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Chapter

Hydrodynamic Methods and Exact Solutions in Application to the Electromagnetic Field Theory in Medium

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Abstract

The new Vavilov-Cherenkov radiation theory which is based on the relativistic generalization of the Landau theory for superfluid threshold velocity and Abraham theory of the electromagnetic field (EMF) in medium is represented. The new exact solution of the Cauchy problem in unbounded space is obtained for the n-dimensional Euler-Helmholtz (EH) equation in the case of a nonzero-divergence velocity field for an ideal compressible medium. The solution obtained describes the inertial vortex motion and coincides with the exact solution to the n-dimensional Hopf equation which simulates turbulence without pressure. Due to the introduction of a fairly large external friction or by introducing an arbitrary small effective volume viscosity, a new analytic solution of the Cauchy problem for the three-dimensional Navier-Stokes (NS) equation is obtained for compressible flows. This gives the positive solution to the Clay problem (www.clamath.org) generalization on the compressible NS equation. This solution also gives the possibility to obtain a new class of regular solutions to the n-dimensional modification of the Kuramoto-Sivashinsky equation, which is ordinarily used for the description of the nonlinear propagation of fronts in active media. The example for potential application of the new exact solution to the Hopf equation is considered in the connection of nonlinear geometrical optics with weak nonlinear medium at the nonlocality of the small action radii.

Keywords: hydrodynamics, compressibility, viscosity, turbulence, vorticity, EMF waves, Abraham theory, photon in medium

1. Introduction

The main subject of the nonlinear optic theory is a nonlinear activity of a medium where electromagnetic field (EMF) is propagated.

In this connection, the analogy between electromagnetic and hydrodynamic phenomena, which was noted yet by Helmholtz and Maxwell [1], is considered. In more recent papers, also different types of this analogy are used [2–4] and give possibility to open new ways for the solution of some nonlinear hydrodynamic problems on the basis of this analogy.

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However up to now, there are only a few examples of the direct mathematical correspondence between hydrodynamics and EMF theory, which gives resolution of the EMF problems on the basis of hydrodynamics [5, 6].

Thus in [5] there is an exact mathematical correspondence between the solutions for the point electric dipole potential and velocity potential obtaining for the rigid sphere moving with constant speed in the ideal incompressible fluid.

In [6] an exact correspondence is established between the mathematical description of the single vortex velocity on the sphere and the Dirac magnetic monopole (DMM) [7] vector potential. Similar analogy with DMM was noted also for the vortices in quantum superfluid He-3A [8–11].

Moreover, in [6], it was proved that the hydrodynamic equations do not allow the existence of a solution in the form of a single isolated vortex on sphere, but allow the exact solution in the form of two antipodal point vortices (which have the same value but different signs of circulation and located on the sphere on the maximal possible distance from each other). This result gives the first theoretical base for the proposition that DMM also cannot exist in the single form, but they must be included in the structure of point magnetic dipole, which is confirmed by all observations and experiment data.

Here we consider some examples of the application of hydrodynamic methods for the problems of EMF interaction with medium which may be important in the field of nonlinear optics.

In Part 1 of the chapter, we give the example for demonstration of the new mechanism of the Vavilov-Cherenkov radiation (VCR), which is obtained only on the basis of relativistic generalization to the Landau theory of superfluid threshold velocity [12]. In analogy with the Landau criterion its relativistic generalization is deduced for the determination of threshold conversion of medium Bose-condensed excitation into Cherenkov's photon. Thus, the VCR arises only due to the reaction of medium on the electric charge moving with super threshold velocity [13–15]:

$$V_0 > V_{th} = c/n_*; n_* = n + \sqrt{n^2 - 1}, n > 1; n_* = \left(1 + \sqrt{1 - n^2}\right)/n, n < 1 \qquad (1)$$

In (1), c is the light speed in vacuum and n is the medium refractive index.

In contraposition to the classic VCR theory [16–18], the new VCR theory in [13–15] and (1) admits the conditions for effective and direct VCR realization even for high-frequency transverse waves of EMF in isotropic plasma when $n < 1$ in (1). This is possible in the new VCR theory only because it is based on the Abraham theory for EMF in a medium where photons have nonzero real mass of rest, which determines necessary (in energy balance equation) energy difference for the medium when the medium emits photon VCR only for the condition (1).

In the second part of this chapter, we consider a new exact solution of nonlinear hydrodynamic equations. This gives corresponding possibility of its application to the problems of nonlinear EMF and other wave propagation in active and dissipative medium, where the Kuramoto-Sivashinsky equation [19–21] is used, giving the generalization of the Korteweg-de Vries (KdV) equation. Indeed, in nonlinear optic the KdF equation may describe the EMF wave propagation (for the case when electric wave E is propagating along axis x):

$$\frac{\partial E}{\partial t} + \sigma E \frac{\partial E}{\partial x} + \sigma_0 \frac{\partial^3 E}{\partial x^3} = 0 \qquad (2)$$

On the other side, the problem of the propagation of a flame front (generated by a self-sustained exothermal chemical reaction) may be considered on the basis of the simplified version of the Sivashinsky equation ∇ [21]:

$$\frac{\partial f}{\partial t} - \frac{1}{2} U_s \left(\vec{\nabla} f \right)^2 = \gamma_0 f \tag{3}$$

In the one-dimensional case, (3) is the same as (2) if $E = \partial f / \partial x$; $U_s = -\sigma$ and if we replace (for the case $\gamma_0 < 0$) $\sigma_0 \partial^3 E / \partial x^3 \to -\gamma_0 E$.

In Eq. (3), the function $x_3 = f(x_1, x_2, t)$ determines the flame front which represents the interface between a combustible matter ($x_3 > 0$) and the combustion products ($x_3 < 0$); U_s and γ_0 are constant positive quantities which characterize the front velocity and the combustion intensity, respectively. For $\gamma_0 = 0$ Eq. (3) coincides with the Hamilton-Jacobi equation for a free nonrelativistic particle. In the two-dimensional case (more exactly, in its modification with account for the external friction with the coefficient μ when $\mu = -\gamma_0$), the exact solution of the n-dimensional Hopf equation modification with $\mu \neq 0$

$$\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} = -\mu u_i; i, k = 1, .., n \tag{4}$$

(for the inertial motion of compressible medium with velocity u_i) gives also the exact solution of Eq. (3) when the velocity of compressible medium $\vec{u} = -U_s \vec{\nabla} f$.

The common solution of 1D, 2D, and 3D equations (4) in Euler variables is first time obtained in [22–26]. On the basis of this solution, we give the positive answer to the generalization of the Clay problem [27] on the case of compressible medium motion with nonzero divergence of velocity field [23–26]. The existence and smoothness of this solution for all time may take place only for super threshold friction $\mu > \mu_{th} = 1/t_0$ (here t_0 is the minimal finite time of singularity realization for solution of the Hopf equation (4)) or for any finite volume viscosity [22–26]. This gives the possibility to obtain also exact solutions in nonlinear optic when equations of Kuramoto-Sivashinsky type are used for EMF wave propagation in nonlinear medium.

