

We are IntechOpen, the world's leading publisher of Open Access books Built by scientists, for scientists

6,900

Open access books available

186,000

International authors and editors

200M

Downloads

Our authors are among the

154

Countries delivered to

TOP 1%

most cited scientists

12.2%

Contributors from top 500 universities



WEB OF SCIENCE™

Selection of our books indexed in the Book Citation Index
in Web of Science™ Core Collection (BKCI)

Interested in publishing with us?
Contact book.department@intechopen.com

Numbers displayed above are based on latest data collected.
For more information visit www.intechopen.com



Obtaining Explicit Formulas and Identities for Polynomials Defined by Generating Functions of the Form $F(t)^x \cdot G(t)^\alpha$

Dmitry Kruchinin, Vladimir Kruchinin and Yuriy Shablya

Abstract

In this chapter, we study properties of polynomials defined by generating functions of the form $A(t, x, \alpha) = F(t)^x \cdot G(t)^\alpha$. Based on the Lagrange inversion theorem and the theorem of logarithmic derivative for generating functions, we obtain new properties related to the compositional inverse generating functions of those polynomials. Also we study the composition of generating functions $R(tA(t))$, where $A(t)$ is the generating function of the form $F(t)^x \cdot G(t)^\alpha$. We apply those results for obtaining explicit formulas and identities for such polynomials as the generalized Bernoulli, generalized Euler, Frobenius-Euler, generalized Sylvester, generalized Laguerre, Abel, Bessel, Stirling, Narumi, Peters, Gegenbauer, and Meixner polynomials.

Keywords: polynomial, identity, generating function, composita, composition, compositional inverse

1. Introduction

Generating functions are a powerful tool for solving problems in number theory, combinatorics, algebra, probability theory, and other fields of mathematics. One of the advantages of generating functions is that an infinite number sequence can be represented in a form of a single expression. Many authors have studied generating functions and their properties and found applications for them (for instance, Comtet [1], Flajolet and Sedgewick [2], Graham et al. [3], Robert [4], Stanley [5], and Wilf [6]).

Generating functions have an important role in the study of polynomials. Vast investigations related to the generating functions for many polynomials can be found in many books and articles (e.g., see [7–17]).

A special place in this area is occupied by research in the field of obtaining new identities for polynomials and special numbers with using their generating functions. Interesting results in the field of obtaining new identities for polynomials can be found in some recent works by Simsek [18–20], Kim et al. [21, 22], and Ryoo [23–25].

Another trend in study of polynomials is getting new representation and explicit formulas for those polynomials. For instance, Qi has recently established explicit

formulas for the generalized Motzkin numbers in [26] and the central Delannoy numbers in [27]. One can find interesting results in papers of Srivastava [28, 29], Cenkci [30], and Boyadzhiev [31].

In this chapter, we obtain some interesting properties of polynomials defined by generating functions of the form $F(t)^x \cdot G(t)^a$. As an application, we give some new identities for the Bernoulli, Euler, Frobenius-Euler, Sylvester, Laguerre, Abel, Bessel, Stirling, Narumi, Peters, Gegenbauer, and Meixner polynomials.

According to Stanley [32], ordinary generating functions are defined as follows:

Definition 1. An ordinary generating function of the sequence $(a_n)_{n \geq 0}$ is the formal power series

$$A(x) = a_0 + a_1x + a_2x^2 + \dots = \sum_{n \geq 0} a_n x^n. \quad (1)$$

Kruchinin et al. [33–35] introduced the mathematical notion of the *composita* of a given generating function, which can be used for calculating the coefficients of a composition of generating functions.

Definition 2. The composita of the generating function $F(x) = \sum_{n \geq 0} f_n x^n$ is the function with two variables

$$F^\Delta(n, k) = \sum_{\pi_k \in C_n} f_{\lambda_1} f_{\lambda_2} \cdots f_{\lambda_k}, \quad (2)$$

where C_n is the set of all compositions of an integer n and π_k is the composition n into k parts such that $\sum_{i=1}^k \lambda_i = n$.

Using the expression of the composita of a given generating function $F^\Delta(n, k)$, we can get powers of the generating function $F(x)$:

$$(F(x))^k = \sum_{n \geq k} F^\Delta(n, k) x^n. \quad (3)$$

Compositae also can be used for calculating the coefficients of generating functions obtained by addition, multiplication, composition, reciprocation, and compositional inversion of generating functions (for details see [33–35]).

By the reciprocal generating function we mean the following [6]:

Definition 1. A reciprocal generating function $A(x)$ of a generating function $B(x) = \sum_{n \geq 0} b_n x^n$ is a power series such that satisfies the following condition:

$$A(x)B(x) = 1. \quad (4)$$

By the compositional inverse generating function we mean the following:

Definition 2. A compositional inverse $\overline{F(x)}$ of generating function $F(x) = \sum_{n \geq 0} f_n x^n$ with $f(1) \neq 0$ is a power series such that satisfies the following condition:

$$F(\overline{F(x)}) = x. \quad (5)$$

Also the compositional inverse can be written as $F^{[-1]}(x)$ or $\overline{F(x)} = \text{Rev}F$.

For example, we will use the following formulas:

If we consider the composition $A(x) = R(F(x)) = \sum_{n \geq 0} a_n x^n$ of generating functions $R(x) = \sum_{n \geq 0} r_n x^n$ and $F(x) = \sum_{n \geq 0} f_n x^n$, then we can get the values of the coefficients a_n by using the following formula ([35], Eq. (17)):

$$a_n = \begin{cases} r_0, & \text{for } n = 0; \\ \sum_{k=1}^n F^\Delta(n, k) r_k, & \text{otherwise.} \end{cases} \quad (6)$$

If we consider the composition $A(x) = R(F(x)) = \sum_{n \geq 0} a_n x^n$ of generating functions $R(x) = \sum_{n \geq 0} r_n x^n$ and $F(x) = \sum_{n \geq 0} f_n x^n$, then we can get the values of the composita $A^\Delta(n, k)$ by using the following formula ([35]):

$$A^\Delta(n, k) = \sum_{m=k}^n F^\Delta(n, m) R^\Delta(m, k). \quad (7)$$

2. Main results

Let us consider a special case of generating functions that can be presented as the product of the powers of generating functions $F(t)^x \cdot G(t)^\alpha$. For such generating functions, we obtain several properties, which are given in the following theorem:

Theorem 1. *If $A(t)$ is a generating function of the following form:*

$$A(t) = F(t)^x \cdot G(t)^\alpha = \sum_{n \geq 0} A_n(x, \alpha) t^n, \quad (8)$$

then:

1. *For the composition of generating functions $D(t) = C(B(t)) = \sum_{n \geq 0} B_n t^n$, where $B(t) = tA(t)$ and $C(t) = \sum_{n \geq 0} C_n t^n$, we have*

$$D_n = D_n(x, \alpha) = \sum_{k=1}^n A_{n-k}(kx, k\alpha) C_k, \quad D_0 = C_0; \quad (9)$$

2. *For the compositional inverse generating function $\bar{B}(t)$ of $B(t) = tA(t)$, we have*

$$\bar{B}(t) = \sum_{n \geq 0} \frac{1}{n} A_{n-1}(-nx, -n\alpha) t^n; \quad (10)$$

3. *We have the following identities*

$$\sum_{m=k}^n A_{n-m}(mx, m\alpha) \frac{k}{m} A_{m-k}(-mx, -m\alpha) = \delta_{n,k} \quad (11)$$

and

$$\sum_{m=k}^n \frac{m}{n} A_{n-m}(-nx, -n\alpha) A_{m-k}(kx, k\alpha) = \delta_{n,k}, \quad (12)$$

where $\delta_{n,k}$ is the Kronecker delta.

