Chapter

A Numerical Investigation on the Structure of the Zeros of the Q-Tangent Polynomials

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Abstract

We introduce q-tangent polynomials and their basic properties including q-derivative and q-integral. By using Mathematica, we find approximate roots of q-tangent polynomials. We also investigate relations of zeros between q-tangent polynomials and classical tangent polynomials.

Keywords: q-tangent polynomials, q-derivative, q-integral, Newton dynamical system, fixed point

2000 Mathematics Subject Classification: 11B68, 11B75, 12D10

1. Introduction

For a long time, studies on q-difference equations appeared in intensive works especially by F. H. Jackson [1, 2], R. D. Carmichael [3], T. E. Mason [4], and other authors [5–26]. An intensive and somewhat surprising interest in q-numbers appeared in many areas of mathematics and applications including q-difference equations, special functions, q-combinatorics, q-integrable systems, variational q-calculus, q-series, and so on. In this paper, we introduce some basic definitions and theorems (see [1–26]).

For any $n \in \mathbb{C}$, the q-number is defined by

$$[n]_q = \frac{1-q^n}{1-q}, \qquad |q| < 1. \tag{1}$$

Definition 1.1. [1, 2, 9, 13] The q-derivative operator of any function f is defined by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \qquad x \neq 0, \tag{2}$$

and $D_q f(0) = f'(0)$. We can prove that f is differentiable at 0, and it is clear that $D_q x^n = [n]_q x^{n-1}$.

Definition 1.2. [1, 2, 9, 13, 17] We define the q-integral as

$$\int_0^b f(x) d_q x = (1-q) b \sum_{j=0}^{\infty} q^j f\left(q^j b\right). \tag{3}$$

If this function, $f(x)$, is differentiable on the point x, the q-derivative in Definition 1.1 goes to the ordinary derivative in the classical analysis when $q \to 1$.

Definition 1.3. [5, 17, 18, 21] The Gaussian binomial coefficients are defined by

$$\binom{m}{r}_q = \begin{bmatrix} m \\ r \end{bmatrix}_q = \begin{cases} 0 & \text{if } r > m \\ \dfrac{(1-q^m)(1-q^{m-1})\cdots(1-q^{m-r+1})}{(1-q)(1-q^2)\cdots(1-q^r)} & \text{if } r \le m \end{cases}, \tag{4}$$

where m and r are non-negative integers. For $r = 0$ the value is 1 since the numerator and the denominator are both empty products. Like the classical binomial coefficients, the Gaussian binomial coefficients are center-symmetric. There are analogues of the binomial formula, and this definition has a number of properties.

Theorem 1.4. Let n, k be non-negative integers. Then we get.

i. $$\prod_{k=0}^{n-1}\left(1+q^k t\right) = \sum_{k=0}^{n} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q t^k, \tag{5}$$

ii. $$\prod_{k=0}^{n-1} \frac{1}{(1-q^k t)} = \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q t^k.$$

Definition 1.5. [5, 26] Let z be any complex number with $|z| < 1$. Two forms of q-exponential functions are defined by

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!}, \quad e_{q^{-1}}(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_{q^{-1}}!} = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{z^n}{[n]_q!}. \tag{6}$$

Bernoulli, Euler, and Genocchi polynomials have been studied extensively by many mathematicians(see [22–25]). In 2013, C. S. Ryoo introduced tangent polynomials and he developed several properties of these polynomials (see [22, 23]). The tangent numbers are closely related to Euler numbers.

Definition 1.6. [22–25] Tangent numbers T_n and tangent polynomials $T_n(x)$ are defined by means of the generating functions

$$\begin{aligned} \sum_{n=0}^{\infty} T_n \frac{t^n}{n!} &= \frac{2}{e^{2t}+1} = 2\sum_{m=0}^{\infty}(-1)^m e^{2mt}, \\ \sum_{n=0}^{\infty} T_n \frac{t^n}{n!} &= \frac{2}{e^{2t}+1} e^{tx} = 2\sum_{m=0}^{\infty}(-1)^m e^{(2m+x)t}. \end{aligned} \tag{7}$$

Theorem 1.7. For any positive integer n, we have

$$T_n(x) = (-1)^n T_n(2-x). \tag{8}$$

Theorem 1.8. For any positive integer $m(=odd)$, we have

$$T_n(x) = m^n \sum_{i=0}^{m-1} (-1)^i T_n\left(\frac{2i+x}{m}\right), \quad n \in \mathbb{Z}_+. \tag{9}$$

Theorem 1.9. For $n \in \mathbb{Z}_+$, we have

$$T_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} T_k(x) y^{n-k}. \tag{10}$$

The main aim of this paper is to extend tangent numbers and polynomials, and study some of their properties. Our paper is organized as follows: In Section 2, we define q-tangent polynomials and find some properties of these polynomials. We consider q-tangent polynomials in two parameters and establish some relations between q-tangent polynomials and q-Euler or Bernoulli polynomials. In Section 3, we observe approximate roots distributions of q-tangent polynomials and demonstrate interesting phenomenon.

2. Some properties of the q-tangent polynomials

In this section we define the q-tangent numbers and polynomials and establish some of their basic properties. we shall also study the q-tangent polynomials involving two parameters. We shall find some important relations between these polynomials and q-other polynomials.

Definition 2.1. For $x, q \in \mathbb{C}$, we define q-tangent polynomials as

$$\sum_{n=0}^{\infty} \mathcal{T}_{n,q}(x) \frac{t^n}{[n]_q!} = \frac{[2]_q}{e_q(2t)+1} e_q(tx), \qquad |t| < \frac{\pi}{2}. \tag{11}$$

From Definition 2.1, it follows that

$$\sum_{n=0}^{\infty} \mathcal{T}_{n,q}(0) \frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} \mathcal{T}_{n,q} \frac{t^n}{[n]_q!} = \frac{[2]_q}{e_q(2t)+1}, \tag{12}$$

where $\mathcal{T}_{n,q}$ is q-tangent number. If $q \rightarrow 1$, then it reduces to the classical tangent polynomial(see [22–25]).

