Chapter

Convexity, Majorization and Time Optimal Control of Coupled Spin Dynamics

Navin Khaneja

Abstract

In this chapter, we study some control problems that derive from time optimal control of coupled spin dynamics in NMR spectroscopy and quantum information and computation. Time optimal control helps to minimize relaxation losses. In a two qubit system, the ability to synthesize, local unitaries, much more rapidly than evolution of couplings, gives a natural time scale separation in these problems. The generators of unitary evolution, $\mathfrak{g}$, are decomposed into fast generators $\mathfrak{k}$ (local Hamiltonians) and slow generators $\mathfrak{p}$ (couplings) as a Cartan decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$. Using this decomposition, we exploit some convexity ideas to completely characterize the reachable set and time optimal control for these problems. The main contribution of the chapter is, we carry out a global analysis of time optimality.

Keywords: Kostant convexity, spin dynamics, Cartan decomposition, Cartan subalgebra, Weyl group, time optimal control

1. Introduction

A rich class of model control problems arise when one considers dynamics of two coupled spin $\frac{1}{2}$. The dynamics of two coupled spins, forms the basis for the field of quantum information processing and computing [1] and is fundamental in multidimensional NMR spectroscopy [2, 3]. Numerous experiments in NMR spectroscopy, involve synthesizing unitary transformations [4–6] that require interaction between the spins (evolution of the coupling Hamiltonian). These experiments involve transferring, coherence and polarization from one spin to another and involve evolution of interaction Hamiltonians [2]. Similarly, many protocols in quantum communication and information processing involve synthesizing entangled states starting from the separable states [1, 7, 8]. This again requires evolution of interaction Hamiltonians between the qubits.

A typical feature of many of these problems is that evolution of interaction Hamiltonians takes significantly longer than the time required to generate local unitary transformations (unitary transformations that effect individual spins only). In NMR spectroscopy [2, 3], local unitary transformations on spins are obtained by application of rf-pulses, whose strength may be orders of magnitude larger than the couplings between the spins. Given the Schróedinger equation for unitary evolution

$$\dot{U} = -i\left[H_c + \sum_{j=1}^{n} u_j H_j\right]U, \quad U(0) = I, \tag{1}$$

where H_c represents a coupling Hamiltonian, and u_j are controls that can be switched on and off. What is the minimum time required to synthesize any unitary transformation in the coupled spin system, when the control generators H_j are local Hamiltonians and are much stronger than the coupling between the spins (u_j can be made large). Design of time optimal rf-pulse sequences is an important research subject in NMR spectroscopy and quantum information processing [4, 9–21], as minimizing the time to execute quantum operations can reduce relaxation losses, which are always present in an open quantum system [22, 23]. This problem has a special mathematical structure that helps to characterize all the time optimal trajectories [4]. The special mathematical structure manifested in the coupled two spin system, motivates a broader study of control systems with the same properties.

The Hamiltonian of a spin $\frac{1}{2}$ can be written in terms of the generators of rotations on a two dimensional space and these are the Pauli matrices $-i\sigma_x, -i\sigma_y, -i\sigma_z$, where,

$$\sigma_z = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \ \sigma_y = \frac{1}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \ \sigma_x = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{2}$$

Note

$$[\sigma_x, \sigma_y] = i\sigma_z, \quad [\sigma_y, \sigma_z] = i\sigma_x, \quad [\sigma_z, \sigma_x] = i\sigma_y, \tag{3}$$

where $[A, B] = AB - BA$ is the matrix commutator and

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \frac{\mathbf{1}}{4}, \tag{4}$$

The Hamiltonian for a system of two coupled spins takes the general form

$$H_0 = \sum a_\alpha \sigma_\alpha \otimes \mathbf{1} \ + \sum b_\beta \mathbf{1} \ \otimes \sigma_\beta + \sum J_{\alpha\beta} \ \sigma_\alpha \otimes \sigma_\beta, \tag{5}$$

where $\alpha, \beta \in \{x, y, z\}$. The Hamiltonians $\sigma_\alpha \otimes \mathbf{1}$ and $\mathbf{1} \otimes \sigma_\beta$ are termed local Hamiltonians and operate on one of the spins. The Hamiltonian

$$H_c = \sum J_{\alpha\beta} \ \sigma_\alpha \otimes \sigma_\beta, \tag{6}$$

is the coupling or interaction Hamiltonian and operates on both the spins. The following notation is therefore common place in the NMR literature.

$$I_\alpha = \sigma_\alpha \otimes \mathbf{1}; \quad S_\beta = \mathbf{1} \ \otimes \sigma_\beta. \tag{7}$$

The operators I_α and S_β commute and therefore $\exp\left(-i\sum_\alpha a_\alpha I_\alpha + \sum_\beta b_\beta S_\beta\right) =$

$$\exp\left(-i\sum_\alpha a_\alpha I_\alpha\right)\exp\left(-i\sum_\beta b_\beta S_\beta\right) = \left(\exp\left(-i\sum_\alpha a_\alpha \sigma_\alpha\right)\otimes \mathbf{1} \ \right)\left(\mathbf{1} \ \otimes \exp\left(-i\sum_\beta b_\beta \sigma_\beta\right)\right), \tag{8}$$

The unitary transformations of the kind

$$\exp\left(-i\sum_\alpha a_\alpha \sigma_\alpha\right)\otimes \exp\left(-i\sum_\beta b_\beta \sigma_\beta\right),$$

obtained by evolution of the local Hamiltonians are called local unitary transformations.

The coupling Hamiltonian can be written as

$$H_c = \sum J_{\alpha\beta} I_\alpha S_\beta. \tag{9}$$

Written explicitly, some of these matrices take the form

$$I_z = \sigma_z \otimes \mathbf{1} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{10}$$

and

$$I_z S_z = \sigma_z \otimes \sigma_z = \frac{1}{4}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{11}$$

The 15 operators,

$$-i\{I_\alpha, S_\beta, I_\alpha S_\beta\},$$

for $\alpha, \beta \in \{x, y, z\}$, form the basis for the Lie algebra $\mathfrak{g} = su(4)$, the 4×4, traceless skew-Hermitian matrices. For the coupled two spins, the generators $-iH_c, -iH_j \in su(4)$ and the evolution operator $U(t)$ in Eq. (1) is an element of $SU(4)$, the 4×4, unitary matrices of determinant 1.

The Lie algebra $\mathfrak{g} = su(4)$ has a direct sum decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$, where

$$\mathfrak{k} = -i\{I_\alpha, S_\beta\}, \quad \mathfrak{p} = -i\{I_\alpha S_\beta\}. \tag{12}$$

Here $\mathfrak{k}$ is a subalgebra of $\mathfrak{g}$ made from local Hamiltonians and $\mathfrak{p}$ nonlocal Hamiltonians. In Eq. (1), we have $-iH_j \in \mathfrak{k}$ and $-iH_c \in \mathfrak{p}$, It is easy to verify that

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{p}. \tag{13}$$

This decomposition of a real semi-simple Lie algebra $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ satisfying (13) is called the Cartan decomposition of the Lie algebra $\mathfrak{g}$ [24].

This special structure of Cartan decomposition arising in dynamics of two coupled spins in Eq. (1), motivates study of a broader class of time optimal control problems.

Consider the following canonical problems. Given the evolution

$$\dot{U} = \left(X_d + \sum_j u_j(t) X_j\right) U, \quad U(0) = \mathbf{1}\ , \tag{14}$$

where $U \in SU(n)$, the special Unitary group (determinant 1, $n \times n$ matrices U such that $UU' = \mathbf{1}$, $'$ is conjugate transpose). Where $X_j \in \mathfrak{k} = so(n)$, skew symmetric matrices and

$$X_d = -i\begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}, \quad \sum \lambda_i = 0.$$

We assume $\{X_j\}_{LA}$, the Lie algebra (X_j and its matrix commutators) generated by generators X_j is all of $so(n)$. We want to find the minimum time to steer this system between points of interest, assuming no bounds on our controls $u_j(t)$. Here again we have a Cartan decomposition on generators. Given $\mathfrak{g} = su(n)$, traceless skew-Hermitian matrices, generators of $SU(n)$, we have $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$, where $\mathfrak{p} = -iA$, where A is traceless symmetric and $\mathfrak{k} = so(n)$. As before, $X_d \in \mathfrak{p}$ and $X_j \in \mathfrak{k}$. We want to find time optimal ways to steer this system. We call this $\frac{SU(n)}{SO(n)}$ problem. For $n = 4$, this system models the dynamics of two coupled nuclear spins in NMR spectroscopy.