Proof. First we get the k -th power of the generating function $B(t) = tA(t)$

$$\begin{aligned} (B(t))^k &= (tA(t))^k = t^k (F(t))^x (G(t))^{\alpha k} = \\ &= t^k \sum_{n \geq 0} A_n(kx, k\alpha) t^n = \sum_{n \geq k} A_{n-k}(kx, k\alpha) t^n. \end{aligned}$$

Hence, the composita of $B(t) = tA(t)$ is

$$B^\Delta(n, k) = A_{n-k}(kx, k\alpha). \quad (13)$$

Using Eqs. (6) and (13), we get Eq. (9).

According to [36], the composita of the compositional inverse generating function $\bar{A}(t)$ of $A(t) = \sum_{n \geq 0} a_n t^n$ is

$$\bar{A}^\Delta(n, k) = \frac{k}{n} R^\Delta(2n - k, n), \quad (14)$$

where $R^\Delta(n, k)$ is the composita of the generating function $R(t) = \frac{t^2}{A(t)}$.

For getting the composita of the compositional inverse generating function $\bar{B}(t)$ of $B(t) = tA(t)$, we need to know the composita of the generating function

$$R(t) = \frac{t^2}{B(t)} = \frac{t^2}{tA(t)} = \frac{t}{A(t)}. \quad (15)$$

Then we get the k -th power of the generating function $R(t) = \frac{t}{A(t)}$

$$\begin{aligned} (R(t))^k &= \left(\frac{t}{A(t)}\right)^k = t^k (F(t))^{-xk} (G(t))^{-\alpha k} = \\ &= t^k \sum_{n \geq 0} A_n(-kx, -k\alpha) t^n = \sum_{n \geq k} A_{n-k}(-kx, -k\alpha) t^n. \end{aligned} \quad (16)$$

Hence, the composita of Eq. (15) is

$$R^\Delta(n, k) = A_{n-k}(-kx, -k\alpha). \quad (17)$$

Using Eqs. (14) and (17), we get

$$\bar{B}^\Delta(n, k) = \frac{k}{n} R^\Delta(2n - k, n) = \frac{k}{n} A_{2n-k-n}(-nx, -n\alpha) = \frac{k}{n} A_{n-k}(-nx, -n\alpha). \quad (18)$$

For $k = 1$, we get Eq. (10).

Applying Eq. (7) for the composition $C(t) = B(\bar{B}(t)) = t$, we get

$$\begin{aligned} C^\Delta(n, k) &= \sum_{m=k}^n \bar{B}^\Delta(n, m) B^\Delta(m, k) = \\ &= \sum_{m=k}^n \frac{m}{n} A_{n-m}(-nx, -n\alpha) A_{m-k}(kx, k\alpha) = \delta_{n,k}. \end{aligned} \quad (19)$$

Applying Eq. (7) for the composition $D(t) = \bar{B}(B(t)) = x$, we get

$$\begin{aligned} D^\Delta(n, k) &= \sum_{m=k}^n B^\Delta(n, m) \bar{B}^\Delta(m, k) = \\ &= \sum_{m=k}^n A_{n-m}(mx, m\alpha) \frac{k}{m} A_{m-k}(-mx, -m\alpha) = \delta_{n,k}. \end{aligned} \quad (20)$$

□

As an application of Theorem 1, we present several examples of its usage for such polynomials as the Bernoulli, Euler, Frobenius-Euler, Sylvester, Laguerre, Abel, Bessel, Stirling, Narumi, Peters, Gegenbauer, and Meixner.

2.1 Generalized Bernoulli polynomials

The generalized Bernoulli polynomials are defined by the following generating function [37, 38]:

$$B(t, x, \alpha) = e^{xt} \left(\frac{t}{e^t - 1} \right)^\alpha = (e^t)^x \left(\frac{t}{e^t - 1} \right)^\alpha = \sum_{n \geq 0} B_n^{(\alpha)}(x) \frac{t^n}{n!}, \quad (21)$$

where

$$B_n^{(\alpha)}(x) = \sum_{i=0}^n \frac{n!}{(n+i)!} \binom{n+\alpha}{n-i} \binom{i+\alpha-1}{i} \sum_{j=0}^i (-1)^j \binom{i}{j} (x+j)^{n+i}. \quad (22)$$

According to Eq. (13), the composita for the generating function $D(t) = tB(t, x, \alpha)$ is

$$D^\Delta(n, k) = \frac{B_{n-k}^{(k\alpha)}(kx)}{(n-k)!}. \quad (23)$$

The triangular form of this composita is

$$\begin{array}{ccccc} & & 1 & & \\ & & \frac{2x - \alpha}{2} & & 1 \\ & & 2 & & \\ & & \frac{12x^2 - 12\alpha x + 3\alpha^2 - \alpha}{24} & & 2x - \alpha & 1 \\ & & 24 & & & \\ & \frac{8x^3 - 12\alpha x^2 + (6\alpha^2 - 2\alpha)x - \alpha^3 + \alpha^2}{48} & \frac{24x^2 - 24\alpha x + 6\alpha^2 - \alpha}{12} & \frac{6x - 3\alpha}{2} & 1 \end{array}$$

Using Eq. (17), the composita for the compositional inverse generating function $\overline{D}(t)$ of $D(t) = tB(t, x, \alpha)$ is

$$\overline{D}^\Delta(n, k) = \frac{k B_{n-k}^{(-n\alpha)}(-nx)}{n (n-k)!}. \quad (24)$$

The triangular form of this composita is

$$\begin{array}{ccccc} & & 1 & & \\ & & \frac{-2x + \alpha}{2} & & 1 \\ & & 2 & & \\ & & \frac{36x^2 - 36\alpha x + 9\alpha^2 + \alpha}{24} & & -2x + \alpha & 1 \\ & & 24 & & & \\ & \frac{-32x^3 + 48\alpha x^2 - (24\alpha^2 + 2\alpha)x + 4\alpha^3 + \alpha^2}{12} & \frac{48x^2 - 48\alpha x + 12\alpha^2 + \alpha}{12} & \frac{-6x + 3\alpha}{2} & 1 \end{array}$$

