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A Volume Integral Equation Method for the Direct/Inverse Problem in Elastic Wave Scattering Phenomena

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1. Introduction

The analysis of elastic wave propagation and scattering is an important issue in fields such as earthquake engineering, nondestructive testing, and exploration for energy resources. Since the 1980s, the boundary integral equation method has played an important role in the analysis of both forward and inverse scattering problems. For example, Colton and Kress (1998) presented a survey of a vast number of articles on forward and inverse scattering analyses. They also presented integral equation methods for acoustic and electromagnetic wave propagation, based on the theory of operators (1983 and 1998). Recently, Guzina, Fata and Bonnet (2003), Fata and Guzina (2004), and Guzina and Chikichev (2007) have dealt with inverse scattering problems in elastodynamics.

The type of volume integral equation known as the Lippmann–Schwinger equation (Colton & Kress, 1998) has been an efficient tool for theoretical investigation in the field of quantum mechanics (see, for example, Ikebe, 1960). Several applications of the volume integral equation to scattering analysis for classical mechanics have also appeared. For example, Hudson and Heritage (1981) used the Born approximation of the solution of the volume integral equation obtained from the background structure of the wave field for the seismic scattering problem. Recently, Zaeytijd, Bogaert, and Franchois (2008) presented the MLFMA-FFT method for analyzing electro-magnetic waves, and Yang, Abubaker, van den Berg et al. (2008) used a CG-FFT approach to solve elastic scattering problems. These methods were used to establish a fast algorithm to solve the volume integral equation via a Fast Fourier transform, which is used for efficient calculation of the convolution integral.

In this chapter, another method for the volume integral equation is presented for the direct forward and inverse elastic wave scattering problems for 3-D elastic full space. The starting point of the analysis is the volume integral equation in the wavenumber domain, which includes the operators of the Fourier integral and its inverse transforms. This starting point yields a different method from previous approaches (for example, Yang et al., 2008). By replacing these operators with discrete Fourier transforms, the volume integral equation in the wavenumber domain can be treated as a Fredholm equation of the 2nd kind with a non-Hermitian operator on a finite dimensional vector space, which is to be solved by the Krylov subspace iterative scheme (Touhei et al, 2009). As a result, the derivation of the coefficient matrix for the volume integral equation is not necessary. Furthermore, by means of the Fast

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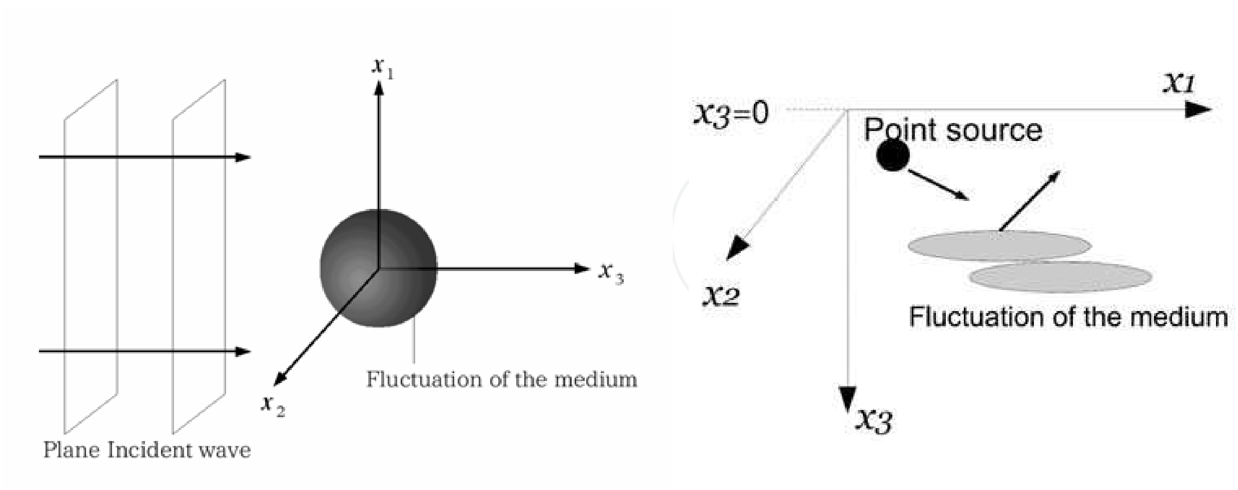
Fourier transform, a fast method for the volume integral equation can be established. The method itself can be extended to the scattering problem for a 3-D elastic half space (Touhei, 2009). This chapter also presents the possibility of the volume integral equation method for 3-D elastic half space by constructing a generalized Fourier transform for the half space.

An important property of the volume integral equation in the wavenumber domain is that it separates the scattered wave field from the fluctuation of the medium. This property yields the possibility of inverse scattering analysis. There are several methods for inverse scattering analysis that make use of the volume integral equation (for example, Kleinman and van den Berg (1992); Colton & Kress (1998)). These methods can be used to investigate the relationship between the far field patterns and the fluctuation of the medium in the volume integral equation in the space domain. Under these circumstances, for the inverse scattering analysis, the possibility of solving the volume integral equation in the wavenumber domain should also be investigated.

In this chapter, basic equations for elastic wave propagation are first presented in order to prepare the formulation. After clarifying the properties of the volume integral equation in the wavenumber domain, a method for solving the volume integral equation is developed.

2. Basic equations for elastic wave propagation

Figures 1(a) and (b) show the concept of the problem discussed in this chapter. Figure 1(a) shows a 3-D elastic full space, in which a plane incident wave is propagating along the x_3 axis towards an inhomogeneous region where material properties fluctuate with respect to their reference values. Figure 1(b) is a 3-D elastic half space. Here, waves from a point source propagate towards an inhomogeneous region. Scattered waves are generated by the interactions between the incident waves and the fluctuating areas. This chapter considers a volume integral equation method for solving the scattering problem for both a 3-D elastic full space and a half space. At this stage, basic equations are presented as the starting point of the formulation.



(a) Scattering problem in a 3-D elastic full space

(b) Scattering problem in a 3-D elastic half space

Fig. 1. Concept of the analyzed model.

A Cartesian coordinate system is used for the wave field. A spatial point in the wave field is expressed as:

$$x = (x_1, x_2, x_3) \tag{1}$$

where the subscript index indicates the component of the vector. For the case in which an elastic half space is considered, x_3 denotes the vertical coordinate with the positive direction downward, where the free surface boundary is denoted by $x_3 = 0$.
The fluctuation of the medium is expressed by the Lamé constants so that:

$$\begin{aligned} \lambda(x) &= \lambda_0 + \tilde{\lambda}(x) \\ \mu(x) &= \mu_0 + \tilde{\mu}(x) \end{aligned} \tag{2}$$

where λ_0 and μ_0 are the background Lamé constants of the wave field, and $\tilde{\lambda}$ and $\tilde{\mu}$ are their fluctuations. The back ground Lamé constants are positive and bounded. The magnitudes of the fluctuations are assumed to satisfy

$$|\tilde{\lambda}(x)| < \lambda_0, \quad |\tilde{\mu}(x)| < \mu_0 \tag{3}$$

Let the time factor of the wave field be $\exp(-i\omega t)$, where ω is the circular frequency and t is the time. Then, the equilibrium equation of the wave field taking into account the effects of a point source is expressed as:

$$\partial_j \sigma_{ij} + \rho\omega^2 u_i = -q_i \delta(x - x_s) \tag{4}$$

where σ_{ij} is the stress tensor, ∂_j is the partial differential operator, ρ is the mass density, u_i is the total displacement field, q_i is the amplitude of the point source, x_s is the position at which the point source is applied, and $\delta(\cdot)$ is the Dirac delta function. The subscript indices i and j in Eq. (4) are the components of the Cartesian coordinate system to which the summation convention is applied. The constitutive equation showing the relationship between the stress and strain tensors is as follows:

$$\sigma_{ij} = \lambda \delta_{ij} \epsilon_{kk} + 2\mu \epsilon_{ij} \tag{5}$$

where δ_{ij} is the Kronecker delta, and ϵ_{ij} is the strain tensor given by

$$\epsilon_{ij} = (1/2)\Big(\partial_i u_j + \partial_j u_i\Big) \tag{6}$$

Substituting Eqs. (6) and (5) into Eq. (4) yields the following governing equation for the current problem:

$$\Big(L_{ij}(\partial_1, \partial_2, \partial_3) + \delta_{ij}\rho\omega^2\Big) u_j(x) = N_{ij}(\partial_1, \partial_2, \partial_3, x) u_j(x) - q_i \delta(x - x_s) \tag{7}$$

where L_{ij} $(\partial_1, \partial_2, \partial_3)$ and N_{ij} $(\partial_1, \partial_2, \partial_3, x)$ are the differential operators constructed by the background Lamé constants and their fluctuations, respectively. The explicit forms of the operators L_{ij} and N_{ij} are given by

$$L_{ij}(\partial_1, \partial_2, \partial_3) \quad = \quad (\lambda_0 + \mu_0)\partial_i \partial_j + \mu_0 \delta_{ij} \partial_k \partial_k \tag{8}$$

$$
\begin{aligned}
N_{ij}(\partial_1, \partial_2, \partial_3, x) &= -\left(\tilde{\lambda}(x) + \tilde{\mu}(x)\right)\partial_i\partial_j - \delta_{ij}\tilde{\mu}(x)\partial_k\partial_k \\
&\quad -\partial_i\tilde{\lambda}(x)\partial_j - \delta_{ij}\partial_k\tilde{\mu}(x)\partial_k - \partial_j\tilde{\mu}(x)\partial_i
\end{aligned} \tag{9}
$$

For the case in which an elastic half space is considered, the free boundary conditions are necessary and are expressed by

$$
P_{ij}u_j(x) = 0, \quad (\text{at } x_3 = 0) \tag{10}
$$

where P_{ij} is the operator describing the free boundary condition having the following components:

$$
[P_{ij}] = \begin{bmatrix} \mu(x)\partial_3 & 0 & \mu(x)\partial_1 \\ 0 & \mu(x)\partial_3 & \mu(x)\partial_2 \\ \lambda(x)\partial_1 & \lambda(x)\partial_2 & (\lambda(x) + 2\mu(x))\partial_3 \end{bmatrix} \tag{11}
$$

3. Method for forward and inverse scattering analysis in the elastic full space based on the volume integral equation

3.1 Definition of the forward and inverse scattering problem

Now, we deal with the concept of the problem shown in Fig. 1(a). The forward and inverse problem for a 3-D elastic full space will be discussed based on the volume integral equation. The forward and inverse scattering problems considered in this section can be described as follows:

Definition 1 *The forward scattering problem is to determine the scattered wave field from information about the regions of fluctuation, the background structure of the wave field, and the plane incident wave.*

Definition 2 *The inverse scattering problem involves reconstructing the fluctuating areas from information about the scattered waves, the background structure of the wave field, and the plane incident wave.*

To clarify the above problems mathematically, the volume integral equation is obtained from Eq. (7). Assume that the right-hand side of Eq. (7) is the inhomogeneous term. Since there is no point source in the wave field shown in Fig. 1(a), the solution of Eq. (7) is expressed by the following volume integral equation:

$$
u_i(x) = F_i(x) - \int_{\mathbb{R}^3} G_{ij}(x,y)N_{jk}(\partial_1, \partial_2, \partial_3, y)u_k(y)dy \tag{12}
$$

where F_i and G_{ij} are the plane incident wave and the Green's function, respectively, which satisfy the following equations:

$$
\left(L_{ij}(\partial_1, \partial_2, \partial_3) + \delta_{ij}\rho\omega^2\right)F_j(x) = 0 \tag{13}
$$

$$
\left(L_{ij}(\partial_1, \partial_2, \partial_3) + \delta_{ij}\rho\omega^2\right)G_{jk}(x,y) = -\delta_{ik}\delta(x-y), \quad (x, y \in \mathbb{R}^3) \tag{14}
$$

It is convenient to express the volume integral equation in terms of the scattered wave field

$$v_i(x) = u_i(x) - F_i(x) \tag{15}$$

which becomes:

$$\begin{aligned} v_i(x) &= -\int_{\mathbb{R}^3} G_{ij}(x,y)N_{jk}(\partial_1,\partial_2,\partial_3,y)F_k(y)dy \\ &\quad -\int_{\mathbb{R}^3} G_{ij}(x,y)N_{jk}(\partial_1,\partial_2,\partial_3,y)v_k(y)dy \end{aligned} \tag{16}$$

By means of Eq. (16), the forward and inverse scattering problems considered in this section can be stated mathematically. The forward scattering problem is to determine v_i after G_{ij}, N_{jk}, and F_k have been obtained. The inverse scattering problem determines $\tilde{\lambda}$ and $\tilde{\mu}$ in N_{jk} in Eq. (16) after G_{ij}, v_i, and F_k have been obtained. In the remainder of this section, a method for dealing with Eq. (16) is described.