In general, U is in a compact Lie group G (such as $SU(n)$), with X_d, X_j in its real semisimple (no abelian ideals) Lie algebra $\mathfrak{g}$ and

$$\dot{U} = \left(X_d + \sum_j u_j(t) X_j \right) U, \quad U(0) = \mathbf{1} \ . \tag{15}$$

Given the Cartan decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$, where $X_d \in \mathfrak{p}$, $\{X_j\}_{LA} = \mathfrak{k}$ and $K = \exp(\mathfrak{k})$ (product of exponentials of $\mathfrak{k}$) a closed subgroup of G, We want to find the minimum time to steer this system between points of interest, assuming no bounds on our controls $u_j(t)$. Since $\{X_j\}_{LA} = \mathfrak{k}$, any rotation (evolution) in subgroup K can be synthesized with evolution of X_j [25, 26]. Since there are no bounds on $u_j(t)$, this can be done in arbitrarily small time [4]. We call this $\frac{G}{K}$ problem.

The special structure of this problem helps in complete description of the reachable set [27]. The elements of the reachable set at time T, takes the form $U(T) \in$

$$S = K_1 \exp\left(T \sum_k \alpha_k \ \ \mathcal{W}_k X_d \mathcal{W}_k^{-1} \right) K_2, \tag{16}$$

where $K_1, K_2, \mathcal{W}_k \in \exp(\mathfrak{k})$, and $\mathcal{W}_k X_d \mathcal{W}_k^{-1}$ all commute, and $\alpha_k > 0$, $\sum \alpha_k = 1$. This reachable set is formed from evolution of K_1, K_2 and commuting Hamiltonians $\mathcal{W}_k X_d \mathcal{W}_k^{-1}$. Unbounded control suggests that $K_1, K_2, \mathcal{W}_k$ can be synthesized in negligible time.

This reachable set can be understood as follows. The Cartan decomposition of the Lie algebra $\mathfrak{g}$, in Eq. (13) leads to a decomposition of the Lie group G [24]. Inside $\mathfrak{p}$ is contained the largest abelian subalgebra, denoted as $\mathfrak{a}$. Any $X \in \mathfrak{p}$ is Ad_K conjugate to an element of $\mathfrak{a}$, i.e. $X = K a_1 K^{-1}$ for some $a_1 \in \mathfrak{a}$.

Then, any arbitrary element of the group G can be written as

$$G = K_0 \exp(X) = K_0 \exp(Ad_K(a_1)) = K_1 \exp(a_1) K_2, \tag{17}$$

for some $X \in \mathfrak{p}$ where $K_i \in K$ and $a_1 \in \mathfrak{a}$. The first equation is a fact about geodesics in G/K space [24], where $K = \exp(\mathfrak{k})$ is a closed subgroup of G. Eq. (17) is called the KAK decomposition [24].

The results in this chapter suggest that K_1 and K_2 can be synthesized by unbounded controls X_i in negligible time. The time consuming part of the evolution

$\exp(a_1)$ is synthesized by evolution of Hamiltonian X_d. Time optimal strategy suggests evolving X_d and its conjugates $\mathcal{W}_k X_d \mathcal{W}_k^{-1}$ where $\mathcal{W}_k X_d \mathcal{W}_k^{-1}$ all commute.

Written as evolution

$$G = K_1 \prod_k \exp\left(t_k \mathcal{W}_k X_d \mathcal{W}_k^{-1}\right) \; K_2 = K_1 \prod_k \mathcal{W}_k \exp\left(t_k X_d\right) \mathcal{W}_k^{-1} \; K_2.$$

where $K_1, K_2, \mathcal{W}_k$ take negligible time to synthesize using unbounded controls u_i and time-optimality is characterized by synthesis of commuting Hamiltonians $\mathcal{W}_k X_d \mathcal{W}_k^{-1}$. This characterization of time optimality, involving commuting Hamiltonians is derived using convexity ideas [4, 28]. The remaining chapter develops these notions.

The chapter is organized as follows. In Section 2, we study the $\frac{SU(n)}{SO(n)}$ problem. In Section 3, we study the general $\frac{G}{K}$ problem. The main contribution of the chapter is, we carry out a *global analysis* of time optimality.

Given Lie algebra $\mathfrak{g}$, we use killing form $\langle x, y\rangle = tr\left(ad_x ad_y\right)$ as an inner product on $\mathfrak{g}$. When $\mathfrak{g} = su(n)$, we also use the inner product $\langle x, y\rangle = tr\left(x' y\right)$. We call this standard inner product.

2. Time optimal control for $SU(n)/SO(n)$ problem

Remark 1. Birkhoff's convexity states, a real $n \times n$ matrix A is doubly stochastic ($\sum_i A_{ij} = \sum_j A_{ij} = 1$, for $A_{ij} \geq 0$) if it can be written as convex hull of permutation matrices P_i (only one 1 and everything else zero in every row and column). Given $\Theta \in SO(n)$ and $X = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$, we have $diag\left(\Theta X \Theta^T\right) = B \;\; diag(X)$ where diag(X) is a column vector containing diagonal entries of X and $B_{ij} = \left(\Theta_{ij}\right)^2$ and hence $\sum_i B_{ij} = \sum_j B_{ij} = 1$, making B a doubly stochastic matrix, which can be written as convex sum of permutations. Therefore B diag$(X) = \sum_i \alpha_i P_i$ diag(X), i.e. diagonal of a symmetric matrix $\Theta X \Theta^T$, lies in convex hull of its eigenvalues and its permutations. This is called Schur convexity.

Remark 2. $G = SU(n)$ has a closed subgroup $K = SO(n)$ and a Cartan decomposition of its Lie algebra $\mathfrak{g} = su(n)$ as $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$, for $\mathfrak{k} = so(n)$ and $p = -iA$ where A is traceless symmetric and $\mathfrak{a}$ is maximal abelian subalgebra of $\mathfrak{p}$, such that $\mathfrak{a} = -i \begin{bmatrix} \lambda_1 & \dots & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix}$, where $\sum_i \lambda_i = 0$. **KAK** decomposition in Eq. (17) states for $U \in SU(n)$, $U = \Theta_1 \exp(\Omega) \Theta_2$ where $\Theta_1, \Theta_2 \in SO(n)$ and

$$\Omega = -i \begin{bmatrix} \lambda_1 & \dots & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix},$$

where $\sum_i \lambda_i = 0$.

Remark 3. We now give a proof of the reachable set (16), for the $\frac{SU(n)}{SO(n)}$ problem. Let $U(t) \in SU(n)$ be a solution to the differential Eq. (14)

$$\dot{U} = \left(X_d + \sum_i u_i X_i\right) U, \quad U(0) = I.$$

To understand the reachable set of this system we make a change of coordinates $P(t) = K^{'}(t)U(t)$, where, $\dot{K} = \left(\sum_i u_i X_i\right) K$. Then

$$\dot{P}(t) = Ad_{K^{'}(t)}(X_d)P(t), \quad Ad_K(X_d) = KXK^{-1}.$$

If we understand reachable set of $P(t)$, then the reachable set in Eq. (14) is easily derived.