Theorem 2.2. Let $x, q \in \mathbb{C}$. Then, the following hold.

$$\text{i. } \mathcal{T}_{n,q} + \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q 2^{n-k} \mathcal{T}_{k,q} = \begin{cases} [2]_q & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}, \tag{13}$$

$$\text{ii. } \mathcal{T}_{n,q}(x) + \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q 2^{n-k} \mathcal{T}_{k,q}(x) = [2]_q x^n.$$

Proof. From the Definition 2.1, we have

$$\begin{aligned} [2]_q &= \left(1 + e_q(2t)\right) \sum_{n=0}^{\infty} \mathcal{T}_{n,q} \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\mathcal{T}_{n,q} + \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q 2^{n-k} \mathcal{T}_{k,q} \right) \frac{t^n}{n!}. \end{aligned} \tag{14}$$

Now comparing the coefficients of t^n we find (i). For (ii) we use the relation

$$\begin{aligned} [2]_q e_q(tx) &= \left(1 + e_q(2t)\right) \sum_{n=0}^{\infty} \mathcal{T}_{n,q}(x) \frac{t^n}{[n]_q!} \\ &= \sum_{n=0}^{\infty} \left(\mathcal{T}_{n,q}(x) + \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q 2^{n-k} \mathcal{T}_{k,q}(x) \right) \frac{t^n}{n!}, \end{aligned} \tag{15}$$

and again compare the coefficients of t^n. □

Theorem 2.3. Let n be a non-negative integer. Then, the following holds

$$\mathcal{T}_{n,q}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{n-k,q} x^k. \tag{16}$$

Proof. From the definition of the q-exponential function, we have

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{T}_{n,q}(x) \frac{t^n}{[n]_q!} &= \frac{[2]_q}{e_q(2t)+1} e_q(tx) = \sum_{n=0}^{\infty} \mathcal{T}_{n,q} \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} x^n \frac{t^n}{[n]_q!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{n-k,q}(x) x^k \right) \frac{t^n}{[n]_q!}. \end{aligned} \tag{17}$$

The required relation now follows on comparing the coefficients of t^n on both sides. □

Theorem 2.4. Let n be a non-negative integer. Then, the following holds

$$\mathcal{T}_{n,q} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^{n-k} q^{\binom{n-k}{2}} \mathcal{T}_{k,q}(x) x^{n-k}. \tag{18}$$

Proof. From the property of q-exponential function, it follows that

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{T}_{n,q} \frac{t^n}{[n]_q!} &= \frac{[2]_q}{e_q(2t)+1} e_q(tx) e_{q^1}(-tx) \\ &= \sum_{n=0}^{\infty} \mathcal{T}_{n,q}(x) \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} q^{\binom{n}{2}} (-1)^n x^n \frac{t^n}{[n]_q!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^{n-k} q^{\binom{n-k}{2}} \mathcal{T}_{k,q}(x) x^{n-k} \right) \frac{t^n}{[n]_q!}. \end{aligned} \tag{19}$$

The required relation now follows immediately. □

In what follows, we consider q-derivative of $e_q(tx)$. Using the Mathematical Induction, we find.

$$\text{i. } k=1: \quad D_q^{(1)} e_q(tx) = \sum_{n=1}^{\infty} x^{n-1} \frac{t^n}{[n-1]_q!}. \tag{20}$$

$$\text{ii. } k=i: \quad D_q^{(i)} e_q(tx) = \sum_{n=i}^{\infty} x^{n-i} \frac{t^n}{[n-i]_q!}.$$

If (ii) is true, then it follows that.

$$\begin{aligned} \text{iii. } k=i+1: \quad D_q^{(i+1)} e_q(tx) &= D_{q;x}^{(1)} \left(\sum_{n=i}^{\infty} x^{n-i} \frac{t^n}{[n-i]_q!} \right) \\ &= \sum_{n=i+1}^{\infty} x^{n-(i+1)} \frac{t^n}{[n-(i+1)]_q!} \\ &= t^{i+1} e_q(tx). \end{aligned} \tag{21}$$

We are now in the position to prove the following theorem.

Theorem 2.5. For $k \in \mathbb{N}$, the following holds

$$D_q^{(k)} \mathcal{T}_{n,q}(x) = \frac{[n]_q!}{[n-k]_q!} \mathcal{T}_{n-k,q}(x). \tag{22}$$

Proof. Considering q-derivative of $e_q(tx)$, we find

$$\begin{aligned} D_q^{(i+1)} \sum_{n=0}^{\infty} \mathcal{T}_{n,q}(x) \frac{t^n}{[n]_q!} &= \sum_{n=0}^{\infty} D_q^{(i+1)} \mathcal{T}_{n,q}(x) \frac{t^n}{[n]_q!} \\ &= \frac{[2]_q}{e_q(2t)+1} D_q^{(i+1)} e_q(tx) \\ &= t^{i+1} \frac{[2]_q}{e_q(2t)+1} e_q(tx) \\ &= \sum_{n=0}^{\infty} [n+(i+1)]_q \cdots [n+2]_q [n+1]_q \\ &\times \mathcal{T}_{n,q}(x) \frac{t^{n+i+1}}{[n+(i+1)]_q!} \\ &= \sum_{n=0}^{\infty} \frac{[n]_q}{[n+(i+1)]_q!} \mathcal{T}_{n-(i+1),q}(x) \frac{t^n}{[n]_q!}, \end{aligned} \tag{23}$$

which immediately gives the required result. □

Theorem 2.6. Let a, b be any real numbers. Then, we have

$$\int_a^b \mathcal{T}_{n,q}(x) d_q x = \sum_{k=0}^{n+1} \frac{1}{[n+1]_q} \left(\mathcal{T}_{n+1,q}(b) - \mathcal{T}_{n+1,q}(a) \right). \tag{24}$$

