Chapter

Optimal Control Promotional Policy for a New Product Incorporating Repeat Purchase in Segmented Market: A Control Theoretic Approach

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Abstract

This chapter considers an optimal control model to obtain dynamic promotional policies for a product considering a segmented market where first-time and additional repeat purchase sales are assumed to be generated through mass and differentiated promotions. Mass promotion is carried out in the whole market which reaches each segment with a fixed spectrum, and differentiated promotion is catered to each segment individually. The firm's finite promotional resources are to be allocated for promoting a product at mass and segment levels of the market in a finite time period. The formulated control problem obtains optimal promotional effort policy for each segment using the maximum principle. The applicability of the proposed control model is illustrated through a numerical example by discretizing the model.

Keywords: market segmentation, innovation-diffusion model, optimal control theory, maximum principle

1. Introduction

In the last few decades, the customers are made available with an increased amount of choices for particular goods or services. In such a situation to ensure that the customer chooses our product among others, it becomes important to communicate and inform about the innovative features and quality offered through the product and make a space in customers' minds. This task is achieved by promoting the product at regular intervals. Promotion plays a major role in raising customer awareness of the product, generates sales, and hence repeats purchases. Repeat purchase is an important phenomenon among the consumers that often measures their loyalty towards a brand. The higher is the repeat purchase value, it can be said that the better a firm is doing to keep customers loyal. This chapter focuses on determining the optimal promotional effort policies for a consumer durable product by assuming that the single purchase and the repeat purchase of a product are generated through the combined effect of mass and differentiated promotions in a segmented market.

Promotional strategies are often targeted to a potential market chosen in accordance with the firm's product type. Once target market is decided, market segmentation is carried out to divide the broad target market into subsets of consumers who have common needs and priorities, and then designing and implementing strategies are done to target them. Market segmentation plays an important role in development of the marketing strategies. Different customers have different needs, and it is impossible to satisfy all customers treating them alike. Promotional policies for the products are built by considering the heterogeneity in the potential market. Firms that identify the specific needs of the groups of customers are able to develop the right offer for the submarkets and obtain a competitive advantage over other firms. The concept of market segmentation emerged, as the market-oriented thought evolved among the firms. Market segmentation has thus become the building block of the effective promotional planning. It partitions the markets into groups of potential customers on the basis of geographic, demographic, and psychographic variables and behavioural customer characteristics.

Once the segmentation process is complete, the next step following it is choosing the targeting strategies that can be implemented. The firm must decide whether they want to choose segment-specific or mass (differentiated) promotional strategies. Mass promotion is implemented by treating the market as homogeneous and giving common message in all the segments through mass communication, the effect of which reaches each of the segments proportionally known as spectrum effect. However, the preferences of customers may differ, and same offering may not affect all potential customers and urge them towards product adoption. If firms ignore these differences, another competing firm can market similar product serving specific groups, and this may lead to losing customers. Segment-specific promotion recognizes this diversified customer base and takes into consideration the varying consumers in different segments. The promotional messages are constructed accordingly here. Both the mass- and segment-specific strategies play important roles and have their own advantages. Firms generally promote their product in the market at both the levels mass and segment. In this chapter, we assume that the evolution of sales of the product is through mass and differentiated promotions and build a control model for determining the promotional policies that maximizes the total profit constrained on the total budget. The promotion effort policies are generated by using the maximum principle. The model proposed is continuous in nature, but in practical the data available is discrete. Also the model is nonlinear and becomes NP-hard in nature. Thus we have used Lingo11 to solve the discretized version and show the model application.

The rest of this chapter is organized as follows. In Section 2 of this chapter, we provide a brief literature review and in Section 3, we introduce the diffusion model with repeat purchasing and discuss its optimal control formulation and develop segmented sales rate under the assumption that the practitioner may choose independently the advertising intensity directed towards each segment as well as combined advertising intensity. The problem is discussed, and it is solved using Pontryagin's maximum principle with particular cases in Section 4. Section 5 gives the numerical illustration for the discretized version of the problem using Lingo11 software and finally in Section 6, we conclude our chapter.

2. Literature review

Few people have worked in optimal control theory considering market segmentation in advertising models [1–3]. A discrete time stochastic model of multiple media selection in a segmented market was analysed by Little and Lodish [1].