1.1 New theory of the Vavilov-Cherenkov radiation (VCR)

The Vavilov-Cherenkov radiation (VCR) phenomenon has justly become an inherent part of modern physics. The VCR in a refractive medium was experimentally discovered by Cherenkov and Vavilov [28] more than half a century ago. This was also the time when Tamm and Frank [16, 17] developed the electromagnetic macroscopical theory of this phenomenon, which, as well as the VCR discovery, was marked later by a Nobel Prize. The Tamm-Frank theory appeared to be very similar to the Heaviside theory, which had been forgotten for a century [29].

The Heaviside-Tamm-Frank (HTF) theory demonstrated that the cylindrically symmetrical EMF, created in a medium by an electron, which moves rectilinearly with the constant velocity V_0, does not exponentially reduce only in the case of the super threshold electron velocity $V_0 \geq c/n$. According to the HTF theory, this field must be identical to the VCR field, observed in the experiment [28].

However, such direct identification is not in agreement with the basic microscopical conception that VCR photons are radiated by a medium and not by an electron itself [16, 30]. The latter can serve only for the initiation of such radiation

by the medium. The phenomenological quantum theory of the VCR, developed by Ginzburg [18] on the basis of the Minkowski EMF theory in medium, still does not take into consideration the changes of the radiating medium energy state, which might be necessary for the VCR realization. As we show the latter, this is so because, in contrast to the Abraham EMF theory, for the momentum of photon in the Minkowski EMF theory, the corresponding photon mass of rest in medium always has only exact imaginary (with zero real part) value and cannot be taken into account in the energy balance equation for the VCR.

Thus, the classic theory of the VCR phenomenon leaves a question of the energy mechanism of the VCR effect open. Indeed, to elaborate this mechanism, we need to find out the necessary possible changes of the energy state of the medium itself, which ensure the VCR effect realization.

The suggested theory is based on directly using the Abraham momentum of photon:

$$\vec{p}_A = \frac{\varepsilon_{ph}}{cn}\vec{k}\,, n > 1; \vec{p_A} = \frac{\varepsilon_{ph} n}{c}\vec{k}\,, n < 1;\ \vec{k} = \frac{\vec{V}_{ph}}{\left|\vec{V}_{ph}\right|} \tag{5}$$

In (5) ε_{ph} is the photon energy and $\vec{V}_{ph}$ its velocity in medium.

For the Minkowski EMF theory, the momentum of photon in medium with n > 1 has the form: $\vec{p}_M = \frac{\varepsilon_{ph} n}{c}\vec{k}$

For (5), the real nonzero photon rest mass m_{ph} is determined from the known relativistic equation $m_{ph}^2 c^2 = \frac{\varepsilon_{ph}^2}{c^2} - p_A^2$, and from (5), we have

$$m_{ph} = \frac{\varepsilon_{ph}}{c^2 n}\sqrt{n^2 - 1}, n > 1; m_{ph} = \frac{\varepsilon_{ph}}{c^2}\sqrt{1 - n^2}, n < 1 \tag{6}$$

In the new VCR quantum theory [13–15], the energy $\Delta E_m = m_{ph}c^2$ may correspond to the energy of a medium long-wave Bose excitation which can transform into the VCR photon only when the super threshold condition (1) takes place. Thus, the value ΔE_m must be taken into account in the energy balance equation for VCR realization possibility (when medium must lose this energy when the VCR photon is arising from it), and this new VCR theory is provided in [13, 14]. In [15] we also give examples where it is easy to obtain experimental and observational evidence of the difference between Abraham's and Minkowski's EMF theories when the VCR may be observed during the electron beam transfer through the medium which is the light of intense laser or when high-energy cosmic rays go through the relict background radiation.

To obtain a relativistic generalization of the Landau criterion [12] for the VCR realization, it is necessary to use the energy balance equation for the VCR (including in it the value of medium energy loss $\Delta E_m = m_{ph}c^2$, where m_{ph} may be taken from (6)) in the coordinate system moving with the initial electron velocity $\vec{V}_0$ [13, 14]:

$$m_e c^2\left[1 - \Gamma_0\Gamma_1\left(1 - \frac{\left(\vec{V}_0\vec{V}_1\right)}{c^2}\right)\right] = \varepsilon_{ph}\Gamma_0\left[1 - \frac{\left(\vec{V}_0\vec{V}_{ph}\right)}{c^2} - \frac{m_{ph}c^2}{\varepsilon_{ph}}\right] \tag{7}$$

where $\vec{V}_1$ is the velocity of electron after VCR photon arising. In (7) $\Gamma_\alpha = 1/\sqrt{1 - \frac{V_\alpha^2}{c^2}}$, where $\alpha = 0$ or $\alpha = 1$ and $m_{ph}c^2/\varepsilon_{ph} = \sqrt{1 - \frac{V_{ph}^2}{c^2}}$ according to (6).

For example, in the case n > 1 in (7), we have $V_{ph} = c/n$ and in the right-hand side of (7) $A = 1 - \frac{\left(\vec{V}_0 \vec{V}_{ph}\right)}{c^2} - \frac{m_{ph}c^2}{\varepsilon_{ph}} = 1 - \frac{V_0}{c}\cos\theta - \frac{\sqrt{n^2-1}}{n}$.

The left-hand side of (7) is always negative (it is zero only for the case when the initial and finite velocity of the electron are the same $\vec{V}_0 = \vec{V}_1$).

In the nonrelativistic limit when $V_0 \ll c$; $V_{ph} \ll c$ from (7) for $\varepsilon_p > 0$, the Landau criterion [12] may be obtained: $\varepsilon_V - \left(\vec{p}\ \vec{V}_0\right) < 0; \varepsilon_V = \varepsilon_p\left(1 - \sqrt{1 - \frac{V_p^2}{c^2}}\right) \cong \frac{\varepsilon_p V_p^2}{2c^2}$. Then $\varepsilon_V = \frac{V_p p}{2}$ is the only kinetic energy of excitation (in [12] these are vorton elementary excitations).

Thus for the possibility of arising VCR photon with positive energy $\varepsilon_{ph} > 0$, it is necessary to have in the right-hand side of (7) the negative value of $A < 0$ or inequality:

$$\cos\theta > \frac{c}{V_0 n_*} \tag{8}$$

where the value $n_*(n) > 1$ for any cases of $n > 1$ or $n < 1$ as it shown in (1). From the condition $|\cos\theta| \le 1$ in (8), the value of threshold velocity in (1) is obtained.

The conditions (8) and (1) give the necessary condition for arising VCR, and from (8) it is possible to obtain the maximal angle of the VCR cone of rays. The classic VCR theory gives good correspondence to experiment only in the determination of position for the maximum of intensity in the VCR cone of rays, but not to the maximal angle of this cone. In [13, 14] it is shown that the new VCR theory gives a better agreement with the experiment [28] than classical VCR theory when describing the threshold edge of the VCR cone of rays.