3.2 The Fourier transform and its application to the volume integral equation

The following Fourier integral and its inverse transforms:

$$\begin{aligned} \left(\mathscr{F}u_i\right)(\xi) &= \frac{1}{\sqrt{2\pi}^3}\int_{\mathbb{R}^3} u_i(x)\exp(-ix\cdot\xi)dx \\ \left(\mathscr{F}^{-1}\hat{u}_i\right)(x) &= \frac{1}{\sqrt{2\pi}^3}\int_{\mathbb{R}^3} \hat{u}_i(\xi)\exp(ix\cdot\xi)d\xi \end{aligned} \tag{17}$$

play an important role in the formulation, where $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ is a point in the wavenumber space, $x \cdot \xi$ is the scalar product defined by

$$x\cdot\xi = x_1\xi_1 + x_2\xi_2 + x_3\xi_3 \tag{18}$$

and $\mathscr{F}$ and $\mathscr{F}^{-1}$ are the operators for the Fourier transform and the inverse Fourier transform, respectively. In the following formulation, the symbol ˆ attached to a function is used to express the Fourier transform of the function. For example, $\hat{u}_i$ denotes the Fourier transform of u_i. The domain of the operators for $\mathscr{F}$ and $\mathscr{F}^{-1}$ defined in Eq. (17) is assumed to be $L^2(\mathbb{R}^3)$, so that the convergence of the integrals should be understood in the sense of the limit in the mean. In the following formulation, the domain of $\mathscr{F}$ and $\mathscr{F}^{-1}$ for the Green's function is assumed to be extended from $L^2(\mathbb{R}^3)$ to the distribution (Hörmander, 1983).
The Fourier transform of the equation for the Green's function defined by Eq. (14) becomes

$$L_{ij}(i\xi_1, i\xi_2, i\xi_3)\hat{G}_{jk}(\xi,y) = -\frac{1}{\sqrt{2\pi}^3}\delta_{ik}\exp(-i\xi\cdot y) \tag{19}$$

Equation (19) yields

$$\hat{G}_{ij}(\xi,y) = \frac{1}{\sqrt{2\pi}^3}\exp(-i\xi\cdot y)\hat{h}_{ij}(\xi) \tag{20}$$

where $\hat{h}_{ij}(\xi)$ is expressed by

$$\begin{aligned}\hat{h}_{ij}(\xi) &= \frac{\delta_{ij}}{\mu_0(|\xi|^2 - k_T^2 - i\epsilon)} \\ &\quad - \frac{\xi_i \xi_j}{2\mu_0(1-\nu_0)} \frac{1}{(|\xi|^2 - k_T^2 - i\epsilon)(|\xi|^2 - k_L^2 - i\epsilon)}\end{aligned} \tag{21}$$

In Eq. (21), ν_0 is the Poisson ratio obtained from the back ground Lamé constants λ_0 and μ_0, k_L, and k_T are the wavenumber of the P and S waves obtained from

$$\begin{aligned}k_L &= \frac{\omega}{c_L} \\ k_T &= \frac{\omega}{c_T}\end{aligned} \tag{22}$$

$|\xi|^2$ is given by

$$|\xi|^2 = \xi_1^2 + \xi_2^2 + \xi_3^2 \tag{23}$$

and ϵ is an infinitesimally small positive number. Note that c_T and c_L in Eq. (22) are the S and P wave velocities, respectively, for the background structure of the wave field defined by

$$c_T = \sqrt{\frac{\mu_0}{\rho}} \tag{24}$$

and

$$c_L = \sqrt{\frac{\lambda_0 + 2\mu_0}{\rho}} \tag{25}$$

respectively.

Next, let us investigate the Fourier transform of function w_i in the following form:

$$w_i(x) = \int_{\mathbb{R}^3} G_{ij}(x,y) f_j(y) dy \tag{26}$$

to obtain the Fourier transform of the volume integral equation. Note that $f_j(y)$ is in $\mathscr{S}(\mathbb{R}^3)$, i.e., the space of rapidly decreasing functions (Reed & Simon, 1975), then changing the order of integration yields

$$\begin{aligned}\hat{w}_i(\xi) &= \frac{1}{\sqrt{2\pi}^3} \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}^3} G_{ij}(x,y) f_j(y) dy \right] \exp(-ix \cdot \xi) dx \\ &= \hat{h}_{ij}(\xi) \frac{1}{\sqrt{2\pi}^3} \int_{\mathbb{R}^3} f_j(y) \exp(-i\xi \cdot y) dy = \hat{h}_{ij}(\xi) \hat{f}_j(\xi)\end{aligned} \tag{27}$$

In particular, the Fourier transform of w_i can be separated into the product of $\hat{h}_{ij}$ and $\hat{f}_j$. As reported in a previous study (Hörmander, 1983), f_j can be extended to distributions with compact support. According to Eq. (27), the Fourier transform of the volume integral equation shown in Eq. (16) becomes:

$$\hat{v}_i(\xi) \quad = \quad -\hat{h}_{ij}(\xi)\Big(\mathscr{F}N_{jk}F_k\Big)(\xi) - \hat{h}_{ij}(\xi)\Big(\mathscr{F}N_{jk}\mathscr{F}^{-1}\hat{v}_k\Big)(\xi) \tag{28}$$

For the case in which an explicit form of the plane incident wave is obtained, $N_{jk}F_k$ on the right-hand side of Eq. (28) can be simplified. As an example, a plane incident pressure (P) wave propagating along the x_3 axis has the following form:

$$F_i(x) = a\partial_i \exp(ik_L x_3) \tag{29}$$

where a is the amplitude of the P wave potential. In this case, $N_{jk}F_k$ can be expressed as

$$N_{jk}F_k(x) = r_j(x)\exp(i\xi_p \cdot x) \tag{30}$$

where

$$\begin{aligned} r_1(x) &= ak_L^2\partial_1\tilde{\lambda}(x) \\ r_2(x) &= ak_L^2\partial_2\tilde{\lambda}(x) \\ r_3(x) &= ak_L^2(\partial_3 + ik_L)\Big(\tilde{\lambda}(x) + 2\tilde{\mu}(x)\Big) \end{aligned} \tag{31}$$

Note that ξ_p is the wavenumber vector of the plane incident wave having the following components:

$$\xi_p = (0, 0, k_L) \tag{32}$$

As a result, Eq. (28) can be rewritten as

$$\hat{v}_i(\xi) \quad = \quad -\hat{h}_{ij}(\xi)\hat{r}_j(\xi - \xi_p) - \hat{h}_{ij}(\xi)\Big(\mathscr{F}N_{jk}\mathscr{F}^{-1}\hat{v}_k\Big)(\xi) \tag{33}$$

A method for forward and inverse scattering analysis is developed in the following based on Eq. (33).

3.3 Method for forward scattering analysis

Let us rewrite Eq. (33) in the following form:

$$\hat{v}_i(\xi) = \hat{f}_i(\xi) - A_{ik}\hat{v}_k \tag{34}$$

where $\hat{f}_i$ is given by

$$\hat{f}_i(\xi) = -\hat{h}_{ij}(\xi)\hat{r}_j(\xi - \xi_p) \tag{35}$$

which can be treated as a given function and A_{ik} is the linear operator such that

$$A_{ik} = \hat{h}_{ij}(\xi)\mathscr{F}N_{jk}\mathscr{F}^{-1} \tag{36}$$

Equation (34) clearly shows a Fredholm integral equation of the second kind, in which the linear operator is constructed by the multiplication operator $\hat{h}_{ij}$, the Fourier transform and the inverse Fourier transform, and the differential operator N_{jk}. For the actual numerical calculations in this chapter, the Fourier transform and its inverse Fourier transform are dealt with by means of the discrete Fourier transform. Naturally, the discrete Fourier transform is evaluated by means of an FFT. Let us denote the operators for the discrete Fourier transforms as $\mathscr{F}_D$ and $\mathscr{F}_D^{-1}$. For the operators $\mathscr{F}_D$ and $\mathscr{F}_D^{-1}$, the subsets in $\mathbb{R}^3$ below are defined as follows:

$$D_x = \left\{(n_1\Delta x_1, n_2\Delta x_2, n_3\Delta x_3) \,\middle|\, n_1 \in \mathbb{N}_1, n_2 \in \mathbb{N}_2, n_3 \in \mathbb{N}_3\right\} \subset \mathbb{R}^3 \tag{37}$$

$$D_\xi = \left\{(n_1\Delta \xi_1, n_2\Delta \xi_2, n_3\Delta \xi_3) \,\middle|\, n_1 \in \mathbb{N}_1, n_2 \in \mathbb{N}_2, n_3 \in \mathbb{N}_3\right\} \subset \mathbb{R}^3 \tag{38}$$

These subsets define a finite number of grid points, where Δx_j , (j = 1, 2, 3) is the interval of the grid in the space domain, $\Delta\xi_j$, (j = 1, 2, 3) is the interval of the grid in the wavenumber space, and $\mathbb{N}_1$, $\mathbb{N}_2$, and $\mathbb{N}_3$ are sets of integers defined by

$$\begin{aligned} \mathbb{N}_1 &= \{n \mid -N_1/2 \le n < N_1/2\} \\ \mathbb{N}_2 &= \{n \mid -N_2/2 \le n < N_2/2\} \\ \mathbb{N}_3 &= \{n \mid -N_3/2 \le n < N_3/2\} \end{aligned} \tag{39}$$

where (N_1,N_2,N_3) defines the number of grid points in $\mathbb{R}^3$. For the discrete Fourier transform, note that there is a relationship between Δx_j and $\Delta\xi_j$ such that

$$\Delta x_j \, \Delta\xi_j = \frac{2\pi}{N_j}, \quad (j = 1, 2, 3) \tag{40}$$

The explicit form of the discrete Fourier transform and the inverse Fourier transform are expressed as

$$\begin{aligned} (\mathscr{F}_D u_{(D)})(\xi^{(l)}) &= \frac{\Delta x}{\sqrt{2\pi}^3} \sum_{k \in \mathbb{N}_1 \times \mathbb{N}_2 \times \mathbb{N}_3} u_{(D)}(x^{(k)}) \exp(-ix^{(k)} \cdot \xi^{(l)}) \\ (\mathscr{F}_D^{-1} \hat{u}_{(D)})(x^{(k)}) &= \frac{\Delta \xi}{\sqrt{2\pi}^3} \sum_{l \in \mathbb{N}_1 \times \mathbb{N}_2 \times \mathbb{N}_3} \hat{u}_{(D)}(\xi^{(l)}) \exp(ix^{(k)} \cdot \xi^{(l)}) \end{aligned} \tag{41}$$

where Δx and $\Delta\xi$ are denoted by

$$\Delta x = \Delta x_1 \, \Delta x_2 \, \Delta x_3, \quad \Delta\xi = \Delta\xi_1 \, \Delta\xi_2 \, \Delta\xi_3 \tag{42}$$

and $x^{(k)}$ and $\xi^{(l)}$ expressed by

$$x^{(k)} = (x_1^{(k)}, x_2^{(k)}, x_3^{(k)}), \quad \xi^{(l)} = (\xi_1^{(l)}, \xi_2^{(l)}, \xi_3^{(l)}) \tag{43}$$

are the points in D_x of the k-th grid and in D_ξ of the l-th grid, respectively. In addition, u_D and $\hat{u}_D$ are the discrete functions defined on the grids D_x and D_ξ.

Based on the discrete Fourier transform, the derivative of a function can be calculated. For example, $\partial_j f(x)$ is expressed by

$$\partial_j f(x) = \left(\mathscr{F}_D^{-1}(i\xi_j \mathscr{F}_D f)\right)(x), \quad x \in D_x, \; \xi \in D_\xi \tag{44}$$

Therefore, treatments for the operator N_{jk} are also made possible by the discrete Fourier transform. Let $N_{(D)jk}$ be the discretization of the operator for N_{jk} by means of the discrete Fourier transform. Then, the discretization for the operator $\mathscr{A}_{ij}$ is defined by

$$A_{(D)ik} = \hat{h}_{ij}(\xi)\mathscr{F}_D N_{(D)jk}\mathscr{F}_D^{-1} \tag{45}$$

As a result of the discretization, Eq. (34) becomes

$$\hat{v}_{(D)i}(\xi) = \hat{f}_{(D)i}(\xi) - A_{(D)ij}\hat{v}_{(D)j}(\xi), \qquad (\xi \in D_\xi) \tag{46}$$

The domain and range of the linear operator in Eq. (45) are in the set of functions defined on a finite number of grids in the wavenumber space D_ξ. Namely, the domain and range for the operator are finite dimensional vector spaces. Note that the operator $N_{(D)jk}$ included in $\mathscr{A}_{(D)ij}$ is bounded because the differential operators are approximated by the discrete Fourier transform. For the case in which the domain and range of the operator are finite dimensional vector spaces, the operator has matrix representations (Kato, 1980). Therefore, a technique for the linear algebraic equation, such as the Krylov subspace iteration method (Barrett et al., 1994), is applicable to Eq. (46). Krylov subspace iteration methods have been developed for systems of algebraic equations in matrix form:

$$A\boldsymbol{x} = \boldsymbol{b} \tag{47}$$

where A is the matrix, and $\boldsymbol{x}$ and $\boldsymbol{b}$ are unknown and given vectors, respectively. The Krylov subspace is defined by

$$K_m = \text{span}\,\{\boldsymbol{b}, A\boldsymbol{b}, A^2\boldsymbol{b}\ldots, A^m\boldsymbol{b}\} \tag{48}$$

where m is the number of iterations. The Krylov subspace iteration method determines the coefficients of the recurrence formula to approximate the solution from the orthonormal basis of K_m during the iterative procedure. Note that matrix A can be regarded as the linear transform on a finite dimensional vector space. In this way, the construction of the Krylov subspace is possible, even if the linear transform is obtained using discrete Fourier transforms. Namely, it is possible to solve Eq. (46) by the Krylov subspace iteration method, where the Krylov subspace is constructed by FFT. As a result, a fast method for the volume integral equation without the derivation of the matrix is expected to be established. The current method is also expected to use less computer memory for numerical analysis. Since the operator $A_{(D)ij}$ is non-Hermitian due to the presence of $N_{(D)jk}$, the Bi-CGSTAB method (Barrett et al., 1994) is selected for the solution of Eq. (46).

