 \right\}, \tag{4}$$

$$^0\mathbf{u}^{pl} = {}^0\mathbf{u}^{pl}(\mathbf{X},t) = {}^i\mathbf{X} - \mathbf{X} = {}^{pl-el}\boldsymbol{\Phi}^{pl}(\mathbf{X},t) - \mathbf{X}, \text{ for } t \geq 0, \text{ and } \mathbf{X} \in {}^0\Omega. \tag{5}$$

In Eqs. (4) and (5) $^i\mathbf{X}$ stands for the position vector of the spatial point $^iP \in {}^i\Omega$, at which the material particle is located when the body has undergone plastic deformations only, and the left superscript $^{pl-el}(\bullet)$ denotes the order of elastic and plastic deformations. Although the Eulerian elastic motion $^{pl-el}\boldsymbol{\Phi}^{el} = {}^{pl-el}\boldsymbol{\Phi}^{el}({}^i\mathbf{X},t)$ and the Eulerian elastic displacement field $^i\mathbf{u}^{el} = {}^i\mathbf{u}^{el}({}^i\mathbf{X},t)$ defined over the intermediate volume of the body $^i\Omega$ have similar forms to the fields above, these

represent spatial vector fields, because the intermediate configuration of the body changes during the deformation process.

$$\left\{ \mathbf{x} | \mathbf{x} = {}^{pl-el}\mathbf{\Phi}^{el}\left({}^{i}\mathbf{X}, t\right), \text{ for } {}^{i}\mathbf{X} \in {}^{i}\Omega, \ \mathbf{x} \in {}^{t}\Omega, \ t \geq 0 \right\}, \tag{6}$$

$${}^{i}\mathbf{u}^{el} = {}^{i}\mathbf{u}^{el}\left({}^{i}\mathbf{X}, t\right) = \mathbf{x} - {}^{i}\mathbf{X} = {}^{pl-el}\mathbf{\Phi}^{el}\left({}^{i}\mathbf{X}, t\right) - {}^{i}\mathbf{X}, \text{ for } t \geq 0 \text{ and } {}^{i}\mathbf{X} \in {}^{i}\Omega. \tag{7}$$

Moreover, because the plastic motion exists, the vector fields have Lagrangian forms too. Then the Lagrangian elastic motion ${}^{pl-el}\mathbf{\Phi}^{el}\left[{}^{pl-el}\mathbf{\Phi}^{pl}(\mathbf{X}, t), t\right]$ and the Lagrangian elastic displacement field ${}^{0}\mathbf{u}^{el} = {}^{0}\mathbf{u}^{el}(\mathbf{X}, t)$, defined over the initial volume of the body, can be expressed as follows:

$$\left\{ \mathbf{x} | \mathbf{x} = {}^{pl-el}\mathbf{\Phi}^{el}\left[{}^{pl-el}\mathbf{\Phi}^{pl}(\mathbf{X}, t), t\right], \text{ for } \mathbf{X} \in {}^{0}\Omega, \ \mathbf{x} \in {}^{t}\Omega, \ t \geq 0 \right\}, \tag{8}$$

$${}^{0}\mathbf{u}^{el} = {}^{0}\mathbf{u}^{el}(\mathbf{X}, t) = \mathbf{x} - {}^{i}\mathbf{X} = {}^{pl-el}\mathbf{\Phi}^{el}\left[{}^{pl-el}\mathbf{\Phi}^{pl}(\mathbf{X}, t), t\right] - {}^{pl-el}\mathbf{\Phi}^{pl}(\mathbf{X}, t), \text{ for } t \geq 0 \text{ and } \mathbf{X} \in {}^{0}\Omega. \tag{9}$$

Eqs. (1) and (8) then imply the following composite function for the overall motion:

$$\mathbf{x} = \mathbf{\Phi}(\mathbf{X}, t) = {}^{pl-el}\mathbf{\Phi}^{el}\left[{}^{pl-el}\mathbf{\Phi}^{pl}(\mathbf{X}, t), t\right]. \tag{10}$$

Moreover, after adding Eqs. (5) and (9) up, the following formula for the overall Lagrangian displacement field can be arrived at

$${}^{0}\mathbf{u}^{el}(\mathbf{X}, t) + {}^{0}\mathbf{u}^{pl}(\mathbf{X}, t) = \mathbf{\Phi}^{el}\left[\mathbf{\Phi}^{pl}(\mathbf{X}, t), t\right] - \mathbf{X} = \mathbf{x} - \mathbf{X} = {}^{0}\mathbf{u}(\mathbf{X}, t). \tag{11}$$

Eq. (11) states that the Lagrangian displacement field can additively be decomposed into a Lagrangian elastic part and a Lagrangian plastic part when the kinematics of motion is considered in accordance with the theory of nonlinear continuum mechanics. The deformation gradient then takes the form

$$\mathbf{F} = \mathbf{F}(\mathbf{X}, t) = \mathbf{I} + \frac{\partial^{0}\mathbf{u}}{\partial \mathbf{X}} = \mathbf{I} + \frac{\partial^{0}\mathbf{u}^{pl}}{\partial \mathbf{X}} + \frac{\partial^{0}\mathbf{u}^{el}}{\partial \mathbf{X}}. \tag{12}$$

It should be noted that Eq. (12) is the simplest form of the deformation gradient, irrespective of whether the additive decomposition of the displacement field in the above or the multiplicative decomposition of the deformation gradient tensor is used as a starting point in its formulation. In the latter case the formulation would modify as follows:

$$\mathbf{F}(\mathbf{X}, t) = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \frac{\partial \mathbf{x}}{\partial^{i}\mathbf{X}} \cdot \frac{\partial^{i}\mathbf{X}}{\partial \mathbf{X}} = {}^{pl-el}\mathbf{F}^{el}(\mathbf{X}, t) \cdot {}^{pl-el}\mathbf{F}^{pl}(\mathbf{X}, t) = \mathbf{I} + \frac{\partial^{0}\mathbf{u}^{el}}{\partial \mathbf{X}} + \frac{\partial^{0}\mathbf{u}^{pl}}{\partial \mathbf{X}}, \tag{13}$$

where

$${}^{pl-el}\mathbf{F}^{el}(\mathbf{X}, t) = \frac{\partial \mathbf{x}}{\partial^{i}\mathbf{X}} = \mathbf{I} + \frac{\partial^{i}\mathbf{u}^{el}}{\partial^{i}\mathbf{X}} = \mathbf{I} + \frac{\partial^{0}\mathbf{u}^{el}}{\partial \mathbf{X}} \cdot \left[{}^{pl-el}\mathbf{F}^{pl}(\mathbf{X}, t)\right]^{-1}, \tag{14}$$

$$
{}^{pl-el}\mathbf{F}^{pl}(\mathbf{X},t) = \frac{\partial^i \mathbf{X}}{\partial \mathbf{X}} = \mathbf{I} + \frac{\partial^0 \mathbf{u}^{pl}}{\partial \mathbf{X}}, \tag{15}
$$

$$
\frac{\partial^i \mathbf{u}^{el}}{\partial^i \mathbf{X}} = \frac{\partial^0 \mathbf{u}^{el}}{\partial \mathbf{X}} \cdot \frac{\partial \mathbf{X}}{\partial^i \mathbf{X}} = \frac{\partial^0 \mathbf{u}^{el}}{\partial \mathbf{X}} \cdot \left[{}^{pl-el}\mathbf{F}^{pl}(\mathbf{X},t)\right]^{-1}, \tag{16}
$$

and where considering that ${}^{0}\mathbf{u}^{el} = {}^{i}\mathbf{u}^{el}$, in the Lagrangian form Eq. (16) of the Eulerian gradient $\partial^i \mathbf{u}^{el}/\partial^i \mathbf{X}$, the inverse of ${}^{pl-el}\mathbf{F}^{pl}(\mathbf{X},t)$ would be expressed as

$$
\mathbf{I} = \frac{\partial^i \mathbf{X}}{\partial^i \mathbf{X}} = \frac{\partial^i \mathbf{X}}{\partial \mathbf{X}} \cdot \frac{\partial \mathbf{X}}{\partial^i \mathbf{X}} = {}^{pl-el}\mathbf{F}^{pl}(\mathbf{X},t) \cdot \frac{\partial \mathbf{X}}{\partial^i \mathbf{X}} \Leftrightarrow \frac{\partial \mathbf{X}}{\partial^i \mathbf{X}} = \left[{}^{pl-el}\mathbf{F}^{pl}(\mathbf{X},t)\right]^{-1}. \tag{17}
$$

Alternatively (see Eq. (10)), the deformation gradient can also be expressed as

$$
\begin{aligned}
\mathbf{F}(\mathbf{X},t) &= \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \frac{\partial^{pl-el}\mathbf{\Phi}^{el}\left[{}^{pl-el}\mathbf{\Phi}^{pl}(\mathbf{X},t),t\right]}{\partial \mathbf{X}} \\
&= \frac{\partial^{pl-el}\mathbf{\Phi}^{el}\left[{}^{pl-el}\mathbf{\Phi}^{pl}(\mathbf{X},t),t\right]}{{}^{pl-el}\mathbf{\Phi}^{pl}(\mathbf{X},t)} \cdot \frac{{}^{pl-el}\mathbf{\Phi}^{pl}(\mathbf{X},t)}{\partial \mathbf{X}} = {}^{pl-el}\mathbf{F}^{el}(\mathbf{X},t) \cdot {}^{pl-el}\mathbf{F}^{pl}(\mathbf{X},t).
\end{aligned} \tag{18}
$$

