Chapter

Some Topological Properties of Intuitionistic Fuzzy Normed Spaces

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Abstract

In 1986, Atanassov introduced the concept of intuitionistic fuzzy set theory which is based on the extensions of definitions of fuzzy set theory given by Zadeh. This theory provides a variable model to elaborate uncertainty and vagueness involved in decision making problems. In this chapter, we concentrate our study on the ideal convergence of sequence spaces with respect to intuitionistic fuzzy norm and discussed their topological and algebraic properties.

Keywords: ideal, intuitionistic fuzzy normed spaces, Orlicz function compact operator, I-convergence

1. Introduction

In recent years, the fuzzy theory has emerged as the most active area of research in many branches of mathematics, computer and engineering [1]. After the excellent work of Zadeh [2], a large number of research work have been done on fuzzy set theory and its applications as well as fuzzy analogues of the classical theories. It has a wide number of applications in various fields such as population dynamics [3], nonlinear dynamical system [4], chaos control [5], computer programming [6], etc. In 2006, Saadati and Park [7] introduced the concept of intuitionistic fuzzy normed spaces after that the concept of statistical convergence in intuitionistic fuzzy normed space was studied for single sequence in [8]. The study of intuitionistic fuzzy topological spaces [9], intuitionistic fuzzy 2-normed space [10] and intuitionistic fuzzy Zweier ideal convergent sequence spaces [11] are the latest developments in fuzzy topology.

First, let us recall some notions, basic definitions and concepts which are used in sequel.

Definition 1.1. (See Ref. [7]). The five-tuple $(X, \mu, \nu, *, \diamond)$ is said to be an intuitionistic fuzzy normed space (for short, IFNS) if X is a vector space, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm, and μ and ν are fuzzy sets on $X \times (0, \infty)$ satisfying the following conditions for every $x, y \in X$ and $s, t > 0$:

(a) $\mu(x,t) + \nu(x,t) \leq 1$,

(b) $\mu(x,t) > 0$,

(c) $\mu(x,t) = 1$ if and only if $x = 0$,

(d) $\mu(\alpha x,t) = \mu\left(x, \frac{t}{|\alpha|}\right)$ for each $\alpha \neq 0$,

(e) $\mu(x,t) * \mu(y,s) \leq \mu(x+y,t+s)$,
(f) $\mu(x,.) : (0,\infty) \to [0,1]$ is continuous,
(g) $\lim_{t\to\infty} \mu(x,t) = 1$ and $\lim_{t\to 0} \mu(x,t) = 0$,
(h) $\nu(x,t) < 1$,
(i) $\nu(x,t) = 0$ if and only if $x = 0$,
(j) $\nu(\alpha x,t) = \nu\left(x, \frac{t}{|\alpha|}\right)$ for each $\alpha \neq 0$,
(k) $\nu(x,t) \diamond \nu(y,s) \geq \nu(x+y,t+s)$,
(l) $\nu(x,.) : (0,\infty) \to [0,1]$ is continuous,
(m) $\lim_{t\to\infty} \nu(x,t) = 0$ and $\lim_{t\to 0} \nu(x,t) = 1$.

In this case (μ,ν) is called an intuitionistic fuzzy norm.

Example 1.1. *Let $(X, \|.\|)$ be a normed space. Denote $a * b = ab$ and $a \diamond b = \min(a+b,1)$ for all $a, b \in [0,1]$ and let μ_0 and ν_0 be fuzzy sets on $X \times (0,\infty)$ defined as follows:*

$$\mu_0(x,t) = \frac{t}{t+\|x\|}, \quad \text{and} \quad \nu_0(x,t) = \frac{\|x\|}{t+\|x\|}$$

for all $t \in \mathbb{R}^+$. Then $(X,\mu,\nu,*,\diamond)$ is an intuitionistic fuzzy normed space.