Theorem 1. Let $P(t) \in SU(n)$ be a solution to the differential equation

$$\dot{P} = Ad_{K(t)}(X_d)P,$$

and $K(t) \in SO(n)$ and $X_d = -i \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & \lambda_n \end{bmatrix}$. The elements of the reachable set at time T, take the form $K_1 \exp(-i\mu T)K_2$, where $K_1, K_2 \in SO(n)$ and $\mu \prec \lambda$ (μ lies in convex hull of λ and its permutations), where $\lambda = (\lambda_1, \dots, \lambda_n)^{'}$.

Proof. As a first step, discretize the evolution of $P(t)$, as piecewise constant evolution, over steps of size τ. The total evolution is then

$$P_n = \prod_i \exp(Ad_{k_i}(X_d)\tau), \tag{18}$$

For $t \in [(n-1)\tau, n\tau]$, choose small step Δ, such that $t + \Delta < n\tau$, then $P(t+\Delta) = \exp(Ad_K(X_d)\Delta)P(t)$.

By KAK, $P(t) = K_1 \underbrace{\begin{bmatrix} \exp(i\phi_1) & 0 & 0 & 0 \\ 0 & \exp(i\phi_2) & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \exp(i\phi_n) \end{bmatrix}}_{A} K_2,$

where $K_1, K_2 \in SO(n)$. To begin with, assume eigenvalues $\phi_j - \phi_k \neq n\pi$, where n is an integer. When we take a small step of size Δ, $P(t)$ changes to $P(t+\Delta)$ as K_1, K_2, A change to

$$K_1(t+\Delta) = \exp(\Omega_1\Delta)K_1, \quad K_2(t+\Delta) = \exp(\Omega_2\Delta)K_2, \quad A(t+\Delta) = \exp(a\Delta)A,$$

where, $\Omega_1, \Omega_2 \in \mathfrak{k}$ and $a \in \mathfrak{a}$. Let $Q(t+\Delta) = K_1(t+\Delta)A(t+\Delta)K_2(t+\Delta)$, which can be written as

$$Q(t+\Delta) = \exp(\Omega_1\Delta)K_1 \exp(a\Delta)A \exp(\Omega_2\Delta)K_2. \tag{19}$$

$$Q(t+\Delta) = \exp(\Omega_1\Delta) \exp\left(K_1 a K_1^{'}\Delta\right) \exp\left(K_1 A \Omega_2 A^{'} K^{'}{}_1\Delta\right)P(t). \tag{20}$$

Observe

$$P(t+\Delta) = \exp(Ad_K(X_d)\Delta)P(t). \tag{21}$$

We equate $P(t+\Delta)$ and $Q(t+\Delta)$ to first order in Δ. This gives,

$$Ad_K(X_d) = \Omega_1 + K_1 a K_1^{'} + K_1 A \Omega_2 A^{'} K^{'}{}_1. \tag{22}$$

Multiplying both sides with $K_1^{'}(\cdot)K_1$ gives

$$Ad_{\overline{K}}(X_d) = \Omega_1^{'} + a + A\Omega_2 A^{'}. \tag{23}$$

where, $\overline{K} = K_1^{'}K$ and $\Omega_1^{'} = K^{'}\Omega K$.

We evaluate $A\Omega_2 A^{\dagger}$, for $\Omega_2 \in so(n)$.

$$\{A\Omega_2 A^{\dagger}\}_{kl} = \exp\{i(\phi_k - \phi_l)\}(\Omega_2)_{kl} = \underbrace{\cos(\phi_k - \phi_l)(\Omega_2)_{kl}}_{S_{kl}} + i\underbrace{\sin(\phi_k - \phi_l)(\Omega_2)_{kl}}_{R_{kl}}. \tag{24}$$

such that S is skew symmetric and R is traceless symmetric matrix with $iR \in \mathfrak{p}$. Note $iR \perp \mathfrak{a}$ and onto $\mathfrak{a}^{\perp}$, by appropriate choice of Ω_2.

Given $Ad_{\overline{K}}(X_d) \in \mathfrak{p}$, we decompose it as

$$Ad_{\overline{K}}(X_d) = P\left(Ad_{\overline{K}}(X_d)\right) + Ad_{\overline{K}}(X_d)^{\perp} = \Omega_1^{'} + a + A\Omega_2 A^{'},$$

with P denoting the projection onto $\mathfrak{a}$ ($\mathfrak{a} = -i\begin{bmatrix} \lambda_1 & \dots & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix}$, where $\sum_i \lambda_i = 0$.) w.r.t to standard inner product and $Ad_{\overline{K}}(X_d)^{\perp}$ to the orthogonal component. In Eq. (24), $\phi_k - \phi_l \neq 0, \pi$, we can solve for $(\Omega_2)_{kl}$ such that $iR = Ad_{\overline{K}}(X_d)^{\perp}$. This gives Ω_2. Let $a = P\left(Ad_{\overline{K}}(X_d)\right)$ and choose $\Omega_1^{'} = Ad_{\overline{K}}(X_d)^{\perp} - A\Omega_2 A^{\dagger} = -S \in \mathfrak{k}$.

With this choice of Ω_1, Ω_2 and a, $P(t+\Delta)$ and $Q(t+\Delta)$ are matched to first order in Δ and

$$P(t+\Delta) - Q(t+\Delta) = o\left(\Delta^2\right).$$

Consider the case, when A is degenerate. Let,

$$A = \begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & A_n \end{bmatrix}, \tag{25}$$

where A_k is n_k fold degenerate (modulo sign) described by $n_k \times n_k$ block. WLOG, we arrange

$$A_k = \exp(i\phi_k)\begin{bmatrix} I_{r\times r} & 0 \\ 0 & -I_{s\times s} \end{bmatrix}. \tag{26}$$

Consider the decomposition

$$Ad_{\overline{K}}(X_d) = P\left(Ad_{\overline{K}}(X_d)\right) + Ad_{\overline{K}}(X_d)^{\perp},$$

where P denotes projection onto $n_k \times n_k$ blocks in Eq. (25) and $Ad_K(X_d)^{\perp}$, the orthogonal complement.

$$P\left(\begin{bmatrix} X_{11} & X_{12} & \dots & X_{1n} \\ X_{21} & X_{22} & \dots & X_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ X_{n1} & X_{n2} & \dots & X_{nn} \end{bmatrix}\right) = \begin{bmatrix} X_{11} & 0 & \dots & 0 \\ 0 & X_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & X_{nn} \end{bmatrix}, \tag{27}$$

where X_{ij} are blocks.

Then we write

$$Q(t+\Delta) = \exp(\Omega_1 \Delta) K_1 \exp\left(P\left(Ad_{\overline{K}}(X_d)\Delta\right)\right) A \exp(\Omega_2 \Delta) K_2. \tag{28}$$

where in Eq. (24) we can solve for $(\Omega_2)_{kl}$ such that $iR = Ad_{\overline{K}}(X_d)^{\perp}$. This gives Ω_2. Choose, $Ad_{\overline{K}}(X_d)^{\perp} - A\Omega_2 A^{\dagger} = \Omega_1^{'} \in \mathfrak{k}$, this gives $\Omega_1 = K_1 \Omega_1^{'} K_1^{'}$. Again $P(t+\Delta) - Q(t+\Delta) = o(\Delta^2)$. We write Eq. (28) slightly differently.