Seidmann et al. [2] proposed a general sales-advertising model in which the state of the system represents a population distribution over a parameter space, and they show that such models are well posed and that there exists an optimal control. Buratto et al. [4] have given some market segmentation concepts into advertising models during the introduction of new product and advertising processes for sales over an infinite horizon. Grosset and Viscolani [3] discussed the optimal advertising policy for a new product introduction considering only the external influence in a segmented market with Nerlove-Arrow's [5] linear goodwill dynamics. Nerlove and Arrow [5] proposed a model in which the effect of advertising on sales is mediated by the goodwill variable. The goodwill state variable represents the effects of the firm investment in advertising, and it affects the demand of the product together with price and other external factors. From past few years, a number of researchers have been working in the area of optimal control models pertaining to advertising expenditure and price in marketing [6]. The simplest diffusion model was due to Bass [7]. Since the landmark work of Bass, the model has been widely used in the diffusion theory. The major limitation of this model is that it does not take into consideration the impact of marketing variables. Many authors have suitably modified the Bass model to study the impact of price on new product diffusion [8–13]. These models incorporate the pricing effects on diffusion. Also there are models that incorporate the effect of advertising on diffusion [9, 14, 15]. Horsky and Simmon [9] incorporated the effects of advertising in the Bass innovation coefficient. Thompson and Teng [16] incorporated learning curve production cost in their oligopoly price-advertising model. Bass et al. [17] included both price and advertising in their generalized Bass model.

Jha et al. [18] used the concept of market segmentation in diffusion model for advertising a new product and studied the optimal advertising effectiveness rate in a segmented market. They discussed the evolution of sales dynamics in the segmented market under two cases. Firstly, they assumed that the firm advertises in each segment independently, and further they took the case of a single advertising channel, which reaches several segments with a fixed spectrum. Manik et al. [19] amalgamated the two problems formulated by Jha et al. [18] and formulated an optimal control problem where they studied the effect of differentiated promotional effort and mass promotional effort on evolution of sales rate for each segment. They obtained the optimal promotional effort policy for the proposed model. Dynamic behaviour of optimal control theory leads to its application in sales-promotion control analysis and provides a powerful tool for understanding the behaviour of sales-promotion system where dynamic aspect plays an important role. Numerous papers on the application of optimal control theory in sales-advertising problem exist in the literature [20, 21]. However the literature missed out the control model to determine the control policies in a segmented market considering repeat purchasers in the sales through mass and differentiated promotions and taking the budget constraint which we try to do in this chapter.

3. Model development

We begin our analysis by stating the following assumption that $M(>1)$ is the total market segments and a discrete variable. The sum $\sum_{i=1}^{M}\overline{X_i}$ denotes the total number of potential customers of the product in all the segments. The firm simultaneously uses mass market promotion and differentiated market promotion to capture the potential market in each segment, respectively. Mass market promotion reaches each segment proportionally called segment-specific spectrum. Let $x_i(t)$ be the number of adopter by time t for the ith segment. During diffusion process,

repeat purchases of the product may also occur, and those adopters who have already adopted may repurchase the product again. Therefore, the number of adopters for a new product can increase due to both first purchase and repeat purchasing. Under the influence of mass market and differentiated market promotion, evolution of sales rate [7] can be described by the following differential equation:

$$\frac{dx_i(t)}{dt} = b_i(t)(u_i(t) + \alpha_i u(t))\left(\overline{X_i} - (1 - g_i)x_i(t)\right), i = 1, 2, ..., M \tag{1}$$

with the initial condition $x_i(0) = x_{i0} \forall i = 1, 2, ..., M,$ where α_i denotes the segment spectrum of mass promotion $\left(\alpha_i > 0 \,\&\, \sum_{i=1}^{M} \alpha_i = 1\right)$; $g_i(0 \le g_i \le 1)$ is susceptible to repeat purchasing, and repeat purchasing is influenced by all factors (both internal and external) affecting first purchase in i^{th} segment by time t; $u_i(t)$ is differentiated promotional effort rate for i^{th} segment at time t; *and* $u(t)$ is mass market promotional effort rate at time t, and $b_i(t)$ is the adoption rate per additional adoption for the ith segment. $b_i(t)$ can be represented either as a function of time or as a function of the number of previous adopters. Since the latter approach is used most widely, it is the one applied here. Therefore, Eq. (1) can be rewritten as follows:

$$\frac{dx_i(t)}{dt} = \left(p_i + q_i \frac{x_i(t)}{\overline{X_i}}\right)(u_i(t) + \alpha_i\, u(t))\left(\overline{X_i} - (1 - g_i)x_i(t)\right), i = 1, 2, ..., M \tag{2}$$

where p_{i} and q_{i} are coefficients of external and internal influences in i^{th} segment, respectively.