Proof. From Theorem 2.3, we find

$$\begin{aligned} \int_a^b \mathcal{T}_{n,q}(x) d_q x &= \int_a^b \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{k,q} x^{n-k} d_q x \\ &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{k,q} \frac{1}{[n-k+1]_q} x^{n-k+1} \Bigg|_a^b \\ &= \sum_{k=0}^{n+1} \frac{\mathcal{T}_{n+1,q}(b) - \mathcal{T}_{n+1,q}(a)}{[n+1]_q}. \end{aligned} \tag{25}$$

□

Definition 2.7. For $x, y \in \mathbb{C}$, we define q-tangent polynomial with two parameters as

$$\sum_{n=0}^{\infty} \mathcal{T}_{n,q}(x,y) \frac{t^n}{[n]_q!} = \frac{[2]_q}{e_q(2t)+1} e_q(tx) e_q(ty), \qquad |t| < \frac{\pi}{2}. \tag{26}$$

From the Definition 2.7, it is clear that

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{T}_{n,q}(x,0)\frac{t^n}{[n]_q!} &= \sum_{n=0}^{\infty}\mathcal{T}_{n,q}(x)\frac{t^n}{[n]_q!} = \frac{[2]_q}{e_q(2t)+1}e_q(tx),\\
\sum_{n=0}^{\infty}\mathcal{T}_{n,q}(0,0)\frac{t^n}{[n]_q!} &= \sum_{n=0}^{\infty}\mathcal{T}_{n,q}\frac{t^n}{[n]_q!} = \frac{[2]_q}{e_q(2t)+1},
\end{aligned} \tag{27}$$

where $\mathcal{T}_{n,q}$ is q-tangent number. We also note that the original tangent number, $\mathcal{T}_n$,

$$\lim_{q\to 1}\sum_{n=0}^{\infty}\mathcal{T}_{n,q}\frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty}T_n\frac{t^n}{n!} = \frac{2}{e^{2t}+1}, \tag{28}$$

where $q \to 1$.

Theorem 2.8. Let x, y be any complex numbers. Then, the following hold.

$$\text{i. } \mathcal{T}_{n,q}(x,y) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{n-k,q}(x)y^k, \tag{29}$$

$$\text{ii. } \mathcal{T}_{n,q}(x,y) = \sum_{l=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{n-l,q}\sum_{k=0}^{l}\begin{bmatrix} l \\ k \end{bmatrix}_q x^{l-k}y^k.$$

Proof. From the Definition 2.7, we have

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{T}_{n,q}(x,y)\frac{t^n}{[n]_q!} &= \frac{[2]_q}{e_q(2t)+1}e_q(tx)e_q(ty)\\
&= \sum_{n=0}^{\infty}\mathcal{T}_{n,q}(x)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}y^n\frac{t^n}{[n]_q!}.
\end{aligned} \tag{30}$$

Using Cauchy's product and the method of coefficient comparison in the above relation, we find (i). Next, we transform q-tangent polynomials in two parameters as

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{T}_{n,q}(x,y)\frac{t^n}{[n]_q!} &= \frac{[2]_q}{e_q(2t)+1}e_q(tx)e_q(ty)\\
&= \sum_{n=0}^{\infty}\mathcal{T}_{n,q}\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}x^n\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}y^n\frac{t^n}{[n]_q!}.
\end{aligned} \tag{31}$$

Now following same procedure as in (i), we obtain (ii). □

Theorem 2.9. Setting $y = 2$ in q-tangent polynomials with two parameters, the following relation holds

$$[2]_q x^n = \mathcal{T}_{n,q}(x,2) + \mathcal{T}_{n,q}(x). \tag{32}$$

Proof. Using q-tangent polynomials and its polynomials with two parameters, we have

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{T}_{n,q}(x,2)\frac{t^n}{[n]_q!} + \sum_{n=0}^{\infty}\mathcal{T}_{n,q}(x)\frac{t^n}{[n]_q!} &= \frac{[2]_q e_q(2t)}{e_q(2t)+1}e_q(tx) + \frac{[2]_q}{e_q(2t)+1}e_q(tx)\\
&= [2]_q e_q(tx)
\end{aligned} \tag{33}$$

Now from the definition of q-exponential function, the required relation follows. □

Theorem 2.9 is interesting as it leads to the relation

$$x^n = \frac{\mathcal{T}_{n,q}(x,2) + \mathcal{T}_{n,q}(x)}{[2]_q}. \tag{34}$$

Theorem 2.10. Let $|q| < 1$. Then, the following holds

$$\mathcal{T}_{n,q}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k \mathcal{T}_{k,\frac{1}{q}}(2) x^{n-k}. \tag{35}$$

Proof. To prove the relation, we note that

$$e_{\frac{1}{q}}(-2t) = \mathcal{E}_q(-2t), \tag{36}$$

where $\mathcal{E}_q(t) = e_{q^{-1}}(t)$. Using the above equation we can represent the q-tangent polynomials as

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathcal{T}_{n,q}(x) \frac{t^n}{[n]_q!} &= \frac{[2]_q}{e_q(2t)+1} e_q(tx) \\
&= \frac{[2]_q}{1+\mathcal{E}_q(-2t)} \mathcal{E}_q(-2t) e_q(tx) \\
&= \frac{[2]_q}{e_{\frac{1}{q}}(-2t)+1} e_{\frac{1}{q}}(-2t) e_q(tx) \\
&= \sum_{n=0}^{\infty} \mathcal{T}_{n,\frac{1}{q}}(2) \frac{(-t)^n}{[n]_q!} \sum_{n=0}^{\infty} x^n \frac{t^n}{[n]_q!} \\
&= \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k T_{k,\frac{1}{q}}(2) x^{n-k} \right\} \frac{t^n}{[n]_q!},
\end{aligned} \tag{37}$$

which leads to the required relation immediately. □

Now we shall find relations between q-tangent polynomials and others polynomials. For this, first we introduce well known polynomials by using q-numbers.