Definition 1.2. Let $(X,\mu,\nu,*,\diamond)$ be an IFNS. Then a sequence $x = (x_k)$ is said to be convergent to $L \in X$ with respect to the intuitionistic fuzzy norm (μ,ν) if, for every $\varepsilon > 0$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\mu(x_k - L,t) > 1-\varepsilon$ and $\nu(x_k - L,t) < \varepsilon$ for all $k \geq k_0$. In this case we write (μ,ν)-$\lim x = L$.

In 1951, the concept of statistical convergence was introduced by Steinhaus [12] and Fast [13] in their papers “Sur la convergence ordinaire et la convergence asymptotique” and “Sur la convergence statistique,” respectively. Later on, in 1959, Schoenberg [14] reintroduced this concept. It is a very useful functional tool for studying the convergence of numerical problems through the concept of density. The concept of ideal convergence, which is a generalization of statistical convergence, was introduced by Kostyrko et al. [15] and it is based on the ideal I as a subsets of the set of positive integers and further studied in [16–20].

Let X be a non-empty set then a family $I \subset 2^X$ is said to be an **ideal** in X if $\emptyset \in I$, I is additive, i.e., for all $A, B \in I \Rightarrow A \cup B \in I$ and I is hereditary, i.e., for all $A \in I, B \subseteq A \Rightarrow B \in I$. A non empty family of sets $\mathcal{F} \subset 2^X$ is said to be a **filter** on X if for all $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$ and for all $A \in \mathcal{F}$ with $A \subseteq B$ implies $B \in \mathcal{F}$. An ideal $I \subset 2^X$ is said to be **nontrivial** if $I \neq 2^X$, this non trivial ideal is said to be admissible if $I \supseteq \{\{x\} : x \in X\}$ and is said to be **maximal** if there cannot exist any nontrivial ideal $J \neq I$ containing I as a subset. For each ideal I, there is a filter $\mathcal{F}(I)$ called as filter associate with ideal I, that is (see [15]),

$$\mathcal{F}(I) = \{K \subseteq X : K^c \in I\}, \quad \text{where} \quad K^c = X \backslash K. \tag{1}$$

A sequence $x = (x_k) \in \omega$ is said to be I-**convergent** [21, 22] to a number L if for every $\varepsilon > 0$, we have $\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\} \in I$. In this case, we write $I - \lim_{xk} = L$.

2. IF-ideal convergent sequence spaces using compact operator

This section consists of some double sequence spaces with respect to intuitionistic fuzzy normed space and study the fuzzy topology on the said spaces. First we recall some basic definitions on compact operator.

Definition 2.1. (See [23]). Let X and Y be two normed linear spaces and $T : \mathcal{D}(T) \to Y$ be a linear operator, where $\mathcal{D} \subset X$. Then, the operator T is said to be bounded, if there exists a positive real k such that

$$\|Tx\| \leq k\|x\|, \quad \text{for all} \quad x \in \mathcal{D}(T).$$

The set of all bounded linear operators $\mathcal{B}(X, Y)$ [24] is a normed linear spaces normed by

$$\|T\| = \sup_{x \in X, \ \|x\|=1} \|Tx\|$$

and $\mathcal{B}(X, Y)$ is a Banach space if Y is a Banach space.

Definition 2.2. (See [23]). Let X and Y be two normed linear spaces. An operator $T : X \to Y$ is said to be a compact linear operator (or completely continuous linear operator), if

(i) T is linear,

(ii) T maps every bounded sequence (x_k) in X on to a sequence $(T(x_k))$ in Y which has a convergent subsequence.

The set of all compact linear operators $\mathcal{C}(X, Y)$ is a closed subspace of $\mathcal{B}(X, Y)$ and $\mathcal{C}(X, Y)$ is Banach space, if Y is a Banach space.