The objective of the firm is to maximize the present value of the profit in a planning horizon for a segmented market by selecting optimal mass and differentiated promotional effort rates for the firm. Thus, the objective function can be represented by

$$\text{Max}\, J = \int_0^T e^{-\gamma t}\left(\sum_{i=1}^{M}[(P_i - C_i(x_i(t))\dot{x}(t) - \phi_i(u_i(t))] - \varphi(u(t))\right) dt \tag{3}$$

where $\phi_i(u_i(t))$ and $\varphi(u(t))$ are differentiated market promotional effort and mass market promotional effort cost, respectively, γ is discounted profit, P_i is sales price for i^{th} segment, and $C_i(x_i(t))$ is production cost per unit for i^{th} segment, that is, continuous and differentiable with assumption $C_i'(.) > 0$ and $P_i - C_i(x_i(t)) > 0$.

During the promotion, differentiated and mass promotions are competing for the limited promotion budget expenditure. Therefore, firms monitor the promotion strategy in all segments closely and allocate their promotional expenditure budget optimally among these segments. The budget constraint for all segments is represented as

$$\int_0^T \left(\sum_{i=1}^{M}\phi_i(u_i(t)) + \varphi(u(t))\right) dt \le W_0 \tag{4}$$

where W_0 is the fixed budget expenditure for all segments over time. Constraint (4) corresponds to the common promotional expenditure capacity that is allocated among all the segments. This constraint couples the segment and prevents us from simply solving M times a single-segment problem. The above problem can be written as an optimal control problem:

$$\left.\begin{aligned} \text{Max}\, J &= \int_0^T e^{-\gamma t}\left(\sum_{i=1}^{M}[(P_i - C_i(x_i(t))\dot{x}(t) - \phi_i(u_i(t))] - \varphi(u(t))\right) dt \\ \frac{dx_i(t)}{dt} &= \left(p_i + q_i\frac{x_i(t)}{\overline{X_i}}\right)(u_i(t) + \alpha_i\, u(t))\left(\overline{X_i} - (1-g_i)x_i(t)\right), i = 1, 2, \ldots, M \\ x_i(0) &= x_{i0} \forall i = 1, 2, \ldots M \\ \int_0^T &\left(\sum_{i=1}^{M}\phi_i(u_i(t)) + \varphi(u(t))\right) dt \le W_0 \end{aligned}\right\} \quad (5)$$

The above formulated optimal control problem consists of $2M + 1$ control variables $(u_i(t), u(t))$ and M state variables $(x_i(t))$.

4. Solution approach

To solve the above optimal control theory problem, we define a new state variable $W(t) = W_0 - \int_0^t \left(\sum_{i=1}^{M}\phi_i(u_i(t)) + \varphi(u(t))\right)dt$ with $W(0) = W_0$ and $W(T) \geq 0$. With new state variable, we rewrite the above optimal control problem (5) as

$$\left.\begin{aligned} \text{Max}\, J &= \int_0^T e^{-\gamma t}\left(\sum_{i=1}^{M}[\,(P_i - C_i(x_i(t)))\dot{x}(t) - \phi_i(u_i(t))] - \varphi(u(t))\right) dt \\ \frac{dx_i(t)}{dt} &= \left(p_i + q_i\frac{x_i(t)}{\overline{X_i}}\right)(u_i(t) + \alpha_i\, u(t))\left(\overline{X_i} - (1-g_i)x_i(t)\right), i = 1, 2, \ldots, M \\ x_i(0) &= x_{i0} \forall i = 1, 2, \ldots M \\ \dot{W}(t) &= -\left(\sum_{i=1}^{M}\phi_i(u_i(t)) + \varphi(u(t))\right), W(0) = W_0, W(T) \geq 0 \end{aligned}\right\} \quad (6)$$

Now, we obtain an optimal control problem with $2M + 1$ control variable and $M + 1$ state variable for all segments. Using the maximum principle [22], Hamiltonian can be defined as

$$H = \begin{pmatrix} \sum_{i=1}^{M}[\,(P_i - C_i(x_i(t)) + \lambda_i(t))\,\dot{x}(t) - \phi_i(u_i(t))] - \varphi(u(t)) \\ -\mu(t)\left(\sum_{i=1}^{M}\phi_i(u_i(t)) + \varphi(u(t))\right) \end{pmatrix} \quad (7)$$

The Hamiltonian represents the overall profit of the various policy decisions with both the immediate and the future effects taken into account. Assuming the existence of an optimal control solution, the maximum principle provides the necessary optimality conditions; there exist piecewise continuously differentiable functions $\lambda_i(t)$ and $\mu(t)$ for all $t \in [0, T]$. The value of $\lambda_i(t)$ and $\mu(t)$ define marginal valuation of state variables $x_i(t)$ and $W(t)$ at time t, respectively. Here, $\lambda_i(t)$ stands for change in future profit as making a small in $x_i(t)$ at time t, and $\mu(t)$ is the future profit of promotional effort per unit promotion effort expenditure at time t. These variables are known as adjoint variables and describe the similar behaviour in optimal control theory as dual variables in nonlinear programming.