Definition 2.11. We define q-Euler polynomials, $E_{n,q}(x)$, and q-Bernoulli polynomials, $B_{n,q}(x)$, as

$$\begin{aligned}
\sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{[n]_q!} &= \frac{[2]_q}{e_q(t)+1} e_q(tx), \qquad |t| < \pi, \\
\sum_{n=0}^{\infty} B_{n,q}(x) \frac{t^n}{[n]_q!} &= \frac{t}{e_q(t)-1} e_q(tx), \qquad |t| < 2\pi.
\end{aligned} \tag{38}$$

Theorem 2.12. For $x, y \in \mathbb{C}$, the following relation holds

$$\mathcal{T}_{n,q}(x,y) = \frac{1}{[2]_q} \sum_{l=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \left(\frac{\mathcal{T}_{n-l,q}(x)}{m^l} + \sum_{k=0}^{n-l} \begin{bmatrix} n-l \\ k \end{bmatrix}_q \frac{\mathcal{T}_{k,q}(x)}{m^{n-k}} \right) E_{l,q}(my). \tag{39}$$

Proof. Transforming q-tangent polynomials containing two parameters, we find

$$\frac{[2]_q}{e_q(2t)+1}e_q(tx)e_q(ty) = \left(\frac{[2]_q}{e_q\left(\frac{t}{m}\right)+1}e_q(ty)\right)\left(\frac{e_q\left(\frac{t}{m}\right)+1}{[2]_q}\right)\left(\frac{[2]_q}{e_q(2t)+1}e_q(tx)\right). \quad (40)$$

Thus, for the relation between q-tangent polynomials of two parameters and q-Euler polynomials, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\mathcal{T}_{n,q}(x,y)\frac{t^n}{[n]_q!} \\
&= \sum_{n=0}^{\infty}E_{n,q}(my)\frac{t^n}{m^n[n]_q!}\sum_{n=0}^{\infty}\mathcal{T}_{n,q}(x)\frac{t^n}{[n]_q!}\left(\sum_{n=0}^{\infty}\frac{1}{[2]_q}\frac{t^n}{m^n[n]_q!}+\frac{1}{[2]_q}\right) \\
&= \frac{1}{[2]_q}\sum_{n=0}^{\infty}\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q E_{l,q}(my)\sum_{k=0}^{n-l}\begin{bmatrix} n-l \\ k \end{bmatrix}_q \frac{\mathcal{T}_{k,q}(x)}{m^{n-k}}\frac{t^n}{[n]_q!} \\
&+ \frac{1}{[2]_q}\sum_{n=0}^{\infty}\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q E_{l,q}(my)\frac{\mathcal{T}_{n-l,q}(x)}{m^l}\frac{t^n}{[n]_q!},
\end{aligned} \quad (41)$$

which on comparing the coefficients immediately gives the required relation. □

Corollary 2.13. From Theorem 2.12, the following hold.

i. $$\mathcal{T}_{n,q}(x,y) = \frac{1}{[2]_q}\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q\left(\frac{\mathcal{T}_{n-l,q}(x)}{m^l}+\sum_{k=0}^{n-l}\begin{bmatrix} n-l \\ k \end{bmatrix}_q\frac{\mathcal{T}_{k,q}(x)}{m^{n-k}}\right)E_{l,q}(my). \quad (42)$$

ii. $$\mathcal{T}_n(x,y) = \frac{1}{2}\sum_{l=0}^{n}\binom{n}{l}\left(\frac{\mathcal{T}_{n-l}(x)}{m^l}+\sum_{k=0}^{n-l}\binom{n-l}{k}\frac{\mathcal{T}_k(x)}{m^{n-k}}\right)E_l(my).$$

Theorem 2.14. For $x, y \in \mathbb{C}$, the following relation holds

$$\mathcal{T}_{n-1,q}(x,y) = \frac{1}{[n]_q}\sum_{l=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q\left(\sum_{k=0}^{n-l}\begin{bmatrix} n-l \\ k \end{bmatrix}_q\frac{\mathcal{T}_{k,q}(x)}{m^{n-k}}-\frac{\mathcal{T}_{n-l,q}(x)}{m^l}\right)B_{l,q}(my). \quad (43)$$

Proof. We note that

$$\frac{[2]_q}{e_q(2t)+1}e_q(tx)e_q(ty) = \left(\frac{t}{e_q\left(\frac{t}{m}\right)-1}e_q(ty)\right)\left(\frac{e_q\left(\frac{t}{m}\right)-1}{t}\right)\left(\frac{[2]_q}{e_q(2t)+1}e_q(tx)\right). \quad (44)$$

Thus as in Theorem 2.12, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\mathcal{T}_{n,q}(x,y)\frac{t^n}{[n]_q!} \\
&= \left(\sum_{n=0}^{\infty}\frac{t^{n-1}}{m^n[n]_q!}-\frac{1}{t}\right)\sum_{n=0}^{\infty}B_{n,q}(my)\frac{t^n}{m^n[n]_q!}\sum_{n=0}^{\infty}\mathcal{T}_{n,q}(x)\frac{t^n}{[n]_q!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q\sum_{k=0}^{n-l}\begin{bmatrix} n-l \\ k \end{bmatrix}_q\frac{\mathcal{T}_{k,q}(x)}{m^{n-k}}B_{l,q}(my)\right)\frac{t^{n-1}}{[n]_q!} \\
&-\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q\frac{\mathcal{T}_{n-l,q}(x)}{m^l}B_{l,q}(my)\right)\frac{t^{n-1}}{[n]_q!}.
\end{aligned} \quad (45)$$