In 2015, Khan et al. [11] introduced the following sequence spaces:

$$\mathcal{Z}^I_{(\mu,\nu)} = \left\{(x_k) \in \omega : \left\{k : \mu\left(x_k^{/} - L, t\right) \leq 1 - \varepsilon \ \ or \ \ \nu\left(x_k^{/} - L, t\right) \geq \varepsilon\right\} \in I\right\},$$
$$\mathcal{Z}^I_{0(\mu,\nu)} = \left\{(x_k) \in \omega : \left\{k : \mu\left(x_k^{/}, t\right) \leq 1 - \varepsilon \ \ or \ \ \nu\left(x_k^{/}, t\right) \geq \varepsilon\right\} \in I\right\}.$$

Motivated by this, we introduce the following sequence spaces with the help of compact operator in intuitionistic fuzzy normed spaces:

$$\mathcal{M}^I_{(\mu,\nu)}(T) = (x_k) \in \ell_\infty : \{k : \mu(T(x_k) - L, t) \leq 1 - \varepsilon \\ \text{or} \ \ \nu(T(x_k) - L, t) \geq \varepsilon \in I\} \tag{2}$$

$$\mathcal{M}^I_{0(\mu,\nu)}(T) = (x_k) \in \ell_\infty : \{k : \mu(T(x_k), t) \leq 1 - \varepsilon \\ \text{or} \ \ \nu(T(x_k), t) \geq \varepsilon \in I\}. \tag{3}$$

Here, we also define an open ball with center x and radius r with respect to t as follows:

$$\mathcal{B}_x(r, t)(T) = (y_k) \in \ell_\infty : \{k : \mu(T(x_k) - T(y_k), t) \leq 1 - \varepsilon \\ \text{or} \ \ \nu(T(x_k) - T(y_k), t) \geq \varepsilon \in I\}. \tag{4}$$

Now, we are ready to state and prove our main results. This theorem is based on the linearity of new define sequence spaces which is stated as follows:

Theorem 2.1. The sequence spaces $\mathcal{M}^I_{(\mu,\nu)}(T)$ and $\mathcal{M}^I_{0(\mu,\nu)}(T)$ are linear spaces.

Proof. Let $x = (x_k), y = (y_k) \in \mathcal{M}^I_{(\mu,\nu)}(T)$ and α, β be scalars. Then for a given $\varepsilon > 0$, we have the sets:

$$P_1 = \left\{k : \mu\left(T(x_k) - L_1, \frac{t}{2|\alpha|}\right) \leq 1 - \varepsilon \ \ \text{or} \ \ \nu\left(T(x_k) - L_1, \frac{t}{2|\alpha|}\right) \geq \varepsilon\right\} \in I;$$
$$P_2 = \left\{k : \mu\left(T(y_k) - L_2, \frac{t}{2|\beta|}\right) \leq 1 - \varepsilon \ \ \text{or} \ \ \nu\left(T(y_k) - L_2, \frac{t}{2|\beta|}\right) \geq \varepsilon\right\} \in I.$$

This implies

$$P_1^c = \left\{k : \mu\left(T(x_k) - L_1, \frac{t}{2|\alpha|}\right) > 1 - \varepsilon \ \ \text{or} \ \ \nu\left(T(x_k) - L_1, \frac{t}{2|\alpha|}\right) < \varepsilon\right\} \in \mathcal{F}(I);$$

$$P_2^c = \left\{ k : \mu\left(T\left(_{yk}\right) - L_2, \frac{t}{2|\beta|}\right) > 1 - \varepsilon \text{ or } \nu\left(\left(T\left(_{yk}\right) - L_2, \frac{t}{2|\beta|}\right) < \varepsilon \right\} \in \mathcal{F}(I).$$

Now, we define the set $P_3 = P_1 \cup P_2$, so that $P_3 \in I$. It shows that P_3^c is a non-empty set in $\mathcal{F}(I)$. We shall show that for each (x_k), $(y_k) \in \mathcal{M}^I_{(\mu,\nu)}(T)$.

$$P_3^c \subset \{k : \mu\left(\left(\alpha T(x_k) + \beta T(y_k)\right)\right) - (\alpha L_1 + \beta L_2), t\right) > 1 - \varepsilon$$
$$\text{or } \nu\left(\left(\alpha T(x_k) + \beta T(y_k)\right)\right) - (\alpha L_1 + \beta L_2), t\right) < \varepsilon\}.$$