It should be noted that by employing the same procedure (Eqs. (13)–(17)), identical formula (Eq. (12)) for the deformation gradient would be arrived at, if the order of elastic and plastic deformations was reversed, although in that case the definitions of the elastic motion ${}^{el-pl}\mathbf{\Phi}^{el}(\mathbf{X},t)$, the plastic motion ${}^{el-pl}\mathbf{\Phi}^{pl}(\mathbf{X},t)$, the elastic ${}^{el-pl}\mathbf{F}^{el}(\mathbf{X},t)$ and plastic ${}^{el-pl}\mathbf{F}^{pl}(\mathbf{X},t)$ parts of the deformation gradient would be different. The corresponding elastic deformation gradient ${}^{el-pl}\mathbf{F}^{el}(\mathbf{X},t)$, which from now on will be denoted as ${}^{el-pl}\mathbf{F}^{el} = \mathbf{F}^{el}$, then would take the following form:

$$
\mathbf{F}^{el} = \mathbf{I} + \frac{\partial^0 \mathbf{u}^{el}}{\partial \mathbf{X}} = \mathbf{F} - \frac{\partial^0 \mathbf{u}^{pl}}{\partial \mathbf{X}}. \tag{19}
$$

When the deformation gradient is in the form of Eq. (12), the material $\dot{\mathbf{E}}$ and the spatial $\mathbf{d} = \mathcal{L}_e(\mathbf{e})$ strain-rate tensors take the forms

$$
\dot{\mathbf{E}} = \frac{1}{2} \cdot \left(\dot{\mathbf{F}}^T \cdot \mathbf{F} + \mathbf{F}^T \cdot \dot{\mathbf{F}}\right) = \dot{\mathbf{E}}^{el} + \dot{\mathbf{E}}^{pl}, \Rightarrow \mathbf{d} = \mathbf{d}^{el} + \mathbf{d}^{pl}, \tag{20}
$$

where

$$
\mathbf{d} = \mathbf{F}^{-T} \cdot \dot{\mathbf{E}} \cdot \mathbf{F}^{-1}, \quad \mathbf{d}^{el} = \mathbf{F}^{-T} \cdot \dot{\mathbf{E}}^{el} \cdot \mathbf{F}^{-1}, \quad \mathbf{d}^{pl} = \mathbf{F}^{-T} \cdot \dot{\mathbf{E}}^{pl} \cdot \mathbf{F}^{-1}, \tag{21}
$$

$$
\dot{\mathbf{E}}^{el} = \frac{1}{2} \cdot \left[\left(\frac{\partial^0 \dot{\mathbf{u}}^{el}}{\partial \mathbf{X}}\right)^T \cdot \mathbf{F} + \mathbf{F}^T \cdot \frac{\partial^0 \dot{\mathbf{u}}^{el}}{\partial \mathbf{X}}\right], \quad \dot{\mathbf{E}}^{pl} = \frac{\dot{\lambda}}{2} \cdot \left[\left(\frac{\partial^P \Psi}{\partial \mathbf{P}}\right)^T \cdot \mathbf{F} + \mathbf{F}^T \cdot \frac{\partial^P \Psi}{\partial \mathbf{P}}\right], \tag{22}
$$

$$
\frac{\partial^0 \dot{\mathbf{u}}^{pl}}{\partial \mathbf{X}} = \dot{\lambda} \cdot \frac{\partial^P \Psi}{\partial \mathbf{P}}, \quad \text{and} \quad \frac{\partial^P \Psi}{\partial \mathbf{P}} \neq \left(\frac{\partial^P \Psi}{\partial \mathbf{P}}\right)^T. \tag{23}
$$

In the above $\mathbf{E} = 1/2 \cdot \left(\mathbf{F}^T \cdot \mathbf{F} - \mathbf{I}\right)$ denotes the Green-Lagrangian strain tensor and $\mathbf{e} = 1/2 \cdot \left(\mathbf{I} - \mathbf{F}^{-T} \cdot \mathbf{F}^{-1}\right)$ the Eulerian-Almansi strain tensor, respectively. The symbols $\dot{\mathbf{E}}^{el}$, $\dot{\mathbf{E}}^{pl}/\mathbf{d}^{el}$, $\mathbf{d}^{pl}$ stand for the elastic and the plastic material/spatial strain-rate tensors, wherein the latter plastic flow is defined by Eq. (23)$_1$ as a product of a plastic multiplier $\dot{\lambda}$ and an appropriate yield surface normal, $\partial^P \Psi/\partial \mathbf{P}$, defined in terms of a first Piola-Kirchhoff stress tensor $\mathbf{P}$. Here the symbol $\mathfrak{L}_e(\mathbf{e}) = \mathbf{F}^{-T} \cdot \left[\partial\left(\mathbf{F}^T \cdot (\mathbf{e}) \cdot \mathbf{F}\right)/\partial t\right] \cdot \mathbf{F}^{-1}$ denotes the Lie derivative of the Eulerian-Almansi strain tensor $\mathbf{e}$. It should also be noted that both elastic and the plastic strain-rate tensors have forms similar to the strain-rate tensor itself. Besides, it can be shown that the plastic flow defined by Eq. (23)$_1$ is not constrained, resulting in Eqs. (22)$_2$ and (21)$_3$, respectively, being the only non-degenerated forms of the material and spatial plastic strain-rate tensors.

A crucial role in the Lagrangian description plays the invariance of the internal mechanical power. It not only defines conjugate pairs of stress measures and strain or deformation rates, but, being the expression of the conservation of internal mechanical energy (first law of thermodynamics), it plays an inevitable role in making sure that the total Lagrangian description and the updated Lagrangian description are equivalent. As a result, appropriate transformations can be found between various stress measures and strain or deformation rates constituting conjugate pairs, when switching from one stress space in one configuration of the body to the other stress space in the same or any other configuration of the body [15–18]. Unfortunately contemporary continuum theory does not cover materials whose constitutive and evolution equations are defined in rate forms. In order to extend the theory, so that it could cover the materials, Cauchy's stress theorem [16] had to be generalised as follows:

$$\mathfrak{L}_{Tr}^{(n)}[\mathbf{T}(\mathbf{X}, t, \mathbf{N})] = \mathfrak{L}_P^{(n)}[\mathbf{P}(\mathbf{X}, t)] \cdot \mathbf{N} \ \text{ and } \ \mathfrak{L}_{tr}^{(n)}[\mathbf{t}(\mathbf{x}, t, \mathbf{n})] = \mathfrak{L}_T^{(n)}[\boldsymbol{\sigma}(\mathbf{x}, t)] \cdot \mathbf{n}, \quad (24)$$

for all $n = 0, 1, ..., n \in N$, where n denotes objective differentiation with respect to time $t \geq 0$ and not an exponent and N the set of natural numbers. In Eq. (24) the variables $\mathfrak{L}_{Tr}^{(n)}[\mathbf{T}(\mathbf{X}, t, \mathbf{N})]$, $\mathfrak{L}_{tr}^{(n)}[\mathbf{t}(\mathbf{x}, t, \mathbf{n})]$ stand for the n$^{\text{th}}$ objective derivatives of the surface traction vectors $\mathbf{T} = \mathbf{T}(\mathbf{X}, t, \mathbf{N})$, $\mathbf{t} = \mathbf{t}(\mathbf{x}, t, \mathbf{n})$ in the initial and current configurations of the body, and $\mathbf{N}$, $\mathbf{n}$ are the corresponding unit outwards surface normal vectors. Similarly, the quantities $\mathfrak{L}_P^{(n)}[\mathbf{P}(\mathbf{X}, t)]$, $\mathfrak{L}_T^{(n)}[\boldsymbol{\sigma}(\mathbf{x}, t)]$ denote the n$^{\text{th}}$ objective derivatives of the first Piola-Kirchhoff stress tensor $\mathbf{P} = \mathbf{P}(\mathbf{X}, t)$ and the Cauchy stress tensor $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{x}, t)$, respectively. Then the requirements of thermodynamic consistency of the Lagrangian formulation are ensured by the following postulates:

Postulate no. 1. The product of a surface traction vector, including all its higher-order objective time derivatives and the surface of an infinitesimal volume element in the initial and current configurations of the body, on which they act, have to be the same during the deformation process, i.e.:

$$\mathfrak{L}_{Tr}^{(n)}(\mathbf{T}) \cdot dS_0 = \mathfrak{L}_P^{(n)}(\mathbf{P}) \cdot \mathbf{N} \cdot dS_0 = \mathfrak{L}_P^{(n)}(\mathbf{P}) \cdot d\mathbf{S}_0 = \mathfrak{L}_{tr}^{(n)}(\mathbf{t}) \cdot ds$$

or $$= \mathfrak{L}_T^{(n)}(\boldsymbol{\sigma}) \cdot \mathbf{n} \cdot ds = \mathfrak{L}_T^{(n)}(\boldsymbol{\sigma}) \cdot d\mathbf{s}, \quad (25)$$

$$\mathfrak{L}_P^{(n)}(\mathbf{P}) \cdot d\mathbf{S}_0 = \mathfrak{L}_T^{(n)}(\boldsymbol{\sigma}) \cdot d\mathbf{s} = J \cdot \mathfrak{L}_T^{(n)}(\boldsymbol{\sigma}) \cdot \mathbf{F}^{-T} \cdot d\mathbf{S}_0, \text{ for all } n = 0, 1, 2, ..., n \in N, \quad (26)$$

where $d\mathbf{S}_0$, $d\mathbf{s} = J \cdot \mathbf{F}^{-T} \cdot d\mathbf{S}_0$ denote the infinitesimal surface elements in the initial and current configurations of the body, where the latter is expressed using the Nanson's formula [16].