3.4 Method for inverse scattering analysis

According to Eq. (31) the explicit form of $\hat{r}_j(\xi - \xi_p)$ shown as the first term on the right-hand side of Eq. (33) becomes:

$$\begin{aligned}
\hat{r}_1(\xi-\xi_p) &= ak_L^2 i\xi_1\hat{\hat{\lambda}}(\xi-\xi_p)\\
\hat{r}_2(\xi-\xi_p) &= ak_L^2 i\xi_2\hat{\hat{\lambda}}(\xi-\xi_p)\\
\hat{r}_3(\xi-\xi_p) &= ak_L^2 i\xi_3\left(\hat{\hat{\lambda}}(\xi-\xi_p)+2\hat{\tilde{\mu}}(\xi-\xi_p)\right)
\end{aligned} \tag{49}$$

Based on Eq. (49), $\hat{r}_j$ is found to be the function describing the fluctuation of the medium. Therefore, the inverse scattering analysis becomes possible if $\hat{r}_j$ is obtained from Eq. (33) after the scattered wave field $\hat{v}_i$ and the background structure of the wave field represented by $\hat{h}_{ij}$ have been provided. We introduce the vector Q_i, such that

$$r_i(x) = ak_L^3\lambda_0 Q_i(x) \tag{50}$$

to obtain the equation for the inverse scattering analysis in dimensionless form. Let us multiply both sides of Eq. (33) by $\left(-\hat{h}_{ij}^{-1}/(a\lambda_0 k_L^3)\right)$, which yields

$$\hat{\gamma}_j(\xi) \quad = \quad \hat{Q}_j(\xi - \xi_p) + \frac{1}{ak_L^3\lambda_0}\left(\mathscr{F}N_{jk}\mathscr{F}^{-1}\hat{v}_k\right)(\xi) \tag{51}$$

where $\hat{\gamma}_j$ is defined by

$$\hat{\gamma}_j(\xi) = -\hat{h}_{ji}^{-1}(\xi)\hat{v}_i(\xi)/(a\lambda_0 k_L^3) \tag{52}$$

Next, let the second term of Eq. (51) be modified to obtain the following:

$$\frac{1}{ak_L^3\lambda_0}\mathscr{F}N_{jk}\mathscr{F}^{-1}\hat{v}_k(\xi) = \mathscr{F}M_{jk}\mathscr{F}^{-1}\hat{Q}_k(\xi) \tag{53}$$

where M_{jk} is the differential operator determined by the scattered wave field. The remainder of this section describes how to obtain an explicit form of M_{jk}, so that Eq. (51) can be used to obtain $\hat{Q}$, which makes the estimation of the fluctuation of the medium possible. In order to obtain the explicit form of M_{jk}, a_j, which is defined as being equal to $N_{jk}v_k$, can be expressed as follows:

$$\alpha_j \quad = \quad -(\tilde{\lambda} + \tilde{\mu})\partial_j\Delta_v - \tilde{\mu}\eta_j - (\partial_j\tilde{\lambda})\Delta_v - 2(\partial_k\tilde{\mu})\epsilon_{kj} \tag{54}$$

where Δ_v and η_j are defined by

$$\begin{aligned} \Delta_v &= \partial_l v_l \\ \eta_j &= (\partial_1^2 + \partial_2^2 + \partial_3^2)v_j \end{aligned} \tag{55}$$

and ϵ_{jk} is the strain tensor due to the scattered wave field defined by Eq. (6). Let the separation of the fluctuation of the medium and the scattered wave field for α_j be denoted by

$$\alpha_j = m_{jk}p_k \tag{56}$$

where p_k is the state vector for the fluctuation of the medium, the components of which are

$$p_1 = \partial_1\tilde{\lambda}(x)/k_L, \quad p_2 = \tilde{\lambda}(x), \quad p_3 = \tilde{\mu}(x) \tag{57}$$

and m_{jk} is the differential operator that includes the effects of the scattered wave field, so that

$$[m_{jk}] = \begin{bmatrix} k_L\Delta_v & \partial_1\Delta_v & \partial_1\Delta_v + \eta_1 + 2\epsilon_{1l}\partial_l \\ 0 & \Delta_v\partial_2 + \partial_2\Delta_v & \partial_2\Delta_v + \eta_2 + 2\epsilon_{2l}\partial_l \\ 0 & \Delta_v\partial_3 + \partial_3\Delta_v & \partial_3\Delta_v + \eta_3 + 2\epsilon_3\partial_l \end{bmatrix} \tag{58}$$

Likewise, let the separation of the fluctuation of the medium and the scattered wave field for Q_j defined by Eq. (50) be denoted as follows:

$$Q_j = \kappa_{jk} p_k \tag{59}$$

where κ_{jk} is the operator that includes the effects of the scattered wave field, so that:

$$[\kappa_{jk}] = \frac{1}{k_L \lambda_0} \begin{bmatrix} k_L & 0 & 0 \\ 0 & \partial_2 & 0 \\ 0 & (\partial_3 + ik_L) & 2(\partial_3 + ik_L) \end{bmatrix} \tag{60}$$

According to Eqs. (59) and (60), the formal representation of the relationship between p_j and Q_j becomes

$$p_j = s_{jk} Q_k \tag{61}$$

where s_{jk} is the inverse of κ_{jk}, the components of which are

$$[s_{jk}] = \lambda_0 k_L \begin{bmatrix} k_L^{-1} & 0 & 0 \\ 0 & \partial_2^{-1} & 0 \\ 0 & -(1/2)\partial_2^{-1} & (1/2)(\partial_3 + ik_L)^{-1} \end{bmatrix} \tag{62}$$

Based on Eqs. (56) and (59), the following relationship can be derived:

$$\alpha_j = N_{jk} v_k = m_{jk} p_k = m_{jk} s_{kl} R_l \tag{63}$$

As a result, the operator M_{jk} defined by Eq. (53) can be constructed as follows:

$$M_{jl} = \frac{1}{a k_L^3 \lambda_0} m_{jk} s_{kl} \tag{64}$$

By means of the operator, Eq. (51) is modified to obtain

$$\hat{\gamma}_i(\xi) = \hat{Q}_i(\xi - \xi_p) + \left(\mathscr{F} M_{ij} \mathscr{F}^{-1} \right) \hat{Q}_j(\xi) \tag{65}$$

At this point, we have two tasks involving Eq. (65). One is to modify Eq. (65) to obtain a Fredholm equation of the second kind. The other task is to clarify the treatment of the operator s_{jk}, which includes ∂_2^{-1} and $(\partial_3 + ik_L)^{-1}$. To modify Eq. (65) to obtain a Fredholm equation of a second kind, the shift operator $S(\xi_p)$ defined by

$$S(\xi_p) \hat{Q}_i(\xi - \xi_p) = \hat{Q}_i(\xi) \tag{66}$$

is introduced. An explicit form of the shift operator can be obtained in terms of the Fourier transform, so that

$$S(\xi_p) = \mathscr{F} \exp(ix \cdot \xi_p) \mathscr{F}^{-1} \tag{67}$$

Application of the shift operator to both sides of Eq. (65) yields

$$S(\xi_p)\,\hat{\gamma}_i(\xi) = \hat{Q}_i(\xi) + \left(\mathscr{F}\exp(ix\cdot\xi_p)M_{ij}\mathscr{F}^{-1}\hat{Q}_j\right)(\xi) \tag{68}$$

which clearly has the form of a Fredholm equation of the second kind.
To clarify the treatments of ∂_2^{-1} in s_{jk}, consider the following equation:

$$\partial_2 u(x) = f(x), \quad f \in \mathscr{S}(\mathbb{R}^3) \tag{69}$$

Formally, it is possible to write the solution of the equation as

$$u(x) = \partial_2^{-1} f(x) = \int_{-\infty}^{\infty} H(x_2 - x_2') f(x_1, x_2', x_3) dx_2' \tag{70}$$

where H is the unit step function. The Fourier transform of H can be expressed as

$$\hat{H}(\xi_2) = -\frac{i}{\sqrt{2\pi}}\left(\mathrm{p.v.}\frac{1}{\xi_2} + i\pi\delta(\xi_2)\right) \tag{71}$$

where p.v. denotes Cauchy's principal value.
Equation (71) can also be expressed as (Friedlander & Joshin, 1998),

$$\hat{H}(\xi_2) = \frac{1}{\sqrt{2\pi}}\frac{1}{i\xi_2 + \epsilon} \tag{72}$$

and, therefore, the Fourier transform for $u(x)$ in Eq. (70) becomes

$$\hat{u}(\xi) = \frac{1}{i\xi_2 + \epsilon}\hat{f}(\xi) \tag{73}$$

The treatment of ∂_2^{-1} is resolved by means of Eq. (73), which is represented by

$$\partial_2^{-1} f = \mathscr{F}^{-1}\frac{1}{i\xi_2 + \epsilon}\mathscr{F} f \tag{74}$$

Likewise, we obtain

$$(\partial_3 + ik_L)^{-1} f = \int_{-\infty}^{\infty} H(x_3 - x_3') \exp\left(ik_L(x_3 - x_3')\right) f(x_1, x_2, x_3') dx_3' \tag{75}$$

which yields

$$(\partial_3 + ik_L)^{-1} f = \mathscr{F}^{-1}\frac{1}{i\xi_3 + ik_L + \epsilon}\mathscr{F} f \tag{76}$$

As can be seen from Eqs. (74) and (76), ∂_2^{-1} and $(\partial_3 + ik_L)^{-1}$ in the operator s_{ij} can be dealt with and resolved in terms of the Fourier transform. As a result of the above procedure, the treatment of the differential operator M_{ij} defined by Eq. (53) can also be handled by the Fourier transform. After all, as in the formulation of the forward scattering problem, Eq. (68) can be discretized into the following form:

$$S_D(\xi_p)\,\hat{\gamma}_{(D)i}(\xi) = \hat{Q}_{(D)i}(\xi) + B_{(D)ij}\hat{Q}_{(D)j}, \quad \xi \in D_\xi \tag{77}$$

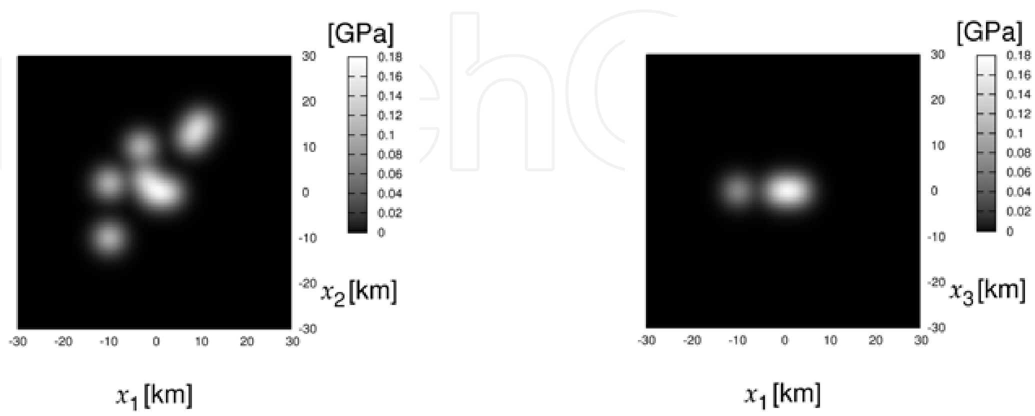
where $B_{(D)ij}$ is the operator expressed by

$$B_{(D)ij} = \mathscr{F}_D \exp(ix \cdot \xi_p) M_{(D)ij} \mathscr{F}_D^{-1} \tag{78}$$

The Krylov subspace iteration technique is also applied to Eq. (77) in the analysis. As a result of the above procedure, a fast method for the analysis of the inverse scattering is expected to be established.