According to [28] the VCR effect is observed in the whole region of angles $0 \le \theta \le \theta_{\max}^{A,B}$ with the maximum of radiation intensity $I(\theta)$ at the angle $\theta = \theta_0^{A,B} < \theta_{\max}^{A,B}$. Here Index A corresponds to gamma rays of ThC'', and the Index B corresponds to the VCR induced by Ra. Thus, $I(\theta) = 0$ when $\theta > \theta_{\max}^{A,B}$. In the [31] the same result was also obtained for VCR realization through the direct use of high-energy electron beam.

In the classic VCR theory in (1) and (8), the value n_* must be replaced with the value n for the case with n > 1.

Let us introduce the values $\beta_*^A; \beta_*^B$ which correspond to $\theta_{\max}^{A,B}$ of experiment [28] when (8) is used for evaluation of parameter $\beta = V_0/c$ and the analogy values $\beta^A; \beta^B$ for the classic VCR theory.

For example, when the medium where the VCR arising is water (H_2O), where $n = 1.333$, $n_* = 2.247$, and for the values $\cos\theta_{\max}^A = 0.6691$; $\cos\theta_{\max}^B = 0.7431$ from (8), we obtain $\beta_*^A = 0.6718; \beta_*^B = 0.6049$ which are smaller than 1, as they need from the relativity theory. For the classic VCR theory, the result is not corresponding to the inequality $\beta = V_0/c < 1$ of the relativity theory because from the classic VCR theory, $\beta^A = 1.1177; \beta^B = 1.0064$ may be obtained. The same results obtained for all other media are considered in the experiment [28, 31] (see [13, 14]).

Thus, the classic VCR theory gives good correspondence with experiment [28] only in the determination of angle $\theta_0^{A,B}$, but not of the angle $\theta_{\max}^{A,B}$. In this connection the classic VCR theory tied only with interference maximum at $\theta = \theta_0^{A,B}$ and does not consider at all the energetic base for threshold arising of this coherent VCR. Actually, this is clearer for the case of plasma with $n < 1$, where the classic VCR theory total excludes the possibility of the VCR in the form of transverse

high-frequency EMF waves. The present new VCR theory gives this possibility due to the transformation of a longitudinal Bose-condensed plasmon into transverse VCR photon, during the scattering of a plasmon on the relativistic electron [14, 37].

Moreover in this new VCR theory, the VCR phenomenon has the same nature as for numerous physical systems where dissipative instability is realized when corresponding excitations in a medium become energetically favorable at some super threshold conditions [12, 32–36].

1.2 Exact solution of hydrodynamic equations

Fundamental turbulence problem was unsolved during many years by virtue of the absence of analytical, time-dependent, smooth-at-all-time solutions of the nonlinear hydrodynamic equations. A few exact solutions are known in hydrodynamics, but none of these solutions is time-dependent and defined in unbounded space or in space with periodic boundary conditions [38–40].

The importance of this problem is determined by stability and predictability problems in all fields of science where solutions and methods of hydrodynamics are used. In this connection in 2000, the problem of the existence of smooth time-dependent hydrodynamic solutions was stated as one of the seven Millennium Prize Problems (MPPs) by the Clay Institute of Mathematics [27]. MPPs relate only to incompressible flows "since it is well known that the behavior of compressible flows is abominable" [41].

Here we show that even for a compressible case, it is possible to obtain exact analytical, time-dependent, smooth-at-all-time solutions of Hopf equation (4) (which gives also new class solution also for vortex typ. 2D and 3D Euler equation) when any viscosity of super threshold friction is taken into account [22–26].

With the aim to introduce effective volume viscosity (in addition to external friction in (4)), let us consider the n-dimensional Hopf equation (4) in the moving with velocity $V_i(t)$ coordinate system, where $V_i(t)$ is a random Gaussian delta-correlated in-time velocity field for which the relations hold:

$$\begin{aligned} &\left\langle V_i(t)V_j(\tau)\right\rangle = 2\nu\delta_{ij}\delta(t-\tau) \\ &\langle V_i(t)\rangle = 0 \end{aligned} \tag{9}$$

In (9) δ_{ij} is the Kronecker delta, δ is Dirac-Heaviside delta function, and the coefficient ν characterizes the action of the viscosity forces. In the general case, the coefficient ν can be a function of time when describing the effective turbulent viscosity, but also it can coincide with the constant kinematic viscosity coefficient when the random velocity field considered corresponds to molecular fluctuations. We will restrict our attention to the consideration of the case of constant coefficient ν in (9).

Thus, the initial equation (4) (for the case $\mu = 0$) takes the form:

$$\frac{\partial u_i}{\partial t} + \left(u_j + V_j(t)\right)\frac{\partial u_i}{\partial x_j} = 0 \tag{10}$$

As shown in Appendix, in the case of an arbitrary dimensionality of the space (n = 1, 2, 3, etc.), Eq. (10) has the following exact solution (see also [22–26]):

$$u_i\left(\vec{x},t\right) = \int d^n\xi u_{0i}\left(\vec{\xi}\right)\delta\left(\vec{\xi} - \vec{x} + \vec{B}\,(t) + t\vec{u_0}\,\left(\vec{\xi}\right)\right)\det\hat{A} \tag{11}$$

where $B_i(t) = \int_0^t dt_1 V_i(t_1)$, $\hat{A} \equiv A_{nm} = \delta_{nm} + t\frac{\partial u_{0n}}{\partial \xi_m}$, $\det\hat{A}$ is the determinant of the matrix $\hat{A}$, and $u_{0i}\left(\vec{x}\right)$ is an arbitrary smooth initial velocity field. The solution (11)

satisfies Eq. (10) only at such times for which the determinant of the matrix $\hat{A}$ is positive for any values of the spatial coordinates $\det\hat{A} > 0$.

In the case of the potential initial velocity field, the solution (11) is potential for all successive instants of time, corresponding to a zero-vortex field. On the contrary, in the case of nonzero initial vortex field, the solution also determines the evolution of velocity with a nonzero vortex field. In [42] the potential solution to the two-dimensional Hopf equation (4) (or when $\vec{B}= 0$ in (12)) was obtained only in the Lagrangian representation which also exactly follows from (11) for $n = 2$. It is important to understand that here in (11) we have a solution in Euler variables, which is firstly obtained in [22] for n = 2 and n = 3. From the solution of (10) or (4) in Lagrangian variables, it is unreal to obtain a solution of (4) or (10) in Euler variables. From the other side, it is easy to obtain a solution in Lagrangian variables if we have a solution in Euler variables as in (11).