Also we can get the following new identities for the generalized Bernoulli polynomials:

$$\sum_{m=k}^n \frac{m}{n} \frac{B_{n-m}^{(-n\alpha)}(-nx)}{(n-m)!} \frac{B_{m-k}^{(k\alpha)}(kx)}{(m-k)!} = \delta_{n,k} \quad (25)$$

and

$$\sum_{m=k}^n \frac{B_{n-m}^{(m\alpha)}(mx)}{(n-m)!} \frac{k}{m} \frac{B_{m-k}^{(-m\alpha)}(-mx)}{(m-k)!} = \delta_{n,k}. \quad (26)$$

2.2 Generalized Euler polynomials

The generalized Euler polynomials are defined by the following generating function [37]:

$$E(t, x, \alpha) = e^{xt} \left(\frac{2}{e^t + 1} \right)^\alpha = (e^t)^x \left(\frac{2}{e^t + 1} \right)^\alpha = \sum_{n \geq 0} E_n^{(\alpha)}(x) \frac{t^n}{n!}, \quad (27)$$

where

$$E_n^{(\alpha)}(x) = \sum_{i=0}^n \frac{1}{2^i} \binom{i+\alpha-1}{i} \sum_{j=0}^i (-1)^j \binom{i}{j} (x+j)^n. \quad (28)$$

According to Eq. (13), the composita for the generating function $D(t) = tE(t, x, \alpha)$ is

$$D^\Delta(n, k) = \frac{E_{n-k}^{(k\alpha)}(kx)}{(n-k)!}. \quad (29)$$

The triangular form of this composita is

$$\begin{array}{ccccc} & & 1 & & \\ & & \frac{2x-\alpha}{2} & & 1 \\ & & 2 & & \\ & & \frac{4x^2-4\alpha x+\alpha^2-\alpha}{8} & & 2x-\alpha & 1 \\ & & 8 & & & \\ \frac{8x^3-12\alpha x^2+(6\alpha^2-6\alpha)x-\alpha^3+3\alpha^2}{48} & \frac{8x^2-8\alpha x+2\alpha^2-\alpha}{4} & \frac{6x-3\alpha}{2} & 1 & \end{array}$$

Using Eq. (17), the composita for the compositional inverse generating function $\overline{D}(t)$ of $D(t) = tE(t, x, \alpha)$ is

$$\overline{D}^\Delta(n, k) = \frac{k E_{n-k}^{(-n\alpha)}(-nx)}{n (n-k)!}. \quad (30)$$

The triangular form of this composita is

$$\begin{array}{ccccc} & & 1 & & \\ & & \frac{-2x+\alpha}{2} & & 1 \\ & & 2 & & \\ & & \frac{12x^2-12\alpha x+3\alpha^2+\alpha}{8} & & -2x+\alpha & 1 \\ & & 8 & & & \\ \frac{-32x^3+48\alpha x^2-(24\alpha^2+6\alpha)x+4\alpha^3+3\alpha^2}{12} & \frac{16x^2-16\alpha x+4\alpha^2+\alpha}{4} & \frac{-6x+3\alpha}{2} & 1 & \end{array}$$

Also we can get the following new identities for the generalized Euler polynomials:

$$\sum_{m=k}^n \frac{m}{n} \frac{E_{n-m}^{(-n\alpha)}(-nx)}{(n-m)!} \frac{E_{m-k}^{(k\alpha)}(kx)}{(m-k)!} = \delta_{n,k} \quad (31)$$

and

$$\sum_{m=k}^n \frac{E_{n-m}^{(m\alpha)}(mx)}{(n-m)!} \frac{k}{m} \frac{E_{m-k}^{(-m\alpha)}(-mx)}{(m-k)!} = \delta_{n,k}. \quad (32)$$

2.3 Frobenius-Euler polynomials

The Frobenius-Euler polynomials are defined by the following generating function [39]:

$$H(t, x, \alpha, \lambda) = e^{xt} \left(\frac{1-\lambda}{e^t - \lambda} \right)^\alpha = (e^t)^x \left(\frac{1-\lambda}{e^t - \lambda} \right)^\alpha = \sum_{n \geq 0} H_n^{(\alpha)}(x, \lambda) \frac{t^n}{n!}, \quad (33)$$

where

$$H_n^{(\alpha)}(x, \lambda) = \sum_{i=0}^n \frac{1}{(1-\lambda)^i} \binom{i+\alpha-1}{i} \sum_{j=0}^i (-1)^j \binom{i}{j} (x+j)^n. \quad (34)$$

According to Eq. (13), the composita for the generating function $D(t) = tH(t, x, \alpha, \lambda)$ is

$$D^\Delta(n, k) = \frac{H_{n-k}^{(k\alpha)}(kx, \lambda)}{(n-k)!}. \quad (35)$$

The triangular form of this composita is

$$\begin{array}{c} 1 \\ \frac{(\lambda-1)x + \alpha}{\lambda-1} \quad 1 \\ \frac{(\lambda^2 - 2\lambda + 1)x^2 + (2\lambda - 2)\alpha x + \alpha^2 + \lambda\alpha}{2\lambda^2 - 4\lambda + 2} \quad \frac{(2\lambda - 2)x + 2\alpha}{\lambda-1} \quad 1 \end{array}$$

Using Eq. (17), the composita for the compositional inverse generating function $\overline{D}(t)$ of $D(t) = tH(t, x, \alpha, \lambda)$ is

$$\overline{D}^\Delta(n, k) = \frac{k}{n} \frac{H_{n-k}^{(-n\alpha)}(-nx, \lambda)}{(n-k)!}. \quad (36)$$

The triangular form of this composita is

$$\begin{array}{c} 1 \\ -\frac{(\lambda-1)x + \alpha}{\lambda-1} \quad 1 \\ \frac{(3\lambda^2 - 6\lambda + 3)x^2 + (6\lambda - 6)\alpha x + 3\alpha^2 - \lambda\alpha}{2\lambda^2 - 4\lambda + 2} \quad -\frac{(2\lambda - 2)x + 2\alpha}{\lambda-1} \quad 1 \end{array}$$

Also we can get the following new identities for the Frobenius-Euler polynomials:

$$\sum_{m=k}^n \frac{m}{n} \frac{H_{n-m}^{(-n\alpha)}(-nx, \lambda)}{(n-m)!} \frac{H_{m-k}^{(k\alpha)}(kx, \lambda)}{(m-k)!} = \delta_{n,k} \quad (37)$$

and

$$\sum_{m=k}^n \frac{H_{n-m}^{(m\alpha)}(mx, \lambda)}{(n-m)!} \frac{k}{m} \frac{H_{m-k}^{(-m\alpha)}(-mx, \lambda)}{(m-k)!} = \delta_{n,k}. \quad (38)$$

2.4 Generalized Sylvester polynomials

The generalized Sylvester polynomials are defined by the following generating function [40]:

$$F(t, x, \alpha) = (1-t)^{-x} e^{\alpha x t} = \left(\frac{e^{\alpha t}}{1-t} \right)^x = \sum_{n \geq 0} F_n(x, \alpha) t^n, \quad (39)$$

where

$$F_n(x, \alpha) = \sum_{i=0}^n \frac{(\alpha x)^{n-i}}{(n-i)!} \binom{i+x-1}{i}. \quad (40)$$