From the necessary optimality conditions [22, 23] of maximum principle, we have

$$H\left(t, x_i^*, u_i^*, u^*, \lambda, \mu\right) = H\left(t, x_i^*, u_i, u, \lambda, \mu\right) \quad (8)$$

$$\frac{\partial H^*}{\partial u_i} = 0 \tag{9}$$

$$\frac{\partial H^*}{\partial u} = 0 \tag{10}$$

$$\frac{d\lambda_i(t)}{dt} = \gamma\,\lambda_i(t) - \frac{\partial H^*}{\partial x_i(t)},\ \lambda_i(T) = 0 \tag{11}$$

$$\frac{d\mu(t)}{dt} = \lambda\,\mu(t) - \frac{\partial H^*}{\partial W(t)},\ \mu(T) \geq 0, \tag{12}$$

$$W(T) + W_0 \geq 0,\ \mu(T)(W(T) + W_0) = 0 \tag{13}$$

Here, $\mu(T) \geq 0$, $W(T) + W_0 \geq 0$, $\mu(T)(W(T) + W_0) = 0$ are called as transversality conditions for $W(t)$. Here, Hamiltonian is independent to $W(t)$, and then we have $\dot{\mu} = \gamma\mu - \frac{\partial H}{\partial W} \Longrightarrow \mu(t) = \mu_T\, e^{\gamma(t-T)}$. Hence, it is clear that the multiplier associated with any integral constraint is constant over time irrespective of their nature (i.e. whether equality or inequality). The Hamiltonian H of each of the segments is strictly concave in $u_i(t)$ and $u(t)$. According to the Mangasarian sufficiency theorem [22, 23], there exist unique values of promotional effort controls $u_i^*(t)$ and $u^*(t)$ for each segment, respectively. From Eqs. (9) and (10), we get

$$u_i^*(t) = \phi_i^{-1}\left(\frac{(P_i - C_i(x_i(t)) + \lambda_i(t))\frac{\partial \dot{x}_i(t)}{\partial u_i} - \frac{\partial C_i}{\partial x_i}\ \frac{\partial x_i}{\partial u_i}\ \dot{x}_i}{1 + \mu_T\, e^{\gamma(t-T)}}\right),\ i = 1, 2, \ldots, M \tag{14}$$

$$u(t) = \varphi^{-1}\left(\frac{\sum_{i=1}^{M}\left((P_i - C_i(x_i(t)) + \lambda_i(t))\frac{\partial \dot{x}_i(t)}{\partial u_i}\right) - \frac{\partial C_i}{\partial x_i}\ \frac{\partial x_i}{\partial u_i}\ \dot{x}_i}{1 + \mu_T\, e^{\gamma(t-T)}}\right),\ i = 1, 2, \ldots, M \tag{15}$$

where ϕ_i^{-1} and φ^{-1} are inverse functions of ϕ_i and φ, respectively. If we assume product cost is independent to $x_i(t)$, i.e. $C_i(x_i(t)) = C_i$, then optimal promotional effort policies for each segment become

$$u_i^*(t) = \phi_i^{-1}\left(\frac{(P_i - C_i + \lambda_i(t))\left(p_i + q_i\frac{x_i(t)}{\overline{X_i}}\right)\left(\overline{X_i} - (1 - g_i)x_i(t)\right)}{1 + \mu_T\, e^{\gamma(t-T)}}\right),\ i = 1, 2, \ldots, M \tag{16}$$

$$u(t) = \varphi^{-1}\left(\frac{\sum_{i=1}^{M}\left((P_i - C_i + \lambda_i(t))\,\alpha_i\left(p_i + q_i\frac{x_i(t)}{\overline{X_i}}\right)\left(\overline{X_i} - (1 - g_i)x_i(t)\right)\right)}{1 + \mu_T\, e^{\gamma(t-T)}}\right),\ i = 1, 2, \ldots, M \tag{17}$$

The optimal control promotional policy shows that when market is almost saturated, then differentiated market promotional expenditure rate and mass market promotional expenditure rate, respectively, should be zero (i.e. there is no need of promotion in the market).