For example, in the one-dimensional case ($n = 1$) in (11), we have $\det\hat{A} = 1 + t\frac{du_{01}}{d\xi_1}$, and the solution (11) coincides exactly with the solutions obtained in [43, 44]. The solution (11) can be obtained if we use the integral representation for the implicit solution of Eq. (10) in the form $u_k\left(\vec{x},t\right) = u_{0k}\left(\vec{x} - \vec{B}\ (t) - t\ \vec{u}\ \left(\vec{x},t\right)\right)$ with the use of the Dirac delta function (see Appendix or [22, 23]).

After averaging over the random field $B_i(t)$ (with the Gaussian probability density), from (11) we can obtain the exact solution in the form:

$$\langle u_i\rangle = \int d^n\xi u_{0i}\left(\vec{\xi}\right)\left|\det\hat{A}\right|\frac{1}{\left(2\sqrt{\pi\nu t}\right)^n}\exp\left[-\frac{\left(\vec{x} - \vec{\xi} - t\vec{u_0}\left(\vec{\xi}\right)\right)^2}{4\nu t}\right] \tag{12}$$

As distinct from (11), the average solution (12) of Eq. (10) is already arbitrarily smooth on any unbounded time interval and not only providing the positiveness of the determinant of the matrix $\hat{A}$.

If, on the other side, we neglect the viscosity forces when $\vec{B}\ (t) = 0$ in (11), the smooth solution (11) is defined, as was already noted, only under the condition $\det\hat{A} > 0$ [22–26] (see Appendix). This condition corresponds to a bounded time interval $0 \le t < t_0$, where the minimum limiting time t_0 of existence of the solution can be determined from the solution to the following nth-order algebraic equation (and successive minimization of the expression obtained, which depends on the spatial coordinates, with respect to these coordinates):

$$\begin{aligned}
&\det\hat{A}(t) = 1 + t\frac{du_{01}(x_1)}{dx_1} = 0, n = 1\\
&\det\hat{A}(t) = 1 + tdiv\vec{u_0} + t^2\det\hat{U}_{012} = 0, n = 2\\
&\det\hat{A}(t) = 1 + tdiv\vec{u_0} + t^2\left(\det\hat{U}_{012} + \det\hat{U}_{013} + \det\hat{U}_{023}\right) + t^3\det\hat{U}_0 = 0, n = 3
\end{aligned} \tag{13}$$

where $\det\hat{U}_0$ is the determinant of the 3×3 matrix $U_{0nm} = \frac{\partial u_{0n}}{\partial x_m}$, and $\det\hat{U}_{012} = \frac{\partial u_{01}}{\partial x_1}\frac{\partial u_{02}}{\partial x_2} - \frac{\partial u_{01}}{\partial x_2}\frac{\partial u_{02}}{\partial x_1}$ is the determinant of a similar matrix in the two-dimensional case for the variables (x_1, x_2). In this case $\det\hat{U}_{013}$, $\det\hat{U}_{023}$ are the determinants of the matrices in the two-dimensional case for the variables (x_1, x_3) and (x_2, x_3), respectively.

In the two-dimensional case, the condition in the form of Eq. (13) exactly coincides with the collapse condition obtained in [42] in connection with the problem of propagation of a flame front investigated on the basis of the Kuramoto-Sivashinsky Eq. (3). In this case for exact coincidence, it is necessary to replace $t \to b(t) = \frac{U_s(\exp(\gamma_0 t)-1)}{\gamma_0}$ in (13).

In the one-dimensional case, when n = 1, from Eq. (13) we can obtain the minimum time of appearance of the singularity $t_0 = \frac{1}{\max\left|\frac{du_{01}(x_1)}{dx_1}\right|} > 0$. In particular, for the initial distribution $u_{01}(x_1) = a\exp\left(-\frac{x_1^2}{L^2}\right), a > 0$, it follows that $t_0 = \frac{L}{a}\sqrt{\frac{e}{2}}$ obtained for the value $x_1 = x_{1\max} = \frac{L}{\sqrt{2}}$. In this case the singularity itself can be implemented only for positive values of the coordinate $x_1 > 0$ when Eq. (13) has a positive solution for time.

This means that the singularity (collapse) of the smooth solution can never occur when the initial velocity field is nonzero only for negative values of the spatial coordinate $x_1 < 0$.

Similarly, we can also determine the vortex wave burst time t_0 for n > 1. For (13) in the two-dimensional case (when the initial velocity field is divergence-free) for the initial stream function in the form $\psi_0(x_1,x_2) = a\sqrt{L_1L_2}\exp\left(-\frac{x_1^2}{L_1^2}-\frac{x_2^2}{L_2^2}\right), a > 0$, we obtain that the minimum time of existence of the smooth solution is equal to $t_0 = \frac{e\sqrt{L_1L_2}}{2a}$.

In the example considered, this minimum time of existence of the smooth solution is implemented for the spatial variables corresponding to points on the ellipse $\frac{x_1^2}{L_1^2}+\frac{x_2^2}{L_2^2} = 1$.

In accordance with (13), the necessary condition of implementation of the singularity is the condition of existence of a real positive solution to a quadratic (when n = 2) or cubic (when n = 3) equation for the time variable t. For example, in the case of two-dimensional flow with the initial divergence-free velocity field $div\,\vec{u_0} = 0$, in accordance with (13), the necessary and sufficient condition of implementation of the singularity (collapse) of the solution in finite time is the condition:

$$\det U_{012} < 0 \tag{14}$$

For the example considered above from (14), there follows the inequality $\frac{x_1^2}{L_1^2}+\frac{x_1^2}{L_2^2} > \frac{1}{2}$. When this inequality is satisfied, for n = 2 there exists a real positive solution to the quadratic equation in (13) for which the minimum collapse time $t_0 = \frac{e\sqrt{L_1L_2}}{2a} > 0$ given above is obtained.

On the contrary, if the initial velocity field is defined in the form of a finite function which is nonzero only in the domain $\frac{x_1^2}{L_1^2}+\frac{x_2^2}{L_2^2} \le \frac{1}{2}$, then the inequality (14) is violated, and the development of the singularity in a finite time turns out already to be impossible, and the solution remains smooth in unbounded time even regardless of the viscosity effects.

The condition of existence of a real positive solution of Eq. (13) (e.g., see (14)) is the necessary and sufficient condition of implementation of the singularity (collapse) of the solution, as distinct from the sufficient but not necessary integral criterion which was proposed in [45] (see formula (38) in [45]) and has the form:

$$\left(\frac{dI}{dt}\right)_{t=0} = -\int d^3x\, div\, \vec{u_0}\ \det{}^2\hat{U}_0 > 0; I = \int d^3x \det{}^2\hat{U} \tag{15}$$

In fact, in accordance with this criterion proposed in [45], the collapse of the solution is not possible in the case of the initial divergence-free velocity field, i.e., when $div\, \vec{u_0} = 0$. However, in this case the violation of criterion (15) does not exclude the possibility of the collapse of the solution by virtue of the fact that the criterion (15) does not determine the necessary condition of implementation of the collapse. Actually, in the example considered above (in determination of the minimum time of implementation of the collapse $t_0 = \frac{e\sqrt{L_1 L_2}}{2a}$) for two-dimensional compressible flow, the initial condition corresponded just to the initial velocity field with $div\, \vec{u_0} = 0$ in (13) when $n = 2$.