According to Eq. (13), the composita for the generating function $D(t) = tF(t, x, \alpha)$ is

$$D^\Delta(n, k) = F_{n-k}(kx, \alpha). \quad (41)$$

The triangular form of this composita is

$$\begin{array}{ccccccc} & & & & & & 1 \\ & & & & & & (\alpha+1)x \\ & & & & & & 1 \\ & & & & & & (\alpha^2+2\alpha+1)x^2+x \\ & & & & & & 2 \\ & & & & & & (2\alpha+2)x \\ & & & & & & 1 \\ & & & & & & (\alpha^3+3\alpha^2+3\alpha+1)x^3+(3\alpha+3)x^2+2x \\ & & & & & & 6 \\ & & & & & & (2\alpha^2+4\alpha+2)x^2+x \\ & & & & & & (3\alpha+3)x \\ & & & & & & 1 \end{array}$$

Using Eq. (17), the composita for the compositional inverse generating function $\overline{D}(t)$ of $D(t) = tF(t, x, \alpha)$ is

$$\overline{D}^\Delta(n, k) = \frac{k}{n} F_{n-k}(-nx, \alpha). \quad (42)$$

The triangular form of this composita is

$$\begin{array}{ccccccc} & & & & & & 1 \\ & & & & & & -(\alpha+1)x \\ & & & & & & 1 \\ & & & & & & (3\alpha^2+6\alpha+3)x^2-x \\ & & & & & & 2 \\ & & & & & & -(2\alpha+2)x \\ & & & & & & 1 \\ & & & & & & (8\alpha^3+24\alpha^2+24\alpha+8)x^3-(6\alpha+6)x^2+x \\ & & & & & & 3 \\ & & & & & & (4\alpha^2+8\alpha+4)x^2-x \\ & & & & & & -(3\alpha+3)x \\ & & & & & & 1 \end{array}$$

Also we can get the following new identities for the generalized Sylvester polynomials:

$$\sum_{m=k}^n \frac{m}{n} F_{n-m}(-nx, \alpha) F_{m-k}(kx, \alpha) = \delta_{n,k} \quad (43)$$

and

$$\sum_{m=k}^n F_{n-m}(mx, \alpha) \frac{k}{m} F_{m-k}(-mx, \alpha) = \delta_{n,k}. \quad (44)$$

2.5 Generalized Laguerre polynomials

The generalized Laguerre polynomials are defined by the following generating function [8]:

$$L(t, x, \alpha) = (1-t)^{-\alpha-1} e^{\frac{xt}{1-t}} = \left(e^{\frac{t}{1-t}} \right)^x \left(\frac{1}{1-t} \right)^{\alpha+1} = \sum_{n \geq 0} L_n^{(\alpha)}(x) t^n, \quad (45)$$

where

$$L_n^{(\alpha)}(x) = \sum_{i=0}^n \frac{(-x)^i}{i!} \binom{n+\alpha}{n-i}. \quad (46)$$

According to Eq. (13), the composita for the generating function $D(t) = tL(t, x, \alpha)$ is

$$D^\Delta(n, k) = L_{n-k}^{(k\alpha+k-1)}(kx). \quad (47)$$

The triangular form of this composita is

$$\begin{array}{c} 1 \\ -x + \alpha + 1 \quad 1 \\ \frac{x^2 - (2\alpha + 4)x + \alpha^2 + 3\alpha + 2}{2} \quad -2x + 2\alpha + 2 \quad 1 \end{array}$$

Using Eq. (17), the composita for the compositional inverse generating function $\overline{D}(t)$ of $D(t) = tL(t, x, \alpha)$ is

$$\overline{D}^\Delta(n, k) = \frac{k}{n} L_{n-k}^{(-n\alpha-n-1)}(-nx). \quad (48)$$

The triangular form of this composita is

$$\begin{array}{c} 1 \\ x - \alpha - 1 \quad 1 \\ \frac{3x^2 - (6\alpha + 4)x + 3\alpha^2 + 5\alpha + 2}{2} \quad 2x - 2\alpha - 2 \quad 1 \end{array}$$

Also we can get the following new identities for the generalized Laguerre polynomials:

$$\sum_{m=k}^n \frac{m}{n} L_{n-m}^{(-n\alpha-n-1)}(-nx) L_{m-k}^{(k\alpha+k-1)}(kx) = \delta_{n,k} \quad (49)$$

and

$$\sum_{m=k}^n L_{n-m}^{(m\alpha+m-1)}(mx) \frac{k}{m} L_{m-k}^{(-m\alpha-m-1)}(-mx) = \delta_{n,k}. \quad (50)$$

2.6 Abel polynomials

The Abel polynomials are defined by the following generating function [8, 41]:

$$A(t, x, \alpha) = e^{\frac{W(\alpha t)x}{\alpha}} = \left(e^{\frac{W(\alpha t)}{\alpha}} \right)^x = \sum_{n \geq 0} A_n(x, \alpha) \frac{t^n}{n!}, \quad (51)$$

where $W(t)$ is the Lambert W function and

$$A_n(x, \alpha) = x(x - \alpha n)^{n-1}. \quad (52)$$

According to Eq. (13), the composita for the generating function $D(t) = tA(t, x, \alpha)$ is

$$D^\Delta(n, k) = \frac{A_{n-k}(kx, \alpha)}{(n-k)!}. \quad (53)$$

The triangular form of this composita is

$$\begin{array}{ccccccc} 1 & & & & & & \\ x & & & & 1 & & \\ \frac{x^2 - 2\alpha x}{2} & & & 2x & & 1 & \\ \frac{x^3 - 6\alpha x^2 + 9\alpha^2 x}{6} & & 2x^2 - 2\alpha x & & 3x & & 1 \\ \frac{x^4 - 12\alpha x^3 + 48\alpha^2 x^2 - 64\alpha^3 x}{24} & \frac{4x^3 - 12\alpha x^2 + 9\alpha^2 x}{3} & \frac{9x^2 - 6\alpha x}{2} & 4x & & 1 & \end{array}$$

Using Eq. (17), the composita for the compositional inverse generating function $\bar{D}(t)$ of $D(t) = tA(t, x, \alpha)$ is

$$\bar{D}^\Delta(n, k) = \frac{k}{n} \frac{A_{n-k}(-nx, \alpha)}{(n-k)!}. \quad (54)$$

The triangular form of this composita is

$$\begin{array}{ccccccc} 1 & & & & & & \\ -x & & & & 1 & & \\ \frac{3x^2 + 2\alpha x}{2} & & & -2x & & 1 & \\ -\frac{16x^3 + 24\alpha x^2 + 9\alpha^2 x}{6} & & 4x^2 + 2\alpha x & & -3x & & 1 \\ \frac{125x^4 + 300\alpha x^3 + 240\alpha^2 x^2 + 64\alpha^3 x}{24} & -\frac{25x^3 + 30\alpha x^2 + 9\alpha^2 x}{3} & \frac{15x^2 + 6\alpha x}{2} & -4x & & 1 & \end{array}$$