For optimal control policy, the optimal sales trajectory using optimal values of differentiated market promotional effort $\left(u_i^*(t)\right)$ and mass market promotional effort $(u^*(t))$ rates for each segment are given by

$$x_i^*(t) = \frac{\overline{X_i}\left(\left(\frac{p_i + q_i\frac{x_i(0)}{\overline{X_i}}}{\overline{X_i} - (1-g_i)x_i(0)}\right)\exp\left((q_i + p_i(1-g_i))\int_0^t (u_i^*(\tau) + \alpha_i u^*(\tau))d\tau\right)\right) - p_i}{\frac{q_i}{\overline{X_i}} + (1-g_i)\left(\frac{p_i + q_i\frac{x_i(0)}{\overline{X_i}}}{\overline{X_i} - (1-g_i)x_i(0)}\right)\exp\left((q_i + p_i(1-g_i))\int_0^t (u_i^*(\tau) + \alpha_i u^*(\tau))d\tau\right)} \tag{18}$$

If $x_i(0) = 0$, then we get the following result:

$$x_i^*(t) = \frac{1 - \exp\left(-(q_i + p_i(1-g_i))\int_0^t (u_i^*(\tau) + \alpha_i u^*(\tau))d\tau\right)}{(1-g_i) + \frac{q_i}{p_i}\exp\left(-(q_i + p_i(1-g_i))\int_0^t (u_i^*(\tau) + \alpha_i u^*(\tau))d\tau\right)}, i = 1, 2, \ldots, M \tag{19}$$

and adjoint trajectory is given as

$$\frac{d\lambda_i(t)}{dt} = \gamma\,\lambda_i(t) - \left\{(P_i - C_i(x_i(t)) + \lambda_i(t))\left(\frac{\partial \dot{x_i}}{\partial x_i}\right) - \dot{x_i}(t)\left(\frac{\partial C_i(x_i(t))}{\partial x_i(t)}\right)\right\} \tag{20}$$

with transversality condition $\lambda_i(T) = 0$. Integrating (20), the value of future profit of having one more unit of sales is

$$\lambda_i(t) = e^{-\gamma t}\int_t^T e^{-\gamma s}\left((P_i - C_i + \lambda_i(t))\left(\frac{\partial \dot{x_i}}{\partial x_i}\right) - \dot{x_i}(t)\left(\frac{\partial C_i}{\partial x_i}\right)\right)dt \tag{21}$$

4.1 Particular cases

4.1.1 When differentiated market promotional effort and mass market promotional effort costs are linear functions

Let us assume that differentiated market promotional effort and mass market promotional effort costs take the following linear forms: $\phi_i(u_i(t)) = \kappa_i\, u_i(t)$ and $\varphi(u(t)) = \kappa\, u(t)$ and $\overline{a}_i \le u_i(t) \le \overline{A}_i$, $\overline{a} \le u(t) \le \overline{A}$, where $\overline{a}_i$, $\overline{A}_i$, $\overline{a}$, and $\overline{A}$ are positive constants which are minimum and maximum acceptable promotional effort rates ($\overline{a}_i$, $\overline{A}_i$, $\overline{a}$, and $\overline{A}$ are determined by the promotional budget) and κ_i is the per unit cost of promotional effort per unit time towards i^{th} segment and κ is the per unit cost of promotional effort per unit time towards mass market. Now, Hamiltonian can be defined as

$$H = \begin{pmatrix} \sum_{i=1}^{M}[(P_i - C_i(x_i(t)) + \lambda_i(t))\,\dot{x_i}(t) - \kappa_i\, u_i(t)] - \kappa\, u(t) \\ -\mu(t)\left(\sum_{i=1}^{M}\kappa_i\, u_i(t) + \kappa\, u(t)\right) \end{pmatrix} \tag{22}$$

Since Hamiltonian is linear in $u_i(t)$ and $u(t)$, optimal differentiated market promotional effort and mass market promotional effort as obtained by the maximum principle are given by

$$u_i^*(t) = \begin{cases} \overline{a}_i \text{ if } B_i \le 0 \\ \overline{A}_i \text{ if } B_i > 0 \end{cases} \tag{23}$$

$$u^*(t) = \begin{cases} \overline{a} \text{ if } D \le 0 \\ \overline{A} \text{ if } D > 0 \end{cases} \tag{24}$$

where $B_i = (P_i - C_i + \lambda_i(t))\left(p_i + q_i \frac{x_i}{\overline{X}_i}\right)(\overline{X}_i - (1-g_i)x_i(t)) - \kappa_i(1+\mu(t))$ and $D = \sum_{i=1}^{M}\left(\alpha_i(P_i - C_i + \lambda_i(t))\left(p_i + q_i \frac{x_i}{\overline{X}_i}\right)(\overline{X}_i - (1-g_i)x_i(t))\right) - \varepsilon(1+\mu(t))$ are promotional effort switching functions and called 'bang-bang' control. However, interior control is possible on an arc along $u_i(t)$ and $u(t)$. Such an arc is known as the 'singular arc' [22].