Postulate no. 2. The rate of change of the internal mechanical energy accumulated in the infinitesimal volume element in the initial and current configurations of the body and all its higher-order time derivatives have to be the same during the deformation process, i.e.:

$$\frac{\partial^n dW}{\partial t^n} = \left[\sum_{k=0}^{n} \binom{n}{k} \cdot \frac{\partial^{n-k}\mathbf{S}}{\partial t^{n-k}} : \frac{\partial^k \dot{\mathbf{E}}}{\partial t^k}\right] \cdot dV_0 = \left[\sum_{k=0}^{n} \binom{n}{k} \cdot \mathfrak{L}_P^{(n-k)}(\mathbf{P}) : \mathfrak{L}_F^{(k)}\left(\frac{\partial^0 \dot{\mathbf{u}}}{\partial \mathbf{X}}\right)\right]$$

$$\cdot dV_0 = \left[\sum_{k=0}^{n} \binom{n}{k} \cdot \mathfrak{L}_O^{(n-k)}(\boldsymbol{\tau}) : \mathfrak{L}_e^{(k)}(\mathbf{d})\right] \cdot dV_0 = \left[\sum_{k=0}^{n} \binom{n}{k} \cdot \mathfrak{L}_T^{(n-k)}(\boldsymbol{\sigma}) : \mathfrak{L}_e^{(k)}(\mathbf{d})\right]$$

$$\cdot dv, \quad \text{for all } n = 0, 1, 2, ..., n \in N,$$

or

(27)

$$\frac{\partial^n dW}{\partial t^n} = \left[\sum_{k=0}^{n} \binom{n}{k} \cdot \frac{\partial^{n-k}\mathbf{S}}{\partial t^{n-k}} : \frac{\partial^k \dot{\mathbf{E}}}{\partial t^k}\right] \cdot dV_0 = \left[\sum_{k=0}^{n} \binom{n}{k} \cdot \mathbf{F} \cdot \left(\frac{\partial^{n-k}\mathbf{S}}{\partial t^{n-k}}\right) : \mathbf{F}^{-T} \cdot \frac{\partial^k (\mathbf{F}^T \cdot \dot{\mathbf{F}})}{\partial t^k}\right]$$

$$\cdot dV_0 = \left[\sum_{k=0}^{n} \binom{n}{k} \cdot \mathbf{F} \cdot \left(\frac{\partial^{n-k}\mathbf{S}}{\partial t^{n-k}}\right) \cdot \mathbf{F}^T : \mathbf{F}^{-T} \cdot \left(\frac{\partial^k \dot{\mathbf{E}}}{\partial t^k}\right) \cdot \mathbf{F}^{-1}\right] \cdot dV_0 =$$

$$= \left[\sum_{k=0}^{n} \binom{n}{k} \cdot \frac{\mathbf{F}}{J} \cdot \left(\frac{\partial^{n-k}\mathbf{S}}{\partial t^{n-k}}\right) \cdot \mathbf{F}^T : \mathbf{F}^{-T} \cdot \left(\frac{\partial^k \dot{\mathbf{E}}}{\partial t^k}\right) \cdot \mathbf{F}^{-1}\right] \cdot dv \quad \text{for all } n = 0, 1, 2, ..., n \in N.$$

(28)

where $dV_0, dv = J \cdot dV_0$ stand for the infinitesimal volume elements in the initial and current configurations of the body and $J = \det(\mathbf{F})$. Then Eqs. (26)–(28) define the following transformations:

$$\mathfrak{L}_P^{(n)}(\mathbf{P}) = \mathbf{F} \cdot \left(\frac{\partial^n \mathbf{S}}{\partial t^n}\right), \ \mathfrak{L}_O^{(n)}(\boldsymbol{\tau}) = \mathbf{F} \cdot \left(\frac{\partial^n \mathbf{S}}{\partial t^n}\right) \cdot \mathbf{F}^T, \ \mathfrak{L}_T^{(n)}(\boldsymbol{\sigma}) = J^{-1} \cdot \mathbf{F} \cdot \left(\frac{\partial^n \mathbf{S}}{\partial t^n}\right) \cdot \mathbf{F}^T,$$

$$\mathfrak{L}_e^{(n)}(\mathbf{d}) = \mathbf{F}^{-T} \cdot \left(\frac{\partial^n \dot{\mathbf{E}}}{\partial t^n}\right) \cdot \mathbf{F}^{-1}, \ \frac{\partial^n \dot{\mathbf{E}}}{\partial t^n} = \left[\mathbf{F}^T \cdot \mathfrak{L}_F^{(n)}(\dot{\mathbf{F}})\right]^{sym} = \left[\frac{\partial^n \left(\mathbf{F}^T \cdot \frac{\partial^0 \dot{\mathbf{u}}}{\partial \mathbf{X}}\right)}{\partial t^n}\right]^{sym},$$

$$\mathfrak{L}_e^{(n)}(\mathbf{d}) = \left[\mathfrak{L}_F^{(n)}(\dot{\mathbf{F}}) \cdot \mathbf{F}^{-1}\right]^{sym} = \left\{\mathbf{F}^{-T} \cdot \left[\frac{\partial^n \left(\mathbf{F}^T \cdot \frac{\partial^0 \dot{\mathbf{u}}}{\partial \mathbf{X}}\right)}{\partial t^n}\right] \cdot \mathbf{F}^{-1}\right\}^{sym} \quad \text{for all } n = 0, 1, .., n \in N,$$

(29)

as the sufficient conditions of thermodynamic consistency, because they ensure that the two postulates above are met. It should also be noted that for $n = 0$ the transformations define the necessary conditions of thermodynamic consistency. In that case the generalised Cauchy's stress theorem Eq. (24) reduces to its well-known form, $\mathbf{T} = \mathbf{P} \cdot \mathbf{N}$ and $\mathbf{t} = \boldsymbol{\sigma} \cdot \mathbf{n}$, , while the transformations Eq. (29) reduce to the already well-known transformations in nonlinear continuum mechanics, defining the relationship between various stress measures and strain or deformation rates constituting the conjugate pairs.

The objective rates, which meet the sufficient conditions of thermodynamic consistency defined by Eq. (29), are already known in nonlinear continuum mechanics as the nth Lie derivative of the first Piola-Kirchhoff stress tensor **P** (Eq. (30)), the nth Lie derivative of the rate of deformation gradient tensor $\dot{\mathbf{F}}$ (Eq. (31)), the nth Oldroyd derivative of the Kirchhoff stress $\boldsymbol{\tau}$ tensor (Eq. (32)), the nth Lie derivative of the spatial strain-rate tensor **d** (Eq. (33)) and the nth Truesdell derivative of the Cauchy stress tensor Eq. (34), respectively:

$$\mathcal{L}_P^{(n)}(\mathbf{P}) = \mathbf{F} \cdot \left[\frac{\partial^n (\mathbf{F}^{-1} \cdot \mathbf{P})}{\partial t^n} \right] = \mathbf{F} \cdot \left(\frac{\partial^n \mathbf{S}}{\partial t^n} \right), \tag{30}$$

$$\mathcal{L}_F^{(n)}(\dot{\mathbf{F}}) = \mathbf{F}^{-T} \cdot \left[\frac{\partial^n (\mathbf{F}^T \cdot \dot{\mathbf{F}})}{\partial t^n} \right] = \mathbf{F}^{-T} \cdot \frac{\partial^n \left(\mathbf{F}^T \cdot \frac{\partial^0 \dot{\mathbf{u}}}{\partial \mathbf{X}} \right)}{\partial t^n}, \tag{31}$$

$$\mathcal{L}_O^{(n)}(\boldsymbol{\tau}) = \mathbf{F} \cdot \left[\frac{\partial^n (\mathbf{F}^{-1} \cdot \boldsymbol{\tau} \cdot \mathbf{F}^{-T})}{\partial t^n} \right] \cdot \mathbf{F}^T = \mathbf{F} \cdot \left(\frac{\partial^n \mathbf{S}}{\partial t^n} \right) \cdot \mathbf{F}^T, \tag{32}$$

$$\mathcal{L}_e^{(n)}(\mathbf{d}) = \mathbf{F}^{-T} \cdot \left[\frac{\partial^n (\mathbf{F}^T \cdot \mathbf{d} \cdot \mathbf{F})}{\partial t^n} \right] \cdot \mathbf{F}^{-1} = \mathbf{F}^{-T} \cdot \left(\frac{\partial^n \dot{\mathbf{E}}}{\partial t^n} \right) \cdot \mathbf{F}^{-1}, \tag{33}$$

$$\mathcal{L}_T^{(n)}(\boldsymbol{\sigma}) = J^{-1} \cdot \mathbf{F} \cdot \left[\frac{\partial^n (J \cdot \mathbf{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \mathbf{F}^{-T})}{\partial t^n} \right] \cdot \mathbf{F}^T = J^{-1} \cdot \mathbf{F} \cdot \left(\frac{\partial^n \mathbf{S}}{\partial t^n} \right) \cdot \mathbf{F}^T. \tag{34}$$