Let $m \in P_3^c$, in this case

$$\mu\left(T(x_m) - L_1, \frac{t}{2|\alpha|}\right) > 1 - \varepsilon \text{ or } \nu\left(T(x_m) - L_1, \frac{t}{2|\alpha|}\right) < \varepsilon$$

and

$$\mu\left(T(y_m) - L_2, \frac{t}{2|\beta|}\right) > 1 - \varepsilon \ \ or \ \ \nu\left(T(y_m) - L_2, \frac{t}{2|\beta|}\right) < \varepsilon.$$

Thus, we have

$$\begin{aligned} &\mu\left(\left(\alpha T(x_m) + \beta T(y_m)\right)\right) - (\alpha L_1 + \beta L_2), t\right) \\ &\geq \mu\left(\alpha T(x_m) - \alpha L_1, \frac{t}{2}\right) * \mu\left(\beta T(x_m) - \beta L_2, \frac{t}{2}\right) \\ &= \mu\left(T(x_m) - L_1, \frac{t}{2|\alpha|}\right) * \mu\left(T(x_m) - L_2, \frac{t}{2|\beta|}\right) \\ &> (1 - \varepsilon) * (1 - \varepsilon) = 1 - \varepsilon. \end{aligned}$$

and

$$\begin{aligned} &\nu\left(\left(\alpha T(x_m) + \beta T(y_m)\right)\right) - (\alpha L_1 + \beta L_2), t\right) \\ &\leq \nu\left(\alpha T(x_m) - \alpha L_1, \frac{t}{2}\right) \diamond \nu\left(\beta T(x_m) - \beta L_2, \frac{t}{2}\right) \\ &= \mu\left(T(x_m) - L_1, \frac{t}{2|\alpha|}\right) \diamond \mu\left(T(x_m) - L_2, \frac{t}{2|\beta|}\right) \\ &< \varepsilon \diamond \varepsilon = \varepsilon. \end{aligned}$$

This implies that

$$P_3^c \subset \{k : \mu\left(\left(\alpha T(x_k) + \beta T(y_k)\right)\right) - (\alpha L_1 + \beta L_2), t\right) > 1 - \varepsilon$$
$$\text{or } \nu\left(\left(\alpha T(x_k) + \beta T(y_k)\right)\right) - (\alpha L_1 + \beta L_2), t\right) < \varepsilon.$$

Therefore, the sequence space $\mathcal{M}^I_{(\mu,\nu)}(T)$ is a linear space.

Similarly, we can proof for the other space. □

In the following theorems, we discussed the convergence problem in the said sequence spaces. For this, firstly we have to discuss about the topology of this space. Define

$$\tau^I_{(\mu,\nu)}(T) = A \subset \mathcal{M}^I_{(\mu,\nu)}(T) : \text{for each } x \in A \text{ there exists } t > 0$$
$$\text{and } r \in (0, 1) \text{ such that } \mathcal{B}_x(r, t)(T) \subset A.$$

Then $\tau^I_{(\mu,\nu)}(T)$ is a topology on $\mathcal{M}^I_{(\mu,\nu)}(T)$.

Theorem 2.2. Let $\mathcal{M}^I_{(\mu,\nu)}(T)$ is an IFNS and $\tau^I_{(\mu,\nu)}(T)$ is a topology on $\mathcal{M}^I_{(\mu,\nu)}(T)$. Then a sequence $(x_k)\in\mathcal{M}^I_{(\mu,\nu)}(T)$, $x_k \to x$ if and only if $\mu(T(x_k) - T(x), t) \to 1$ and $\nu(T(x_k) - T(x), t) \to 0$ as $k \to \infty$.

Proof. Fix $t_0>0$. Suppose $x_k \to x$. Then for $r\in(0,1)$, there exists $n_0\in\mathbb{N}$ such that $(x_k)\in\mathcal{B}_x(r,t)(T)$ for all $k\geq n_0$. So, we have

$$\mathcal{B}_x(r,t_0)(T) = \{k : \mu(T(x_k) - T(x), t)\leq 1 - r \ \text{ or } \ \nu(T(x_k) - T(x), t_0)\geq r\}\in I,$$

such that $\mathcal{B}^c_x(r,t_0)(T)\in\mathcal{F}(I)$. Then $1 - \mu(T(x_k) - T(x), t_0) < r$ and $\nu(T(x_k) - T(x), t_0) < r$. Hence $\mu(T(x_k) - T(x), t_0) \to 1$ and $\nu(T(x_k) - T(x), t_0) \to 0$ as $k \to \infty$.