Let H_1 be a rotation formed from block diagonal matrix

$$H_1 = \begin{bmatrix} \Theta_1 & 0 & \dots & 0 \\ 0 & \Theta_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \Theta_n \end{bmatrix}, \tag{29}$$

where Θ_k is $n_k \times n_k$ sub-block in $SO(n_k)$. $H_1 = \exp(h_1)$ is chosen such that

$$H_1^{'} P\left(Ad_{\overline{K}}(X_d)\right) H_1 = a$$

is a diagonal matrix. Let $H_2 = \exp(\underbrace{A^{-1} h_1 A}_{h_2})$, where h_2 is skew symmetric, such that

$$h_1 = \begin{bmatrix} \theta_1 & 0 & \dots & 0 \\ 0 & \theta_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \theta_n \end{bmatrix}, h_2 = \begin{bmatrix} \hat{\theta}_1 & 0 & \dots & 0 \\ 0 & \hat{\theta}_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \hat{\theta}_n \end{bmatrix}, \tag{30}$$

where

$\theta_k, \hat{\theta}_k$ is $n_k \times n_k$ sub-block in $so(n_k)$, related by (see 26)

$$\hat{\theta}_k = A_{k^{'}} \theta_k A_k, \quad \theta_k = \begin{bmatrix} \overbrace{\theta_{11}}^{r\times r} & \theta_{12} \\ -\theta_{12}^{\dagger} & \underbrace{\theta_{22}}_{s\times s} \end{bmatrix}, \hat{\theta}_k = \begin{bmatrix} \theta_{11} & -\theta_{12} \\ \theta_{12}^{\dagger} & \theta_{22} \end{bmatrix} \tag{31}$$

Note $H_1^{'} P(Ad_k(X_d)) H_1 = a$ lies in convex hull of eigenvalues of X_d. This is true if we look at the diagonal of $H_1^{'} Ad_K(X_d) H_1$, it follows from Schur Convexity. The diagonal of $H_1^{'} Ad_k(X_d)^{\perp} H_1$ is zero as its inner product

$$tr\left(a_1 H_1^{'} Ad_k(X_d)^{\perp} H_1\right) = tr\left(H_1 a_1 H_1^{'} Ad_k(X_d)^{\perp}\right) = 0.$$

as $H_1 a_1 H_1'$ has block diagonal form which is perpendicular to $Ad_k(X_d)^{\perp}$. Therefore diagonal of $H_1' P(Ad_k(X_d))H_1$ is same as diagonal of $H_1' Ad_K(X_d)H_1$.

Now using $H_1 A H_2^{\dagger} = A$, from 28, we have

$$Q(t+\Delta) = \exp(\Omega_1 \Delta) K_1 \exp\left(P\left(Ad_{\overline{K}}(X_d)\Delta\right)\right) H_1 A H_2^{\dagger} \exp(\Omega_2 \Delta) K_2. \tag{32}$$

$$Q(t+\Delta) = \exp(\Omega_1 \Delta) K_1 H_1 \exp(a\Delta) A H_2^{\dagger} \exp(\Omega_2 \Delta) K_2. \tag{33}$$

where the above expression can be written as

$$Q(t+\Delta) = \exp(\Omega_1 \Delta) \exp\left(K_1 H_1 a H_1' K_1' \Delta\right) \exp\left(K_1 A \Omega_2 A' K'{}_1 \Delta\right) P(t).$$

where Ω_1, H_1, a, Ω_2, are chosen such that

$$\left(\Omega_1 + K_1 H_1 a H_1' K_1' + K_1 A \Omega_2 A' K'{}_1\right) = Ad_K(X_d).$$

$$\left(\Omega_1' + H_1 a H_1' + A \Omega_2 A'\right) = Ad_{\overline{K}}(X_d).$$

$$Q(t+\Delta) - P(t+\Delta) = o\left(\Delta^2\right) P(t).$$

$$Q(t+\Delta) = \left(I + o\left(\Delta^2\right)\right) P(t+\Delta).$$

$$\begin{aligned} Q(t+\Delta) Q(t+\Delta)^T &= \left(I + o\left(\Delta^2\right)\right) P(t+\Delta) P^T(t+\Delta) \left(I + o\left(\Delta^2\right)\right) \\ &= P(t+\Delta) P^T(t+\Delta) \left[I + o\left(\Delta^2\right)\right]. \end{aligned}$$

$$P(t+\Delta) P^T(t+\Delta) = K_1 \begin{bmatrix} \exp(i2\phi_1) & 0 & \dots & 0 \\ 0 & \exp(i2\phi_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \exp(i2\phi_n) \end{bmatrix} K_1^T.$$

Let $F = P(t+\Delta)P^T(t+\Delta)$ and $G = Q(t+\Delta)Q^T(T+\Delta)$ we relate the eigenvalues, of F and G. Given F, G, as above, with $|F - G| \le \varepsilon$, and a ordered set of eigenvalues of F, denote $\lambda(F) = \begin{bmatrix} \exp(i2\phi_1) \\ \exp(i2\phi_2) \\ \vdots \\ \exp(i2\phi_n) \end{bmatrix}$, there exists an ordering (correspondence) of eigenvalues of G, such that $|\lambda(F) - \lambda(G)| < \varepsilon$.

Choose an ordering of $\lambda(G)$ call μ that minimizes $|\lambda(F) - \lambda(G)|$.

$F = U_1 D(\lambda) U_1'$ and $G = U_2 D(\mu) U_{2'}$, where $D(\lambda)$ is diagonal with diagonal as λ, let $U = U_1' U_2$,

$$|F - G|^2 = \left|D(\lambda) - UD(\mu)U'\right|^2 = |\lambda|^2 + |\mu|^2 - tr\left(D(\lambda)' UD(\mu)U' + (UD(\mu)U)' D(\lambda)\right),$$

By Schur convexity,

$$tr\left(D(\lambda)' UD(\mu)U' + \left(UD(\mu)U'\right)' D(\lambda)\right) = \sum_i \alpha_i \left(\lambda' P_i(\mu) + P_i(\mu)' \lambda\right),$$

where P_i are permutations. Therefore $|F - G|^2 > |\lambda - \mu|^2$.

Therefore,

$$\lambda\left(QQ^T(t+\Delta)\right) = \lambda\left(PP^T(t+\Delta)\right) + o\left(\Delta^2\right).$$

The difference

$$o\left(\Delta^2\right) = \underbrace{\exp\left(\left(\Omega_1 + K_1 H_1 a H_1' K_1' + K_1 A \Omega_2 A' K'_1\right)\Delta\right)}_{\exp\left(Ad_K(X_d)\Delta\right)}$$
$$- \exp\left(\Omega_1\Delta\right)\exp\left(K_1 H_1 a H_1' K_1'\Delta\right)\exp\left(K_1 A\Omega_2 A' K'_1\Delta\right),$$

is regulated by size of Ω_2, which is bounded by $|\Omega_2| \le \frac{\|X_d\|}{\sin\left(\phi_i - \phi_j\right)}$, where $\sin\left(\phi_i - \phi_j\right)$ is smallest non-zero difference. Δ is chosen small enough such that $|o\left(\Delta^2\right)| < \varepsilon\Delta$.

For each point $t \in [0, T]$, we choose an open nghd $N(t) = (t - N_t, t + N_t)$, such that $o_t\left(\Delta^2\right) < \varepsilon\Delta$ for $\Delta \in N(t)$. $N(t)$ forms a cover of $[0, T]$. We can choose a finite subcover centered at $t_1, \ldots, t_n$ (see **Figure 1A**). Consider trajectory at points $P(t_1), \ldots, \ldots P(t_n)$. Let $t_{i,i+1}$ be the point in intersection of $N(t_i)$ and $N(t_{i+1})$. Let $\Delta_i^+ = t_{i,i+1} - t_i$ and $\Delta_{i+1}^- = t_{i+1} - t_{i,i+1}$. We consider points $P(t_i), P(t_{i+1}), P(t_{i,i+1}), \underbrace{Q\left(t_i + \Delta_i^+\right)}_{Q_{i+}}, \underbrace{Q\left(t_{i+1} - \Delta_{i+1}^-\right)}_{Q_{(i+1)-}}$ as shown in **Figure 1B**.

Then we get the following recursive relations.

$$\lambda\left(Q_{i+} Q_{i+}^T\right) = \exp\left(2a_i^+\Delta_i^+\right)\ \lambda\left(P_i P_i^T\right) \tag{34}$$

$$\lambda\left(P_{i,i+1} P_{i,i+1}^T\right) = \lambda\left(Q_{i+} Q_{i+}^T\right) + o\left(\left(\Delta_i^+\right)^2\right) \tag{35}$$

$$\lambda\left(Q_{(i+1)-} Q_{(i+1)-}^T\right) = \lambda\left(P_{i,i+1} P_{i,i+1}^T\right) + o\left(\left(\Delta_{i+1}^-\right)^2\right) \tag{36}$$

$$\exp\left(-2a_{i+1}^-\Delta_{i+1}^-\right)\ \lambda\left(P_{i+1} P_{i+1}^T\right) = \lambda\left(Q_{(i+1)-} Q_{(i+1)-}^T\right) \tag{37}$$

where a_i^+ and a_{i+1}^- correspond to a in Eq. (33) and lie in the convex hull of the eigenvalues X_d.