It should also be noted here that Eq. (27) is the result of straightforward manipulation of the nth time derivative of the internal mechanical power (see Eq. (28)), whose last terms formally define the formulas for evaluating the nth objective derivative of the power with respect to time in the first Piola-Kirchhoff, Kirchhoff and Cauchy stress spaces. Then Eq. (27) defines not only conjugate pairs of stress measures and strain or deformations rates but also conjugate pairs of objective differentiation operators and derivatives. Moreover, we used intentionally the term 'requirements of thermodynamic consistency' for the transformations Eq. (29), because without the invariance of the internal mechanical power (first law of thermodynamics) and its higher-order time derivatives (Eq. (27) or (28)), no formulation is thermodynamically consistent, in spite of the fact that in thermodynamics the term is associated with the second law of thermodynamics to show that the constitutive equation of a material is compatible with the second law.

2.2 Modelling of the plastic flow in the material

In order to modify the nonlinear continuum theory for finite deformations of elastoplastic media, it is assumed that the yield surface of the material has definitions $^S\Psi = {}^S\Psi \left[{}^S\sigma_{eq}(\mathbf{S}), \mathbf{q} \right], {}^P\Psi = {}^P\Psi \left[{}^P\sigma_{eq}(\mathbf{P}), \mathbf{q} \right], {}^{\tau}\Psi = {}^{\tau}\Psi \left[{}^{\tau}\sigma_{eq}(\boldsymbol{\tau}), \mathbf{q} \right], {}^{\sigma}\Psi = {}^{\sigma}\Psi$ $\left[{}^{\sigma}\sigma_{eq}(\boldsymbol{\sigma}), \mathbf{q} \right]$ in terms of the second Piola-Kirchhoff stress tensor $\mathbf{S}$, the first Piola-Kirchhoff stress tensor $\mathbf{P}$, the Kirchhoff stress tensor $\boldsymbol{\tau}$, the Cauchy stress tensor $\boldsymbol{\sigma}$ and a vector of internal variables $\mathbf{q}$ in the second Piola-Kirchhoff, first Piola-Kirchhoff, Kirchhoff and Cauchy stress spaces, where $^S\sigma_{eq}(\mathbf{S})$, $^P\sigma_{eq}(\mathbf{P})$, $^{\sigma}\sigma_{eq}(\boldsymbol{\tau})$, $^{\sigma}\sigma_{eq}(\boldsymbol{\sigma})$ are the corresponding equivalent stresses. After changing the physical interpretation of the plastic flow and applying push-forward and pull-back operations to the material gradient of the plastic velocity field, Eq. (23)$_1$ is as follows:

$$\frac{\partial \dot{\mathbf{u}}^{pl}}{\partial \mathbf{x}} = \frac{\partial^0 \dot{\mathbf{u}}^{pl}}{\partial \mathbf{X}} \cdot \mathbf{F}^{-1} = \dot{\lambda} \cdot \frac{\partial^{\sigma} \Psi}{\partial \boldsymbol{\sigma}}, \frac{\partial \dot{\mathbf{u}}^{pl}}{\partial \mathbf{x}} = \frac{\partial^0 \dot{\mathbf{u}}^{pl}}{\partial \mathbf{X}} \cdot \mathbf{F}^{-1} = \dot{\lambda} \cdot \frac{\partial^{\tau} \Psi}{\partial \boldsymbol{\tau}}, \mathbf{F}^T \cdot \frac{\partial^0 \dot{\mathbf{u}}^{pl}}{\partial \mathbf{X}} = \dot{\lambda} \cdot \frac{\partial^S \Psi}{\partial \mathbf{S}}, \tag{35}$$

where it can be shown that the definitions of the yield surface are not independent of each other but are related, and the following formulas hold true:

$$\frac{\partial^P \Psi}{\partial \mathbf{P}} \cdot \mathbf{F}^{-1} = \frac{\partial^\sigma \Psi}{\partial \boldsymbol{\sigma}}, \quad \frac{\partial^P \Psi}{\partial \mathbf{P}} \cdot \mathbf{F}^{-1} = \frac{\partial^\tau \Psi}{\partial \boldsymbol{\tau}}, \quad \mathbf{F}^T \cdot \frac{\partial^P \Psi}{\partial \mathbf{P}} = \frac{\partial^S \Psi}{\partial \mathbf{S}}. \tag{36}$$

As a result, one of the definitions of the yield surfaces has to be chosen as a reference to define the material model, and the rest of them can be calculated by solving the differential equations in Eq. (36). Moreover, when $^\sigma \Psi$ or $^\tau \Psi$ is used as the reference definition of the yield surface in the current configuration of the body, the contemporary flow plasticity models will be recovered. It also can be verified that the various definitions of the yield surface and their equivalent stresses $^P \sigma_{eq}, {}^S \sigma_{eq}, {}^\tau \sigma_{eq}, {}^\sigma \sigma_{eq}$, which also meet the transformations defined by Eq. (36), have the following properties:

$$^P \sigma_{eq} = {}^S \sigma_{eq} = {}^\tau \sigma_{eq} = J \cdot {}^\sigma \sigma_{eq}, \tag{37}$$

$$\frac{\partial^S \Psi}{\partial \mathbf{S}} : \mathbf{S} = \frac{\partial^P \Psi}{\partial \mathbf{P}} : \mathbf{P} = \frac{\partial^\tau \Psi}{\partial \boldsymbol{\tau}} : \boldsymbol{\tau} = J \cdot \frac{\partial^\sigma \Psi}{\partial \boldsymbol{\sigma}} : \boldsymbol{\sigma}, \tag{38}$$

$$\frac{\partial^S \Psi}{\partial \mathbf{S}} : \dot{\mathbf{S}} = \frac{\partial^P \Psi}{\partial \mathbf{P}} : \mathfrak{L}_P(\mathbf{P}) = \frac{\partial^\tau \Psi}{\partial \boldsymbol{\tau}} : \mathfrak{L}_O(\boldsymbol{\tau}) = J \cdot \frac{\partial^\sigma \Psi}{\partial \boldsymbol{\sigma}} : \mathfrak{L}_T(\boldsymbol{\sigma}), \tag{39}$$

where Eqs. (38) and (39) represent 'normality rules', which from the physical point of view are equivalent with the following equations:

$$dW^{pl} = \dot{\mathbf{E}}^{pl} : \mathbf{S} \cdot dV_0 = \frac{\partial^0 \dot{\mathbf{u}}^{pl}}{\partial \mathbf{X}} : \mathbf{P} \cdot dV_0 = \mathbf{d}^{pl} : \boldsymbol{\tau} \cdot dV_0 = \mathbf{d}^{pl} : \boldsymbol{\sigma} \cdot dv, \tag{40}$$

$$\dot{\mathbf{E}}^{pl} : \dot{\mathbf{S}} \cdot dV_0 = \frac{\partial^0 \dot{\mathbf{u}}^{pl}}{\partial \mathbf{X}} : \mathfrak{L}_P(\mathbf{P}) \cdot dV_0 = \mathbf{d}^{pl} : \mathfrak{L}_O(\boldsymbol{\tau}) \cdot dV_0 = \mathbf{d}^{pl} : \mathfrak{L}_T(\boldsymbol{\sigma}) \cdot dv. \tag{41}$$

where W^{pl} is the internal plastic power.