On the basis of the solution (11), using (13) and the Lagrangian variables $\vec{a}$ (where $\vec{x} = \vec{x}\left(t, \vec{a}\right) = \vec{a} + t\vec{u_0}\left(\vec{a}\right)$), we can represent the expression for the matrix of the first derivatives of the velocity $\hat{U}_{im} = \frac{\partial u_i}{\partial x_m}$ in the form:

$$\hat{U}_{im}\left(\vec{a}, t\right) = \hat{U}_{0ik}\left(\vec{a}\right) A^{-1}_{km}\left(\vec{a}, t\right) \tag{16}$$

In this case the expression (16) exactly coincides with the formula (30) given in [45] for the Lagrangian time evolution of the matrix of the first derivatives of the velocity which must satisfy the three-dimensional Hopf equation (10) (when $\vec{B}(t) = 0$ in (10)). In particular, in the one-dimensional case when $n = 1$, in the Lagrangian representation from (11) and (13), we obtain a particular case of the formula (16):

$$\left(\frac{\partial u(x,t)}{\partial x}\right)_{x=x(a,t)} = \frac{\frac{du_0(a)}{da}}{1 + t\frac{du_0(a)}{da}} \tag{17}$$

where a is the coordinate of a fluid particle at the initial time $t = 0$.

The solution (17) also coincides with the formula (14) in [45] and describes the catastrophic process of collapse of a simple wave in a finite time t_0 whose estimate is given above on the basis of the solution to Eq. (13) in the case $n = 1$ with the use of the Euler variables.

Let us take into account only the external friction. For this purpose it is necessary to consider the case with $\mu > 0$ in Eq. (4). In this case we can also obtain the exact solution from the expression (11) (for the case when in (11) $\vec{B} = 0$) changing in them the time variable t by the variable $\tau = \frac{1 - \exp(-t\mu)}{\mu}$ (see (31) in Appendix and [22, 23]). The new time variable τ now varies within the finite limits from $\tau = 0$ (when $t = 0$) to $\tau = \frac{1}{\mu}$ (as $t \to \infty$). This leads to the fact that in the case of fulfillment of the inequality

$$\mu > \frac{1}{t_0} \tag{18}$$

for given initial conditions, the quantity $\det\hat{A} > 0$ for all times since the necessary and sufficient condition of implementation of the singularity (13) will be not satisfied because the change $t \to \tau(t)$ must also be carried out in the condition (13).

Providing (18), the solution to the n-dimensional EH equation is smooth on an unbounded interval of time t. The corresponding analytic vortical solution to the three-dimensional Navier–Stokes equation also remains smooth for any $t \geq 0$ if the condition (18) is satisfied [22–26].

Note that under the formal coincidence of the parameters $\mu = -\gamma_0$ (see the Sivashinsky equation (3) in Introduction), the equality $\tau(t) = b(t)$ takes place providing the implementation of the singularity (13) when n = 2 and in accordance with the solution of the Kuramoto-Sivashinsky equation in [42] and the regularization of this solution for all times if (18) takes place.

Moreover the example of interesting prosperity for the direct application for solution (11) (see also (12)–(18)) may be done in the connection of the results [46], where the description of light propagation in a nonlinear medium on the basis of the Burgers-Hopf equation is done.

Indeed, in [46], the model of light propagation in weak nonlinear 3D Coul-Coul's medium with small action radii of nonlocality is represented. In [46], it was stated that in the geometric optic approach, this model is integrated and described by the Veselov-Novikov equation which has a 1D reduction in the form of the Burgers-Hopf equation. The last equation is considered in connection with nonlinear geometrical optics when 1D reduction is made for the case when the refractive index has no dependence on one of the space coordinates. It is important when the property of nonlinear wave finite-time breakdown for Burgers-Hopf solutions is considered in the application to the case of nonlinear geometrical optics. These solutions are useful for modeling of dielectrics which have impurities which induced sharp variations of the refractive index. Indeed, in the points of breakdown, the curvature of the light rays obtained discontinues property as it takes place at the boundary between different media [46].

In [46], the only hodograph method is used for the Burgers-Hopf (or Hopf equation which is obtained from the Burgers' equation in the limit of zero viscosity) equation solution in this connection. Thus the direct analytical description of the 1D–3D solutions to the Hopf equation in the form (11) gives the new possibility also for the nonlinear optic problem which is considered in [46]. For example, according to this solution, it is possible to obtain the important effect of avoidance of finite-time singularities when viscosity or friction forces are taken into account (when condition (18) takes place for the case of external friction).

2. Conclusions

Here we represent some examples where hydrodynamic methods and solutions may be useful for different problems in nonlinear optics. In these examples, the medium itself has the first degree of importance in realization of all mentioned phenomena. Indeed, the main future of the Vavilov-Cherenkov radiation is that the medium is the source of this radiation instead of any kinds of bremsstrahlung radiations by moving charged particles. The VCR theory presented here for the first time takes into account the real mechanism of VCR by the medium itself, excited by a sufficiently fast electron. It can also be shown only from the microscopic theory, but not from the macroscopic one stated in [16]. The first step in this direction was made in [47] also on the basis of the Abraham theory where it is proposed that the Vavilov-Cherenkov radiation is emitted by the medium in a nonequilibrium polarization state which is arising due to the parametric resonance interaction of the medium with a fast-charged particle.

The second example, which is represented here, also gives new perspectives on the basis of the new exact solution (in the Euler variables) for n-dimensional Hopf equation because this equation is known as the possible model for weak nonlinear optic problems [46]. The importance of the new solution is connected with its Euler form in dependence from space variables, which are not represented in the solution of the Burgers-Hopf equation well known before (see [45] and others).

A. Exact solution of n-D Hopf equation (n = 1, 2, 3)

The Appendix presents a procedure for deriving the exact solution of the 3D Hopf equation.

The Hopf equation in the n-dimensional space (n = 1..3) is as follows:

$$\frac{\partial u_i}{\partial t} + u_l \frac{\partial u_i}{\partial x_l} = 0 \tag{19}$$

When the external friction coefficient tends to zero in Eq. (4), $\mu \to 0$, Eq. (4) also coincides with the Hopf equation (19).