Adding the above equations,

$$\lambda\left(P_{i+1} P_{i+1}^T\right) = \exp\left(o\left(\Delta^2\right)\right)\exp\left(2\left(a_i^+\Delta_i^+ + a_{i+1}^-\Delta_{i+1}^-\right)\ \lambda\left(P_i P_i^\dagger\right).\right. \tag{38}$$

$$\lambda\left(P_n P_n^T\right) = \exp\left(\underbrace{\sum o\left(\Delta^2\right)}_{\le \varepsilon T}\right)\exp\left(2\sum_i a_i^+\Delta_i^+ + a_{i+1}^-\Delta_{i+1}^-\right)\ \lambda\left(P_1 P_1^T\right). \tag{39}$$

where $o\left(\Delta^2\right)$ in Eq. (38) is diagonal.

$$\lambda\left(P_n P_n^T\right) = \exp\left(\underbrace{\sum o\left(\Delta^2\right)}_{\le \varepsilon T}\right)\ \exp\left(2T\sum_k \alpha_k P_k(\lambda)\right)\ \lambda\left(P_1 P_1^T\right) = \exp\left(\underbrace{\sum o\left(\Delta^2\right)}_{\le \varepsilon T}\right)\exp\left(2\mu T\right)\ \lambda\left(P_1 P_1^T\right), \tag{40}$$

where $\mu < \lambda$ and $P_1 = I$.

$$P_n = K_1 \exp\left(\frac{1}{2}\underbrace{\sum o\left(\Delta^2\right)}_{\le \varepsilon T}\right)\ \exp\left(\mu T\right)\ K_2. \tag{41}$$

Note, $|P_n - K_1 \exp\left(\mu T\right) K_2| = o(\varepsilon)$. This implies that P_n belongs to the compact set $K_1 \exp\left(\mu T\right) K_2$, else it has minimum distance from this compact set and by making $\Delta \to 0$ and hence $\varepsilon \to 0$, we can make this arbitrarily small. In Eq. (18), $P_n \to P(T)$ as $\tau \to 0$. Hence $P(T)$ belongs to compact set $K_1 \exp(\mu T) K_2$. **q.e.d.**

Corollary 1. Let $U(t) \in SU(n)$ be a solution to the differential equation

$$\dot{U} = \left(X_d + \sum_i u_i X_i \right) U,$$

where $\{X_i\}_{LA}$, the Lie algebra generated by X_i, is $so(n)$ and $X_d = -i \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$. The elements of reachable set at time T, takes the form $U(T) \in K_1 \exp(-i\mu T) K_2$, where $K_1, K_2 \in SO(n)$ and $\mu \prec \lambda$, where $\lambda = (\lambda_1, \dots, \lambda_n)'$ and the set $S = K_1 \exp(-i\mu T) K_2$ belongs to the closure of reachable set.

Proof. Let $V(t) = K'(t) U(t)$, where, $\dot{K} = \left(\sum_i u_i X_i\right) K$. Then

$$\dot{V}(t) = Ad_{K'(t)}(X_d) V(t).$$

From Theorem 1, we have $V(T) \in K_1 \exp(-i\mu T) K_2$. Therefore $U(T) \in K_1 \exp(-i\mu T) K_2$. Given

$$\begin{aligned} U = K_1 \exp(-i\mu T) K_2 &= K_1 \exp\left(-i \sum_j \alpha_j P_j(\lambda) T \right) K_2 \\ &= K_1 \prod_j \exp\left(-i t_j X_d\right) K_j, \quad \sum t_j = T. \end{aligned}$$

We can synthesize K_j in negligible time, therefore $|U(T) - U| < \varepsilon$, for any desired ε. Hence U is in closure of reachable set. **q.e.d.**

Remark 4. We now show how Remark 2 and Theorem 1 can be mapped to results on decomposition and reachable set for coupled spins/qubits. Consider the transformation

$$W = \exp\left(-i\pi I_y S_y\right) \exp\left(-i\frac{\pi}{2} I_z\right)$$

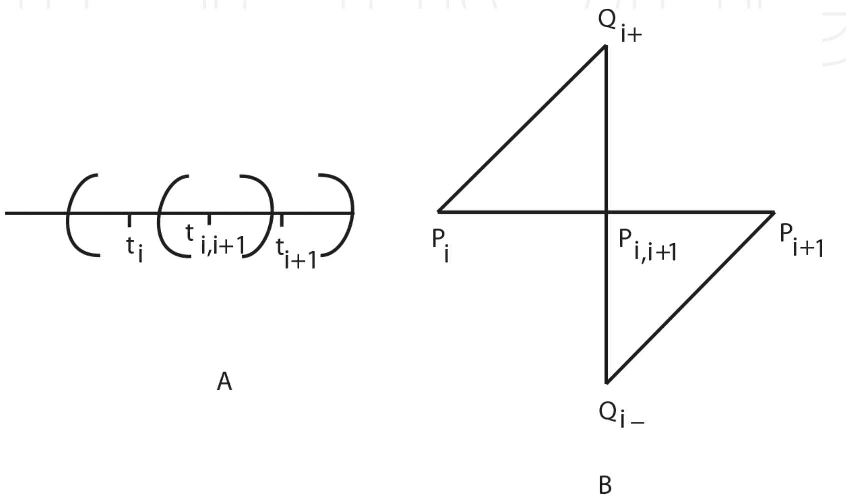

Figure 1.
A. Collection of overlapping neighborhoods forming the finite subcover. B. Depiction of P_i, P_{i+1}, Q_{i+}, Q_{i-}, $P_{i,i+1}$ *as in proof of Theorem 1.*

The transformation maps the algebra $\mathfrak{k} = su(2) \times su(2) = \{I_\alpha, S_\alpha\}$ to $\mathfrak{k}_1 = so(4)$, four dimensional skew symmetric matrices, i.e., $Ad_W(\mathfrak{k}) = \mathfrak{k}_1$. The transformation maps $\mathfrak{p} = \{I_\alpha S_\beta\}$ to $\mathfrak{p}_1 = -iA$, where A is traceless symmetric and maps $\mathfrak{a} = -i\{I_x S_x, I_y S_y, I_z S_z\}$ to $\mathfrak{a}_1 = -i\{-\frac{S_z}{2}, \frac{I_z}{2}, I_z S_z\}$, space of diagonal matrices in $\mathfrak{p}_1$, such that $a_x I_x S_x + a_y I_y S_y + a_z I_z S_z$ gets mapped to the four vector (the diagonal) $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (a_y + a_z - a_x, a_x + a_y - a_z, -(a_x + a_y + a_z), a_x + a_z - a_y)$.

Corollary 2. Canonical decomposition. Given the decomposition of SU(4) from Remark 2, we can write

$$U = \exp(\Omega_1) \exp\left(-i \begin{bmatrix} \lambda_1 & \dots & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_4 \end{bmatrix}\right) \exp(\Omega_2),$$

where $\Omega_1, \Omega_2 \in so(4)$. We write above as

$$U = \exp(\Omega_1) \exp\left(-i\left(-\frac{a_x}{2} S_z + \frac{a_y}{2} I_z + a_z I_z S_z\right)\right) \exp(\Omega_2),$$

Multiplying both sides with $W^{'}(.)W$ gives

$$W^{'} U W = K_1 \exp(-i a_x I_x S_x + a_y I_y S_y + a_z I_z S_z) K_2,$$

where $K_1, K_2 \in SU(2) \times SU(2)$ local unitaries and we can rotate to $a_x \geq a_y \geq |a_z|$.