This optimal control advertising policy shows that when market is almost saturated, then our switching functions $B_i = (P_i - C_i + \lambda_i(t))\left(p_i + q_i \frac{x_i}{\overline{X}_i}\right)(\overline{X}_i - (1-g_i)x_i(t)) - \kappa_i(1+\mu(t))$ and $D = \sum_{i=1}^{M}\left(\alpha_i(P_i - C_i + \lambda_i(t))\left(p_i + q_i \frac{x_i}{\overline{X}_i}\right)(\overline{X}_i - (1-g_i)x_i(t))\right) - \varepsilon(1+\mu(t))$ become negative or zero. Therefore, optimal advertising policy shows that there is no need to spend money, time, or resources on advertising, i.e. we do the advertising with minimum effectiveness rate.

There are four possible sets of optimal control values of differentiated market promotional effort $(u_i^*(t))$ and mass market promotional effort $(u^*(t))$ rate (**Figures 1** and **2**): (1) $u_i^*(t) = \overline{a}_i, u^*(t) = \overline{a},$ (2)

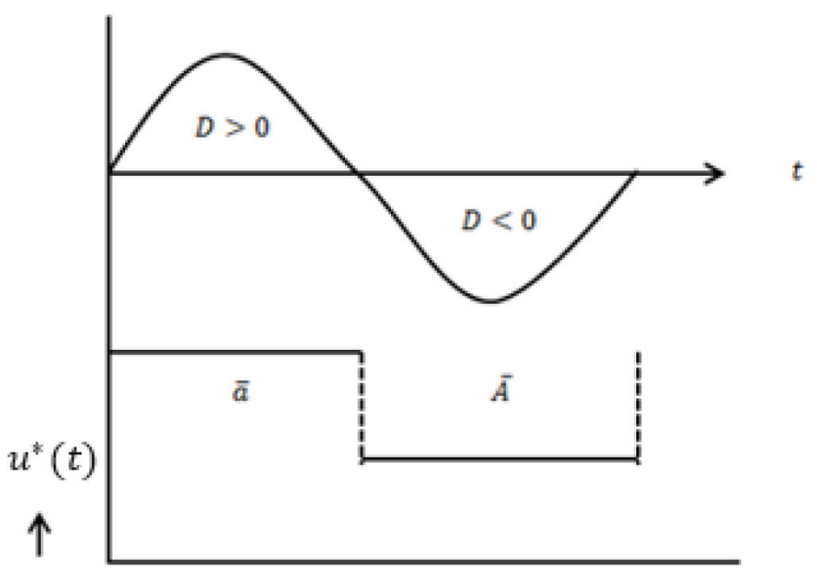

Figure 1.
Optimal promotional effort allocation policy for mass market promotional effort.

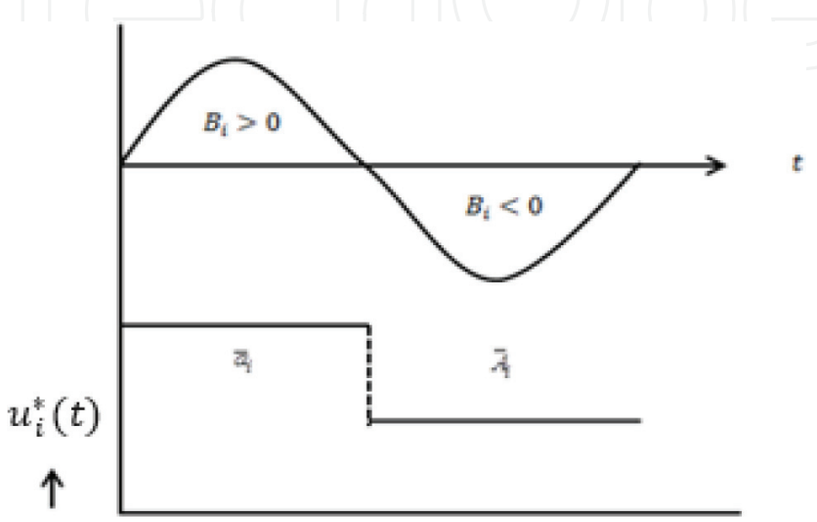

Figure 2.
Optimal promotional effort allocation policy for differentiated market promotional.