2.3 The constitutive equations of the material

Proper formulation of a material model for finite-strain elastoplasticity allows for the definition of the model in all stress spaces in any configuration of the body. These, however, have to comply with the principles of material modelling, particularly to meet the requirements of material objectivity and be thermodynamically consistent in order that they would define the same material. Finite-strain computational plasticity distinguishes between two major types of material models known as hypoelastic-plastic-based material models and hyperelastic-plastic-based material models. Moreover, hypoelastic-plastic-based material models exist in rate forms only, because the additive decomposition of the strain-rate tensor Eqs. (20)–(23) and (36) exists either in rate forms only. In this research we have modified our former material model with internal damping, capable of imitating even ductile-to-brittle failure mode transition at high strain rates, to model our hypoelastic-plastic-based material [21]. The rate form of the constitutive equation of the material then can take any of the following forms:

$$\dot{\mathbf{S}} = {}^{mat}\mathbb{C}^{el} : \left(\dot{\mathbf{E}} - xx \cdot \dot{\mathbf{E}}^{pl} \right) + {}^{mat}\mathbb{C}^{vis} : \left[\ddot{\mathbf{E}} - (1 - xx) \cdot \ddot{\mathbf{E}}^{pl} \right], \tag{42}$$

$$\mathfrak{L}_P(\mathbf{P}) = \mathbf{F} \cdot \dot{\mathbf{S}} = {}^{mix}\mathbb{C}^{el} : \left(\dot{\mathbf{F}} - xx \cdot \frac{\partial^0 \dot{\mathbf{u}}^{pl}}{\partial \mathbf{X}} \right) + {}^{mix}\mathbb{C}^{vis} : \left[\mathfrak{L}_F(\dot{\mathbf{F}}) - (1 - xx) \cdot \mathfrak{L}_F \left(\frac{\partial^0 \dot{\mathbf{u}}^{pl}}{\partial \mathbf{X}} \right) \right],$$

(43)

$$\mathfrak{L}_O(\boldsymbol{\tau}) = \mathbf{F} \cdot \dot{\mathbf{S}} \cdot \mathbf{F}^T = J \cdot {}^{spat}\mathbb{C}^{el} : \left(\mathbf{d} - xx \cdot \mathbf{d}^{pl} \right) + J \cdot {}^{spat}\mathbb{C}^{vis} : \left[\mathfrak{L}_e(\mathbf{d}) - (1 - xx) \cdot \mathfrak{L}_e \left(\mathbf{d}^{pl} \right) \right],$$

(44)

$$\mathfrak{L}_T(\boldsymbol{\sigma}) = J^{-1} \cdot \mathbf{F} \cdot \dot{\mathbf{S}} \cdot \mathbf{F}^T = {}^{spat}\mathbb{C}^{el} : \left(\mathbf{d} - xx \cdot \mathbf{d}^{pl} \right) + {}^{spat}\mathbb{C}^{vis} : \left[\mathfrak{L}_e(\mathbf{d}) - (1 - xx) \cdot \mathfrak{L}_e \left(\mathbf{d}^{pl} \right) \right],$$

(45)

where

$${}^{mat}\mathbb{C}^{el} = 2 \cdot G \cdot \mathbb{I} + \lambda^{el} \cdot \mathbf{I} \otimes \mathbf{I}, \qquad {}^{mat}\mathbb{C}^{vis} = 2 \cdot G^{vis} \cdot \mathbb{I} + \lambda^{vis} \cdot \mathbf{I} \otimes \mathbf{I},$$

(46)

$$G = \frac{E}{2 \cdot (1 + \nu)}, \quad \lambda^{el} = \frac{\nu \cdot E}{(1 + \nu) \cdot (1 - 2 \cdot \nu)},$$

(47)

$$G^{vis} = \frac{E^{vis}}{2 \cdot (1 + \nu^{vis})}, \quad \lambda^{vis} = \frac{\nu^{vis} \cdot E^{vis}}{(1 + \nu^{vis}) \cdot (1 - 2 \cdot \nu^{vis})},$$

(48)

$${}^{mix}\mathbb{C}^{el}_{ijkl} = F_{im} \cdot \delta_{jn} \cdot F_{ko} \cdot \delta_{lp} \cdot {}^{mat}\mathbb{C}^{el}_{mnop}, \quad {}^{mix}\mathbb{C}^{vis}_{ijkl} = F_{im} \cdot \delta_{jn} \cdot F_{ko} \cdot \delta_{lp} \cdot {}^{mat}\mathbb{C}^{vis}_{mnop},$$

(49)

$${}^{spat}\mathbb{C}^{el}_{ijkl} = J^{-1} \cdot F_{im} \cdot F_{jn} \cdot F_{ko} \cdot F_{lp} \cdot {}^{mat}\mathbb{C}^{el}_{mnop}, \quad {}^{spat}\mathbb{C}^{vis}_{ijkl} = J^{-1} \cdot F_{im} \cdot F_{jn} \cdot F_{ko} \cdot F_{lp} \cdot {}^{mat}\mathbb{C}^{vis}_{mnop}.$$

(50)

In Eqs. (42)–(50) the symbols $\dot{\mathbf{S}}$, $\mathfrak{L}_P(\mathbf{P})$, $\mathfrak{L}_O(\boldsymbol{\tau})$, $\mathfrak{L}_T(\boldsymbol{\sigma})$ denote the time derivative of the second Piola-Kirchhoff stress tensor, the Lie derivative of the first Piola-Kirchhoff stress tensor, the Oldroyd rate of the Kirchhoff stress and the Truesdell rate of the Cauchy stress, respectively. Here the fourth-order material elasticity tensor ${}^{mat}\mathbb{C}^{el}$ and the fourth-order material viscosity tensor ${}^{mat}\mathbb{C}^{vis}$ have similar forms as the fourth-order elasticity tensor of the St. Venant-Kirchhoff material [17] using two independent material parameters E, ν and E^{vis}, ν^{vis}, respectively, where $\mathbb{I}$ denotes the symmetric fourth-order identity tensor and $\mathbf{I}$ the second-order identity tensor. The fourth-order mixed spatial-material elasticity and viscosity tensors ${}^{mat}\mathbb{C}^{el}$, ${}^{mat}\mathbb{C}^{vis}$ then can be determined in accordance with Eq. (49), where δ_{ij} is the Kronecker delta, and the fourth-order spatial elasticity and viscosity tensors in accordance with Eq. (50). The variable xx denotes the ratio of ductile and total damage increment [21]. It should be noted that the objective rates $\dot{\mathbf{S}}$, $\mathfrak{L}_P(\mathbf{P})$, $\mathfrak{L}_O(\boldsymbol{\tau})$, $\mathfrak{L}_T(\boldsymbol{\sigma})$ transform in the same way from one form into another as do the stress tensors $\mathbf{S}$, $\mathbf{P}$, $\boldsymbol{\tau}$, $\boldsymbol{\sigma}$, which ensure that the formulation is thermodynamically consistent (see also Eq. (29)). Then, the corresponding rate forms of loading/unloading discrete Kuhn-Tucker plastic optimization conditions in the second Piola-Kirchhoff (Eq. (51)), first Piola-Kirchhoff (Eq. (52)), Kirchhoff (Eq. (53)) and Cauchy (Eq. (54)) stress spaces are modified as follows:

$$\dot{\lambda} \geq 0, \qquad {}^{S}\Psi \leq 0, \qquad \dot{\lambda} \cdot {}^{S}\dot{\Psi} = 0, \tag{51}$$

$$\dot{\lambda} \geq 0, \qquad {}^{P}\Psi \leq 0, \qquad \dot{\lambda} \cdot \mathfrak{L}_{P}\left({}^{P}\Psi\right) = 0, \tag{52}$$

$$\dot{\lambda} \geq 0, \qquad {}^{\tau}\Psi \leq 0, \qquad \dot{\lambda} \cdot \mathfrak{L}_{O}\left({}^{\tau}\Psi\right) = 0, \tag{53}$$

$$\dot{\lambda} \geq 0, \qquad {}^{\sigma}\Psi \leq 0, \qquad \dot{\lambda} \cdot \mathfrak{L}_{T}\left({}^{\sigma}\Psi\right) = 0. \tag{54}$$

It should be noted here that the above conditions resulted from the invariance of the internal mechanical power and its first time derivative (see Eq. (27)), which now also define conjugate pairs of differentiation operators and derivatives in all stress spaces, and that all yield surface is the expression of the conservation of internal plastic power, which will be shown later.

Hyperelastic-plastic-based material models are essentially elastic material models. The starting point in their development is Eq. (19), wherein the constitutive equation of the material the elastic Green-Lagrangian strain tensor and its time derivative are modified as follows:

$$*\mathbf{E}^{el} = \frac{1}{2} \cdot \left[\left(\mathbf{F}^{el}\right)^{T} \cdot \mathbf{F}^{el} - \mathbf{I}\right], \qquad *\dot{\mathbf{E}}^{el} = \frac{1}{2} \cdot \left[\left(\dot{\mathbf{F}}^{el}\right)^{T} \cdot \mathbf{F}^{el} + \left(\mathbf{F}^{el}\right)^{T} \cdot \dot{\mathbf{F}}^{el}\right]. \tag{55}$$

In our research, the St. Venant hyperelastic material with internal damping was used, whose constitutive equation then takes any of the following forms:

$$\mathbf{S} = {}^{mat}\mathbb{C}^{el} : *\mathbf{E}^{el} + {}^{mat}\mathbb{C}^{vis} : *\dot{\mathbf{E}}^{el}, \tag{56}$$

$$\mathbf{P} = \mathbf{F} \cdot \mathbf{S}, \qquad \boldsymbol{\tau} = \mathbf{F} \cdot \mathbf{S} \cdot \mathbf{F}^{T}, \qquad \boldsymbol{\sigma} = J^{-1} \cdot \mathbf{F} \cdot \mathbf{S} \cdot \mathbf{F}^{T}. \tag{57}$$