3.5 Numerical example

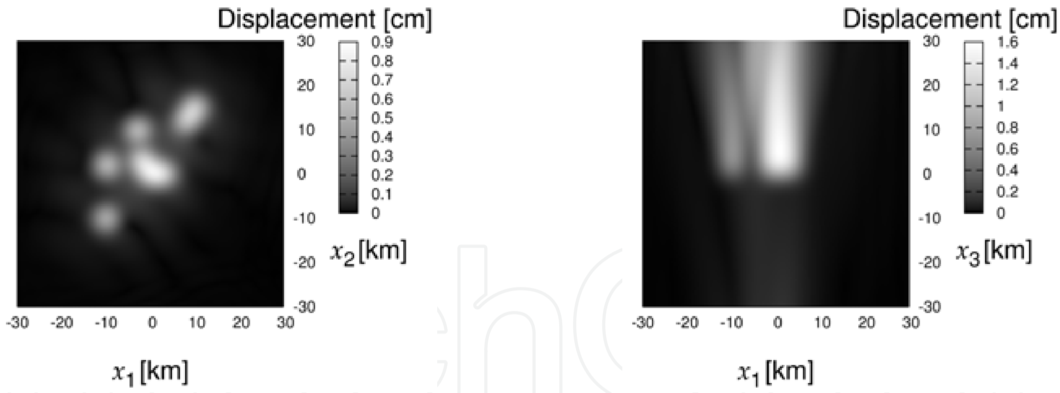
A numerical example for a multiple scattering problem in a 3-D elastic full space is presented. The fluctuations in the x_1–x_2 and x_1–x_3 planes are shown in Figs. 2(a) and 2(b), respectively, where the maximum amplitudes of $\tilde{\lambda}$ and $\tilde{\mu}$ are 0.18 GPa. These fluctuations are smooth, so that they have continuous spatial derivatives. The background structure of the wave field for the Lamé constants is set such that λ_0 = 4 GPa and μ_0 = 2 GPa, and the mass density is set to ρ = 2 g/cm^3. The background velocity of the P and S waves are 2 and 1 km/s, respectively. The analyzed frequency is f = 1 Hz, and the amplitude of the potential for the incident P wave is $a = 1.0 \times 10^5$ cm^2. The intervals of the grids in the space domain for the discrete Fourier transform are set by Δx_j = 0.25(km), (j = 1, 2, 3), and the number of intervals of the grids in the space domain for the discrete Fourier transform are set by N_j = 256, (j = 1, 2, 3). As a result, the intervals of the grid in the wavenumber space become $\Delta\xi_j = 2\pi/(N_j \times \Delta x_j) \approx 0.098$ km^{-1}. In addition, ϵ for the Green's function in the wavenumber domain shown in Eq. (21) is set to 0.2.
Figures 3(a) and 3(b) show the amplitudes of the scattered waves in the $x_1 - x_2$ and $x_1 - x_3$ planes, respectively. According to Fig. 3(a), the scattered waves are prominent in the regions in which fluctuations of the medium are present. The regions for the high amplitudes of the scattered waves are found to be separated due to the locations of the fluctuations of the medium. Therefore, the effects of multiple scattering are not very significant here. The reflection of the waves due to the incident wave is found to be small because of the smooth fluctuations. According to Fig. 3(b), forward scattering is noticeable with the narrow directionality in the x_3 direction. Interference of the scattered waves can be observed in the far field range of regions of the fluctuation.



(a) Fluctuations of Lamé constants $\tilde{\lambda}$ and $\tilde{\mu}$ in the $x_1 - x_2$ plane.

(b) Fluctuations of Lamé constants $\tilde{\lambda}$ and $\tilde{\mu}$ in the $x_1 - x_3$ plane.

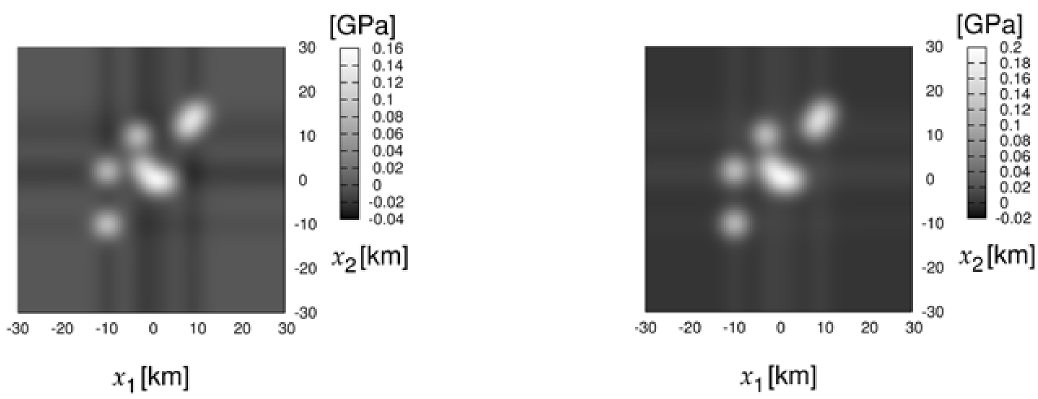
Fig. 2. Analyzed model of smooth fluctuations.



(a) Amplitudes of scattered waves in the $x_1 - x_2$ plane.

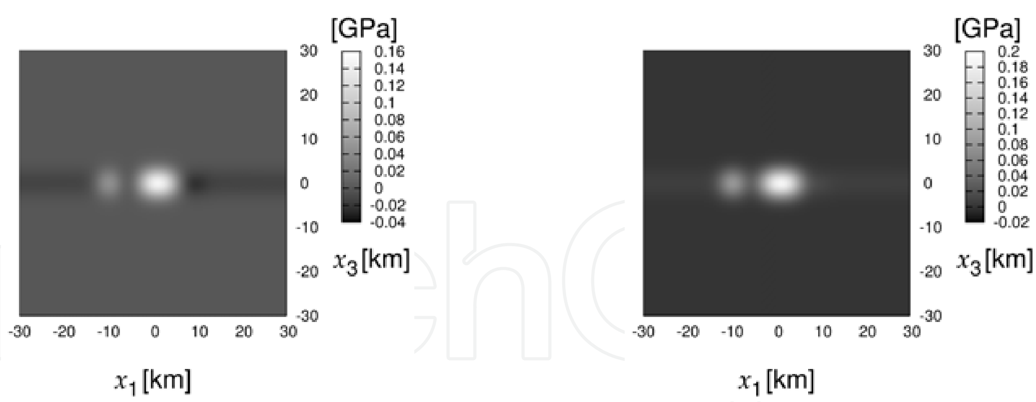
(b) Amplitudes of the scattered waves in the $x_1 - x_3$ plane.

Fig. 3. Results of the forward scattering analysis due to smooth fluctuations.



(a) Reconstruction of $\tilde{\lambda}$ in the $x_1 - x_2$ plane.

(b) Reconstruction of $\tilde{\mu}$ in the $x_1 - x_2$ plane.



(c) Reconstruction of $\tilde{\lambda}$ in the $x_1 - x_3$ plane.

(d) Reconstruction of $\tilde{\mu}$ in the $x_1 - x_3$ plane.

Fig. 4. Results of the inverse scattering analysis due to smooth fluctuations.

The results of the inverse scattering analysis in the $x_1 - x_2$ and $x_1 - x_3$ planes are shown in Figs. 4(a) through 4(d). For the analysis, ϵ for expressing ∂_2^{-1} and $(\partial_3 + ik_L)^{-1}$ in the operator M_{jk} was set to 0.01. Figures 4(a) through 4(d) show that the amplitudes and locations for the fluctuations were successfully reconstructed from the scattered wave field. Namely, Eq. (77) is effective and available for the inverse scattering analysis for the case in which the entire scattered wave field is provided.

For an AMD Opteron 2.4 GHz processor, and using the ACML library for the FFT and Bi-CGSTAB method for the Krylov subspace iteration technique, the required CPU time was only two minutes, where two iterations of the Bi-CGSTAB method were needed to obtain the solutions.

4. Volume integral equation method for an elastic half space

In this section, we deal with the concept of the analyzed model shown in Fig. 1(b), which is the scattering problem in an elastic half space. As shown in Eq. (16), the volume integral equation for the problem in terms of the scattered wave field can be expressed by

$$v_i(x) = -\int_{\mathbb{R}^3_+} G_{ij}(x,y)N_{jk}(y)F_k(y)dy - \int_{\mathbb{R}^3_+} G_{ij}(x,y)N_{jk}(y)v_k(y)dy \tag{79}$$

where G is the Green's function in an elastic half space, and F_i is the wave from the point source, expressed as

$$F_i(x) = G_{ij}(x, x_s)q_j \tag{80}$$

Equation (79) can be solved by means of the Fourier transform constructed for elastic wave propagation for a half space. This section explains this transform for the integral equation for an elastic half space and its application to the volume integral equation.

4.1 Transforms for the elastic wave equation in a half space for horizontal components

First, in order to determine an appropriate transform for the elastic wave equation in a half space, the following equation:

$$\left[L_{ij}(\partial_1, \partial_2, \partial_3) + \rho\omega^2\delta_{ij}\right]u_j(x) = -f_i(x), \quad (x \in \mathbb{R}^2 \times \mathbb{R}_+ = \mathbb{R}^3_+) \tag{81}$$

together with the following boundary condition:

$$P^{(0)}_{ij}u_j(x) = 0, \quad (\text{at } x_3 = 0) \tag{82}$$

are investigated, where $P^{(0)}_{ij}$ is the operator describing the free boundary condition, the components of which are

$$[P^{0)}_{ij}] = \begin{bmatrix} \mu_0\partial_3 & 0 & \mu_0\partial_1 \\ 0 & \mu_0\partial_3 & \mu_0\partial_2 \\ \lambda_0\partial_1 & \lambda_0\partial_2 & (\lambda_0 + 2\mu_0)\partial_3 \end{bmatrix} \tag{83}$$

The force density f_i and the displacement field u_i are assumed to be in $L_2(\mathbb{R}^3_+)$. The scalar product of the function in $L_2(\mathbb{R}^3_+)$ is defined as

$$\Big(u_i, v_i\Big)_{L_2(\mathbb{R}^3_+)} = \int_{\mathbb{R}^3_+} [u_1^*(x)v_1(x) + u_2^*(x)v_2(x) + u_3^*(x)v_3(x)]dx \tag{84}$$

The following Fourier integral transform for the displacement field for the horizontal components is introduced for Eq. (81):

$$\begin{aligned}\left(\mathscr{F}^{(h)}u_i\right)(\xi_1,\xi_2,x_3) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_i(x_1,x_2,x_3)\exp\Big[-i(x_1\xi_1+x_2\xi_2)\Big]dx_1dx_2 \\ \left(\mathscr{F}^{(h)^{-1}}\hat{u}_i\right)(x_1,x_2,x_3) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{u}_i(\xi_1,\xi_2,x_3)\exp\Big[i(x_1\xi_1+x_2\xi_2)\Big]d\xi_1d\xi_2\end{aligned} \tag{85}$$

where ξ_1 and ξ_2 are the horizontal coordinates of the wavenumber space. Note that the convergence of the integrals shown in Eq. (85) should be understood in the sense of the limit in the mean. According to the Fourier transform given by Eq. (85), Eq. (81) is transformed into the following:

$$L_{ij}(i\xi_1,i\xi_2,\partial_3)\hat{u}_j(\hat{x}) = -\hat{f}_i(\hat{x}) \tag{86}$$

where $\hat{u}_j$ and $\hat{f}_i$ in this section are define by

$$\hat{u}_j = \mathscr{F}^{(h)}u_j, \quad \hat{f}_i = \mathscr{F}^{(h)}f_i \tag{87}$$

respectively, and $\hat{x}$ is given by

$$\hat{x} = (\xi_1,\xi_2,x_3) \tag{88}$$

The Stokes-Helmholtz decomposition (Aki & Richards, 1980) is introduced in order to make the treatments for Eq. (86) more comprehensive. In general, the Stokes-Helmholtz decomposition of the displacement field u_i is expressed as:

$$u_i(x) = \partial_i\phi + \epsilon_{ijk}\epsilon_{kl3}\partial_j\partial_l\psi + \epsilon_{ij3}\partial_j\chi \tag{89}$$

where ϕ, ψ, and χ are the scalar potentials for the P, SV, and SH waves, respectively, and ϵ_{ijk} is the Eddington epsilon. The Fourier transform of Eq. (89) is as follows:

$$\begin{aligned}\hat{u}(\hat{x}) &= (i\xi_1\phi + i\xi_1\partial_3\psi + i\xi_2\chi)e_1 \\ &\quad +(i\xi_2\phi + i\xi_2\partial_3\psi - i\xi_1\chi)e_2 \\ &\quad +(\partial_3\phi + \xi_r^2\psi)e_3\end{aligned} \tag{90}$$

where e_j, (j = 1, 2, 3) are the base vectors for the 3-D displacement field, $\hat{u}$ = $\hat{u}_j e_j$, and $\xi_r = \sqrt{\xi_1^2+\xi_2^2}$. From Eq. (90), the wave field can be decomposed into the P-SV and SH waves by introducing the new base vectors e_i' defined by e_i' = $T_{ij}e_j$, where T_{ij} is expressed as

$$[T_{ij}] = \begin{bmatrix} 0 & 0 & 1 \\ ic & is & 0 \\ is & -ic & 0 \end{bmatrix} \tag{91}$$

where $c = \xi_1/\xi_r$ and $s = \xi_2/\xi_r$ for the case in which $\xi_r \neq 0$ and c = 1 and s = 0 for the case ξ_r = 0. Note that it is possible to impose arbitrary values on c and s when ξ_r = 0, because, based on Eq. (90), $\hat{u}_1 = \hat{u}_2 = 0$.