$u_i^*(t) = \overline{a}_i, u^*(t) = \overline{A}$, (3) $u_i^*(t) = \overline{A}_i, u^*(t) = \overline{a}$, and (4) $u_i^*(t) = \overline{A}_i, u^*(t) = \overline{A}$. Using these optimal values of differentiated market promotional effort ($u_i^*(t)$) and mass market promotional effort ($u^*(t)$) for each segment, we can obtain the optimal sales trajectories and adjoint trajectories. If we consider optimal values $u_i^*(t) = \overline{A}_i, u^*(t) = \overline{A}$, then the optimal sales and adjoint values can be described as

$$x_i^*(t) = \frac{\overline{X_i}\left(\left(\frac{p_i + q_i \frac{x_i(0)}{\overline{X_i}}}{\overline{X_i} - (1-g_i)x_i(0)}\right) \exp\left((q_i + p_i(1-g_i))(\overline{A}_i + \alpha_i \overline{A}_i)t\right)\right) - p_i}{\frac{q_i}{\overline{X_i}} + (1-g_i)\left(\frac{p_i + q_i \frac{x_i(0)}{\overline{X_i}}}{\overline{X_i} - (1-g_i)x_i(0)}\right) \exp\left((q_i + p_i(1-g_i))(\overline{A}_i + \alpha_i \overline{A}_i)t\right)} \forall i = 1, 2, 3, \ldots, M \tag{25}$$

If $x_i(0) = 0$, then we get the following result

$$x_i^*(t) = \frac{1 - \exp\left(-(q_i + p_i(1-g_i))(\overline{A}_i + \alpha_i \overline{A}_i)t\right)}{(1-g_i) + \frac{q_i}{p_i} \exp\left(-(q_i + p_i(1-g_i))(\overline{A}_i + \alpha_i \overline{A}_i)t\right)}, i = 1, 2, \ldots, M \tag{26}$$

which is similar to Bass model [7] sales trajectory with repeat purchasing, and the adjoint variable is given by

$$\frac{d\lambda_i(t)}{dt} = \rho\lambda_i(t) - \left((P_i - C_i + \lambda_i)(\overline{A}_i + \alpha_i \overline{A}\,)\left((1-g_i)(2x_i^* - X_i) - X_i\right)\right), \lambda_i(T) = 0 \tag{27}$$

The value of $\lambda_i(t)$ stands for per unit change in future profit of having one more unit of variable $x_i(t)$.

4.1.2 When differentiated market promotional effort and mass market promotional effort costs are quadratic functions

Promotional efforts towards differentiated market and mass market are costly. Let us assume that differentiated market promotional effort and mass market promotional effort costs take the following quadratic forms $\phi_i(u_i(t)) = \frac{\kappa_i}{2} u_i^2(t)$ and $\varphi(u(t)) = \frac{\kappa}{2} u^2(t)$ where $\kappa_i > 0$ and $\kappa > 0$ are positive constants and represent the magnitude of promotional effort rate per unit time towards i^{th} segment and towards mass market, respectively. This assumption is common in literature [24], where promotion cost is quadratic. Now, Hamiltonian can be defined as

$$H = \left(\begin{array}{r} \sum_{i=1}^{M}\left[(P_i - C_i(x_i(t)) + \lambda_i(t))\,\dot{x}(t) - \frac{\kappa_i}{2} u_i^2(t)\right] - \frac{\kappa}{2} u^2(t) \\ -\mu(t)\left(\sum_{i=1}^{M} \frac{\kappa_i}{2} u_i^2(t) + \frac{\kappa}{2} u^2(t)\right) \end{array}\right) \tag{28}$$

From the optimality necessary conditions (6), the optimal differentiated market promotional effort and mass market promotional effort are given by

$$u_i^*(t) = \frac{1}{\kappa_i}\left(\frac{(P_i - C_i + \lambda_i(t))\left(p_i + q_i \frac{x_i(t)}{\overline{X_i}}\right)\left(\overline{X_i} - (1-g_i)x_i(t)\right)}{1 + \mu_T e^{\gamma(t-T)}}\right) \tag{29}$$

$$u(t) = \frac{1}{\kappa}\left(\frac{\sum_{i=1}^{M}\left((P_i - C_i + \lambda_i(t))\,\alpha_i\left(p_i + q_i\frac{x_i(t)}{\overline{X_i}}\right)\left(\overline{X_i} - (1-g_i)x_i(t)\right)\right)}{1 + \mu_T\, e^{\gamma(t-T)}}\right) \tag{30}$$