In the unbounded space, the general Cauchy problem solution for Eq. (19) under arbitrary smooth initial conditions $\vec{u}_0\left(\vec{x}\right)$ may be obtained as follows (see also in [22, 23]):

Eq. (19) may be represented in an implicit form as follows:

$$u_i\left(\vec{x},t\right) = u_{0i}\left(\vec{x} - t\,\vec{u}\,\left(\vec{x},t\right)\right) = \int d^n\xi u_{0i}\left(\vec{\xi}\right)\delta\left(\vec{\xi} - \vec{x} + t\,\vec{u}\,\left(\vec{x},t\right)\right) \tag{20}$$

In (20), δ is the Dirac delta function. Using known (see farther) properties of the delta function, it is possible to express the delta function in (20) with the help of an identity true for the very velocity field meeting Eq. (19):

$$\delta\left(\vec{\xi} - \vec{x} + t\,\vec{u}\,\left(\vec{x},t\right)\right) \equiv \delta\left(\vec{\xi} - \vec{x} + t\vec{u}_0\,\left(\vec{\xi}\right)\right)\left|\det \hat{A}\right| \tag{21}$$

In (21), the matrix $\hat{A}$ depends only on the initial velocity field and is as follows:

$$\hat{A} \equiv A_{km} = \delta_{km} + t\frac{\partial u_{0k}\left(\vec{\xi}\right)}{\partial \xi_m} \tag{22}$$

To infer (21), it is necessary to use the following delta-function property that is true for any smooth function $\vec{\Phi}\left(\vec{\xi}\right)$:

$$\delta\left(\vec{\Phi}\left(\vec{\xi}\right)\right) = \frac{\delta\left(\vec{\xi} - \vec{\xi}_0\right)}{\left|\det\left(\frac{\partial \Phi_k}{\partial \xi_m}\right)_{\vec{\xi}=\vec{\xi}_0}\right|} \tag{23}$$

In (23), the values $\vec{\xi}_0$ are defined from the solution of the equation

$$\vec{\Phi}\left(\vec{\xi}_0\right) = 0 \tag{24}$$

To prove (23), it is necessary to use Taylor series decomposition wrt $\vec{\xi}$ near $\vec{\xi}=\vec{\xi}_0$ for the argument of the delta function $\vec{\Phi}\left(\vec{\xi}\right)$ when in the limit $\vec{\xi}\to\vec{\xi}_0$ taking into account (24), we get

$$\delta\left(\Phi_k\left(\vec{\xi}_0\right)+\left(\frac{\partial\Phi_k}{\partial\xi_m}\right)_{\vec{\xi}=\vec{\xi}_0}(\xi_m-\xi_{0m})+O\left(\vec{\xi}-\vec{\xi}_0\right)^2\right)=\delta\left(\left(\frac{\partial\Phi_k}{\partial\xi_m}\right)_{\vec{\xi}=\vec{\xi}_0}(\xi_m-\xi_{0m})\right) \tag{25}$$

Using variable substitution in the argument of the right-hand side of (25) (of the type $\hat{A}\vec{x}=\vec{y}$ and taking into account that $d\vec{x}=\frac{d\vec{y}}{\left|\det\hat{A}\right|}$ [48]), we get from the right-hand side of (25) the right-hand side of (23).

When in (23), $\vec{\Phi}\left(\vec{\xi}\right)\equiv\vec{\xi}-\vec{x}+t\vec{u}_0\left(\vec{\xi}\right)$ and $\det\frac{\partial\Phi_k}{\partial\xi_m}=\det A_{km}$ where A_{km} is from (22); then Eq. (24) is reduced to the following equation:

$$\vec{\xi}_0-\vec{x}+t\vec{u}_0\left(\vec{\xi}_0\right)=0 \tag{26}$$

The solution of Eq. (26) is as follows:

$$\vec{\xi}_0=\vec{x}-t\vec{u}\left(\vec{x},t\right) \tag{27}$$

This can be verified substituting (27) into (26) and taking into account that the general implicit solution of the equation (19) can be represented as $\vec{u}\left(\vec{x},t\right)=\vec{u}_0\left(\vec{x}-t\vec{u}(\overline{x},t)\right)$ that is used in (20).

Let us use a known property of the delta function that for any smooth function $\vec{f}\left(\vec{x}\right)$, the following equality $\vec{f}\left(\vec{x}\right)\delta\left(\vec{x}-\vec{x}_0\right)=\vec{f}\left(\vec{x}_0\right)\delta\left(\vec{x}-\vec{x}_0\right)$ holds. That is why, in the general case, it is possible to multiply both sides of (23) by $\left|\det\frac{\partial\Phi_k\left(\vec{\xi}\right)}{\partial\xi_m}\right|$ getting the following:

$$\delta\left(\vec{\xi}-\vec{\xi}_0\right)=\delta\left(\vec{\Phi}\left(\vec{\xi}\right)\right)\left|\det\frac{\partial\Phi_k\left(\vec{\xi}\right)}{\partial\xi_m}\right| \tag{28}$$

From (28) and (27), identical holding of the equality (21) follows.

Taking into account (21), from (20), we get an exact general (for any smooth initial velocity fields) solution of the Cauchy problem for Eq. (19) as

$$u_i\left(\vec{x},t\right)=\int d^n\xi u_{0i}\left(\vec{\xi}\right)\delta\left(\vec{\xi}-\vec{x}+t\vec{u}_0\left(\vec{\xi}\right)\right)\det\hat{A}, \tag{29}$$

where $\det\hat{A}=\det\left(\delta_{mk}+t\frac{\partial u_{0m}\left(\vec{\xi}\right)}{\partial\xi_k}\right)$. That solution of Eq. (19) is considered under the following condition:

$$\det\hat{A}>0 \tag{30}$$

That is why, sign of $\det\hat{A}$ is absent in (29). The condition (30) provides smoothness of the solution only on the finite-time interval defined above from (13).

We can check that the very (29) under condition (30) exactly satisfies Eq. (19) by direct substitution of (29) in (19). The solution (29) describes not only potential but also vortex solutions of Eq. (19) in two- and three-dimensional cases for any smooth initial velocity field $\vec{u_0}\left(\vec{x}\right)$ that was not known earlier for the solutions of Eq. (19) [22–26].

The solution (29) of Eq. (19) allows getting an exact solution of Eq. (10) if in (29) to make a substitution: $\vec{x} \to \vec{x} - \vec{B}(t)$ that yields Eq. (10) representation as in (11).

The solution (29) also can be described as an exact solution of Eq. (4) for $\mu > 0$ if in (29) to substitute:

$$t \to \frac{1 - \exp\left(-t\mu\right)}{\mu} \tag{31}$$

A.1 The direct validation of the solution

To verify the solution (29) satisfies Eq. (19), let us substitute (29) in Eq. (19). Then we get from (19):

$$\int d^n\xi \left[u_{0i}\left(\vec{\xi}\right) \frac{\partial \det \hat{A}}{\partial t} \delta\left(\vec{\xi} - \vec{x} + t\vec{u_0}\left(\vec{\xi}\right)\right) - u_{0i}u_{0m} \det \hat{A} \frac{\partial \delta\left(\vec{\xi} - \vec{x} + t\vec{u_0}\left(\vec{\xi}\right)\right)}{\partial x_m} \right] + \int d^n\xi \int d^n\xi_1 F = 0 \tag{32}$$

where

$F \equiv u_{0m}\left(\vec{\xi_1}\right) \det \hat{A}\left(\vec{\xi_1}\right) \delta\left(\vec{\xi_1} - \vec{x} + t\vec{u_0}\left(\vec{\xi_1}\right)\right) u_{0i}\left(\vec{\xi}\right) \det \hat{A}\left(\vec{\xi}\right) \frac{\partial \delta\left(\vec{\xi} - \vec{x} + t\vec{u_0}\left(\vec{\xi}\right)\right)}{\partial x_m}$.