The linear transform T_{ij} defined by Eq. (91) is unitary and has the property whereby $T^*_{ik}T_{ij} = \delta_{kj}$. Equations (81) and (82) are transformed as follows by means of T_{ij} and the Fourier transform shown in Eq. (85):

$$\left(-\mathscr{A}_{ij} + \rho\omega^2\delta_{ij}\right)\hat{\hat{u}}_j(\hat{x}) = -T^*_{ik}\ \hat{f}_i(\hat{x}) \tag{92}$$

$$\mathscr{P}_{ij}\hat{\hat{u}}_j(\hat{x}) = 0, \quad \text{at } x_3 = 0 \tag{93}$$

where the operators $\mathscr{A}_{ij}$ and $\mathscr{P}_{ij}$ are obtained from:

$$\begin{aligned} \mathscr{A}_{ij} &= -T^*_{im}L_{mk}(i\xi_1, i\xi_2, \partial_3)T_{jk} \\ \mathscr{P}_{ij} &= T^*_{im}P_{mk}T_{jk} \end{aligned} \tag{94}$$

The relationship between $\hat{u}_i$ and $\hat{\hat{u}}_i$ is as follows:

$$\hat{\hat{u}}_i = T^*_{im}\hat{u}_m, \quad \hat{u}_m = T_{jm}\hat{\hat{u}}_j \tag{95}$$

The components of the operators $\mathscr{A}_{ij}$ and $\mathscr{P}_{ij}$ are

$$[\mathscr{A}_{ij}] = \begin{bmatrix} -(\lambda_0+2\mu_0)\partial_3^2 + \mu\xi_r^2 & (\lambda_0+\mu_0)\xi_r\partial_3 & 0 \\ -(\lambda_0+\mu_0)\xi_r\partial_3 & -\mu_0\partial_3^2 + (\lambda_0+2\mu_0)\xi_r^2 & 0 \\ 0 & 0 & -\mu_0\partial_3^2 + \mu_0\xi_r^2 \end{bmatrix} \tag{96}$$

$$[\mathscr{P}_{ij}] = \begin{bmatrix} (\lambda_0+2\mu_0)\partial_3 & -\lambda_0\xi_r & 0 \\ \mu_0\xi_r & \mu_0\partial_3 & 0 \\ 0 & 0 & \mu_0\partial_3 \end{bmatrix} \tag{97}$$

In Eqs. (96) and (97), the matrices are separated into 2×2 and 1×1 minor matrices, which makes the procedures for the operator much easier. Note that the 2 × 2 minor matrix is for the P-SV wave components and that the 1 × 1 minor matrix is for the SH wave component.

4.2 Self-adjointness of the operator $\mathscr{A}_{ij}$

In this section, we discuss the self-adjointness of the operator $\mathscr{A}_{ij}$ and its spectral representation. The domain of the operator $\mathscr{A}_{ij}$ is set by

$$D(\mathscr{A}_{ij}) = \{\psi_i \in L_2(\mathbb{R}_+) \mid \mathscr{A}_{ij}\psi_j \in L_2(\mathbb{R}_+), \mathscr{P}_{ij}\psi_j = 0 \text{ at } x_3 = 0\} \tag{98}$$

with the scalar product

$$\left(u_i, v_i\right)_{L_2(\mathbb{R}_+)} = \int_{\mathbb{R}_+}\left[u_1^*(x_3)v_1(x_3) + u_2^*(x_3)v_2(x_3) + u_3^*(x_3)v_3(x_3)\right]dx_3 \tag{99}$$

for $u_i,\ v_i \in D(\mathscr{A}_{ij})$. The operation for the differentiation in $\mathscr{A}_{ij}$ is carried out in the sense of the distribution. It is not difficult to show the following:

Lemma 1 *The operator $\mathscr{A}_{ij}$ is symmetric and non-negative.*

[Proof]

Let $u_i,\ v_i \in D(\mathscr{A}_{ij})$. Then,

$$
\begin{aligned}
&\Big(u_i, \mathscr{A}_{ij} v_j\Big)_{L_2(\mathbb{R}_+)} \\
&= -\Big[u_i^* \mathscr{P}_{ij} v_j\Big]_0^\infty + \Big[(\mathscr{P}_{ji} u_i^*) v_j\Big]_0^\infty + \Big(\mathscr{A}_{ji} u_i, v_j\Big)_{L_2(\mathbb{R}_+)} \\
&= \Big(\mathscr{A}_{ji} u_i, v_j\Big)_{L_2(\mathbb{R}_+)}
\end{aligned}
\tag{100}
$$

$$
\begin{aligned}
\Big(u_i, \mathscr{A}_{ij} u_j\Big)_{L_2(\mathbb{R}_+)} &= \int_0^\infty \Big[\partial_3 u_1^* (\lambda_0 + 2\mu_0) \partial u_1 + \partial_3 u_2^* \mu_0 \partial_3 u_2 + \partial_3 u_3^* \mu_0 \partial_3 u_3\Big] dx_3 \\
&\quad + \int_0^\infty \Big[u_1^* \mu_0 \xi_r^2 u_1 + u_2^* (\lambda_0 + 2\mu_0) \xi_r^2 u_2 + u_3^* \mu_0 \xi_r^2 u_3\Big] dx_3 \\
&\quad + \int_0^\infty \Big[-\partial_3 u_1^* \lambda_0 \xi_r u_2 - u_2^* \lambda_0 \xi_r \partial_3 u_1 + \partial_3 u_2^* \mu_0 \xi_r u_1 + u_1^* \mu_0 \xi_r \partial_3 u_2\Big] dx_3 \\
&= \int_0^\infty \Big[2\mu_0 |\partial_3 u_1|^2 + \mu_0 |\partial_3 u_3|^2\Big] dx_3 \\
&\quad + \int_0^\infty \Big[2\mu_0 \xi_r^2 |u_2|^2 + \mu_0 \xi_r^2 |u_3|^2\Big] dx_3 \\
&\quad + \int_0^\infty \Big[\lambda_0 |\partial_3 u_1 - \xi_r u_2|^2 + \mu_0 |\partial_3 u_2 + \xi_r u_1|^2\Big] dx_3 \geq 0
\end{aligned}
\tag{101}
$$

□

Next, the following function is defined:

$$
\Big(\mathscr{A}_{ij} - \eta^2 \mu_0 \delta_{ij}\Big) g_{jk}(x_3, y_3, \xi_r, \eta) = \delta_{ik} \delta(x_3 - y_3), \quad (\eta \in \mathbb{C}) \tag{102}
$$

together with the boundary condition

$$
\mathscr{P}_{ij} g_{jk}(x_3, y_3, \xi_r, \eta) = 0, \quad \text{at } x_3 = 0 \tag{103}
$$

where $\mathbb{C}$ is a set of complex numbers. The solution of Eq. (102) for for $\eta \in \mathbb{C} \setminus B$ has the following properties:

$$
\begin{aligned}
\sup_{x_3 \in \mathbb{R}_+} \int_{\mathbb{R}_+} |g_{ij}(x_3, y_3, \xi_r, \eta)| dy_3 &< \infty \\
\sup_{y_3 \in \mathbb{R}_+} \int_{\mathbb{R}_+} |g_{ij}(x_3, y_3, \xi_r, \eta)| dx_3 &< \infty
\end{aligned}
\tag{104}
$$

where B is defined by

$$
B = B_p \cup B_c \tag{105}
$$

in which

$$
\begin{aligned}
B_p &= \{\eta \in \mathbb{R} \mid F_R(\xi_r, \eta) = 0\} \\
B_c &= \{\eta \in \mathbb{R} \mid |\eta| \geq \xi_r\}
\end{aligned}
\tag{106}
$$

Note that F_R in Eq. (106) is the Rayleigh function given by

$$F_R(\xi_r, \eta) = (2\xi_r^2 - \eta^2)^2 - 4\xi_r^2 \gamma\nu \tag{107}$$

where

$$\begin{aligned} \nu &= \sqrt{\xi_r^2 - \eta^2} \\ \gamma &= \sqrt{\xi_r^2 - (c_T/c_L\eta^2} \end{aligned} \tag{108}$$

Lemma 2 *For $f_i \in L_2(\mathbb{R}_+)$ and $\eta \in \mathbb{C} \setminus B$*

$$u_i(x_3) = \int_{\mathbb{R}_+} g_{ij}(x_3, y_3, \xi_r, \eta) f_j(y_3) dy_3 \in L_2(\mathbb{R}_+) \tag{109}$$

[Proof]
First, fix i and j and define

$$v_i(x_3) = \int_{\mathbb{R}_+} g_{ij}(x_3, y_3, \xi_r, \eta)\ f_j(y_3) dy_3 \tag{110}$$

Then, the following is obtained by means of the Schwarz inequality:

$$\begin{aligned} |v_i(x_3)| &\leq \left[\int_{\mathbb{R}_+} |g_{ij}(x_3, y_3, \xi_r, \eta)|\ |f_j(y_3)|^2 dy_3\right]^{1/2} \left[\int_{\mathbb{R}_+} |g_{ij}(x_3, y_3, \xi_r, \eta)| dy_3\right]^{1/2} \\ &\leq \left[\int_{\mathbb{R}_+} |g_{ij}(x_3, y_3, \xi_r, \eta)|\ |f_j(y_3)|^2 dy_3\right]^{1/2} M_1 \end{aligned} \tag{111}$$

where

$$M_1 = \sup_{x_3 \in \mathbb{R}_+} \left[\int_{\mathbb{R}_+} |g_{ij}(x_3, y_3, \xi_r, \eta)| dy_3\right]^{1/2} \tag{112}$$

As a result, the following is obtained:

$$\begin{aligned} \int_{\mathbb{R}_+} |v_i(x_3)|^2 dx_3 &\leq M_1^2 \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} |g_{ij}(x_3, y_3, \xi_r, \eta)|\ |f_j(y_3)|^2 dy_3\ dx_3 \\ &\leq M_1^2 M_2 \|f_j\|^2_{L_2(\mathbb{R}_+)} \end{aligned} \tag{113}$$

where

$$M_2 = \sup_{y_3 \in \mathbb{R}_+} \int_{\mathbb{R}_+} |g_{ij}(x_3, y_3, \xi_r, \eta)| dx_3 \tag{114}$$

Equation (113) concludes the proof. □

Theorem 1 *The operator $\mathscr{A}_{ij}$ with the domain $D(\mathscr{A}_{ij})$ is self-adjoint.*
[Proof]
It is sufficient to prove that $\forall f_i \in L_2(\mathbb{R}_+)$, there exist $u_i^{(+)}, u_i^{(-)} \in D(\mathscr{A}_{ij})$ satisfying

$$\Big(\mathscr{A}_{ij} + ip\mu_0\delta_{ij}\Big)u_j^{(+)}(x_3) \quad = \quad f_i(x_3) \tag{115}$$

$$\Big(\mathscr{A}_{ij} - ip\mu_0\delta_{ij}\Big)u_j^{(-)}(x_3) \quad = \quad f_i(x_3) \tag{116}$$

where p is a positive real number. This fact is based on the results of a previous study (Theorem 3.1, Berthier, 1982).
For the construction of $u_i^{(+)}$, define

$$u_i^{(+)}(x_3) = \int_{\mathbb{R}_+} g_{ij}(x_3, y_3, \xi_r, \eta) f_j(y_3) dy_3 \tag{117}$$

where η is chosen such that $\eta^2 = ip$. Note that $\eta \in \mathbb{C} \setminus B$. The following equation:

$$\begin{aligned}
&\int_{\mathbb{R}_+} \varphi_i(x_3)(\mathscr{A}_{ij} + ip\mu_0\delta_{ij})u_j(x_3)dx_3 \\
&= \int_{\mathbb{R}_+} \Big[(\mathscr{A}_{ji} + ip\mu_0\delta_{ji})\varphi_i(x_3)\Big] u_j(x_3)dx_3 \\
&= \int_{\mathbb{R}_+} (\mathscr{A}_{ji} + ip\mu_0\delta_{ji})\varphi_i(x_3) \int_{\mathbb{R}_+} g_{jk}(x_3, y_3, \xi_r, \eta) f_k(y_3)dy_3 dx_3 \\
&= \int_{\mathbb{R}_+} \Big[\int_{\mathbb{R}_+} \Big((\mathscr{A}_{ji} + ip\mu_0\delta_{ji})\varphi_i(x_3)\Big) g_{jk}(x_3, y_3, \xi_r, \eta)dx_3\Big] f_k(y_3)dy_3 \\
&= \Big(\varphi_k, f_k\Big)_{L_2(\mathbb{R}_+)} \quad , \ (\varphi_i \in \mathscr{D}(\mathbb{R}_+))
\end{aligned} \tag{118}$$

yields Eq. (115), where $\mathscr{D}(\mathbb{R}_+)$ is the Schwartz space. During the derivation of Eq. (118), the following equation:

$$\int_{\mathbb{R}_+} \Big[(\mathscr{A}_{ji} + ip\mu_0\delta_{ij})\varphi_i(x_3)\Big] g_{jk}(x_3, y_3, \xi_r, \eta)dx_3 \quad = \quad \varphi_k(y_3), \quad (\varphi_i \in \mathscr{D}(\mathbb{R}_+)) \tag{119}$$

is based on the following properties of $g_{ij}(x_3, y_3, \xi_r, \eta)$ at $x_3 = y_3$

$$\begin{aligned}
g_{ik}(y_3 + \epsilon, y_3, \xi_r, \eta) &= g_{ik}(y_3 - \epsilon, y_3, \xi_r, \eta) \\
[\mathscr{P}_{ij} g_{jk}(x_3, y_3, \xi_r, \eta)]_{x_3 = y_3 - \epsilon}^{x_3 = y_3 + \epsilon} &= \delta_{ik}
\end{aligned} \tag{120}$$