Using optimal differentiated market promotional effort and mass market promotional effort rates from above Eqs. (29) and (30), we can obtain the optimal sales trajectories. Due to cumbersome analytical expression and an aim to illustrate the applicability of the formulated model through a numerical example, the discounted continuous optimal problem (5) is transformed into equivalent discrete problem [25] which can be solved using differential evolution. The equivalent discrete optimal control of the budgetary problem can be written as follows:

$$\left.\begin{aligned}
&\text{Max } J = \sum_{k=1}^{T}\left(\left(\left[\sum_{i=1}^{M}(P_i - C_i(k))(x_i(k+1) - x_i(k) - \phi_i(u_i(k)))\right] - \varphi(u(k))\right)\left(\frac{1}{(1+\gamma)^{k-1}}\right)\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{subjected to}\\
&x_i(k+1) = x_i(k) + \left(p_i + q_i\frac{x_i(k)}{\overline{X_i}}\right)(u_i(k) + \alpha_i\, u(k))\left(\overline{X_i} - (1-g_i)x_i(k)\right),\ i = 1, 2, \ldots, M\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \sum_{k=1}^{T}\left(\sum_{i=1}^{M}(\phi_i(u_i(k))) + \varphi(u(k))\right) \le W_0
\end{aligned}\right\} \tag{31}$$

The discretized version of the model is NP-hard; therefore, we use Lingo11 [26] to solve the discrete formulation.

5. Numerical illustration

To validate the model formulation, we consider a case of a company that has to find the optimal advertising policies for its consumer durable product. The company advertises at both national and regional levels of the market. To find the advertising policy for four segments, the values of the parameters, price, and cost of the product are given in **Table 1**.

	S1	S2	S3	S4
$\overline{N}_i$	279106.6	152460.1	97580.78	215868.5
p_i	0.000766	0.001161	0.00138	0.000549
q_i	0.137605	0.480576	0.540395	0.31362
α_i	0.3	0.19	0.189	0.320568
g_i	0.05	0.0265	0.0878	0.047644
κ_i (in ₹)	243,961	388,753	336,791	517,530
ε(in ₹)	1,153,922			
P_i	400,000	440,000	420,000	450,000
C_i	340,000	370,000	340,000	390,000
Initial sales$_i$	8969	8000	8000	8000

Table 1.
Parameters.

The discrete optimal control problem developed in this chapter is solved using differential evolution. Total promotional budget is assumed to be ₹ 3,000,000,000 which has to be allocated for mass market promotion and segment-specific promotion in four segments of the market. The time horizon has been divided into 12 equal time periods. The number of market segments is four (i.e. $M = 4$). The problem is coded in Lingo11 and solved.

Optimal allocation of promotional effort resources by solving each segment is given in **Table 2** for both mass and differentiated promotions, and the corresponding sales is tabulated in **Table 3**.

	Differentiated				Mass
	S1	**S2**	**S3**	**S4**	
T1	13.61	2.14	5.16	1.00	12.62
T2	14.69	1.30	2.42	1.00	13.20
T3	14.71	1.57	5.61	1.09	14.13
T4	17.06	6.87	1.56	1.50	15.02
T5	7.56	2.07	6.03	1.00	15.91
T6	19.63	2.32	1.96	1.00	16.80
T7	10.02	2.56	2.16	1.00	17.67
T8	21.99	4.88	6.62	3.06	16.61
T9	11.17	3.01	2.54	1.32	19.35
T10	24.28	8.27	7.00	2.74	20.19
T11	13.54	3.43	2.90	3.11	21.01
T12	26.44	8.68	7.34	10.74	28.78

Table 2.
Optimal differentiated and mass promotional allocations (in units).

	S1	**S2**	**S3**	**S4**
T1	8969	8000	8000	8000
T2	33,386	25,338	39,111	20,804
T3	112,850	64,724	105,406	52,397
T4	292,429	142,818	112,330	124,026
T5	296,492	201,723	98,941	235,744
T6	291,852	59,131	135,301	217,610
T7	297,762	157,208	35,605	235,055
T8	289,245	155,499	106,680	216,804
T9	305,967	158,657	108,227	241,642
T10	276,207	151,979	103,967	203,831
T11	345,626	178,184	121,081	263,316
T12	178,359	90,396	61,631	131,523

Table 3.
Optimal sales from potential market.

In the above case, we have solved the discretized problem by taking differentiated and mass promotional efforts as a linear function.

6. Conclusion

This chapter formulates an optimal control problem to find the optimal promotional policies for a consumer durable product in a segmented market where the sales are evolved through the combination of two promotion strategies: mass and differentiated promotions. The sales include the first-time purchase and the repeat purchases built through loyalty towards the product. Also to make the problem more realistic, we take a total budget constraint. The objective is to maximize the total profit through promotion. Maximum principle has been used to obtain the solution of the proposed problem. After discretizing the problem with linear costs, a numerical example has been solved using Lingo11 to illustrate the applicability of the approach. The developed optimal control model can be further extended in several ways. For instance, factors such as price, quality, and cost can be incorporated along with differentiated and mass market promotional effort expenditures. Further this monopolistic model can also be extended to competitive duopolistic or oligopolistic markets. Also the model can be extended to obtain optimal control policies for two and/or more generations' products in the market.

Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this chapter.

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