Conversely, if for each $t>0$, $\mu(T(x_k) - T(x), t) \to 1$ and $\nu(T(x_k) - T(x), t) \to 0$ as $k \to \infty$, then for $r\in(0,1)$, there exists $n_0\in\mathbb{N}$, such that $1 - \mu(T(x_k) - T(x), t) < r$ and $\nu(T(x_k) - T(x), t) < r$, for all $k\geq n_0$. It shows that $\mu(T(x_k) - T(x), t)>1 - r$ and $\nu(T(x_k) - T(x), t) < r$ for all $k\geq n_0$. Therefore $(x_k)\in\mathcal{B}^c_x(r,t)(T)$ for all $k\geq n_0$ and hence $x_k \to x$.

There are some facts that arise in connection with the convergence of sequences in these spaces. Let us proceed to the next theorem on Ideal convergence of sequences in these new define spaces.

Theorem 2.3. A sequence $x = (x_k)\in\mathcal{M}^I_{(\mu,\nu)}(T)$ is I-convergent if and only if for every $\varepsilon>0$ and $t>0$ there exists a number $N = N(x,\varepsilon,t)$ such that

$$\left\{N : \mu\left(T(x_N) - L, \frac{t}{2}\right)>1 - \varepsilon \ \text{ or } \ \nu\left(T(x_N) - L, \frac{t}{2}\right) < \varepsilon\right\}\in\mathcal{F}(I).$$

Proof. Suppose that $I_{(\mu,\nu)} - \lim x = L$ and let $t>0$. For a given $\varepsilon>0$, choose $s>0$ such that $(1-\varepsilon)*(1-\varepsilon)>1 - s$ and $\varepsilon\diamond\varepsilon < s$. Then for each $x\in\mathcal{M}^I_{(\mu,\nu)}(T)$,

$$R = \left\{k : \mu\left(T(x_k) - L, \frac{t}{2}\right)\leq 1 - \varepsilon \ \text{ or } \ \nu\left(T(x_k) - L, \frac{t}{2}\right)\geq\varepsilon\right\}\in I,$$

which implies that

$$R^c = \left\{k : \mu\left(T(x_k) - L, \frac{t}{2}\right)>1 - \varepsilon \ \text{ or } \ \nu\left(T(x_k) - L, \frac{t}{2}\right) < \varepsilon\right\}\in\mathcal{F}(I).$$

Conversely, let us choose $N\in R^c$. Then

$$\mu\left(T(x_N) - L, \frac{t}{2}\right)>1 - \varepsilon \ \ or \ \ \nu\left(T(x_N) - L, \frac{t}{2}\right) < \varepsilon.$$

Now, we want to show that there exists a number $N = N(x,\varepsilon,t)$ such that

$$\{k : \mu(T(x_k) - T(x_N), t)\leq 1 - s \ \text{ or } \ \nu(T(x_k) - T(x_N), t)\geq s\}\in I.$$

For this, we define for each $x\in\mathcal{M}^I_{(\mu,\nu)}(T)$

$$S = \{k : \mu(T(x_k) - T(x_N), t)\leq 1 - s \ \text{ or } \ \nu(T(x_k) - T(x_N), t)\geq s\}\in I.$$

So, we have to show that $S\subset R$. Let us suppose that $S\subsetneq R$, then there exists $n\in S$ and $n\notin R$. Therefore, we have

$$\mu(T(x_n) - T(x_N), t)\leq 1 - s \ \text{ or } \ \mu\left(T(x_n) - L, \frac{t}{2}\right)>1 - \varepsilon.$$