Corollary 3. Digonalization. Given $-iH_c = -i\sum_{\alpha\beta} J_{\alpha\beta} I_\alpha S_\beta$, there exists a local unitary K such that

$$K(-iH_c)K^{'} = -i(a_x I_x S_x + a_y I_y S_y + a_z I_z S_z), a_x \geq a_y \geq |a_z|.$$

Note $W(-iH_c)W^{'} \in \mathfrak{p}_1$. Then choose $\Theta \in SO(n)$ such that $\Theta W(-iH_c)W^{'}\Theta^{'} = -i(-\frac{a_x}{2} S_z + \frac{a_y}{2} I_z + a_z I_z S_z)$ and hence

$$(W^{'} \exp(\Omega) W)(-iH_c)(W \exp(\Omega) W^{'})^{'} = -i(a_x I_x S_x + a_y I_y S_y + a_z I_z S_z).$$

where $K = W^{'} \exp(\Omega) W$ is a local unitary. We can rotate to ensure $a_x \geq a_y \geq |a_z|$.

Corollary 4. Given the evolution of coupled qubits $\dot{U} = -i\left(H_c + \sum_j u_j H_j\right)U$, we can diagonalize $H_c = \sum_{\alpha\beta} J_{\alpha\beta} I_\alpha S_\beta$ by local unitary $X_d = K^{'} H_c K = a_x I_x S_x + a_y I_y S_y + a_z I_z S_z$, $a_x \geq a_y \geq |a_z|$, which we write as triple (a_x, a_y, a_z). From this, there are 24 triples obtained by permuting and changing sign of any two by local unitary. Then $U(T) \in S$ where

$$S = K_1 \exp\left(T \sum_i \alpha_i (a_i, b_i, c_i)\right) K_2, \quad \alpha_i > 0 \quad \sum_i \alpha_i = 1.$$

Furthermore S belongs to the closure of the reachable set. Alternate description of S is

$$U = K_1 \exp(-i(\alpha I_x S_x + \beta I_y S_y + \gamma I_z S_z)) K_2, \quad \alpha \geq \beta \geq |\gamma|,$$

$\alpha \leq a_x T$ and $\alpha + \beta \pm \gamma \leq (a_x + a_y \pm a_z)T$.

Proof. Let $V(t) = K^{'}(t) U(t)$, where $\dot{K} = \left(-i\sum_j u_j X_j\right)K$. Then

$$\dot{V}(t) = Ad_{K'(t)}(-iX_d)V(t).$$

Consider the product

$$V = \prod_i \exp\left(Ad_{K_i}(-iX_d)\Delta t\right)$$

where $K_i \in SU(2) \otimes SU(2)$ and $X_d = a_x I_x S_x + a_y I_y S_y + a_z I_z S_z$, where $a_x \geq a_y \geq |a_z|$. Then,

$$WVW' = \prod_i \exp\left(Ad_{WK_iW'}\left(-iWX_dW'\right)\Delta t\right)$$

Observe $WK_iW' \in SO(4)$ and $WX_dW' = \text{diag}(\lambda_1, \lambda_2, ..., \lambda_4)$. Then using results from Theorem 1, we have

$$WVW' = J_1 \exp\left(-i\mu\right)J_2 = J_1 \exp\left(-i\sum_j \alpha_j P_j(\lambda)\right)J_2, \quad J_1, J_2 \in SO(4), \quad \mu \prec \lambda T$$

Multiplying both sides with $W'(\cdot)W$, we get

$$V = K_1 \exp\left(T\sum_i \alpha_i(a_i, b_i, c_i)\right)K_2, \quad \alpha_i > 0 \quad \sum_i \alpha_i = 1.$$

which we can write as

$$V = K_1 \exp\left(-i\left(\alpha I_x S_x + \beta I_y S_y + \gamma I_z S_z\right)\right)K_2, \quad \alpha \geq \beta \geq |\gamma|,$$

where using $\mu \prec \lambda T$, we get,

$$\alpha + \beta - \gamma \leq \left(a_x + a_y - a_z\right)T \tag{42}$$

$$\alpha \leq a_x T \tag{43}$$

$$\alpha + \beta + \gamma \leq \left(a_x + a_y + a_z\right)T. \tag{44}$$

Furthermore $U = KV$. Hence the proof. **q.e.d.**

3. Time optimal control for G/K problem

Remark 5. Stabilizer: Let $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ be Cartan decomposition of real semisimple Lie algebra $\mathfrak{g}$ and $\mathfrak{a} \in \mathfrak{p}$ be its Cartan subalgebra. Let $a \in \mathfrak{a}$. $ad_a^2 : \mathfrak{p} \rightarrow \mathfrak{p}$ is symmetric in basis orthonormal wrt to the killing form. We can diagonalize ad_a^2. Let Y_i be eigenvectors with nonzero (negative) eigenvalues $-\lambda_i^2$. Let $X_i = \frac{[a, Y_i]}{\lambda_i}$, $\lambda_i > 0$.

$$ad_a(Y_i) = \lambda_i X_i, \quad ad_a(X_i) = -\lambda_i Y_i.$$

X_i are independent, as $\sum \alpha_i X_i = 0$ implies $-\sum \alpha_i \lambda_i Y_i = 0$. Since Y_i are independent, X_i are independent. Given $X \perp X_i$, then $[a, X] = 0$, otherwise we can decompose it in eigenvectors of ad_a^2, i.e., $[a, X] = \sum_i \alpha_i a_i + \sum_j \beta_j Y_j$, where a_i are zero eigenvectors of ad_a^2. Since $0 = \langle X[a[a, X]] \rangle = -\|[a, X]\|^2$, which means $[a, X] = 0$. This is a contradiction. Y_i are orthogonal, implies X_i are orthogonal,

$\langle [a, Y_i][a, Y_j] \rangle = \langle [a, [a, Y_i]Y_j \rangle = \lambda_i^2 \langle Y_i Y_j \rangle = 0$. Let $\mathfrak{k}_0 \in \mathfrak{k}$ satisfy $[a, \mathfrak{k}_0] = 0$. Then $\mathfrak{k}_0 = \{X_i\}^{\perp}$.

$\tilde{Y}_i$ denote eigenvectors that have λ_i as non-zero integral multiples of π. $\tilde{X}_i$ are ad_a related to $\tilde{Y}_i$. We now reserve Y_i for non-zero eigenvectors that are not integral multiples of π.

Let

$$\mathfrak{f} = \{a_i\} \oplus \tilde{Y}_i, \qquad \mathfrak{h} = \mathfrak{k}_0 \oplus \tilde{X}_i,$$

$\tilde{X}_i, X_l, k_j$ where k_j forms a basis of $\mathfrak{k}_0$, forms a basis of $\mathfrak{k}$. Let $A = \exp(a)$.
$AkA^{-1} = A\left(\sum_i \alpha_i X_i + \sum_l \alpha_l \tilde{X}_l + \sum_j \alpha_j k_j\right) A^{-}$, where $k \in \mathfrak{k}$

$$AkA^{-1} = \textstyle\sum_i \alpha_i [\cos(\lambda_i) X_i - \sin(\lambda_i) Y_i] + \sum_l \pm \alpha_l \tilde{X}_l + \sum_j \alpha_j k_j$$

The range of $A(\cdot)A^{-1}$ in $\mathfrak{p}$, is perpendicular to $\mathfrak{f}$. Given $Y \in \mathfrak{p}$ such that $Y \in \mathfrak{f}^{\perp}$. The norm $\|X\|$ of $X \in \mathfrak{k}$, such that $\mathfrak{p}$ part of $AXA^{-1}\big|_{\mathfrak{p}} = Y$ satisfies

$$\|X\| \le \frac{\|Y\|}{\sin \lambda_s}. \tag{45}$$

where λ_s^2 is the smallest nonzero eigenvalue of $-ad_a^2$ such that λ_s is not an integral multiple of π.