DOI: http://dx.doi.org/10.5772/intechopen.83497

The required relation now follows on comparing the coefficients. □

Corollary 2.15. From the Theorem 2.14, the following relations hold.

i. $$\mathcal{T}_{n-1,q}(x,y) = \frac{1}{[n]_q}\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q \left(\sum_{k=0}^{n-l}\begin{bmatrix} n-l \\ k \end{bmatrix}_q \frac{\mathcal{T}_{k,q}(x)}{m^{n-k}} - \frac{\mathcal{T}_{n-l,q}(x)}{m^l}\right) B_{l,q}(my). \quad (46)$$

ii. $$\mathcal{T}_{n-1}(x,y) = \frac{1}{n}\sum_{l=0}^{n}\binom{n}{l}\left(\sum_{k=0}^{n-l}\binom{n-l}{k}\frac{\mathcal{T}_k(x)}{m^{n-k}} - \frac{\mathcal{T}_{n-l}(x)}{m^l}\right) B_l(my).$$

3. The observation of scattering zeros of the q-tangent polynomials

In this section, we will find the approximate structure and shape of the roots according to the changes in n and q. We will extend this to identify the fixed points and try to understand the structure of the composite function using the Newton method.

The first five q-tangent polynomials are:

$$\begin{aligned}
\mathcal{T}_{0,q}(x) &= \frac{1+q}{2}, \\
\mathcal{T}_{1,q}(x) &= \frac{1}{2}(1+q)(-1+x), \\
\mathcal{T}_{2,q}(x) &= \frac{1}{2}(1+q)\left(1+q(-1+x)+x-x^2\right), \\
\mathcal{T}_{3,q}(x) &= \frac{1}{2}(1+q)\left(-1+q(2-(-2+q)q)-x+q^3x-\left(1+q+q^2\right)x^2+x^3\right), \\
\mathcal{T}_{4,q}(x) &= \frac{1}{2}(1+q)((-1+q)(1+q)(1+(-4+q)q)\left(1+q+q^2\right) \\
&\quad -(1+q)^2(1+(-3+q)q)(1+q^2)x \\
&\quad +(-1+q)(1+q^2)(1+q+q^2)x^2-(1+q)(1+q^2)x^3+x^4).
\end{aligned} \quad (47)$$

Using Mathematica, we will examine the approximate movement of the roots. In **Figure 1**, the x-axis means the numbers of real zeros and the y-axis means the numbers of complex zeros in the q-tangent polynomials. When it moves from left to right, it changes to n = 30, 40, 50, and when it is fixed at q = 0.1, the approximate shape of the root appears to be almost circular. The center is identified as the origin, and it has 2.0 as an approximate root, which is unusual.

Figure 2 shows the shape of the approximate roots when n is changed to the above conditions and fixed at q = 0.5.

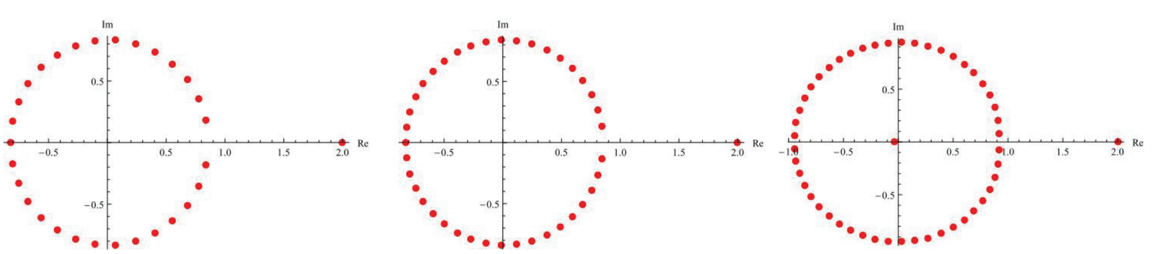

Figure 1.
Zeros of $\mathcal{T}_{n,0.1}(x)$ for n = *30, 40, 50.*

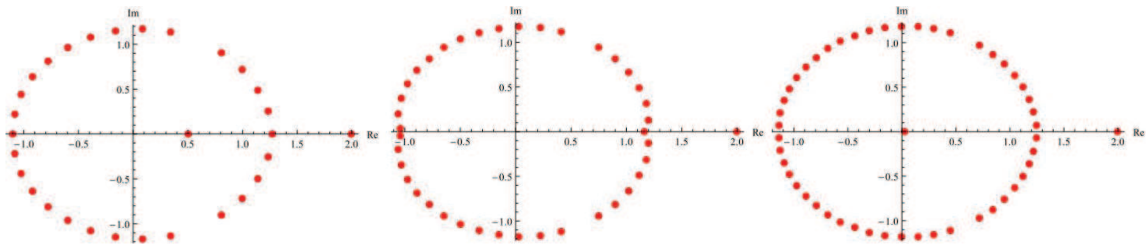

Figure 2.
Zeros of $T_{n,0.5}(x)$ *for* n = *30, 40, 50.*

In **Figure 2**, the shape of the root changes to an ellipse, unlike the $q = 0.1$ condition, and the widening phenomenon appears when the real number is 0.5. In addition, like the previous **Figure 1**, we can see that it has a common approximate root at 2.0. In the following **Figure 3**, n of the far-left figure is 30, and it increases by 10 while moving to the right, and the far-right figure shows the shape of the root when $n = 50$ and is fixed at $q = 0.9$.