Also we can get the following new identities for the Abel polynomials:

$$\sum_{m=k}^n \frac{m}{n} \frac{A_{n-m}(-nx, \alpha)}{(n-m)!} \frac{A_{m-k}(kx, \alpha)}{(m-k)!} = \delta_{n,k} \quad (55)$$

and

$$\sum_{m=k}^n \frac{A_{n-m}(mx, \alpha)}{(n-m)!} \frac{k}{m} \frac{A_{m-k}(-mx, \alpha)}{(m-k)!} = \delta_{n,k}. \tag{56}$$

2.7 Bessel polynomials

The Bessel polynomials are defined by the following generating function [8]:

$$B(t, x) = e^{x(1-\sqrt{1-2t})} = \left(e^{1-\sqrt{1-2t}} \right)^x = \sum_{n \geq 0} B_n(x) \frac{t^n}{n!}, \tag{57}$$

where

$$B_n(x) = \begin{cases} 1, & n = 0; \\ \sum_{k=1}^n \frac{(2n-k-1)!}{(n-k)!(k-1)!} \frac{x^k}{2^{n-k}}, & n > 0. \end{cases} \tag{58}$$

According to Eq. (13), the composita for the generating function $D(t) = tB(t, x)$ is

$$D^\Delta(n, k) = \frac{B_{n-k}(kx)}{(n-k)!}. \tag{59}$$

The triangular form of this composita is

$$\begin{array}{ccccccc} & & & & & & 1 \\ & & & & & & 2 \\ & & & & & & \frac{x^2+x}{2} \\ & & & & & & 2x & & 1 \\ & & & & & & \frac{x^3+3x^2+3x}{6} & & 2x^2+x & & 3x & & 1 \\ & & & & & & \frac{x^4+6x^3+15x^2+15x}{24} & & \frac{4x^3+6x^2+3x}{3} & & \frac{9x^2+3x}{2} & & 4x & & 1 \end{array}$$

Using Eq. (17), the composita for the compositional inverse generating function $\overline{D}(t)$ of $D(t) = tB(t, x)$ is

$$\overline{D}^\Delta(n, k) = \frac{k}{n} \frac{B_{n-k}(-nx)}{(n-k)!}. \tag{60}$$

The triangular form of this composita is

$$\begin{array}{ccccccc} & & & & & & 1 \\ & & & & & & -x \\ & & & & & & \frac{3x^2-x}{2} \\ & & & & & & -2x & & 1 \\ & & & & & & \frac{16x^3-12x^2+3x}{6} & & 4x^2-x & & -3x & & 1 \\ & & & & & & \frac{125x^4-150x^3+75x^2-15x}{24} & & \frac{25x^3-15x^2+3x}{3} & & \frac{15x^2-3x}{2} & & -4x & & 1 \end{array}$$

Also we can get the following new identities for the Bessel polynomials:

$$\sum_{m=k}^n \frac{m}{n} \frac{B_{n-m}(-nx)}{(n-m)!} \frac{B_{m-k}(kx)}{(m-k)!} = \delta_{n,k} \quad (61)$$

and

$$\sum_{m=k}^n \frac{B_{n-m}(mx)}{(n-m)!} \frac{k}{m} \frac{B_{m-k}(-mx)}{(m-k)!} = \delta_{n,k}. \quad (62)$$

2.8 Stirling polynomials

The Stirling polynomials are defined by the following generating function [8, 42]:

$$S(t, x) = \left(\frac{t}{1 - e^{-t}} \right)^x = \sum_{n \geq 0} S_n(x) \frac{t^n}{n!}, \quad (63)$$

where

$$S_n(x) = \sum_{i=0}^n \binom{x+i}{i} \sum_{j=0}^i \frac{j!}{(n+j)!} (-1)^{n+j} \binom{i}{j} \left\{ \begin{matrix} n+j \\ j \end{matrix} \right\}. \quad (64)$$

According to Eq. (13), the composita for the generating function $D(t) = tS(t, x)$ is

$$D^\Delta(n, k) = S_{n-k}(kx + k - 1). \quad (65)$$

The triangular form of this composita is

$\frac{1}{x+1}$	1		
$\frac{2}{3x^2+5x+2}$	$x+1$	1	
$\frac{24}{x^3+2x^2+x}$	$\frac{6x^2+11x+5}{12}$	$\frac{3x+3}{2}$	1
$\frac{48}{15x^4+30x^3+5x^2-18x-8}$	$\frac{2x^3+5x^2+4x+1}{12}$	$\frac{9x^2+17x+8}{8}$	$2x+2$
5760	12	8	1

Using Eq. (17), the composita for the compositional inverse generating function $\overline{D}(t)$ of $D(t) = tS(t, x)$ is

$$\overline{D}^{\Delta}(n, k) = \frac{k}{n} S_{n-k}(-nx - n - 1). \quad (66)$$

The triangular form of this composita is

$$\begin{array}{r}
 \frac{1}{x+2} \\
 - \frac{2}{2} \\
 \hline
 9x^2 + 19x + 10 \\
 \frac{24}{24} \\
 - \frac{4x^3 + 13x^2 + 14x + 5}{12} \\
 \hline
 1875x^4 + 8250x^3 + 13525x^2 + 9798x + 2648 \\
 \hline
 5760
 \end{array}
 \qquad
 \begin{array}{r}
 1 \\
 -x-1 \\
 \hline
 12x^2 + 25x + 13 \\
 \frac{12}{12} \\
 - \frac{25x^3 + 80x^2 + 85x + 30}{24}
 \end{array}
 \qquad
 \begin{array}{r}
 1 \\
 \hline
 3x+3 \\
 - \frac{2}{2} \\
 \hline
 15x^2 + 31x + 16 \\
 \frac{8}{8}
 \end{array}
 \qquad
 \begin{array}{r}
 1 \\
 -2x-2 \\
 \hline
 1
 \end{array}$$

Also we can get the following new identities for the Stirling polynomials:

$$\sum_{m=k}^n \frac{m}{n} S_{n-m}(-nx - n - 1) S_{m-k}(kx + k - 1) = \delta_{n,k} \quad (67)$$

and

$$\sum_{m=k}^n S_{n-m}(mx + m - 1) \frac{k}{m} S_{m-k}(-mx - m - 1) = \delta_{n,k}. \quad (68)$$

2.9 Narumi polynomials

The Narumi polynomials are defined by the following generating function [8]:

$$S(t, x, \alpha) = \left(\frac{t}{\ln(1+t)} \right)^\alpha (1+t)^x = (1+t)^x \left(\frac{t}{\ln(1+t)} \right)^\alpha = \sum_{n \geq 0} S_n(x, \alpha) \frac{t^n}{n!}, \quad (69)$$

where

$$S_n(x, \alpha) = n! \sum_{i=0}^n \binom{x}{n-i} \sum_{j=0}^i \binom{j+\alpha-1}{j} \sum_{l=0}^j (-1)^l \binom{j}{l} \frac{l!}{(l+i)!} \begin{bmatrix} l+i \\ l \end{bmatrix}. \quad (70)$$