In particular $\mu\left(T(x_N) - L, \frac{t}{2}\right) > 1 - \varepsilon$. Therefore, we have

$$1 - s \geq \mu(T(x_n) - T(x_N), t) \geq \mu\left(T(x_n) - L, \frac{t}{2}\right) * \mu\left(T(x_N) - L, \frac{t}{2}\right) \geq (1 - \varepsilon) * (1 - \varepsilon) > 1 - s,$$

which is not possible. On the other hand

$$\nu(T(x_n) - T(x_N), t) \geq s \quad or \quad \nu\left(T(x_n) - L, \frac{t}{2}\right) < \varepsilon.$$

In particular $\nu\left(T(x_N) - L, \frac{t}{2}\right) < \varepsilon$. So, we have

$$s \leq \nu(T(x_n) - T(x_N), t) \leq \nu\left(T(x_n) - L, \frac{t}{2}\right) \diamond \nu\left(T(x_N) - L, \frac{t}{2}\right) \leq \varepsilon \diamond \varepsilon < s,$$

which is not possible. Hence $S \subset R$. $R \in I$ which implies $S \in I$. □

3. IF-ideal convergent sequence spaces using Orlicz function

In this section, we have discussed the ideal convergence of sequences in Intuitionistic fuzzy I-convergent sequence spaces defined by compact operator and Orlicz function. We shall now define the concept of Orlicz function, which is basic definition in our work.

Definition 3.1. An Orlicz function is a function $F : [0, \infty) \to [0, \infty)$, which is continuous, non-decreasing and convex with $F(0) = 0$, $F(x) > 0$ for $x > 0$ and $F(x) \to \infty$ as $x \to \infty$. If the convexity of Orlicz function F is replaced by $F(x + y) \leq F(x) + F(y)$, then this function is called modulus function.

Remark 3.1. If F is an Orlicz function, then $F(\lambda x) \leq \lambda F(x)$ for all λ with $0 < \lambda < 1$.

In 2009, Mohiuddine and Lohani [18] introduced the concept of statistical convergence in intuitionistic fuzzy normed spaces in their paper published in Chaos, Solitons and Fractals. This motivated us to introduced some sequence spaces defined by compact operator and Orlicz function which are as follows:

$$\begin{aligned} \mathcal{M}^I_{(\mu,\nu)}(T, F) = \Bigg\{ (x_k) \in \ell_\infty : \{k : F\left(\frac{\mu(T(x_k) - L, t)}{\rho}\right) \leq 1 - \varepsilon \\ \text{or } F\left(\frac{\nu(T(x_k) - L, t)}{\rho}\right) \geq \varepsilon\} \in I \Bigg\}; \end{aligned} \tag{5}$$

$$\begin{aligned} \mathcal{M}^I_{0(\mu,\nu)}(T, F) = \Bigg\{ (x_k) \in \ell_\infty : \{k : F\left(\frac{\mu(T(x_k), t)}{\rho}\right) \leq 1 - \varepsilon \\ \text{or } F\left(\frac{\nu(T(x_k), t)}{\rho}\right) \geq \varepsilon\} \in I \Bigg\}. \end{aligned} \tag{6}$$

We also define an open ball with center x and radius r with respect to t as follows:

$$\begin{aligned} \mathcal{B}_x(r, t)(T, F) = \Bigg\{ (y_k) \in \ell_\infty : k : F\left(\frac{\mu(T(x_k) - T(y_k), t)}{\rho}\right) \leq 1 - \varepsilon \\ \text{or } F\left(\frac{\nu(T(x_k) - T(y_k), t)}{\rho}\right) \geq \varepsilon \in I \Bigg\}. \end{aligned} \tag{7}$$

We shall now consider some theorems of these sequence spaces and invite the reader to verify the linearity of these sequence spaces.

Theorem 3.1. Every open ball $\mathcal{B}_x(r, t)(T, F)$ is an open set in $\mathcal{M}^I_{(\mu,\nu)}(T, F)$.