In addition, the following is obtained:

$$\begin{aligned}
\mathscr{P}_{ij} u_j(x_3) &= \mathscr{P}_{ij} \int_{\mathbb{R}_+} g_{jk}(x_3, y_3, \xi_r, \eta) f_k(y_3)dy_3 \\
&= \int_{\mathbb{R}_+} \mathscr{P}_{ij} g_{jk}(x_3, y_3, \xi_r, \eta) f_k(y_3)dy_3
\end{aligned} \tag{121}$$

The order of the integral and differential operators of the properties of function g_{ij} are changed such that

$$\lim_{y_3\to\infty} |y_3{}^n \mathscr{P}_{ij} g_{jk}(x_3, y_3, \xi_r, \eta) f_k(y_3)| = 0 \tag{122}$$

for an arbitrary positive integer n. According to Eq. (121), we have

$$\mathscr{P}_{ij} u_j(x_3) = 0, \quad \text{at } x_3 = 0 \tag{123}$$

It has been shown that $u_j \in L_2(\mathbb{R}_+)$ from **Lemma 2**, so that $u_i^{(+)} \in D(\mathscr{A}_{ij})$. The construction of $u_i^{(-)} \in D(\mathscr{A}_{ij})$ is also possible. As a result, the following conclusion is obtained. □

4.3 Generalized Fourier transform for an elastic wave field in a half space

The operator $\mathscr{A}_{ij}$ has been found to be self-adjoint and non-negative, which yields the following spectral representation:

$$\mathscr{A}_{ij} = \int_0^\infty \zeta \, dE_{ij}(\zeta) \tag{124}$$

where E_{ij} is the spectral family. The spectral family is connected with the resolvent by means of the Stone theorem (Wilcox, 1976):

$$\begin{aligned}
&\left(\Big[(E_{ij}(b) + E_{ij}(b-)) - (E_{ij}(a) + E_{ij}(a-))\Big] u_j, v_i\right)_{L_2(\mathbb{R}_+)} \\
&= \lim_{\epsilon\downarrow 0} \frac{1}{\pi i} \int_a^b d\zeta \left(\Big[R_{ij}(\zeta + i\epsilon) - R_{ij}(\zeta - i\epsilon)\Big] u_j, v_i\right)_{L_2(\mathbb{R}_+)}
\end{aligned} \tag{125}$$

for u_i, $v_i \in L_2(\mathbb{R}_+)$. Note that R_{ij} is the resolvent of the operator $\mathscr{A}_{ij}$ and $R_{ij}(\zeta)u_j$ is defined by

$$\Big(R_{ij}(\zeta) u_j\Big)(x_3) = \int_0^\infty g_{ij}\left(x_3, y_3, \xi_r, \sqrt{\zeta/\mu_0}\right) u_j(y_3) dy_3 \tag{126}$$

Let $0 < a < \mu_0 \eta_R^2$ and $b > \mu \xi_r^2$. Then, the right-hand side of Eq. (125) for the integral becomes

$$\begin{aligned}
&\int_a^b d\zeta \left(\Big[R_{ij}(\zeta + i\epsilon) - R_{ij}(\zeta - i\epsilon)\Big] u_j, v_i\right)_{L_2(\mathbb{R}_+)} \\
&= (-2\pi i) \operatorname*{Res}_{\eta=\eta_R} (2\eta\mu_0) \Big(R_{ij}(\zeta) u_j, v_i\Big)_{L_2(\mathbb{R}_+)} \\
&\quad + \int_{\xi_r}^{\sqrt{b/\mu}} d\eta \, (2\eta\mu_0) \left(\Big[R_{ij}(\zeta + i\epsilon) - R_{ij}(\zeta - i\epsilon)\Big] u_j, v_i\right)_{L_2(\mathbb{R}_+)}
\end{aligned} \tag{127}$$

where $\zeta = \mu_0\eta^2$ and η_R is defined by $F_R(\xi_r, \eta_R) = 0$. The path of integration in the complex η plane shown in Fig. 5 is used for the evaluation of the integral.

In the following, the relationship between the right-hand side of Eq. (127) and the eigenfunctions is presented. Let $v_i(x_3, \xi_r, \eta) \in D(\mathscr{A}_{ij})$ satisfy

$$(\mathscr{A}_{ij} - \mu_0 \eta^2 \delta_{ij}) v_j(x_3, \xi_r, \eta) = 0 \tag{128}$$

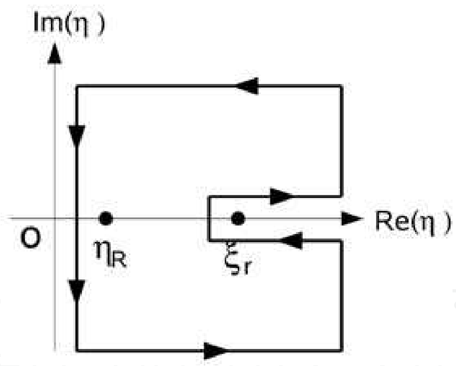


Fig. 5. Path of the integral.

and define the scalar function $W(\eta)$ such that

$$\begin{aligned} W(\eta) &= \left(v_i, (\mathscr{A}_{ij} - \mu_0\eta^2\delta_{ij})v_j\right)_{L_2(\mathbb{R}_+)} \\ &\quad -v_i^*(0,\xi_r,\eta)\mathscr{P}_{ij}v_j(0,\xi_r,\eta) \end{aligned} \tag{129}$$

It is easy to derive the following properties of $W(\eta)$ by means of the boundary conditions for v_i:

$$\begin{aligned} W(\eta) &= 0, \quad (\eta = \eta_R) \\ W(\eta) &\neq 0, \quad (\eta \neq \eta_R) \end{aligned} \tag{130}$$

Note that $v_i(x_3, \xi_r, \eta_R)$ becomes the eigenfunction (Rayleigh wave mode) satisfying the free boundary conditions. Otherwise, $v_i(x, \xi_r, \eta)$, $(\eta \neq \eta_R)$ cannot satisfy the free boundary conditions. As a result, Eq. (130) is established. Integration by parts of Eq. (129) yields

$$W(\eta) = I_1(\eta) - \mu_0\eta^2 I_2(\eta) \tag{131}$$

where

$$\begin{aligned} I_1(\eta) &= \int_0^\infty \Big[2\mu_0|\partial_3 v_1|^2 + \mu_0|\partial_3 v_3|^2 + 2\mu_0\xi_r^2|v_2|^2 + \mu_0\xi_r^2|v_3|^2 \\ &\quad +\lambda_0|\partial_3 v_1 - \xi_r v_2|^2 + \mu_0|\partial_3 v_2 + \xi_r v_1|^2\Big]dx_3 \\ I_2(\eta) &= \int_0^\infty \Big[|v_1|^2 + |v_2|^2 + |v_3|^2\Big]dx_3 \end{aligned} \tag{132}$$

The following lemma can then be obtained:

Lemma 3 *The residue of g_{ij}* at $\eta = \eta_R$ *can be expressed in terms of the eigenfunction such that*

$$\underset{\eta=\eta_R}{Res}\, g_{ij}(x_3, y_3, \xi_r, \eta) = -\frac{\psi_{im}(x_3,\xi)\psi_{jm}(y_3,\xi)}{2\mu_0\eta_R} \tag{133}$$

where $\xi = (\xi_1, \xi_2, \eta_R)$ *and* $\psi_{im}(x_3, \xi)$ *is the eigenfunction defined by*

$$\mathscr{A}_{ij}\psi_{jm}(x_3,\eta_R) = \mu_0\eta_R^2\psi_{im}(x_3,\eta_R), \quad \psi_{im}(x_3,\eta_R) \in D(\mathscr{A}_{ij}) \tag{134}$$

[Proof]
Based on a previous study (Aki & Richards, 1980), the function g_{ij} can be constructed by

$$g_{ij}(x_3, y_3, \xi_r, \eta) = w_{ij}(x_3, y_3, \xi_r, \eta) + v_i(x_3, \xi_r, \eta)\Delta_j(\eta) \tag{135}$$

where w_{ij} is defined by

$$\begin{aligned} w_{ij}(x_3, y_3, \xi_r, \eta) &= 0, \ \ (x_3 > y_3) \\ \mathscr{P}_{ij} w_{jk}(x_3, y_3, \xi_r, \eta) &= +\delta_{ik}, \ (x_3 + \epsilon = y_3, \ \epsilon \downarrow 0) \end{aligned} \tag{136}$$

In addition, Δ_j is defined such that g_{ij} satisfies the free boundary condition:

$$\mathscr{P}_{ij} w_{jk}(x_3, y_3, \xi_r, \eta) + \mathscr{P}_{ij} v_j(x_3, \xi_r, \eta)\Delta_j(\eta) = 0, \ \ (\text{at } x_3 = 0) \tag{137}$$

The definition of $W(\eta)$ shown in Eq. (129) implies that the expression is valid:

$$W(\eta) = -v_i^*(0, \xi_r, \eta)\mathscr{P}_{ij} v_j(0, \xi_r, \eta) \tag{138}$$

Equations (137) and (138) yield

$$\begin{aligned} & g_{ik}(x_3, y_3, \xi_r, \eta) \\ = \ & w_{ik}(x_3, y_3, \xi_r, \eta) + \frac{v_i(x_3, \xi_r, \eta)}{W(\eta)}[v_l^*(0, \xi_r, \eta) P_{lj} w_{jk}(0, y_3, \xi_r, \eta)] \end{aligned} \tag{139}$$

Now, let η approach η_R. Due to reciprocity, it is found that

$$v_l^*(0, \xi_r, \eta_R) P_{lj} w_j(0, y_3, \xi_r, \eta_R) = v_l^*(y_3, \xi_r, \eta_R)\delta_{lk} \tag{140}$$

Therefore,

$$v_i(x_3, \xi_r, \eta_R)[v_l^*(0, s) P_{lj} w_{jk}(0, y_3, \xi_r, \eta)] = \psi_{im}(x_3, \xi)\psi_{lm}(y_3, \xi)\delta_{ik} \tag{141}$$

The residue of the resolvent kernel is expressed as

$$\operatorname*{Res}_{\eta=\eta_R} g_{ij}(x_3, y_3, \xi_r, \eta) = \frac{\psi_{im}(x_3, \xi)\psi_{jm}(y_3, \xi)}{W'(\eta_R)} \tag{142}$$

For the case in which the eigenfunction is normalized as $I_2(\eta_R) = 1$, we have $W'(\eta_R) = -2\mu\eta_R$, which concludes the proof. □

Next, the function g_{ij}:

$$g_{ij}(x, y, \xi_r, s + i\epsilon) - g_{ij}(x, y, \xi_r, s - i\epsilon) \tag{143}$$

is investigated for the case in which $s = \mathrm{Re}(\eta) > \xi_r$. The function g_{ij} for this case is constructed by

$$g_{ij}(x, y, \xi_r, s \pm i\epsilon) = w_{ij}(x, y, \xi_r, s \pm i\epsilon) + v_{ik}(x, \xi_r, s \pm i\epsilon)\Delta_{kj}(s \pm i\epsilon) \tag{144}$$

where v_{ik} is the definition function of the improper eigenfunction (Touhei, 2002). The definition function v_{ik} satisfies the following:

$$(\mathscr{A}_{ij} - \mu_0 s^2 \delta_{ij}) v_{jk}(x_3, \xi_r, s) = 0 \tag{145}$$

The relationship between the improper eigenfunction and the definition function is given as

$$\psi_{ik}(x, \xi) = v_{ik}(x, y, \xi_r, s + i\epsilon) - v_{ik}(x, y, \xi_r, s - i\epsilon), \quad (\epsilon \to 0) \tag{146}$$

Next, let us define the following function:

$$W_{kl}(\xi_r, s \pm i\epsilon) = -\psi_{ik}(0, \xi) P_{ij} v_{jl}(0, y, \xi_r, s \pm i\epsilon) \tag{147}$$

Substitution of the explicit forms of the eigenfunction and definition function of Eq. (147) yields the following:

$$W_{kl}(\xi_r, s \pm i\epsilon) = -\frac{\mu_0 i}{\pi} s\, \delta_{kl}, \quad (\epsilon \to 0) \tag{148}$$

In addition, note that

$$w_{ik}(x_3, y_3, \xi_r, s + i\epsilon) - w_{ik}(x_3, y_3, \xi_r, s - i\epsilon) \to 0, \quad (\epsilon \to 0) \tag{149}$$

which is obtained from the definition of w_{ik} shown in Eq. (136). Based on Eqs. (148) and (149), the following lemma is obtained.

Lemma 4 *For the region of $s > \xi_r$, the function g_{ij} satisfies the following equation:*

$$\begin{aligned} & g_{ij}(x_3, y_3, \xi_r, s + i\epsilon) - g_{ij}(x_3, y_3, \xi_r, s - i\epsilon) \\ = \;& \pi i \frac{\psi_{im}(x_3, \xi) \psi_{jm}(y_3, \xi)}{\mu_0 s}, \quad (\epsilon \to 0) \end{aligned} \tag{150}$$

where $\psi_{im}(x_3, \xi)$ is the improper eigenfunction.