According to Eq. (13), the composita for the generating function $D(t) = tS(t, x, \alpha)$ is

$$D^\Delta(n, k) = \frac{S_{n-k}(kx, k\alpha)}{(n-k)!}. \quad (71)$$

The triangular form of this composita is

$$\begin{array}{ccccccc} & & 1 & & & & \\ & & \frac{2x+\alpha}{2} & & & & \\ & & 2 & & & & \\ & & 12x^2 + (12\alpha - 12)x + 3\alpha^2 - 5\alpha & & & & \\ & & 24 & & & & \\ & & 8x^3 + (12\alpha - 24)x^2 + (6\alpha^2 - 22\alpha + 16)x + \alpha^3 - 5\alpha^2 + 6\alpha & & & & \\ & & 48 & & & & \\ & & 24x^2 + (24\alpha - 12)x + 6\alpha^2 - 5\alpha & & & & \\ & & 12 & & & & \\ & & 6x + 3\alpha & & & & \\ & & 2 & & & & \\ & & 1 & & & & \end{array}$$

Using Eq. (17), the composita for the compositional inverse generating function $\overline{D}(t)$ of $D(t) = tS(t, x, \alpha)$ is

$$\overline{D}^\Delta(n, k) = \frac{k S_{n-k}(-nx, -n\alpha)}{n (n-k)!}. \quad (72)$$

The triangular form of this composita is

$$\begin{array}{ccccccc} & & 1 & & & & \\ & & -\frac{2x+\alpha}{2} & & & & \\ & & 2 & & & & \\ & & 36x^2 + (36\alpha + 12)x + 9\alpha^2 - 5\alpha & & & & \\ & & 24 & & & & \\ & & 64x^3(96\alpha + 48)x^2 + (48\alpha^2 + 44\alpha + 8)x + 8\alpha^3 + 10\alpha^2 + 3\alpha & & & & \\ & & 24 & & & & \\ & & 48x^2 + (48\alpha + 12)x + 12\alpha^2 + 5\alpha & & & & \\ & & 12 & & & & \\ & & -\frac{6x+3\alpha}{2} & & & & \\ & & 1 & & & & \end{array}$$

Also we can get the following new identities for the Narumi polynomials:

$$\sum_{m=k}^n \frac{m}{n} \frac{S_{n-m}(-nx, -n\alpha)}{(n-m)!} \frac{S_{m-k}(kx, k\alpha)}{(m-k)!} = \delta_{n,k} \quad (73)$$

and

$$\sum_{m=k}^n \frac{S_{n-m}(mx, m\alpha)}{(n-m)!} \frac{k}{m} \frac{S_{m-k}(-mx, -m\alpha)}{(m-k)!} = \delta_{n,k}. \quad (74)$$

2.10 Peters polynomials

The Peters polynomials are defined by the following generating function [8]:

$$S(t, x, \mu, \lambda) = \left(1 + (1+t)^\lambda\right)^{-\mu} (1+t)^x = (1+t)^x \left(\frac{1}{1 + (1+t)^\lambda}\right)^\mu = \sum_{n \geq 0} S_n(x, \mu, \lambda) \frac{t^n}{n!}, \quad (75)$$

where

$$S_n(x, \mu, \lambda) = n! \sum_{i=0}^n \binom{x}{n-i} \sum_{j=0}^i \frac{1}{2^{j+\mu}} \binom{j+\mu-1}{j} \sum_{l=0}^j (-1)^l \binom{j}{l} \binom{l\lambda}{i}. \quad (76)$$

According to Eq. (13), the composita for the generating function $D(t) = tS(t, x, \mu, \lambda)$ is

$$D^\Delta(n, k) = \frac{S_{n-k}(kx, k\mu, \lambda)}{(n-k)!}. \quad (77)$$

The triangular form of this composita is

$$\begin{array}{ccc} 2^{-\mu} & & \\ 2^{-\mu-1}(2x - \lambda\mu) & & 2^{-2\mu} \\ 2^{-\mu-3}(4x^2 - (4\lambda\mu + 4)x + \lambda^2\mu^2 - \lambda^2\mu + 2\lambda\mu) & 2^{-2\mu}(2x - \lambda\mu) & 2^{-3\mu} \end{array}$$

Using Eq. (17), the composita for the compositional inverse generating function $\overline{D}(t)$ of $D(t) = tS(t, x, \mu, \lambda)$ is

$$\overline{D}^\Delta(n, k) = \frac{k}{n} \frac{S_{n-k}(-nx, -n\mu, \lambda)}{(n-k)!}. \quad (78)$$

The triangular form of this composita is

$$\begin{array}{ccc} 2^\mu & & \\ 2^{2\mu-1}(-2x + \lambda\mu) & & 2^{2\mu} \\ 2^{3\mu-3}(12x^2 + (4 - 12\lambda\mu)x + 3\lambda^2\mu^2 + \lambda^2\mu - 2\lambda\mu) & 2^{3\mu}(\lambda\mu - 2x) & 2^{3\mu} \end{array}$$

Also we can get the following new identities for the Peters polynomials:

$$\sum_{m=k}^n \frac{m}{n} \frac{S_{n-m}(-nx, -n\mu, \lambda)}{(n-m)!} \frac{S_{m-k}(kx, k\mu, \lambda)}{(m-k)!} = \delta_{n,k} \quad (79)$$

and

$$\sum_{m=k}^n \frac{S_{n-m}(mx, m\mu, \lambda)}{(n-m)!} \frac{k}{m} \frac{S_{m-k}(-mx, -m\mu, \lambda)}{(m-k)!} = \delta_{n,k}. \quad (80)$$

2.11 Gegenbauer polynomials

The Gegenbauer polynomials are defined by the following generating function [43]:

$$C(t, x, \alpha) = (1 - 2xt + t^2)^{-\alpha} = \left(\frac{1}{1 - 2xt + t^2} \right)^\alpha = \sum_{n \geq 0} C_n^{(\alpha)}(x) t^n, \quad (81)$$

where

$$C_n^{(\alpha)}(x) = \sum_{i=0}^n (-1)^{n-i} \binom{i}{n-i} \binom{i+\alpha-1}{i} (2x)^{2i-n}. \quad (82)$$

According to Eq. (13), the composita for the generating function $D(t) = tC(t, x, \alpha)$ is

$$D^\Delta(n, k) = C_{n-k}^{(k\alpha)}(x). \quad (83)$$

The triangular form of this composita is

$$\begin{array}{ccccccc} & & 1 & & & & \\ & & 2\alpha x & & & 1 & \\ & & (2\alpha^2 + 2\alpha)x^2 - \alpha & & 4\alpha x & & 1 \\ & & \frac{(4\alpha^3 + 12\alpha^2 + 8\alpha)x^3 - (6\alpha^2 + 6\alpha)x}{3} & & (8\alpha^2 + 4\alpha)x^2 - 2\alpha & 6\alpha x & 1 \end{array}$$