$A^2 k A^{-2}$ stabilizes $\mathfrak{h} \in \mathfrak{k}$ and $\mathfrak{f} \in \mathfrak{p}$. If $k \in \mathfrak{k}$, is stabilized by $A^2(\cdot)A^{-2}$, $\lambda_i = n\pi$, i.e., $k \in \mathfrak{h}$. This means $\mathfrak{h}$ is an subalgebra, as the Lie bracket of $[y, z] \in \mathfrak{k}$ for $y, z \in \mathfrak{h}$ is stabilized by $A^2(\cdot)A^{-2}$.

Let $H = \exp(\mathfrak{h})$, be an integral manifold of $\mathfrak{h}$. Let $\tilde{H} \in K$ be the solution to $A^2 \tilde{H} A^{-2} = \tilde{H}$ or $A^2 \tilde{H} - \tilde{H} A^{-2} = 0$. $\tilde{H}$ is closed, $H \in \tilde{H}$. We show that $\tilde{H}$ is a manifold. Given element $H_0 \in \tilde{H} \in K$, where K is closed, we have a $\exp\left(B_\delta^{\mathfrak{k}}\right)$ nghd of H_0, in $\exp(B_\delta)$ ball nghd of H_0, which is one to one. For $x \in B_\delta^{\mathfrak{k}}$,
$A^2 \exp(x) A^{-2} = \exp(x)$, implies,

$$A^2 \exp\left(\sum_i \alpha_i X_i + \sum_l \beta_l \tilde{X}_l + \sum_j \gamma_j k_j\right) H_0 A^{-2} = \exp\left(\sum_i \alpha_i \cos(2\lambda_i) X_i - \sin(2\lambda_i) Y_i \right.$$

$$\left. + \sum_l \beta_l \tilde{X}_l + \sum_j \gamma_j k_j\right) H_0 = \exp\left(\sum_i \alpha_i X_i + \sum_l \beta_l \tilde{X}_l + \sum_j \gamma_j k_j\right) H_0,$$

then by one to one property of $\exp(B_\delta)$, we get $\alpha_i = 0$ and $x \in \mathfrak{h}$. Therefore $\exp\left(B_\delta^{\mathfrak{h}}\right) H_0$ is a nghd of H_0.

Given a sequence $H_i \in \exp(\mathfrak{h})$ converging to H_0, for n large enough $H_n \in \exp\left(B_\delta^{\mathfrak{h}}\right) H_0$. Then H_0 is in invariant manifold $\exp(\mathfrak{h})$. Hence $\exp(\mathfrak{h})$ is closed and hence compact.

Let $y \in \mathfrak{f}$, then there exists a $h_0 \in \mathfrak{h}$ such that $\exp(h_0) y \exp(-h_0) \in \mathfrak{a}$. We maximize the function $\langle a_r, \exp(h) y \exp(h) \rangle$, over the compact group $\exp(\mathfrak{h})$, for regular element $a_r \in \mathfrak{a}$ and $\langle ., . \rangle$ is the killing form. At the maxima, we have at $t = 0$, $\frac{d}{dt} \langle a_r, \exp(h_1 t)(\exp(h_0) y \exp(-h_0)) \exp(-h_1 t) \rangle = 0$.

$$\langle a_r, [h_1 \exp(h_0) y \exp(-h_0)] \rangle = -\langle h_1, [a_r \exp(h_0) y \exp(-h_0)] \rangle,$$

if $\exp(h_0)y\exp(-h_0) \neq \mathfrak{a}$, then $[a_r, \quad \exp(h_0)y\exp(-h_0)] \in \mathfrak{k}$. The bracket $[a_r, \quad \exp(h_0)y\exp(-h_0)]$ is Ad_{A^2} invariant and, hence, belongs to $\mathfrak{h}$. We can choose h_1 so that gradient is not zero. Hence $\exp(h_0)y\exp(-h_0) \in \mathfrak{a}$. For $z \in \mathfrak{p}$ such that $z \in \mathfrak{f}^{\perp}$, we have $\exp(h_0)z\exp(-h_0) \in \mathfrak{a}^{\perp}$.

$$\langle \mathfrak{a}, \exp(h_0)z\exp(-h_0)\rangle = \langle \exp(-h_0)\mathfrak{a}\exp(h_0), z\rangle = 0,$$

as $\exp(-h_0)\mathfrak{a}\exp(h_0)$ is Ad_{A^2} invariant, hence $\exp(-h_0)\mathfrak{a}\exp(h_0) \in \mathfrak{f}$. In above, we worked with killing form. For $\mathfrak{g} = su(n)$, we may use standard inner product.

Remark 6. Kostant's convexity: [28] Given the decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$, let $\mathfrak{a} \subset \mathfrak{p}$ and $X \in \mathfrak{a}$,. Let $\mathcal{W}_i \in \exp(\mathfrak{k})$ such that $\mathcal{W}_i X \mathcal{W}_i \in \mathfrak{a}$ are distinct, Weyl points. Then projection (w.r.t killing form) of $Ad_K(X)$ on $\mathfrak{a}$ lies in convex hull of these Weyl points. The $\mathcal{C}$ be the convex hull and let projection $P(Ad_K(X))$ lie outside this Hull. Then there is a separating hyperplane a, such that $\langle Ad_K(X), a\rangle < \langle \mathcal{C}, a\rangle$. W.L.O.G we can take a to be a regular element. We minimize $\langle Ad_K(X), a\rangle$, with choice of K and find that minimum happens when $[Ad_K(X), a] = 0$, i.e. $Ad_K(X)$ is a Weyl point. Hence $P(Ad_K(X)) \in \sum_i \alpha_i \mathcal{W}_i X \mathcal{W}_i^{-1}$, for $\alpha_i > 0$ and $\sum_i \alpha_i = 1$. The result is true with a projection w.r.t inner product that satisfies $\langle x, [y, z]\rangle = \langle [x, y], z]\rangle$, like standard inner product on $\mathfrak{g} = su(n)$.

Theorem 2 Given a compact Lie group G and Lie algebra $\mathfrak{g}$. Consider the Cartan decomposition of a real semisimple Lie algebra $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$. Given the control system

$$\dot{X} = Ad_{K(t)}(X_d)X, \quad P(0) = \mathbf{1}$$

where $X_d \in \mathfrak{a}$, the Cartan subalgebra $\mathfrak{a} \in \mathfrak{p}$ and $K(t) \in \exp\mathfrak{k}$, a closed subgroup of G. The end point

$$P(T) = K_1 \exp\left(T \sum_i \alpha_i \mathcal{W}_i(X_d)\right) K_2,$$

where $K_1, K_2 \in \exp(\mathfrak{k})$ and $\mathcal{W}_i(X_d) \in \mathfrak{a}$ are Weyl points, $\alpha_i > 0$ and $\sum_i \alpha_i = 1$.

Proof. As in proof of Theorem 1, we define

$$P(t + \Delta) = \exp(Ad_K(X_d)\Delta)P(t) = \exp(Ad_K(X_d)\Delta)K_1\exp(a)K_2$$

and show that

$$\exp(Ad_K(X_d)\Delta)K_1AK_2 = K_a\exp\left(a_0\Delta + C\Delta^2\right)AK_b = K_a\exp\left(a + a_0\Delta + C\Delta^2\right)K_b, \quad (46)$$

where for $\overline{K} = K_1^{-1}K$,

$$Ad_{\overline{K}}(X_d) = \underbrace{P\left(Ad_{\overline{K}}(X_d)\right)}_{a_0} + Ad_{\overline{K}}(X_d)^{\perp}.$$

where P is projection w.r.t killing form and $a_0 \in \mathfrak{f}$, the centralizer in $\mathfrak{p}$ as defined in Remark 5, $C\Delta^2 \in \mathfrak{f}$ is a second order term that can be made small by choosing Δ. $K_a, K_b \in \exp(\mathfrak{k})$.