[Proof]

The requirement of the free boundary condition for g_{ij} yields the following expression of $\Delta_{k\beta}$:

$$\Delta_{kj}(s \pm i\epsilon) = [W_{lk}(\xi_r, s \pm i\epsilon)]^{-1} \psi_{il}(0, \xi) \mathscr{P}_{im} w_{mj}(0, y_3, \xi_r, s \pm i\epsilon) \tag{151}$$

Incorporating the following reciprocity relation:

$$\psi_{il}(0, \xi) \mathscr{P}_{im} w_{mj}(0, y_3, \xi_r, s \pm i\epsilon) = \psi_{il}(y_3, \xi) \delta_{i\beta} \tag{152}$$

into Eq. (151) yields

$$\Delta_{kj}(s \pm i\epsilon) = [W_{lk}(\xi_r, s \pm i\epsilon)]^{-1} \psi_{jl}(y_3, \xi) \tag{153}$$

Therefore, the following is obtained:

$$\begin{aligned} g_{ij}(x_3, y_3, \xi_r, s \pm i\epsilon) = \;& w_{ij}(x_3, y_3, \xi_r, s \pm i\epsilon) \\ & + v_{ik}(x_3, y_3, \xi_r, s \pm i\epsilon) [W_{lk}(\xi_r, s \pm i\epsilon)]^{-1} \psi_{jl}(y_3, \xi) \end{aligned} \tag{154}$$

Thus, Eqs. (146), (148), and (149) conclude the proof. □

Next, let us again consider Eq. (127). Equation (127) holds for an arbitrary $v_i \in L_2(\mathbb{R}_+)$, so that the following equation can be obtained by incorporating the results of Lemmas 3 and 4:

$$\begin{aligned}\int_a^b d\zeta \Big[R_{ij}(\zeta+i\epsilon) - R_{ij}(\zeta-i\epsilon)\Big] u_j \\ = \quad (2\pi i)\sum_{\xi\in\sigma_p} \psi_{im}(x_3,\xi)\Big(\psi_{jm}(\cdot,\xi), u_j(\xi_r,\cdot\,))\Big)_{L_2(\mathbb{R}_+)} \\ +(2\pi i)\int_{\xi_r}^{\sqrt{b/\mu}} d\xi_3\, \psi_{im}(x_3,\xi)\Big(\psi_{jm}(\cdot,\xi), u_j(\xi_r,\cdot)\Big)_{L_2(\mathbb{R}_+)}\end{aligned} \tag{155}$$

where $u_j(\xi_r,\ \cdot) \in L_2(\mathbb{R}_+)$, $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3_+$ and

$$\sigma_p = \{\xi = (\xi_1,\xi_2,\xi_3) \in \mathbb{R}^3_+ \mid F(\xi_r,\xi_3) = 0\} \tag{156}$$

As mentioned earlier, $0 < a < \mu\eta_R^2$ and $b > \mu\xi_r^2$, so that

$$\begin{aligned} E_{ij}(a) &= E_{ij}(a-) = 0 \\ E_{ij}(b) &= E_{ij}(b-) \end{aligned} \tag{157}$$

Therefore, Eqs. (125) and (155) yield

$$\begin{aligned}\Big(E_{ij}(b)u_j\Big)(\xi_r,x_3) \quad = \quad \sum_{\xi\in\sigma_p}\Big(\psi_{jm}(\,\cdot,\xi), u_j(\xi_r,\cdot\,)\Big)_{L_2(\mathbb{R}_+)} \psi_{im}(x_3,\xi) \\ + \int_{\xi_r}^{\sqrt{b/\mu}}\Big(\psi_{jm}(\,\cdot,\xi), u_j(\xi_r,\xi_3)\Big)_{L_2(\mathbb{R}_+)} \psi_{im}(x_3,\xi)\; d\xi_3\end{aligned} \tag{158}$$

Let b in Eq. (158) approach infinity. Then, the following eigenfunction expansion form of u_i is obtained:

$$\begin{aligned}u_i(\xi_r,x_3) \quad = \quad \sum_{\xi\in\sigma_p}\Big(\psi_{jm}(\,\cdot,\xi), u_j(\xi_r,\cdot\,)\Big)_{L_2(\mathbb{R}_+)} \psi_{im}(x_3,\xi) \\ + \int_{\xi_r}^{\infty}\Big(\psi_{jm}(\cdot,\xi), u_j(\xi_r,\cdot\,)\Big)_{L_2(\mathbb{R}_+)} \psi_{im}(x_3,\xi\,)d\xi_3\end{aligned} \tag{159}$$

Note that the eigenfunction expansion form shown in Eq. (159) is that for $u_i(\xi_r,\ \cdot)$ having the compact support. This result can be extended to all $u_i(\xi_r,\ \cdot\ ) \in L_2(\mathbb{R}_+)$ by a limiting procedure, namely,

$$\Big(\psi_{jm}(\cdot,\xi), u_j(\xi_r,\cdot\,)\Big)_{L_2(\mathbb{R}_+)} \quad = \quad \underset{M\to\infty}{\mathrm{l.i.m.}} \int_0^M \psi_{jm}(x_3,\xi)u_j(\xi_r,x_3)dx_3 \tag{160}$$

where the convergence is in $L_2(\mathbb{R}_+)$. The transform of the function in $L_2(\mathbb{R}_+)$ obtained here can be summarized as follows:

$$\begin{aligned}
\grave{u}_m(\xi) &= \left(\mathscr{F}^{(v)}_{mj} u_j\right)(\xi) = \int_0^\infty \psi_{jm}(x_3,\xi) u_j(\xi_r, x_3) dx_3 \\
u_i(\xi_r, x_3) &= \left(\mathscr{F}^{(v)^{-1}}_{im} \grave{u}_m\right)(\xi_r, x_3) \\
&= \sum_{\xi\in\sigma_p} \psi_{im}(x_3,\xi)\grave{u}_m(\xi) + \int_{\xi_r}^\infty \psi_{im}(x_3,\xi)\grave{u}_m(\xi) d\xi_3
\end{aligned} \tag{161}$$

At this point, the transformation of the elastic wave field in a half space can be presented. Let us define the subset of the wavenumber space as follows:

$$\sigma_c \;=\; \{\xi = (\xi_1,\xi_2,\xi_3) \in \mathbb{R}^3_+ \mid \xi_3 > \xi_r\} \tag{162}$$

The following theorem is obtained based on Eqs. (85), (95), and (158):
Theorem 2 *There exists a map satisfying the free boundary condition of the elastic half space of the wave field from* $L_2(\mathbb{R}^3_+)$ to $L_2(\sigma_p) \oplus L_2(\sigma_c)$ *defined by*

$$\begin{aligned}
\hat{u}_j(\xi) &= \left(\mathscr{U}_{ij} u_j\right)(\xi) \\
&= \left(\mathscr{F}^{(v)}_{im}\, T^*_{mj}\, \mathscr{F}^{(h)} u_j\right)(\xi),
\end{aligned} \tag{163}$$

the inverse of which is

$$\begin{aligned}
u_i(x) &= \left(\mathscr{U}^{-1}_{ij} \hat{u}_j\right)(x) \\
&= \left(\mathscr{F}^{(h)^{-1}}\, T_{mi}\, \mathscr{F}^{(v)^{-1}}_{mj}\, \hat{u}_j\right)(x)
\end{aligned} \tag{164}$$

Here, $\mathscr{U}_{ij} u_j$ *and* $\mathscr{U}^{-1}_{ij} u_j$ *are expressed as follows:*

$$\begin{aligned}
\left(\mathscr{U}_{ij} u_j\right)(\xi) &= \int_{\mathbb{R}^3_+} \Lambda^*_{ji}(\xi,x) u_j(x) dx, \quad (\xi \in \sigma_p \cup \sigma_c) \\
\left(\mathscr{U}^{-1}_{ij} \hat{u}_j\right)(x) &= \int_{\mathbb{R}^2} \sum_{\xi\in\sigma_p} \Lambda_{ij}(x,\xi)\hat{u}_j(\xi) d\xi_1 d\xi_2 \\
&\quad + \int_{\mathbb{R}^2}\int_{\xi_r}^\infty \Lambda_{ij}(x,\xi)\hat{u}_j(\xi) d\xi_3 d\xi_1 d\xi_2, \quad (x \in \mathbb{R}^3_+)
\end{aligned} \tag{165}$$

where

$$\Lambda_{ij}(\xi,x) = \frac{1}{2\pi}\exp(i(\xi_1 x_1 + \xi_2 x_2))\psi_{lj}(x_3,\xi) T_{li} \tag{166}$$

Here, $\mathscr{U}_{ij} u_j$ is referred to as the generalized Fourier transform of u_j, and $\mathscr{U}^{-1}_{ij}\hat{u}_j$ is referred to as the generalized inverse Fourier transform of $\hat{u}_j$. Based on the literature (Reed and Simon, 1975), the domain of the operators $\mathscr{U}_{ij}$ and $\mathscr{U}^{-1}_{ij}$ could be extended from L_2 to the space of tempered distributions $\mathscr{S}'$.

4.4 Method for the volume integral equation

We have obtained the transform for elastic waves in a 3-D half space, which is to be applied to the volume integral equation. Preliminary to showing the application of the transform to the volume integral equation, we have to construct the Green's function for the elastic half space based on the proposed transform. The definition of the Green's function for the half space is expressed as

$$\begin{aligned}\Big(L_{ij}(\partial_1,\partial_2,\partial_3)+\delta_{ij}\rho\omega^2\Big)G_{jk}(x,y) &= -\delta_{ik}\delta(x-y)\\ P^{(0)}_{ij}G_{jk}(x,y) &= 0 \ \text{ at } \ x_3=0, \quad (x,y\in\mathbb{R}^3_+)\end{aligned} \tag{167}$$

The application of the generalized Fourier transform to Eq. (167) yields

$$(\mu_0\xi_3^2-\rho\omega^2)\hat{G}_{kj}(\xi,y)=\Lambda^*_{jk}(\xi,y) \tag{168}$$

where $\hat{G}_{ij}$ is the generalized Fourier transform of the Green's function. Therefore, as a result of Eq. (168), the Green's function for a half space can be represented as

$$\begin{aligned}G_{ij}(x,y) &= \int_{\mathbb{R}^2} d\xi_1 d\xi_2 \sum_{\xi\in\sigma_p}\frac{\Lambda_{ik}(\xi,x)\Lambda^*_{jk}(\xi,y)}{\mu_0\xi_3^2-\rho\omega^2-i\epsilon}\\ &\quad+\int_{\mathbb{R}^2} d\xi_1 d\xi_2\int_{\xi_r}^{\infty}\frac{\Lambda_{ik}(\xi,x)\Lambda^*_{jk}(\xi,y)}{\mu_0\xi_3^2-\rho\omega^2-i\epsilon}d\xi_3\end{aligned} \tag{169}$$

Next, let the function $w_i(x)$ be given in the following form:

$$w_i(x)=\int_{\mathbb{R}^3_+}G_{ij}(x,y)f_j(y)dy \tag{170}$$

The formal calculation reveals that

$$\begin{aligned}\Big(\mathscr{U}_{ki}w_i\Big)(\xi) &= \int_{\mathbb{R}^3_+}\Lambda^*_{ik}(\xi,x)\left[\int_{\mathbb{R}^3_+}G_{ij}(x,y)f_j(y)dy\right]dx\\ &= \int_{\mathbb{R}^3_+}\left[\int_{\mathbb{R}^3_+}\Lambda^*_{ik}(\xi,x)G_{ij}(x,y)dx\right]f_j(y)dy\\ &= \int_{\mathbb{R}^3_+}\hat{h}(\xi)\Lambda^*_{jk}(\xi,y)f_j(y)dy\\ &= \hat{h}(\xi)\Big(\mathscr{U}_{kj}f_j\Big)(\xi)\end{aligned} \tag{171}$$

where $\hat{h}$ denotes

$$\hat{h}(\xi)=\frac{\mathbf{1}(\xi_1)\,\mathbf{1}(\xi_2)}{\mu_0\xi_3^2-\rho\omega^2-i\epsilon} \tag{172}$$