Using Eq. (17), the composita for the compositional inverse generating function $\overline{D}(t)$ of $D(t) = tC(t, x, \alpha)$ is

$$\overline{D}^\Delta(n, k) = \frac{k}{n} C_{n-k}^{(-n\alpha)}(x). \quad (84)$$

The triangular form of this composita is

$$\begin{array}{ccccccc} & & 1 & & & & \\ & & -2\alpha x & & & 1 & \\ & & (6\alpha^2 - 2\alpha)x^2 + \alpha & & -4\alpha x & & 1 \\ & & \frac{(64\alpha^3 - 48\alpha^2 + 8\alpha)x^3 + (24\alpha^2 - 6\alpha)x}{3} & & (16\alpha^2 - 4\alpha)x^2 + 2\alpha & -6\alpha x & 1 \end{array}$$

Also we can get the following new identities for the Gegenbauer polynomials:

$$\sum_{m=k}^n \frac{m}{n} C_{n-m}^{(-n\alpha)}(x) C_{m-k}^{(k\alpha)}(x) = \delta_{n,k} \quad (85)$$

and

$$\sum_{m=k}^n C_{n-m}^{(m\alpha)}(x) \frac{k}{m} C_{m-k}^{(-m\alpha)}(x) = \delta_{n,k}. \quad (86)$$

2.12 Meixner polynomials of the first kind

The Meixner polynomials of the first kind are defined by the following generating function [8, 44]:

$$M(t, x, \beta, c) = \left(1 - \frac{t}{c}\right)^x (1-t)^{-x-\beta} = \left(\frac{c-t}{c(1-t)}\right)^x \left(\frac{1}{1-t}\right)^\beta = \sum_{n \geq 0} M_n(x, \beta, c) \frac{t^n}{n!}, \quad (87)$$

where

$$M_n(x, \beta, c) = (-1)^n n! \sum_{i=0}^n \binom{x}{i} \binom{-x-\beta}{n-i} c^{-i}. \quad (88)$$

According to Eq. (13), the composita for the generating function $D(t) = tM(t, x, \beta, c)$ is

$$D^\Delta(n, k) = \frac{M_{n-k}(kx, k\beta, c)}{(n-k)!}. \quad (89)$$

The triangular form of this composita is

$$\begin{array}{c} 1 \\ \frac{(c-1)x + \beta c}{c} \quad 1 \\ \frac{(c^2 - 2c + 1)x^2 + ((2\beta + 1)c^2 - 2\beta c - 1)x + (\beta^2 + \beta)c^2}{2c^2} \quad \frac{(2c-2)x + 2\beta c}{c} \quad 1 \end{array}$$

Using Eq. (17), the composita for the compositional inverse generating function $\overline{D}(t)$ of $D(t) = tM(t, x, \beta, c)$ is

$$\overline{D}^\Delta(n, k) = \frac{k}{n} \frac{M_{n-k}(-nx, -n\beta, c)}{(n-k)!}. \quad (90)$$

The triangular form of this composita is

$$\begin{array}{c} 1 \\ \frac{(1-c)x - \beta c}{c} \quad 1 \\ \frac{(3c^2 - 6c + 3)x^2 + ((6\beta - 1)c^2 - 6\beta c + 1)x + (3\beta^2 - \beta)c^2}{2c^2} \quad \frac{(2-2c)x - 2\beta c}{c} \quad 1 \end{array}$$

Also we can get the following new identities for the Meixner polynomials of the first kind:

$$\sum_{m=k}^n \frac{m}{n} \frac{M_{n-m}(-nx, -n\beta, c)}{(n-m)!} \frac{M_{m-k}(kx, k\beta, c)}{(m-k)!} = \delta_{n,k} \quad (91)$$

and

$$\sum_{m=k}^n \frac{M_{n-m}(mx, m\beta, c)}{(n-m)!} \frac{k}{m} \frac{M_{m-k}(-mx, -m\beta, c)}{(m-k)!} = \delta_{n,k}. \quad (92)$$

3. Conclusions and future developments

In this chapter, we find new explicit formulas and identities for such polynomials as the generalized Bernoulli, generalized Euler, Frobenius-Euler, generalized

Sylvester, generalized Laguerre, Abel, Bessel, Stirling, Narumi, Peters, Gegenbauer, and Meixner polynomials that are defined by generating functions of the form $A(t, x, \alpha) = F(t)^x \cdot G(t)^\alpha$.

A lot of studies have recently showed that polynomials are a solution for practical problems related to modeling, quantum mechanics, and other areas. So a study of obtaining explicit formulas and representations of polynomials will be important and influential. Also the further research can be conducted to find practical means of obtained properties.

Acknowledgements


This work was partially funded by the Russian Foundation for Basic Research and the government of the Tomsk region of Russian Federation (Grants No. 18-41-703006) and the Ministry of Education and Science of Russian Federation (Government Order No. 2.8172.2017/8.9, TUSUR).

Author details

Dmitry Kruchinin*, Vladimir Kruchinin and Yuriy Shablya
Tomsk State University of Control Systems and Radioelectronics, Tomsk, Russia

*Address all correspondence to: kruchinindm@gmail.com

IntechOpen

© 2019 The Author(s). Licensee IntechOpen. This chapter is distributed under the terms of the Creative Commons Attribution License (<http://creativecommons.org/licenses/by/3.0>), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited. 

References

- [1] Comtet L. Advanced Combinatorics. Dordrecht, Holland: D. Reidel Publishing Company; 1974. 343p
- [2] Flajolet P, Sedgewick R. Analytic Combinatorics. United Kingdom: Cambridge University Press; 2009. 810p. DOI: 10.1017/CBO9780511801655
- [3] Graham RL, Knuth DE, Patashnik O. Concrete Mathematics. USA: Addison-Wesley; 1989. 657p
- [4] Robert AM. A Course in p-adic Analysis. New York, USA: Springer; 2000. DOI: 10.1007/978-1-4757-3254-2. 438p
- [5] Stanley RP. Generating functions, in MAA Studies in Combinatorics. In: Rota G-C, editor. Studies in Combinatorics. Mathematical Association of America, Washington DC. 1978. pp. 100-141
- [6] Wilf HS. Generating functionology. USA: Academic Press; 1994. 226p
- [7] Boas RP Jr, Buck RC. Polynomial Expansions of Analytic Functions. Berlin, Heidelberg, Germany: Springer-Verlag; 1964. DOI: 10.1007/978-3-662-25170-6. 77p
- [8] Roman S. The Umbral Calculus. New York, USA: Academic Press; 1984. 195p
- [9] Srivastava HM, Manocha HL. A Treatise on Generating Functions. New York, USA: Halsted Press; 1984. 569p
- [10] Srivastava HM, Choi J. Zeta and q-Zeta Functions and Associated Series and Integrals. Amsterdam, The Netherlands: Elsevier; 2012. 674p
- [11] Srivastava HM. Some generalizations and basic (or q -) extensions of the Bernoulli, Euler and Genocchi polynomials. Applied Mathematics & Information Sciences. 2011;5(3):390-444
- [12] Acikgoz M, Simsek Y. A new generating function of $(q-)$ Bernstein-type polynomials and their interpolation function. Abstract and Applied Analysis. 2010;2010:1-12. DOI: 10.1155/2010/769095
- [13] Ozden H, Simsek Y, Srivastava HM. A unified presentation of the generating functions of the generalized Bernoulli, Euler and Genocchi polynomials. Computers & Mathematics with Applications. 2010;60(10):2779-2787. DOI: 10.1016/j.camwa.2010.09.031
- [14] Simsek Y, Kim D. Identities and recurrence relations of special numbers and polynomials of higher order by analysis of their generating functions. Journal of Inequalities and Applications. 2018;2018:1-9. DOI: 10.1186/s13660-018-1815-7
- [15] Simsek Y. On generating functions for the special polynomials. Univerzitet u Nišu. 2017;31(1):9-16. DOI: 10.2298/FIL1701009S
- [16] Ozdemir G, Simsek Y, Milovanovic GV. Generating functions for special polynomials and numbers including Apostol-type and Humbert-type polynomials. Mediterranean Journal of Mathematics. 2017;14(3):1-17. DOI: 10.1007/s00009-017-0918-6
- [17] Kim T, Kim DS, Jang G-W, Jang L-C. A generalization of some results for Appell polynomials to Sheffer polynomials. Journal of Computational Analysis and Applications. 2019;26(5):889-898
- [18] Simsek Y. Special numbers and polynomials including their generating functions in umbral analysis methods. Axioms. 2018;7(2):1-12. DOI: 10.3390/axioms7020022