In the above equations and also in (19), the incremental form of the material gradient of the plastic displacement field is determined as follows:

$$\frac{\partial^{n+1,0}\mathbf{u}^{pl}}{\partial \mathbf{X}} = \Delta t \cdot \frac{\partial^{0}\dot{\mathbf{u}}^{pl}}{\partial \mathbf{X}} + \frac{\partial^{n,0}\mathbf{u}^{pl}}{\partial \mathbf{X}}, \tag{58}$$

where the material gradient of the plastic velocity field $\partial^{0}\dot{\mathbf{u}}pl/\partial\mathbf{X}$ is calculated in accordance with Eq. $(23)_1$ or Eq. (36), depending on the reference definition of the yield surface. The corresponding loading/unloading discrete Kuhn-Tucker plastic optimization conditions then take the forms:

$$\dot{\lambda} \geq 0, \qquad {}^{S}\Psi \leq 0, \qquad \dot{\lambda} \cdot {}^{S}\Psi = 0, \tag{59}$$

$$\dot{\lambda} \geq 0, \qquad {}^{P}\Psi \leq 0, \qquad \dot{\lambda} \cdot {}^{P}\Psi = 0, \tag{60}$$

$$\dot{\lambda} \geq 0, \qquad {}^{\tau}\Psi \leq 0, \qquad \dot{\lambda} \cdot {}^{\tau}\Psi = 0, \tag{61}$$

$$\dot{\lambda} \geq 0, \qquad {}^{\sigma}\Psi \leq 0, \qquad \dot{\lambda} \cdot {}^{\sigma}\Psi = 0. \tag{62}$$

2.4 The reference definition of the yield surface

Objective and thermodynamically consistent formulation of the plastic flow allows for the development of consistent material models. As a result, the material model can by formulated in any stress space and in whatever configuration of the body, though we intentionally omitted to give an example of the constitutive equation of our hypoelastic-plastic-based and hyperelastic-plastic-based material models in the intermediate configuration of the body, as it is just a matter of proper stress transformation using the multiplicative split of the deformation. Moreover, one of the formulations of the yield surface has to be a reference, from which other definitions of the yield surface in the second Piola-Kirchhoff, first Piola-Kirchhoff, Kirchhoff and Cauchy stress spaces can be calculated by solving Eq. (36). The various definitions of the yields surface then have the properties Eqs. (37)–(39), from which the 'normality rules' (Eqs. (38) and (39)) (whose physical meaning is defined by Eqs. (40) and (41)) are used in the return mapping/rate form of the return mapping algorithms to calculate the plastic multiplier. In this study the reference yield surface was defined in the first Piola-Kirchhoff stress space, because the corresponding plastic flow Eq. $(23)_1$ is the only not constrained. Then the generalised J_2 flow plasticity theory with isotropic hardening is defined by Eqs. (63)–(67). It should be noted that the $^P J_2(\mathbf{P}) = \mathbf{P} : \mathbf{P}$ invariant in the definition of the equivalent stress no longer bases on the deviatoric part of the first Piola-Kirchhoff stress tensor. The change was implied by the objectivity requirements, since the first Piola-Kirchhoff stress tensor transforms under the change of the observer as $\mathbf{P}^+ = \mathbf{Q}_R \cdot \mathbf{P}$ and $^P J_2(\mathbf{P})$ is the only invariant, which is not affected by the change, i.e. $^P J_2(\mathbf{P}) = {}^P J_2(\mathbf{P}^+)$, where $\mathbf{Q}_R$ is a rotating tensor expressing the relative rotation of the coordinate systems of an arbitrarily moving observer with respect to the reference coordinate system. The resulting yield surface is then no longer a cylinder but a sphere:

$$^P\Psi = {}^P\sigma_{eq} - {}^P\sigma_y \leq 0, \quad \text{where} \quad {}^P\sigma_{eq} = {}^P\sigma_{eq}(\mathbf{P}) = \sqrt{{}^P J_2(\mathbf{P})} = \sqrt{\mathbf{P} : \mathbf{P}}, \quad (63)$$

$$^P\sigma_y = F_{UT11} \cdot \sqrt{r^2 - [a \cdot e^{pl} - \text{center}]^2}, r = \sigma_y + Q, \quad \text{center} = \sqrt{r^2 - \sigma_y^2} \quad \text{and} \quad a = \frac{\text{center} + r}{b},$$

$$(64)$$

$$^{spring}\dot{e}^{pl} = {}^{spring}\dot{e}^{pl}\left(xx \cdot \frac{\partial^0 \dot{\mathbf{u}}^{pl}}{\partial \mathbf{X}}\right) = \sqrt{xx \cdot \frac{\partial^0 \dot{\mathbf{u}}^{pl}}{\partial \mathbf{X}} : xx \cdot \frac{\partial^0 \dot{\mathbf{u}}^{pl}}{\partial \mathbf{X}}} = xx \cdot \dot{\lambda},$$

$$^{spring}e^{pl} = \int_0^t {}^{spring}\dot{e}^{pl} \cdot dt = \int_0^t xx \cdot \dot{\lambda} \cdot dt, \quad \text{with} \quad xx \in \langle 0; 1 \rangle,$$

$$(65)$$

$$^{damper}\dot{e}^{pl} = {}^{damper}\dot{e}^{pl}\left((1-xx) \cdot \frac{\partial^0 \dot{\mathbf{u}}^{pl}}{\partial \mathbf{X}}\right) = \sqrt{(1-xx) \cdot \frac{\partial^0 \dot{\mathbf{u}}^{pl}}{\partial \mathbf{X}} : (1-xx) \cdot \frac{\partial^0 \dot{\mathbf{u}}^{pl}}{\partial \mathbf{X}}} = (1-xx) \cdot \dot{\lambda},$$

$$^{damper}e^{pl} = \int_0^t {}^{damper}\dot{e}^{pl} \cdot dt = \int_0^t (1-xx) \cdot \dot{\lambda} \cdot dt, \quad \text{with} \quad xx \in \langle 0; 1 \rangle,$$

$$(66)$$

$$\dot{e}^{pl} = \dot{e}^{pl}\left(\dot{\mathbf{F}}^{pl}\right) = \sqrt{\frac{\partial^0 \dot{\mathbf{u}}^{pl}}{\partial \mathbf{X}} : \frac{\partial^0 \dot{\mathbf{u}}^{pl}}{\partial \mathbf{X}}} = \dot{\lambda}, \quad e^{pl} = \int_0^t \dot{e}^{pl} \cdot dt = \int_0^t \dot{\lambda} \cdot dt, \quad \frac{\partial^0 \dot{\mathbf{u}}^{pl}}{\partial \mathbf{X}} = \dot{\lambda} \cdot \frac{\partial^P \Psi}{\partial \mathbf{X}}.$$

(67)

The actual yield stress $^P\sigma_y$, which is a first Piola-Kirchhoff stress measure, determines the radius of the yield surface and is defined by Eq. (64)$_1$. It is the only nonzero component of a stress tensor $\mathbf{P}_{UT}$ (i.e. $^P\sigma_y = P_{UT11} = [\mathbf{P}_{UT}]_{11}$) coming from a uniaxial tensile test of the modelled material, where the operator $[(\bullet)]_{11}$ extracts the element in the first row and the first column of a second-order tensor, $(\bullet)$, is written as a 3×3 square matrix. The corresponding deformation gradient and the Jacobian of deformation are denoted as $\mathbf{F}_{UT}, J_{UT}$, where $F_{UT11} = [\mathbf{F}_{UT}]_{11}$ and $J_{UT} = \det(\mathbf{F}_{UT})$. Please also note that the only nonzero element of the related second Piola-Kirchhoff stress tensor $\mathbf{S}_{UT}$, coming from the tensile test of the material, is $S_{UT11} = [\mathbf{S}_{UT}]_{11} = {}^S\sigma_y = \sqrt{r^2 - [a \cdot e^{pl} - \text{center}]^2}$. The equation defines an arc of a circle using three material parameters: the constant yield stress of the material σ_y, the maximum hardening stress Q by which the material can harden and the maximum accumulated plastic strain value $b = e^{pl}_{\max}$, at which the material loses its integrity, i.e. $^S\sigma_y = 0$. The relationship between the corresponding stress measures then can be written in tensor form as $\mathbf{P}_{UT} = \mathbf{F}_{UT} \cdot \mathbf{S}_{UT}$, in which the parameters σ_y, Q are second Piola-Kirchhoff stresses and $e^{pl} \subset \langle 0, b\rangle$. It should also be noted that the definitions of the accumulated plastic strain rate $\dot{e}^{pl}$ (an equivalent strain rate defined by Eq. (67)$_1$); the accumulated plastic strain e^{pl} (Eq. (67)$_2$), which controls the hardening and denotes the plastic damage; and the equivalent stress $^P\sigma_{eq}$ (Eq. (63)$_2$), respectively, which have the following physical meaning in the uniaxial tensile test of the material, $\dot{e}^{pl} = \partial^0 \dot{u}^{pl}_{UTX}/\partial X, e^{pl} = \partial^0 u^{pl}_{UTX}/\partial X, {}^P\sigma_{eq} = P_{UT11} = F_{UT11} \cdot S_{UT11}$, have changed. The changes were required by consistency conditions, so that the model could work properly in either a one-dimensional (1D) stress state or a three-dimensional (3D) stress state, which may occur at a material particle during the analysis. In the above equations, the variables $^{spring}e^{pl}$ (see Eq. (65)) and $^{damper}e^{pl}$ (see Eq. (66)) denote the ductile and brittle damage defined in terms of the ratio of ductile and total damage increment xx [21].