Proof. Let $\mathcal{B}_x(r,t)(T,F)$ be an open ball with center x and radius r with respect to t. That is

$$\mathcal{B}_x(r,t)(T,F) = \left\{y = (y_k) \in \ell_\infty : \left\{k : F\left(\frac{\mu(T(x_k) - T(y_k),t)}{\rho}\right) \leq 1 - r \right.\right.$$
$$\left.\left. \text{or } F\left(\frac{\nu(T(x_k) - T(y_k),t)}{\rho}\right) \geq r \right\} \in I\right\}.$$

Let $y \in \mathcal{B}_x^c(r,t)(T,F)$, then $F\left(\frac{\mu(T(x_k)-T(y_k),t)}{\rho}\right) > 1 - r$ and $F\left(\frac{\nu(T(x_k)-T(y_k),t)}{\rho}\right) < r$. Since $F\left(\frac{\mu(T(x_k)-T(y_k),t)}{\rho}\right) > 1 - r$, there exists $t_0 \in (0,t)$ such that $F\left(\frac{\mu(T(x_k)-T(y_k),t_0)}{\rho}\right) > 1 - r$ and $F\left(\frac{\nu(T(x_k)-T(y_k),t_0)}{\rho}\right) < r$.

Putting $r_0 = F\left(\frac{\mu(T(x_k)-T(y_k),t_0)}{\rho}\right)$, so we have $r_0 > 1 - r$, there exists $s \in (0,1)$ such that $r_0 > 1 - s > 1 - r$. For $r_0 > 1 - s$, we have $r_1, r_2 \in (0,1)$ such that $r_0 * r_1 > 1 - s$ and $(1 - r_0) \diamond (1 - r_0) \leq s$. Putting $r_3 = \max\{r_1, r_2\}$. Now we consider a ball $\mathcal{B}_y^c(1 - r_3, t - t_0)(T,F)$. And we prove that

$$\mathcal{B}_y^c(1 - r_3, t - t_0)(T,F) \subset \mathcal{B}_x^c(r,t)(T,F).$$

Let $z = (z_k) \in \mathcal{B}_y^c(1 - r_3, t - t_0)(T,F)$, then $F\left(\frac{\mu(T(y_k)-T(z_k),t-t_0)}{\rho}\right) > r_3$ and $F\left(\frac{\nu(T(y_k)-T(z_k),t-t_0)}{\rho}\right) < 1 - r_3$. Therefore, we have

$$F\left(\frac{\mu(T(x_k) - T(z_k),t)}{\rho}\right) \geq F\left(\frac{\mu(T(x_k) - T(y_k),t_0)}{\rho}\right) * F\left(\frac{\mu(T(y_k) - T(z_k),t - t_0)}{\rho}\right)$$
$$\geq (r_0 * r_3) \geq (r_0 * r_1) \geq (1 - s) \geq (1 - r)$$

and

$$F\left(\frac{\nu(T(x_k) - T(z_k),t)}{\rho}\right) \leq F\left(\frac{\nu(T(x_k) - T(y_k),t_0)}{\rho}\right) \diamond F\left(\frac{\nu(T(y_k) - T(z_k),t - t_0)}{\rho}\right)$$
$$\leq (1 - r_0) \diamond (1 - r_3) \leq (1 - r_0) \diamond (1 - r_2) \leq s \leq r.$$

Thus $z \in \mathcal{B}_x^c(r,t)(T,F)$ and hence, we get

$$\mathcal{B}_y^c(1 - r_3, t - t_0)(T,F) \subset \mathcal{B}_x^c(r,t)(T,F).$$

Remark 3.2. $\mathcal{M}^I_{(\mu,\nu)}(T,F)$ is an IFNS.

Define

$$\tau^I_{(\mu,\nu)}(T,F) = A \subset \mathcal{M}^I_{(\mu,\nu)}(T,F) : \text{ for each } x \in A \text{ there exists } t > 0$$
$$\text{and } r \in (0,1) \text{ such that } \mathcal{B}_x(r,t)(T,F) \subset A.$$

Then $\tau^I_{(\mu,\nu)}(T,F)$ is a topology on $\mathcal{M}^I_{(\mu,\nu)}(T,F)$.