- [19] Simsek Y. New families of special numbers for computing negative order Euler numbers and related numbers and polynomials. *Applicable Analysis and Discrete Mathematics*. 2018;**12**(1):1-35. DOI: 10.2298/AADM1801001S
- [20] Simsek Y. Construction method for generating functions of special numbers and polynomials arising from analysis of new operators. *Mathematical Methods in the Applied Sciences*. 2018;**41**(16):6934-6954. DOI: 10.1002/mma.5207
- [21] Kim DS, Kim T. On degenerate Bell numbers and polynomials. *Iranian Journal of Science and Technology*. 2017;**41**(3):749-753. DOI: 10.1007/s13398-016-0304-4
- [22] Kim T, Kim DS, Kwon H-I. Some identities of Carlitz degenerate Bernoulli numbers and polynomials. *Rev. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas RACSAM*. 2017;**111**(2):435-446. DOI: 10.1007/s40995-017-0286-x
- [23] Kim T, Ryoo CS. Some identities for Euler and Bernoulli polynomials and their zeros. *Axioms*. 2018;**7**(3):1-19. DOI: 10.3390/axioms7030056
- [24] Ryoo CS. Some identities involving generalized degenerate tangent polynomials arising from differential equations. *Journal of Computational Analysis and Applications*. 2019;**26**(6): 975-984
- [25] Ryoo CS. Some identities involving modified degenerate tangent numbers and polynomials. *Global Journal of Pure and Applied Mathematics*. 2016;**12**(3): 2621-2630
- [26] Zhao J-L, Qi F. Two explicit formulas for the generalized Motzkin numbers. *Journal of Inequalities and Applications*. 2017;**2017**(1):1-8. DOI: 10.1186/s13660-017-1313-3
- [27] Qi F, Cernanova V, Shi X-T, Guo B-N. Some properties of central Delannoy numbers. *Journal of Computational and Applied Mathematics*. 2018;**328**:101-115. DOI: 10.1016/j.cam.2017.07.013
- [28] Liu G-D, Srivastava HM. Explicit formulas for the Norlund polynomials $B_n^{(x)}$ and $b_n^{(x)}$. *Computers & Mathematics with Applications*. 2006;**51**(9-10):1377-1384. DOI: 10.1016/j.camwa.2006.02.003
- [29] Srivastava HM, Todorov PG. An explicit formula for the generalized Bernoulli polynomials. *Journal of Mathematical Analysis and Applications*. 1988;**130**(2):509-513. DOI: 10.1016/0022-247X(88)90326-5
- [30] Cenkci M. An explicit formula for generalized potential polynomials and its applications. *Discrete Mathematics*. 2009;**309**:1498-1510. DOI: 10.1016/j.disc.2008.02.021
- [31] Boyadzhiev KN. Derivative polynomials for Tanh, Tan, Sech and Sec in explicit form. *The Fibonacci Quarterly*. 2007;**45**(4):291-303
- [32] Stanley RP. *Enumerative Combinatorics*. Vol. 2. Cambridge University Press; 1999. 600p
- [33] Kruchinin DV, Kruchinin VV. A method for obtaining generating functions for central coefficients of triangles. *Journal of Integer Sequences*. 2012;**15**:1-10
- [34] Kruchinin DV, Kruchinin VV. Application of a composition of generating functions for obtaining explicit formulas of polynomials. *Journal of Mathematical Analysis and Applications*. 2013;**404**(1):161-171. DOI: 10.1016/j.jmaa.2013.03.009
- [35] Kruchinin VV, Kruchinin DV. Composita and its properties. *Journal of*

Analysis and Number Theory. 2014;
2:37-44

[36] Kruchinin DV, Shablya YV, Kruchinin VV, Shelupanov AA. A method for obtaining coefficients of compositional inverse generating functions. In: International Conference of Numerical Analysis and Applied Mathematics (ICNAAM 2015); 23–29 September 2015; Rodos: American Institute of Physics Inc.; 2016. pp. 1-4

[37] Boutiche MA, Rahmani M, Srivastava HM. Explicit formulas associated with some families of generalized Bernoulli and Euler polynomials. *Mediterranean Journal of Mathematics*. 2017;**14**(2):1-10. DOI: 10.1007/s00009-017-0891-0

[38] Elezovic N. Generalized Bernoulli polynomials and numbers, revisited. *Mediterranean Journal of Mathematics*. 2016;**13**(1):141-151. DOI: 10.1007/s00009-014-0498-7

[39] Srivastava HM, Boutiche MA, Rahmani M. Some explicit formulas for the Frobenius-Euler polynomials of higher order. *Applied Mathematics & Information Sciences*. 2017;**11**(2): 621-626. DOI: 10.18576/amis/110234

[40] Agarwal AK. A note on generalized Sylvester polynomials. *Indian Journal of Pure and Applied Mathematics*. 1984;**15**: 431-434

[41] Kim DS, Kim T, Lee S-H, Rim S-H. Some identities of Bernoulli, Euler and Abel polynomials arising from umbral calculus. *Adv. Difference Equ.* 2013; **2013**:1-8. DOI: 10.1186/1687-1847-2013-15

[42] Kang JY, Ryoo CS. A research on the some properties and distribution of zeros for Stirling polynomials. *Journal of Nonlinear Sciences and Applications*. 2016;**9**(4):1735-1747

[43] Prajapati JC, Choi J, Kachhia KB, Agarwal P. Certain properties of Gegenbauer polynomials via Lie algebra. *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas RACSAM*. 2017;**111**(4): 1031-1037. DOI: 10.1007/s13398-016-0343-x

[44] Kruchinin DV, Shablya YV. Explicit formulas for Meixner polynomials. *International Journal of Mathematics and Mathematical Sciences*. 2015;**2015**: 1-5. DOI: 10.1155/2015/620569