In **Figure 3**, the roots have a general tangent polynomial shape with similar properties (see [22–25]). If each approximate root obtained in the previous step is piled up according to the value of n, it will appear as shown in **Figure 4**. The left **Figure 4** is $q = 0.1$ with n from 1 to 50. The middle **Figure 4** is $q = 0.5$ with n from 1 to 50. The right **Figure 4** is $q = 0.9$ with n from 1 to 50.

Let $f : D \to D$ be a complex function, with D as a subset of $\mathbb{C}$. We define the iterated maps of the complex function as the following:

$$f_r : z_0 \mapsto \underbrace{f(f(\cdots(f}_{r}(z_0)\cdots))) \tag{48}$$

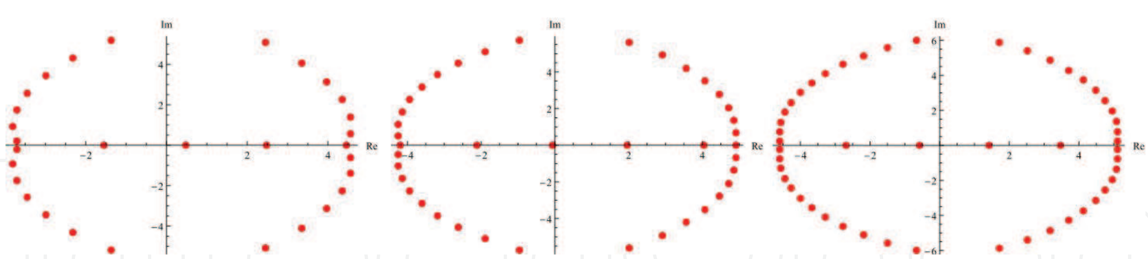

Figure 3.
Zeros of $T_{n,0.9}(x)$ *for* n = *30, 40, 50.*

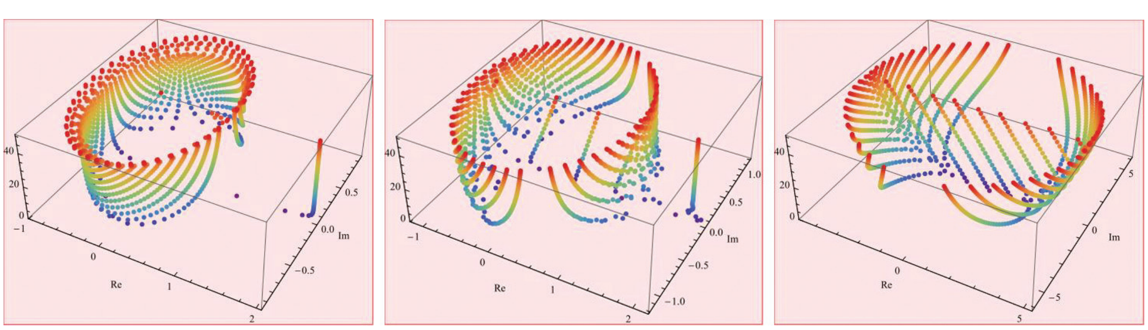

Figure 4.
Zeros of $T_{n,q}(x)$ *for* q = *0.1, 0.5, 0.9, 1* ≤ n ≤ *50.*

The iterates of f are the functions $f, f \circ f, f \circ f \circ f$, ..., which are denoted f^1, f^2, f^3, ... If $z \in \mathbb{C}$, and then the orbit of z_0 under f is the sequence $< z_0, f(z_0), f(f(z_0)), \cdots >$.

We consider the Newton's dynamical system as follows [12, 15, 20]:

$$\left\{ \mathbb{C}_\infty : R(x) = x - \frac{\mathcal{T}(x)}{\mathcal{T}'(x)} \right\}. \tag{49}$$

R is called the Newton iteration function of $\mathcal{T}$. It can be considered that the fixed points of R are the zeros of $\mathcal{T}$ and all the fixed points of R are attracting. R may also have one or more attracting cycles.

For $x \in \mathbb{C}$, we consider $\mathcal{T}_{4,q}(x)$, and then this polynomial has four distinct complex numbers, $a_i (i = 1, 2, 3, 4)$ such that $\mathcal{T}_{4,q}(a_i) = 0$. Using a computer, we obtain the approximate zeros (**Table 1**) as follows:

In Newton's method, the generalized expectation is that a typical orbit $\{R(x)\}$ will converge to one of the roots of $\mathcal{T}_{4,q}(x)$ for $x_0 \in \mathbb{C}$. If we choose x_0, which is sufficiently close to a_i, then this proves that

$$\lim_{r \to \infty} R(x_0) = a_i, \text{ for } i = 1, 2, 3, 4. \tag{50}$$

When it is given a point x_0 in the complex plane, we want to determine whether the orbit of x_0 under the action of $R(x)$ converges to one of the roots of the equation. The orbit of x_0 under the action of R also appears by calculating until 30 iterations or the absolute difference value of the last two iterations is within 10^{-6}.

The output in **Figure 5** is the last calculated orbit value. We construct a function, which assigns one of four colors for each point according to the outcome of R in the plane. If an orbit of x_0 for $q = 0.1$ converges to -0.672809, $-0.0821877 - 0.710388i$, $-0.0821877 + 0.710388i$ and 1.94818, then we denote the red, blue, yellow, and sky-blue, respectively(the left figure). For example, the yellow region for the left figure represents the part of the basin of attraction of $a_3 = -0.0821877 + 0.710388i$.

i	$q = 0.1$	$q = 0.5$	$q = 0.9$
1	-0.672809	$-0.581881 - 0.412941i$	-1.10249
2	$-0.0821877 - 0.710388i$	$-0.581881 + 0.412941i$	-0.158841
3	$-0.0821877 + 0.710388i$	0.907024	1.84004
4	1.94818	2.13174	2.86029

Table 1.
Approximate zeros of $\mathcal{T}_{4,q}(x)$.