Note that $\mathbf{1}(\cdot)$ in Eq. (172) is defined such that

$$\mathbf{1}(\xi_1)=1,\quad(\forall\xi_1\in\mathbb{R}) \tag{173}$$

At this point, the application of the generalized Fourier transform to the volume integral equation becomes possible and is achieved as follows:

$$\begin{aligned}\hat{v}_i(\xi) &= -\hat{h}(\xi)\mathscr{U}_{ij}N_{jk}\mathscr{U}_{kl}^{-1}\hat{F}_l(\xi)\\ &\quad -\hat{h}(\xi)\mathscr{U}_{ij}N_{jk}\mathscr{U}_{kl}^{-1}\hat{v}_l(\xi)\end{aligned} \tag{174}$$

where $\hat{v}_i$ is the generalized Fourier transform of v_i and $\hat{F}_l$ is the incident wave field due to the point source expressed by

$$\hat{F}_l(\xi) = \hat{h}(\xi)\Lambda^*_{ml}(\xi, x_s)q_m \tag{175}$$

The volume integral equation for the elastic wave equation in the wavenumber domain in a half space has the same structure as that in a full space. Therefore, almost the same numerical scheme based on the Krylov subspace iteration technique is available. Note that the difference in the numerical scheme between that for the elastic full space and that for the half space lies in the discretization of the wavenumber space. The discretization of the wavenumber space for elastic half space is as follows:

$$\begin{aligned}D_\xi &= \left\{(n_1\Delta\xi_1, n_2\Delta\xi_2, \eta_R)\,\middle|\, n_1 \in \mathbb{N}_1, n_2 \in \mathbb{N}_2, F(\xi_r, \eta_R) = 0\right\} \oplus\\ &\quad \left\{(n_1\Delta\xi_1, n_2\Delta\xi_2, \xi_r + n_3\Delta\xi_3)\,\middle|\, n_1 \in \mathbb{N}_1, n_2 \in \mathbb{N}_2, n_3 \in \mathbb{N}_3\right\}\end{aligned} \tag{176}$$

where $\Delta\xi_j$, (j = 1, 2, 3) are the intervals of the grids in the wavenumber space,

$$\xi_r = \sqrt{n_1^2\Delta\xi_1^2 + n_2^2\Delta\xi_2^2} \tag{177}$$

and $\mathbb{N}_1$, $\mathbb{N}_2$, and $\mathbb{N}_3$ compose the set of integers defined by

$$\begin{aligned}\mathbb{N}_1 &= \{n \mid -N_1/2 \le n < N_1/2\}\\ \mathbb{N}_2 &= \{n \mid -N_2/2 \le n < N_2/2\}\\ \mathbb{N}_3 &= \{n \mid 0 \le n < N_3 - 1\}\end{aligned} \tag{178}$$

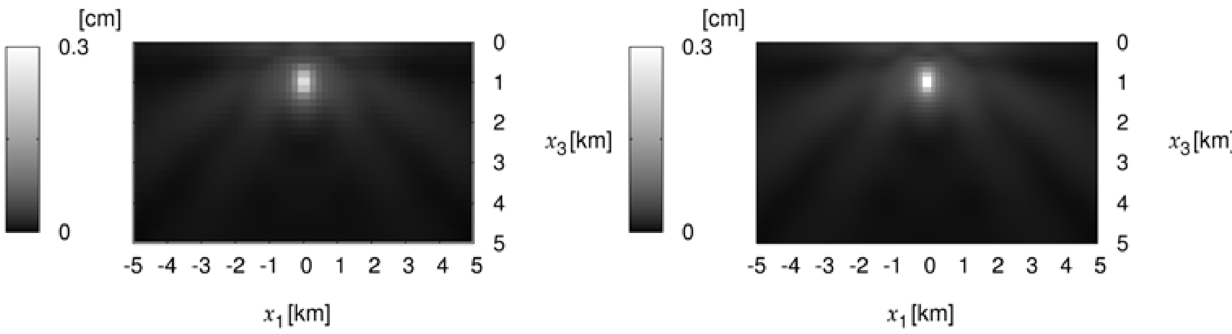
where (N_1, N_2, N_3) defines the number of grids in the wavenumber space. Note that Eq. (176) corresponds to the decomposition of the Rayleigh and body waves.

4.5 Numerical example

For the numerical analysis of an elastic half space, the Lam´e constants of the background structure is set such that λ_0 = 4 GPa, μ_0 = 2 GPa and the mass density is set at ρ =2 g/cm^3. Therefore, the background velocity of the P and S waves are 2 km/s and 1 km/s, respectively. and that for the Rayleigh wave velocity is 0.93 km/s. In addition, the analyzed frequency is f = 1 Hz.

First, let us investigate the accuracy of the generalized Fourier transform by composing the Green's function. For the calculation of the generalized Fourier transform, $N_1 = N_2 = N_3$ = 256, $\Delta x_1 = \Delta x_2$ = 0.25 km, and Δx_3=0.125 km are chosen to define D_x and D_ξ. The parameter ϵ for the Green's function is set at 0.6.

Figures 6(a) and 6(b) show the Green's function calculated by the generalized Fourier transform and the Hankel transform. The distributions of the absolute displacements are shown is these figures. For the calculation of the Green's function, the point source is set at a depth of 1 km from the free surface. The direction of the excitation force is vertical, and the excitation force has an amplitude of 1.0 × 10^7 kN. Comparison of these figures reveals good agreement between the calculated results, which confirms the accuracy of the generalized Fourier transform.



(a) Generalized Fourier transform (b) Hankel transform

Fig. 6. Comparison of the Green's function calculated by the generalized Fourier transform and the Hankel transform.

The following example shows the solution of the volume integral equation. The fluctuation of the elastic wave field is set as follows:

$$\tilde{\lambda}(x) = A_\lambda \exp(-\zeta_\lambda |x - x_c|^2) \tag{179}$$

$$\tilde{\mu}(x) = A_\mu \exp(-\zeta_\mu |x - x_c|^2), \tag{180}$$

where A_λ and A_μ describe the amplitude of the fluctuation, ζ_λ and ζ_μ describe the spread of the fluctuation in the space, and x_c is the center of the fluctuation. These parameters are set at $A_\lambda = A_\mu = 0.6$ GPa, $\zeta_\lambda = \zeta_\mu = 0.3$ km^{-2} and

$$x_c = (0, 0, 1) \ \ \text{km} \tag{181}$$

The fluctuation of the medium in the $x_2 - x_3$ plane at $x_1 = 0$ [km] is shown in Fig. 7.

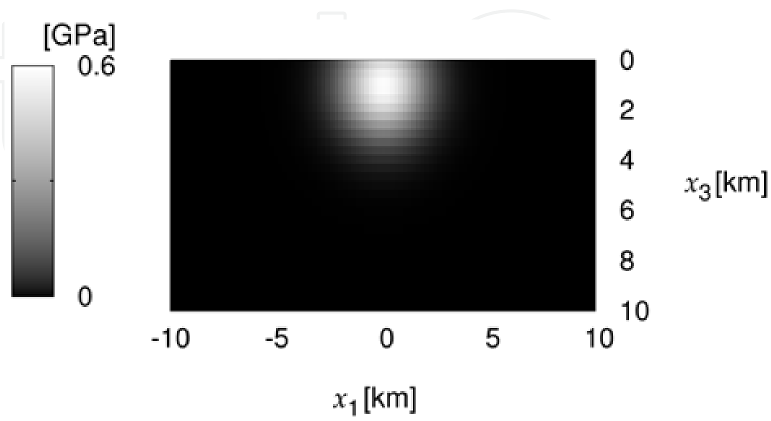


Fig. 7. Fluctuation of the medium

In order to generate the scattered wave field, the location of the point source is set at x_s = (5, 0, 0) km. The direction of the excitation force is vertical, and the excitation force has an amplitude of 1×10^7 kN. Bi-CGSTAB method (Barrett et al., 1994) is used for the Krylov subspace iteration technique. Figures 8 and 9 show the propagation of scattered waves at the free surface and the amplitudes of the scattered waves in the $x_1 - x_3$ plane, respectively. According to Fig. 8, the amplitudes of the scattered waves are larger in the forward region of the fluctuating area, where $x_1 < 0$. Figure 9 shows that the propagation of the Rayleigh waves as the scattered wvaes in the forwrad region. The amplitude of the scattered waves are smaller in the fluctuating area. The scattered waves are found to be reflected at the fluctuating area, thereby generating Rayleigh waves. The above numerical results explain well the propagation of the scattered elastic waves in the half space.

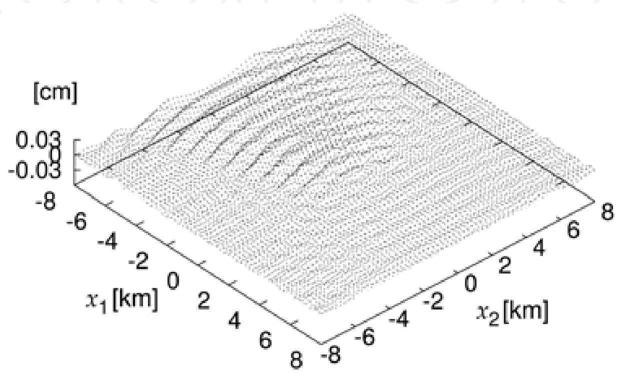


Fig. 8. Scattering of waves at the free surface.

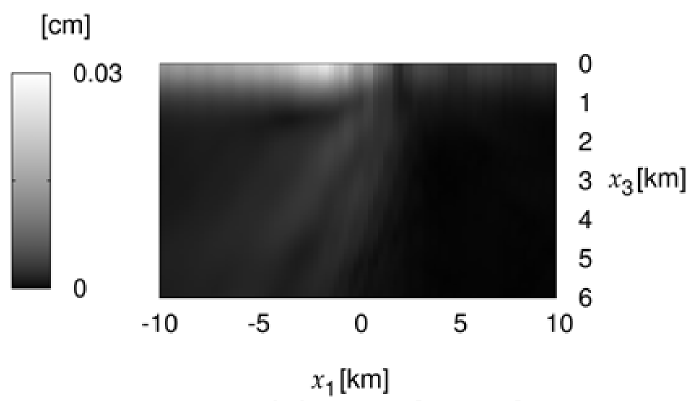


Fig. 9. Distribution of scattered waves in the vertical plane.

The numerical calculations were carried out using a computer with an AMD Opteron 2.4-GHz processor. The CPU time needed for iteration in Bi-CGSTAB was five hours, which is due primarily to the calculation of the generalized Fourier transforms. Note that the 2-D FFT for the horizontal coordinate system was used for the generalized Fourier transform. The transform for the vertical coordinate required a large CPU time. The reduction of this large CPU time requirement should be investigated in the future. The development of a fast algorithm for the generalized Fourier transforms may be required. It is also important to formulate the inverse scattering analysis method and to carry out the analysis.

5. Conclusions

In this chapter, a volume integral equation method was developed for elastic wave propagation for 3-D elastic full and half spaces. The developed method did not require the derivation of a coefficient matrix. Instead, the Fourier transform and the Krylov subspace iterative technique were used for the integral equation. The starting point of the formulation was the volume integral equation in the wavenumber domain. The Fourier transform and the inverse Fourier transform were repeatedly applied during the Krylov iterative process. Based on this procedure, a fast method was realized for both forward and inverse scattering analysis in a 3-D elastic full space via the fast Fourier transform and Bi-CGSTAB method. For example, if the number of iterations was two, the CPU time to obtain accurate solutions was only two minutes. Furthermore, for the inverse scattering problem, the reconstruction of inhomogeneities of the wave field was also successful, even for the multiple scattering problem.

The ordinary Fourier transform is not valid for an elastic half space due to the boundary conditions at the free surface. The generalized Fourier transform and the inverse Fourier transform for elastic wave propagations in a half space were developed for the integral equation based on the spectral theory. The generalized Fourier transform composing the Green's function was also verified numerically. The properties of the scattered wave field in a half space were found to be well explained by the proposed method. At present, the proposed method for an elastic half space requires a large amount of CPU time, which was five hours for the present numerical model. As such, a fast algorithm for the generalized Fourier transforms should be developed in the future.

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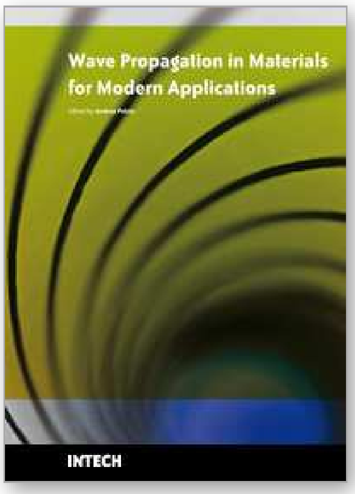

Wave Propagation in Materials for Modern Applications
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In the recent decades, there has been a growing interest in micro- and nanotechnology. The advances in nanotechnology give rise to new applications and new types of materials with unique electromagnetic and mechanical properties. This book is devoted to the modern methods in electrodynamics and acoustics, which have been developed to describe wave propagation in these modern materials and nanodevices. The book consists of original works of leading scientists in the field of wave propagation who produced new theoretical and experimental methods in the research field and obtained new and important results. The first part of the book consists of chapters with general mathematical methods and approaches to the problem of wave propagation. A special attention is attracted to the advanced numerical methods fruitfully applied in the field of wave propagation. The second part of the book is devoted to the problems of wave propagation in newly developed metamaterials, micro- and nanostructures and porous media. In this part the interested reader will find important and fundamental results on electromagnetic wave propagation in media with negative refraction index and electromagnetic imaging in devices based on the materials. The third part of the book is devoted to the problems of wave propagation in elastic and piezoelectric media. In the fourth part, the works on the problems of wave propagation in plasma are collected. The fifth, sixth and seventh parts are devoted to the problems of wave propagation in media with chemical reactions, in nonlinear and disperse media, respectively. And finally, in the eighth part of the book some experimental methods in wave propagations are considered. It is necessary to emphasize that this book is not a textbook. It is important that the results combined in it are taken “from the desks of researchers“. Therefore, I am sure that in this book the interested and actively working readers (scientists, engineers and students) will find many interesting results and new ideas.

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