To transform sub-integral expression in (32), the following identities shall be used:

$$\frac{\partial \delta\left(\vec{\xi} - \vec{x} + t\vec{u_0}\left(\vec{\xi}\right)\right)}{\partial x_m} = -A^{-1}_{km} \frac{\partial \delta\left(\vec{\xi} - \vec{x} + t\vec{u_0}\left(\vec{\xi}\right)\right)}{\partial \xi_k} \tag{33}$$

$$\frac{\partial \det \hat{A}}{\partial t} \equiv \frac{\partial u_{0m}}{\partial \xi_k} A^{-1}_{km} \det \hat{A} \tag{34}$$

$$\frac{\partial}{\partial \xi_k}\left(A^{-1}_{km} \det \hat{A}\right) \equiv 0 \tag{35}$$

The identity (33) is obtained from the relationship (obtained by differentiating the delta function having argument as a given function of $\vec{\xi}$) $\frac{\partial \delta\left(\vec{\xi} - \vec{x} + t\vec{u_0}\left(\vec{\xi}\right)\right)}{\partial \xi_k} = -\frac{\partial \delta\left(\vec{\xi} - \vec{x} + t\vec{u_0}\left(\vec{\xi}\right)\right)}{\partial x_l} A_{lk}$ after multiplying it both sides by the inverse matrix A^{-1}_{km} (where $A_{lk}A^{-1}_{km} = \delta_{lm}$и and δ_{lm} is the unity matrix or the Kronecker delta).

The validity of the identities (34) and (35) is proved by the direct checking. In the one-dimensional case, when $\hat{A} = 1 + t\frac{du_{01}}{d\xi_1} = \det \hat{A}; \hat{A}^{-1} = \left(\det \hat{A}\right)^{-1}$, it obviously follows directly from (34) and (35). Further, in Item 3, the proof of the identities (34) and (35) of the two- and three-dimensional cases is given.

Taking into account (33)–(35), from (32), we get

$$\int d^n\xi\delta\left(\vec{\xi}-\vec{x}+t\vec{u_0}\left(\vec{\xi}\right)\right)A_{km}^{-1}\det\hat{A}\left(u_{0i}\frac{\partial u_{0m}}{\partial\xi_k}-\frac{\partial}{\partial\xi_k}\left(u_{0i}u_{0m}\right)\right)+\int d^n\xi\int d^n\xi_1F_1=0 \tag{36}$$

where the sub-integral expression in the second term of the left-hand side of (36) is as follows:

$$F_1=u_{0m}\left(\vec{\xi}_1\right)\frac{\partial u_{0i}\left(\vec{\xi}\right)}{\partial\xi_k}\det\hat{A}\left(\vec{\xi}_1\right)\det\hat{A}\left(\vec{\xi}\right)A_{km}^{-1}\left(\vec{\xi}\right)\delta\left(\vec{\xi}-\vec{x}+t\vec{u_0}\left(\vec{\xi}\right)\right)\delta\left(\vec{\xi}_1-\vec{x}+t\vec{u_0}\left(\vec{\xi}_1\right)\right) \tag{37}$$

To transform (37), it is necessary to use the following identities:

$$\begin{aligned}&\delta\left(\vec{\xi}-\vec{x}+t\vec{u_0}\left(\vec{\xi}\right)\right)\delta\left(\vec{\xi}_1-\vec{x}+t\vec{u_0}\left(\vec{\xi}_1\right)\right)\\&\equiv\delta\left(\vec{\xi}-\vec{x}+t\vec{u_0}\left(\vec{\xi}\right)\right)\delta\left(\vec{\xi}_1-\vec{\xi}+t\left(\vec{u_0}\left(\vec{\xi}_1\right)-\vec{u_0}\left(\vec{\xi}\right)\right)\right)\end{aligned} \tag{38}$$

$$\delta\left(\vec{\xi}_1-\vec{\xi}+t\left(\vec{u_0}\left(\vec{\xi}_1\right)-\vec{u_0}\left(\vec{\xi}\right)\right)\right)\equiv\frac{\delta\left(\vec{\xi}_1-\vec{\xi}\right)}{\det\hat{A}} \tag{39}$$

In (39), as it is noted above, $\det\hat{A}>0$, and that is why the sign is not used in the denominator of (39).

The identity (38) is a consequence of the noted above property of the delta function (see discussion before the formula (28)).

To infer the identity (39), it is necessary to consider in the argument of the delta function a Taylor series decomposition of the function $u_{0k}\left(\vec{\xi}_1\right)=u_{0k}\left(\vec{\xi}\right)+\left(\partial u_{0k}\left(\frac{\vec{\xi}_1)}{\partial\xi_{1m}}\right)_{\vec{\xi}_1=\vec{\xi}}(\xi_{1m}-\xi_m)+O\left(\vec{\xi}_1-\vec{\xi}\right)^2$ near the point $\vec{\xi}_1=\vec{\xi}$. Then the left-hand side of (39) has the form $\delta\left(\hat{A}\left(\vec{\xi}_1-\vec{\xi}\right)\right)$ similar to that of the right-hand side of (25), and according to (23), we get from here the identity (39).

After the application of the identity (39) to the expression (37), defining the form of the second term in (36), from (36), we get

$$\int d^n\xi\delta\left(\vec{\xi}-\vec{x}+t\vec{u_0}\left(\vec{\xi}\right)\right)A_{km}^{-1}\det\hat{A}\left[u_{0i}\frac{\partial u_{0m}}{\partial\xi_k}-\frac{\partial}{\partial\xi_k}\left(u_{0i}u_{0m}\right)+u_{0m}\frac{\partial u_{0i}}{\partial\xi_k}\right]=0 \tag{40}$$

Equality (40) holds identically due to the identical equality to zero of the expression in the brackets in the sub-integral expression in (40).

Thus, we have proved that (29) exactly satisfies the Hopf equation (19) for any smooth initial velocity fields on the finite-time interval under condition $\det\hat{A}>0$ in (13).