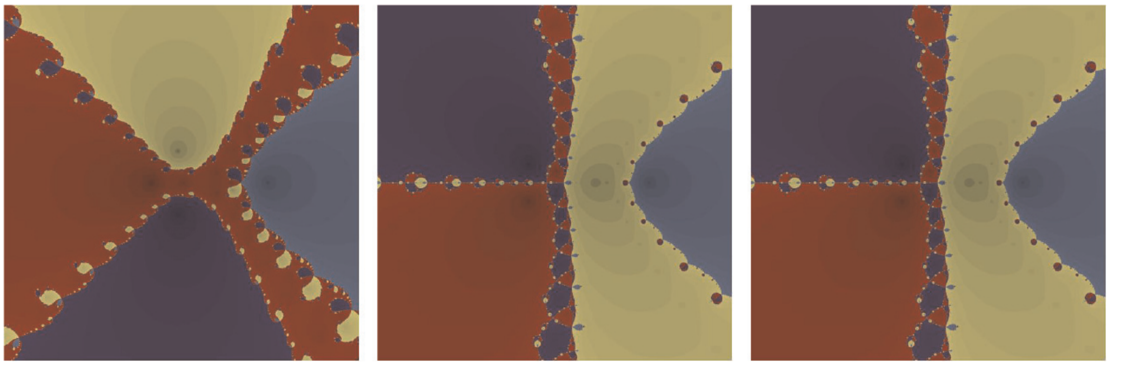

Figure 5.
Orbit of x_0 under the action of R for $\mathcal{T}_{4,q}(x)$ for q = 0.1, 0.5, 0.9.

If we use $\mathcal{T}_{3,0.1}(x)$ to draw a figure using the Newton method, we can obtain **Figure 6**. The picture on the left shows three roots, and the colors are blue, red, and ivory in the counterclockwise direction. When we examine the area closely, we can see that it converges to an approximate value in each color area. The convergence value in the blue area is $-0.379202 + 0.523651i$, that in the red area is $-0.379202 - 0.523651i$, and that in the ivory area is 1.8684. We can also see that it shows self-similarity at the boundary point as divided into three areas. The figure on the right is obtained by 2-times iterated q-tangent polynomials, $\mathcal{T}^2_{3,0.1}(x)$, and the area is divided into nine colors "gray ($x = 2.31831$), scarlet ($x = 1.76736 + 0.216319i$), light brown ($x = 0.137247 + 0.59473i$), sky blue ($x = -0.604153 + 1.19884i$), blue ($x = -0.794606 + 0.378411i$), red ($x = -0.794606 - 0.378411i$), ivory ($x = -0.604153 - 1.19884i$), green ($x = 0.137247 - 0.59473i$), and navy blue ($x = 1.76736 - 0.216319i$) in the counterclockwise direction. This also shows self-similarity at the boundary.

In **Figure 7**, we express the coloring for $\mathcal{T}^2_{3,0.1}(x)$.

Conjecture 3.1. The q-tangent polynomials always have self-similarity at the boundary.

We know that the fixed point is divided as follows. Suppose that the complex function f is analytic in a region D of $\mathbb{C}$, and f has a fixed point at $z_0 \in D$. Then z_0 is said to be (see [6, 16, 20]):

an attracting fixed point if $|f'(z_0)| < 1$;
a repelling fixed point if $|f'(z_0)| > 1$;
a neutral fixed point if $|f'(z_0)| = 1$.

For example, $\mathcal{T}_{3,0.1}(x)$ has three points satisfying $\mathcal{T}_{3,0.1}(x) = x$.
That is, $x_0 = -0.967484, \ -0.33466, 2.41214$. Since

$$\left|\frac{d}{dt}\mathcal{T}_{3,0.1}(-0.967484)\right| = 0 < 1, \quad \left|\frac{d}{dt}\mathcal{T}_{3,0.1}(-0.33466)\right| = 0 < 1 \tag{51}$$

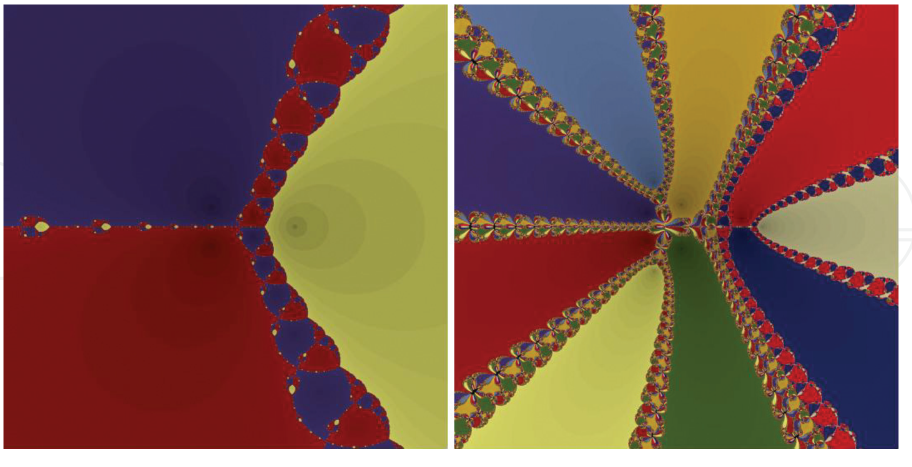

Figure 6.
Orbit of x_o *under the action of* R *for* $\mathcal{T}_{3,0.1}(x)$, $\mathcal{T}^2_{3,0.1}(x)$.

Figure 7.
Palette for escaping points.