2.5 Calculation of the plastic multiplier

A thermodynamically consistent formulation of the plastic flow allows for the calculation of the plastic multiplier in whatever (second Piola-Kirchhoff, first Piola-Kirchhoff, Kirchhoff and Cauchy) stress spaces using the corresponding definition of the yield surface. There are altogether two types of return mapping procedures for plastic multiplier calculation, which result in a thermodynamically consistent material model.

The first type of return mapping procedure, which is best suited for hyperelastic-plastic-based materials, is the ordinary return mapping procedure. Its thermodynamically consistent form in terms of the normality rules is defined by Eq. (38). Here the equation of the yield surface is solved directly for the plastic multiplier value, which in the case of our hyperelastic-plastic material takes the form (see Eq. (63)$_1$)

$$^P\sigma_{eq} - {}^P\sigma_y = 0. \tag{68}$$

In order to show that the return mapping is thermodynamically consistent, the equivalent stress in the material model is manipulated as follows:

$$^P\sigma_{eq} = \sqrt{\mathbf{P} : \mathbf{P}} = \frac{\mathbf{P}}{\sqrt{\mathbf{P} : \mathbf{P}}} : \mathbf{P} = \frac{\partial^P\Psi}{\partial\mathbf{P}} : \mathbf{P}. \tag{69}$$

Then after multiplying Eq. (68) by the plastic multiplier $\dot{\lambda}$ and the infinitesimal volume element dV_0 in the initial configuration of the body, the first term of Eq. (68) becomes the internal plastic power at a material constituent of the model $\dot{\lambda} \cdot {}^P\sigma_{eq} \cdot dV_0 = \dot{\lambda} \cdot \partial^P\Psi/\partial\mathbf{P} : \mathbf{P} \cdot dV_0 = \partial^0\dot{\mathbf{u}}pl/\partial\mathbf{X} : \mathbf{P} \cdot dV_0 = dW^{pl}$. Similarly, the second term of Eq. (68) becomes $\dot{\lambda} \cdot {}^P\sigma_y \cdot dV_0 = \dot{e}^{pl} \cdot {}^P\sigma_y \cdot S_0 \cdot dX = \partial^0\dot{u}^{pl}_{UTX}/\partial X \cdot {}^P\sigma_y \cdot S_0 \cdot dX = dW^{pl}$, which is just the internal plastic power at a material constituent of the specimen coming from the tensile test of the material, where $S_0 = S_0(X)$ is the cross-sectional area of the specimen.

The second type of return mapping procedure, which is best suited for hypoelastic-plastic-based materials, is the rate form of the return mapping procedure. Its thermodynamically consistent form in terms of the normality rules is defined by Eq. (39). Here the objective time derivative of the yield surface is used for the plastic multiplier calculation, which in the case of our hypoelastic-plastic material model takes the form

$$\frac{\partial^P\Psi}{\partial\mathbf{P}} : \mathfrak{L}_{\mathrm{P}}(\mathbf{P}) - [\mathfrak{L}_{\mathrm{P}}(\mathbf{P}_{UT})]_{11} = 0, \tag{70}$$

where $\mathfrak{L}_{\mathrm{P}}(\mathbf{P})$ is then replaced by the rate form of the constitutive equation of the material Eq. (43) and the second term on the LHS of Eq. (70) by the form $[\mathfrak{L}_{\mathrm{P}}(\mathbf{P}_{UT})]_{11} = F_{UT11} \cdot \left\{ \left[-a \cdot (a \cdot e^{pl} - \mathrm{center}) \right] / \sqrt{r^2 - [a \cdot e^{pl} - \mathrm{center}]^2} \right\} \cdot \dot{e}^{pl}$. It should be noted that the first term of Eq. (70) can be replaced by any other term of Eq. (39), because the formulation is thermodynamically consistent. Moreover, multiplying Eq. (70) by the plastic multiplier $\dot{\lambda}$ and the infinitesimal volume element dV_0 in the initial configuration of the body, it is easy to see that by solving Eq. (70), the following conservation equation is enforced:

$$\frac{\partial^0\dot{\mathbf{u}}^{pl}}{\partial\mathbf{X}} : \mathfrak{L}_{\mathrm{P}}(\mathbf{P}) \cdot dV_0 - \frac{\partial^0\dot{u}^{pl}_{UTX}}{\partial X} \cdot [\mathfrak{L}_{\mathrm{P}}(\mathbf{P}_{UT})]_{11} \cdot S_0 \cdot dX = 0. \tag{71}$$

Therefore, both return mapping algorithms in the above result in such a plastic multiplier calculation, during which the internal plastic power density of the models becomes the same as the internal plastic power density of the specimen, coming from the uniaxial tensile test of the material. It should also be noted that Eq. (70) is just the objective and thermodynamically consistent rate form of the Eq. (68) resulting from the invariance of the internal mechanical power and its first derivative Eq. (27), which now defines not only conjugate pairs of stress measures and strain or deformation rates but also conjugate differentiation operators and derivatives. Moreover, using Eqs. (20)–(23) and (40) and considering the fact that the physical meaning of Eqs. (68)–(71) is the conservation of plastic energy, the

overall power at a material particle of the body dW can be decomposed into an elastic part dW^{el} and a plastic part dW^{pl} as follows:

$$dW = \dot{\mathbf{E}} : \mathbf{S} \cdot dV_0 = \frac{\partial^0 \dot{\mathbf{u}}}{\partial \mathbf{X}} : \mathbf{P} \cdot dV_0 = \mathbf{d} : \boldsymbol{\tau} \cdot dV_0 = \mathbf{d} : \boldsymbol{\sigma} \cdot dv = dW^{el} + dW^{pl}, \quad (72)$$

where

$$dW^{el} = \dot{\mathbf{E}}^{el} : \mathbf{S} \cdot dV_0 = \frac{\partial^0 \dot{\mathbf{u}}^{el}}{\partial \mathbf{X}} : \mathbf{P} \cdot dV_0 = \mathbf{d}^{el} : \boldsymbol{\tau} \cdot dV_0 = \mathbf{d}^{el} : \boldsymbol{\sigma} \cdot dv, \quad (73)$$

$$dW^{pl} = \dot{\lambda} \cdot \frac{\partial^S \Psi}{\partial \mathbf{S}} : \mathbf{S} \cdot dV_0 = \dot{\lambda} \cdot \frac{\partial^P \Psi}{\partial \mathbf{P}} : \mathbf{P} \cdot dV_0 = \dot{\lambda} \cdot \frac{\partial^\tau \Psi}{\partial \boldsymbol{\tau}} : \boldsymbol{\tau} \cdot dV_0 = \dot{\lambda} \cdot \frac{\partial^\sigma \Psi}{\partial \boldsymbol{\sigma}} : \boldsymbol{\sigma} \cdot dv, \quad (74)$$

which also proves that the formulation of the material model is thermodynamically consistent.

2.6 The ratio of ductile and total damage increment

The idea of the ratio of ductile and total damage increment xx was first introduced by Écsi and Élesztős in order to take into account the internal damping properly during plastic deformation of the hypoelastic-plastic material, where xx allowed for the redistribution of the plastic flow between the spring and the damper of a 1D frictional device representing the rheological model of the material [21]. The ratio is determined in an elastic corrector phase during return mapping, and its value is then kept constant. Since the return mapping procedure in our material model is carried out in the first Piola-Kirchhoff stress space, we had to modify the definition of the ratio as follows:

$$xx = \frac{\left\langle \mathbf{N} : \mathbf{F} \cdot \left({}^{mat}\mathbb{C}^{el} : \dot{\mathbf{E}} \right) \right\rangle}{\left\langle \mathbf{N} : \mathbf{F} \cdot \left({}^{mat}\mathbb{C}^{el} : \dot{\mathbf{E}} \right) \right\rangle + \left\langle \mathbf{N} : \mathbf{F} \cdot \left({}^{mat}\mathbb{C}^{vis} : \ddot{\mathbf{E}} \right) \right\rangle}, \quad (75)$$

where

$$\frac{\partial^P \Psi}{\partial \mathbf{P}} = \mathbf{N}, \qquad \mathbf{N} = \frac{\mathbf{P}}{\sqrt{\mathbf{P} : \mathbf{P}}} = \frac{\mathbf{P}}{\|\mathbf{P}\|}, \qquad \langle y \rangle = \frac{y + |y|}{2} \geq 0. \quad (76)$$

Eqs. (75) and (76) $\langle y \rangle$ denote the McCauly's brackets, which return zero if $y < 0$ and where we also used the transformation $\mathcal{L}_P(\mathbf{P}) = \mathbf{F} \cdot \dot{\mathbf{S}}$. Please also note that all terms of Eq. (75) are objective stress rates, so that the value of xx is not affected by the change of the observer.