A.2 The validation of identities (34) and (35)

In the two-dimensional case, the elements of the inverse matrix A_{km}^{-1} and the determinant of the matrix $\hat{A}$ are

$$A_{11}^{-1} = \frac{1 + t\partial u_{02}/\partial \xi_2}{\det \hat{A}}; A_{12}^{-1} = -\frac{t\partial u_{01}/\partial \xi_2}{\det \hat{A}}; A_{21}^{-1} = -\frac{t\partial u_{02}/\partial \xi_1}{\det \hat{A}}; A_{22}^{-1} = \frac{1 + t\partial u_{01}/\partial \xi_1}{\det \hat{A}} \quad (41)$$

$$\det \hat{A} = 1 + t\left(\frac{\partial u_{01}}{\partial \xi_1} + \frac{\partial u_{02}}{\partial \xi_2}\right) + t^2\left(\frac{\partial u_{01}}{\partial \xi_1}\frac{\partial u_{02}}{\partial \xi_2} - \frac{\partial u_{01}}{\partial \xi_2}\frac{\partial u_{02}}{\partial \xi_1}\right) \quad (42)$$

Here, (42) corresponds to the formula (13) for n = 2.

Using (41), it is possible to show that the following equality holds (in the left-hand side of (43), summation is assumed on the repeating indices from 1 to 2):

$$\frac{\partial u_{0m}}{\partial \xi_k} A_{km}^{-1} \det \hat{A} = \frac{\partial u_{01}}{\partial \xi_1} + \frac{\partial u_{02}}{\partial \xi_2} + 2t\left(\frac{\partial u_{01}}{\partial \xi_1}\frac{\partial u_{02}}{\partial \xi_2} - \frac{\partial u_{01}}{\partial \xi_2}\frac{\partial u_{02}}{\partial \xi_1}\right) \quad (43)$$

From (42), it follows that the right-hand side of (43) exactly matches $\frac{\partial \det \hat{A}}{\partial t}$ obtained when differentiating over time in (42). This proves the identity of (34) in the two-dimensional case.

To prove the identity (35), let us introduce

$$B_m = \frac{\partial}{\partial \xi_k}\left(A_{km}^{-1} \det \hat{A}\right) \quad (44)$$

Using (41), one gets from (44)

$$B_1 = \frac{\partial}{\partial \xi_1}\left(1 + t\frac{\partial u_{02}}{\partial \xi_2}\right) - \frac{\partial}{\partial \xi_2}\left(t\frac{\partial u_{02}}{\partial \xi_1}\right) \equiv 0 \quad (45)$$

$$B_2 = \frac{\partial}{\partial \xi_1}\left(-t\frac{\partial u_{01}}{\partial \xi_2}\right) + \frac{\partial}{\partial \xi_2}\left(1 + t\frac{\partial u_{01}}{\partial \xi_1}\right) \equiv 0 \quad (46)$$

The identities (45) and (46) confirm the truth of the identity (35) in the two-dimensional case.

Similarly, the identity (35) is proved in the three-dimensional case. For that, we need the following representation of the entries of the inverse matrix $\hat{A}^{-1}$ [49]:

$$\begin{aligned}
A_{11}^{-1} &= \frac{1}{\det \hat{A}}\left[\left(1 + t\frac{\partial u_{02}}{\partial \xi_2}\right)\left(1 + t\frac{\partial u_{03}}{\partial \xi_3}\right) - t^2\frac{\partial u_{02}}{\partial \xi_3}\frac{\partial u_{03}}{\partial \xi_2}\right];\\
A_{12}^{-1} &= \frac{1}{\det \hat{A}}\left[t^2\frac{\partial u_{01}}{\partial \xi_3}\frac{\partial u_{03}}{\partial \xi_2} - t\left(1 + t\frac{\partial u_{03}}{\partial \xi_3}\right)\frac{\partial u_{01}}{\partial \xi_2}\right];\\
A_{13}^{-1} &= \frac{1}{\det \hat{A}}\left[t^2\frac{\partial u_{01}}{\partial \xi_2}\frac{\partial u_{02}}{\partial \xi_3} - t\left(1 + t\frac{\partial u_{02}}{\partial \xi_2}\right)\frac{\partial u_{01}}{\partial \xi_3}\right];\\
A_{21}^{-1} &= \frac{1}{\det \hat{A}}\left[t^2\frac{\partial u_{02}}{\partial \xi_3}\frac{\partial u_{03}}{\partial \xi_1} - t\left(1 + t\frac{\partial u_{03}}{\partial \xi_3}\right)\frac{\partial u_{02}}{\partial \xi_1}\right];\\
A_{22}^{-1} &= \frac{1}{\det \hat{A}}\left[\left(1 + t\frac{\partial u_{01}}{\partial \xi_1}\right)\left(1 + t\frac{\partial u_{03}}{\partial \xi_3}\right) - t^2\frac{\partial u_{01}}{\partial \xi_3}\frac{\partial u_{03}}{\partial \xi_1}\right];\\
A_{23}^{-1} &= \frac{1}{\det \hat{A}}\left[t^2\frac{\partial u_{01}}{\partial \xi_3}\frac{\partial u_{02}}{\partial \xi_1} - t\left(1 + t\frac{\partial u_{01}}{\partial \xi_1}\right)\frac{\partial u_{02}}{\partial \xi_3}\right];\\
A_{31}^{-1} &= \frac{1}{\det \hat{A}}\left[t^2\frac{\partial u_{02}}{\partial \xi_1}\frac{\partial u_{03}}{\partial \xi_2} - t\left(1 + t\frac{\partial u_{02}}{\partial \xi_2}\right)\frac{\partial u_{03}}{\partial \xi_1}\right];\\
A_{32}^{-1} &= \frac{1}{\det \hat{A}}\left[t^2\frac{\partial u_{01}}{\partial \xi_2}\frac{\partial u_{03}}{\partial \xi_1} - t\left(1 + t\frac{\partial u_{01}}{\partial \xi_1}\right)\frac{\partial u_{03}}{\partial \xi_2}\right]\\
A_{33}^{-1} &= \frac{1}{\det \hat{A}}\left[\left(1 + t\frac{\partial u_{01}}{\partial \xi_1}\right)\left(1 + t\frac{\partial u_{02}}{\partial \xi_2}\right) - t^2\frac{\partial u_{01}}{\partial \xi_2}\frac{\partial u_{02}}{\partial \xi_1}\right]
\end{aligned} \quad (47)$$

From (44), in the three-dimensional case, we get on the basis of (47) that all three components of the vector $B_m \equiv 0$. For each $m = 1, 2, 3$, we get identical zeroing separately for the sum of terms proportional to t and separately for the sum of the terms proportional to t^2.

For example, in the expression for B_1 the sum of terms proportional to the first degree of time has the form $t\left[\frac{\partial}{\partial\xi_1}\left(\frac{\partial u_{02}}{\partial\xi_2}+\frac{\partial u_{03}}{\partial\xi_3}\right)-\frac{\partial^2 u_{02}}{\partial\xi_2\partial\xi_1}-\frac{\partial^2 u_{03}}{\partial\xi_3\partial\xi_1}\right]\equiv 0$, and similarly we can show the vanishing of the sum of twelve terms proportional to the square of time. Thus, the identity (35) is also proved in the three-dimensional case.

Proof of the identity (34) also is possible in the 3D case on the basis of (47) and (13) but is related to the cumbersome transformations.

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