Degree n	Attractor	Repellor	Neutral
1	0	1	0
2	1	1	0
3	2	1	0
4	1	3	0
5	1	4	0

Table 2.
Numbers of fixed points of $\mathcal{T}_{n,0.1}(x)$.

r	$\mathbf{R}_{\mathcal{T}^r_{3,0.1}(x)}$	$\mathbf{RF}_{\mathcal{T}^r_{3,0.1}(x)}$
1	3	2
2	3	2
3	3	2
4	23	2
5	2	2
6	1	1

Table 3.
The numbers of $\boldsymbol{R}_{\mathcal{T}^r_{3,0.1}(x)}$ and $\boldsymbol{RF}_{\mathcal{T}^r_{3,0.1}(x)}$ for $1 \leq r \leq 6$.

Theorem 3.2. $\mathcal{T}_{3,0.1}(x)$ for $q = 0.1$ has two attracting fixed points.

Using Mathematica, we can separate the numerical results for fixed points of $\mathcal{T}_{n,0.1}(x)$. From **Table 2**, we know that $\mathcal{T}_{n,0.1}(x)$ have no neutral fixed point for $1 \leq n \leq 4$. We can also reach Conjecture 3.3.

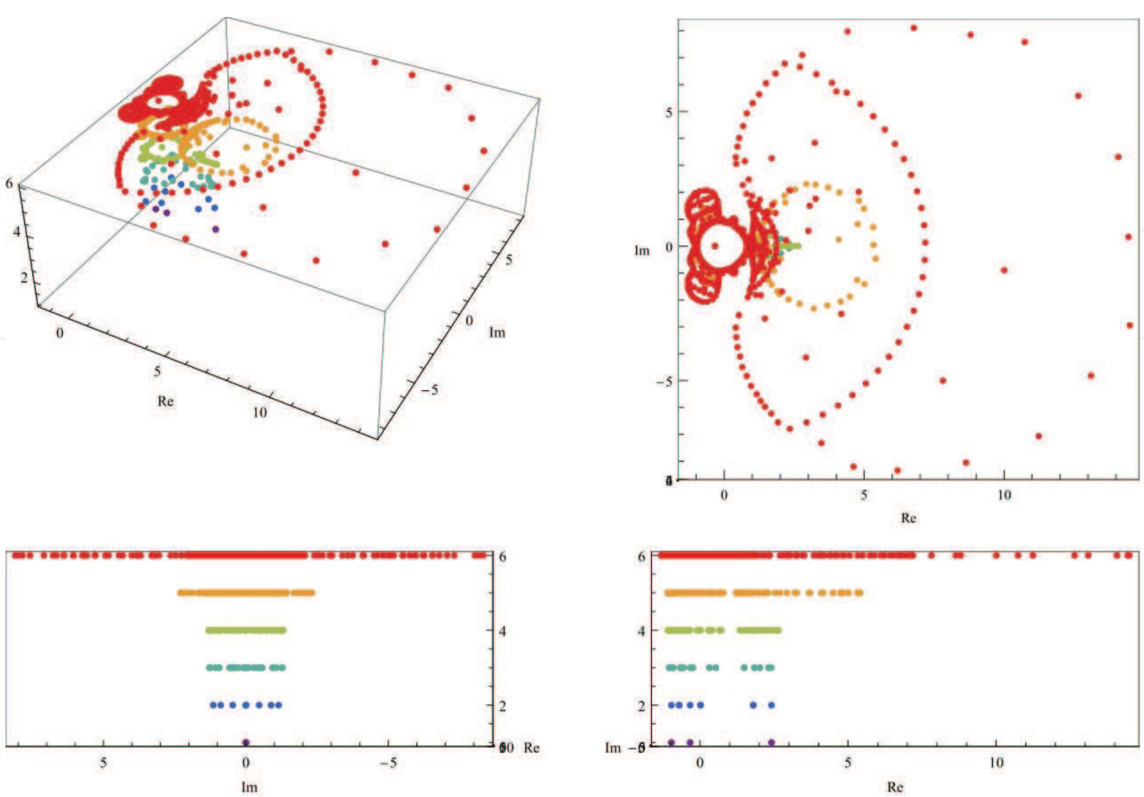

Figure 8.
Stacks of fixed point of $\mathcal{T}^r_{3,0.1}(x)$ for $1 \leq r \leq 6$.

Conjecture 3.3. The q-tangent polynomials for $n \geq 2$ have at least one attracting fixed point except for infinity.

In **Table 3**, we denote $\mathbf{R}_{\mathcal{T}^r_{n,q}(x)}$ as the numbers of real zeros for rth iteration and $\mathbf{RF}_{\mathcal{T}^r_{n,q}(x)}$ as the numbers of attracting fixed point on real number. From this table, we can know that number of real fixed points of $\mathcal{T}^r_{3,q}(x)$ are less than two. Here, we can suggest Conjecture 3.4.

Conjecture 3.4. The q-tangent polynomials that are iterated, $\mathcal{T}^r_{3,0.1}(x)$, have real fixed point, $\alpha = -0.33466$.

In the top-left of **Figure 8**, we can see the forms of 3D structure related to stacks of fixed points of $\mathcal{T}^r_{3,0.1}(x)$ for $1 \leq r \leq 6$. When we look at the top-left of **Figure 8** in the below position, we can draw the top-right figure. The bottom-left of **Figure 8** shows that image and n-axes exist but not real axis in three dimensions. In three dimensions, the bottom-right of **Figure 8** is the right orthographic viewpoint for the top-left figure,-that is, there exist real and n-axes but there is no image axis (**Figure 8**).

4. Conclusion

We can see that when q comes closer to 0, the approximate shape of the roots become increasingly more circular. Also in this situation, we can observe scattering of zeros in q-tangent polynomials around 2 in three-dimension. When q comes closer to 1, it has properties that are more symmetrical. We can also assume that the property that appears when iterating $\mathcal{T}_{n,q}(x)$ has self-similarity. By iterating, we can conjecture some properties about fixed points. This property warrants further study so that we can create a new property.

Acknowledgements

This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Science, ICT and Future Planning (No. 2017R1E1A1A03070483).

Conflict of interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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