3. Numerical experiment

In our numerical experiment a cantilever, size 50 mm × 50 mm × 600 mm was studied applying dynamic pressure on 1/3 of its upper surface near the cantilever free end. The loading was defined as a product of a constant p = 3 MPa pressure and the Heaviside step function. The analysis was run as dynamic using $\Delta t = 5 \cdot 10^{-6}$ s time step size. **Table 1** lists the used material properties, and **Figure 2** depicts the spatially discretized model of the cantilever.

	Hypoelastic-plastic material with damping	Hyperelastic-plastic material with/ without damping
E [Pa]	7.31×10^{10}	7.31×10^{10}
E^{vis} [Pa $\cdot$ s]	7.31×10^{2}	$7.31 \times 10^{2}/0.0$
$\nu = \nu^{vis}$ [-]	0.33	0.33
σ_y [Pa]	220.0×10^{6}	220.0×10^{6}
Q [Pa]	110.0×10^{6}	110.0×10^{6}
b [-]	10^{64}	10^{64}
ρ_0 [kg $\cdot$ m^{-3}]	2770.0	2770.0

Table 1.
Material properties.
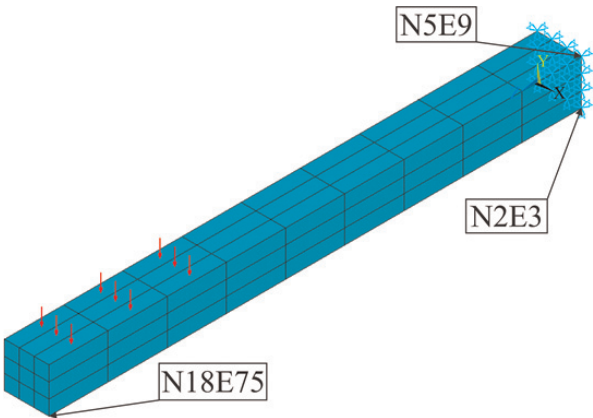
Figure 2.
The spatially discretized model.

In order to assess the value of the axial component of the deformation gradient coming from the tensile stress of the material F_{UT11}, we solved the one-dimensional (1D) rate form of the constitutive equation of the material (Eq. (42)) for the unknown component of the derivative of the axial elastic displacement field with respect to the axial material coordinate $\partial^0 u_x^{el}/\partial X$. The rate form of the constitutive equation of this specific 1D stress analysis, after neglecting the internal damping in the material, can be expressed in the following finite-strain form:

$$^S\dot{\sigma}_y = \dot{S}_{11} = E \cdot \left[\frac{\partial^0 \dot{u}_x^{el}}{\partial X} \cdot \left(1 + \frac{\partial^0 u_x^{el}}{\partial X} + \frac{\partial^0 u_x^{pl}}{\partial X} \right) \right], \qquad (77)$$

where $^S\dot{\sigma}_y = \dot{S}_{11}$ is the axial component of the second Piola-Kirchhoff stress rate tensor from the tensile test of the material and E is the Young's modulus. Furthermore considering that the accumulated plastic strain rate Eq. (67)$_1$ in this 1D stress state is $\dot{e}^{pl} = \partial^0 \dot{u}_x^{pl}/\partial X = \dot{\lambda}$, and that its integral is $e^{pl} = \partial^0 u_x^{pl}/\partial X$ (Eq. (67)$_2$), one can find F_{UT11} as a function of the accumulated plastic strain e^{pl} only in the following form:

$$F_{UT11} = 1 + \frac{\partial^0 u_x^{el}}{\partial X} + \frac{\partial^0 u_x^{pl}}{\partial X} = 1 + \left[-(1+e^{pl}) + \sqrt{(1+e^{pl})^2 + 2 \cdot \frac{^S\sigma y}{E}} \right] + e^{pl}, \quad (78)$$

where $^S\sigma y = {^S\sigma y}(e^{pl})$ see also Eq. (64)$_1$.

4. Numerical results

Figure 3 depicts a few selected analysis results. These are the first principal Cauchy stress and the accumulated plastic strain distributions over the body, which is similar in the case of both materials, the vertical displacement time history curves and the accumulated plastic strain time history curves for the used material at selected nodes (see **Figure 2** for the location of the nodes).

It should be noted that in order to avoid problems with convergence with the used hypoelastic-plastic material, the value of the b parameter had to be set extraordinarily high. As a result, the isotropic hardening curve becomes flat in the range of the accumulated plastic strain value that occurred in the analysis, i.e. no isotropic hardening took place in the analysis. Moreover, for the same reasons, a very small viscous damping parameter has to be used with the model. The viscous damping, however, did not affect the analysis result at this kind of intensive loading, resulting in the plastic collapse of the beam, which can be seen in the time history curves. The analysis results otherwise seem to be reasonable.
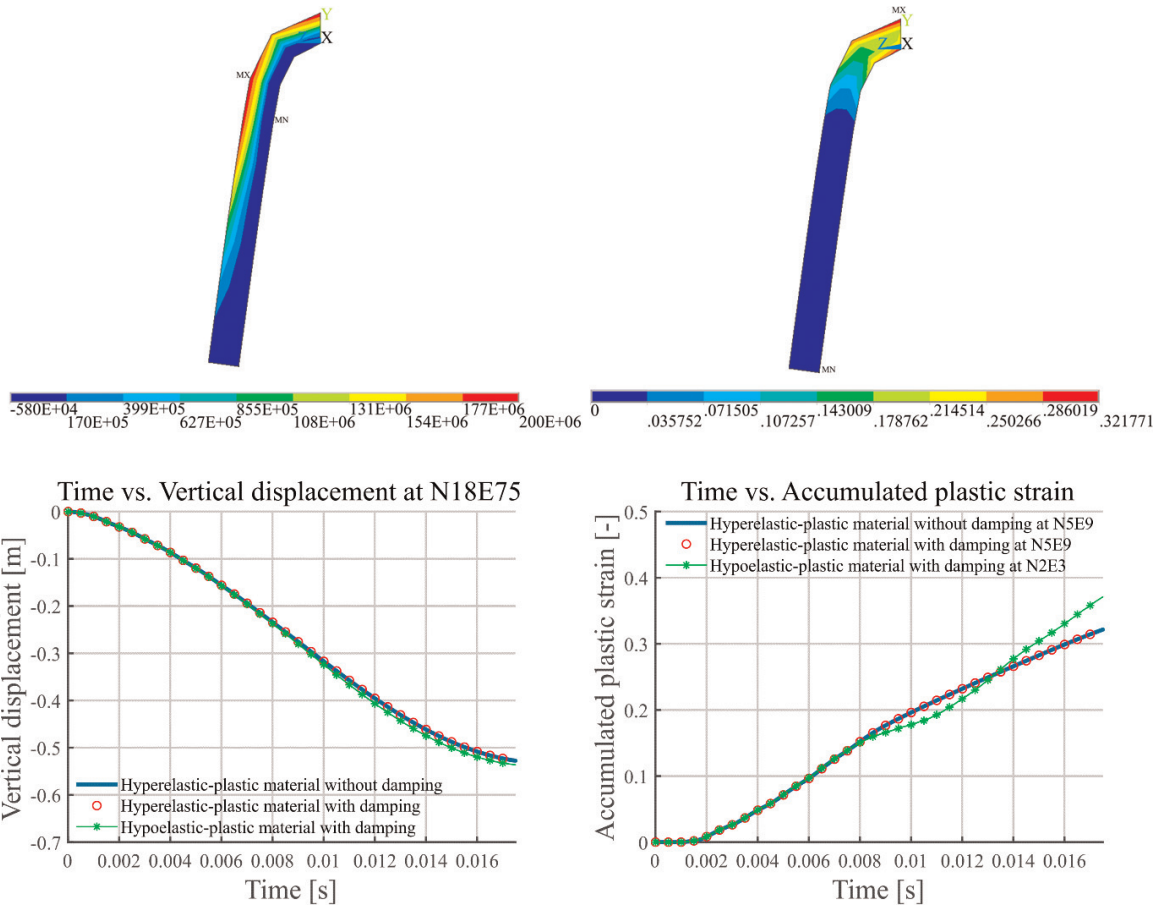

Figure 3.
Selected results from the analyses.

5. Conclusions

In this chapter an alternative framework for developing objective and thermo-dynamically consistent hypoelastic-plastic- and hyperelastic-plastic-based material models was presented using the first nonlinear continuum theory for finite deformations of elastoplastic media. The related material models were demonstrated in

numerical experiments. The most important implication of the presented theory is that the analysis results of the related models are no longer affected by the description and the particularities of the mathematical formulation. The nonlinear continuum theory was also briefly presented, while the thermodynamic consistency of the formulation was in detail discussed. Another important implication of the theory is that the dissipated plastic power density of the model can directly be related to the dissipated plastic power density of the specimen coming from the uniaxial tensile stress of the modelled material. Moreover, contemporary tensile testing for material parameter determination will also have to be extended by determination of the deformation gradient of the specimen of the modelled material, as it is an important entry for the presented material models.

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