In the above result we can easily verify that the open sets in these spaces are open ball in the same spaces. This theorem itself will have various applications in our future work.

Theorem 3.2. The topology $\tau^I_{(\mu,\nu)}(T,F)$ on $\mathcal{M}^I_{0(\mu,\nu)}(T,F)$ is first countable.

Proof. $\left\{\mathcal{B}_x\left(\frac{1}{n},\frac{1}{n}\right)(T,F) : n = 1,2,3,\ldots\right\}$ is a local base at x, the topology $\tau^I_{(\mu,\nu)}(T,F)$ on $\mathcal{M}^I_{0(\mu,\nu)}(T,F)$ is first countable. □

Theorem 3.3. $\mathcal{M}^I_{(\mu,\nu)}(T,F)$ and $\mathcal{M}^I_{0(\mu,\nu)}(T,F)$ are Hausdorff spaces.

Proof. Let $x, y \in \mathcal{M}^I_{(\mu,\nu)}(T,F)$ such that $x \neq y$. Then $0 < F\left(\frac{\mu(T(x)-T(y),t)}{\rho}\right) < 1$ and $0 < F\left(\frac{\nu(T(x)-T(y),t)}{\rho}\right) < 1$.

Putting $r_1 = F\left(\frac{\mu(T(x)-T(y),t)}{\rho}\right)$, $r_2 = F\left(\frac{\nu(T(x)-T(y),t)}{\rho}\right)$ and $r = \max\{r_1, 1-r_2\}$. For each $r_0 \in (r,1)$ there exists r_3 and r_4 such that $r_3 * r_4 \geq r_0$ and $(1-r_3)\diamond(1-r_4) \leq (1-r_0)$.

Putting $r_5 = \max\{r_3, 1-r_4\}$ and consider the open balls $\mathcal{B}_x\left(1-r_5,\frac{t}{2}\right)$ and $\mathcal{B}_y\left(1-r_5,\frac{t}{2}\right)$. Then clearly $\mathcal{B}^c_x\left(1-r_5,\frac{t}{2}\right)\cap\mathcal{B}^c_y\left(1-r_5,\frac{t}{2}\right) = \phi$. For if there exists $z \in \mathcal{B}^c_x\left(1-r_5,\frac{t}{2}\right)\cap\mathcal{B}^c_y\left(1-r_5,\frac{t}{2}\right)$, then

$$r_1 = F\left(\frac{\mu(T(x)-T(y),t)}{\rho}\right) \geq \left(\frac{\mu\left(T(x)-T(z),\frac{t}{2}\right)}{\rho}\right) * F\left(\frac{\mu\left(T(z)-T(y),\frac{t}{2}\right)}{\rho}\right)$$
$$\geq r_5 * r_5 \geq r_3 * r_3 \geq r_0 > r_1$$

and

$$r_2 = F\left(\frac{\nu(T(x)-T(y),t)}{\rho}\right) \leq F\left(\frac{\nu\left(T(x)-T(z),\frac{t}{2}\right)}{\rho}\right) \diamond F\left(\frac{\nu\left(T(z)-T(y),\frac{t}{2}\right)}{\rho}\right)$$
$$\leq (1-r_5)\diamond(1-r_5) \leq (1-r_4)\diamond(1-r_4) \leq (1-r_0) < r_2$$

which is a contradiction. Hence, $\mathcal{M}^I_{(\mu,\nu)}(T,F)$ is Hausdorff. Similarly the proof follows for $\mathcal{M}^I_{0(\mu,\nu)}(T,F)$. □

4. Conclusion

The concept of defining intuitionistic fuzzy ideal convergent sequence spaces as it generalized the fuzzy set theory and give quite useful and interesting applications in many areas of mathematics and engineering. This chapter give brief introduction to intuitionistic fuzzy normed spaces with some basic definitions of convergence applicable on it. We have also summarized different types of sequence spaces with the help of ideal, Orlicz function and compact operator. At the end of this chapter some theorems and remarks based on these new defined sequence spaces are discussed for proper understanding.

Conflict of interest

The authors declare that they have no competing interests.

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