To show Eq. (46), we show there exists $K_{1'}^{'}, K_{2''} \in K$ such that

$$\underbrace{\exp\left(k_1''\right)}_{K_1''}\ \exp\left(Ad_{\overline{K}}(X_d)\Delta\right)\underbrace{\exp\left(Ak_2''A^{-1}\right)}_{K_2''} = \exp\left(a_0\Delta + C\Delta^2\right), \tag{47}$$

where K_1'' and K_2'' are constructed by a iterative procedure as described in the proof below.

Given X and Y as $N \times N$ matrices, considered elements of a matrix Lie algebra $\mathfrak{g}$, we have,

$$\log\left(e^X e^Y\right) - (X+Y) = \sum_{n>0} \frac{(-1)^{n-1}}{n} \sum_{1\le i\le n} \frac{\left[X^{r_1}Y^{s_1}\dots X^{r_n}Y^{s_n}\right]}{\sum_{i=1}^{n}(r_i+s_i)r_1!s_1!\dots r_n!s_n!}, \tag{48}$$

where $r_i + s_i > 0$.

We bound the largest element (absolute value) of $\log\left(e^X e^Y\right) - (X+Y)$, denoted as $\left|\log\left(e^X e^Y\right) - (X+Y)\right|_0$, given $|X|_0 < \Delta$ and $|Y|_0 < b_0\Delta^k$, where $k \ge 1$, $\Delta < 1$, $b_0\Delta < 1$.

$$\left|\log\left(e^X e^Y\right) - (X+Y)\right|_0 \le \sum_{n=1} Nb_0 e\Delta^{k+1} + \sum_{n>1} \frac{1}{n} \frac{\left(2Ne^2\right)^n b_0 \Delta^{n+k-1}}{n} \tag{49}$$

$$\le Nb_0 e\Delta^{k+1} + \left(Ne^2\right)^2 b_0 \Delta^{k+1}\left(1 + 2Ne^2\Delta + \dots\right) \tag{50}$$

$$\le Nb_0 e\Delta^{k+1} + \frac{\left(Ne^2\right)^2 b_0 \Delta^{k+1}}{1 - 2Ne^2\Delta} \le \tilde{M} b_0 \Delta^{k+1} \tag{51}$$

where $2N\Delta < 1$ and $\tilde{M}\Delta < 1$.

Given decomposition of $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$, $\mathfrak{p} \perp \mathfrak{k}$ with respect to the negative definite killing form $B(X, Y) = tr(ad_X ad_Y)$. Furthermore there is decomposition of $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{a}^\perp$.

Given

$$U_0 = \exp\left(a_0\Delta + b_0\Delta + c_0\Delta\right),$$

where $a_0 \in \mathfrak{a}$, $b_0 \in \mathfrak{a}^\perp$ and $c_0 \in \mathfrak{k}$, such that $|a_0|_0 + |b_0|_0 + |c_0|_0 < 1$, which we just abbreviate as $a_0 + b_0 + c_0 < 1$ (we follow this convention below).

We describe an iterative procedure

$$U_n = \Pi_{k=1}^{n} \exp\left(-c_k\Delta\right)\ U_0\ \Pi_{k=0}^{n} \exp\left(-b_k\Delta\right), \tag{52}$$

where $c_k \in \mathfrak{k}$ and $b_k \in \mathfrak{a}^\perp$, such that the limit

$$n \to \infty \quad U_n = \exp\left(a_0\Delta + C\Delta^2\right), \tag{53}$$

where $a_0, C \in \mathfrak{a}$.

$$\begin{aligned} U_1 &= \exp\left(-c_0\Delta\right)\exp\left(a_0\Delta + b_0\Delta + c_0\Delta\right)\exp\left(-b_0\Delta\right) \\ &= \exp\left(a_0\Delta + b_0\Delta + c_{0'}\Delta^2\right)\exp\left(-b_0\Delta\right) \\ &= \exp\left(a_0\Delta + b_{0'}\Delta^2 + c_{0'}\Delta^2\right) \\ &= \exp\left((a_1 + b_1 + c_1)\Delta\right) \end{aligned}$$

Note b_0' and c_0' are elements of $\mathfrak{g}$ and need not be contained in $\mathfrak{a}^\perp$ and $\mathfrak{k}$.

Where, using bound in $c_0' \leq \tilde{M} c_0$, which gives $a_0 + b_0 + c_0'\Delta \leq a_0 + b_0 + c_0$. Using the bound again, we obtain, $b_0' \leq \tilde{M} b_0$. We can decompose, $(b_0' + c_0')\Delta$, into subspaces $a_0'' + b_1 + c_1$, where $a_0'' \leq M(b_0' + c_0')\Delta$, $b_1 \leq M(b_0' + c_0')\Delta$ and $c_1 \leq M(b_0' + c_0')\Delta$, where $-B(X,X) \leq \lambda_{max}|X|^2$, where $|X|$ is Frobenius norm and $-B(X,X) \geq \lambda_{min}|X|^2$. Let $M = \frac{N\lambda_{max}}{\lambda_{min}}$.

This gives, $a_0'' \leq M(b_0' + c_0')\Delta$, $b_1 \leq M(b_0' + c_0')\Delta$ and $c_1 \leq M(b_0' + c_0')\Delta$. This gives

$$a_1 \leq a_0 + \tilde{M}M(b_0 + c_0)\Delta \; b_1 \leq \tilde{M}M(b_0 + c_0)\Delta \; c_1 \leq \tilde{M}M(b_0 + c_0)\Delta$$

For $4\tilde{M}M\Delta<1$, we have, $a_1 + b_1 + c_1 \leq a_0 + b_0 + c_0$. Continuing and using $(b_k + c_k) \leq 2\tilde{M}M\Delta(b_{k-1} + c_{k-1}) \leq (2\tilde{M}M\Delta)^k (b_0 + c_0)$.

Similarly,

$$|a_k - a_{k-1}|_0 \leq (2\tilde{M}M\Delta)^k (b_0 + c_0)$$

Note, (a_k, b_k, c_k) is a Cauchy sequences which converges to $(a_\infty, 0, 0)$, where

$$|a_\infty - a_0|_0 \leq (b_0 + c_0) \sum_{k=1}^{\infty} (2\tilde{M}M\Delta)^k \leq \frac{2M\tilde{M}\Delta(b_0 + c_0)}{1 - 2\tilde{M}M\Delta} \leq C\Delta,$$

where $C = 4\tilde{M}M(b_0 + c_0)$.

The above exercise was illustrative. Now we use an iterative procedure as above to show Eq. (47).

Writing

$$Ad_{\overline{K}}(X_d) = \underbrace{P\left(Ad_{\overline{K}}(X_d)\right)}_{a_0} + \underbrace{Ad_{\overline{K}}(X_d)^{\perp}}_{b_0},$$

where $a_0 \in \mathfrak{f}$ and $b_0 \in \mathfrak{f}^{\perp}$, consider again the iterations

$$\begin{aligned} U_0 &= \exp(-\bar{c}_0\Delta)\exp(a_0\Delta + b_0\Delta)\exp(-b_0\Delta + \bar{c}_0\Delta) \\ &= \exp(-\bar{c}_0\Delta)\exp\left(a_0\Delta + \bar{c}_0\Delta + b_{0'}\Delta^2\right) \\ &= \exp\left(a_0\Delta + b_{0'}\Delta^2 + c_{0'}\Delta^2\right) \\ &= \exp(a_1\Delta + b_1\Delta + c_1\Delta) \end{aligned}$$

We refer to Remark 5, Eq. (45). Given $b_0\Delta \in \mathfrak{p}$ such that $b_0\Delta \in \mathfrak{f}^{\perp}$. If $Ak'A' = -b_0\Delta + \bar{c}_0\Delta$, then $\|k'\| \leq h\|b_0\Delta\|$ (killing norm).

$\bar{c}_0 \in \mathfrak{k}$, is bounded $\bar{c}_0 \leq Mhb_0$, where M as before converts between two different norms. Using bounds derived above $b_0' \leq \tilde{M}(Mh+1)b_0$, and $c_0' \leq \tilde{M}Mhb_0$, $2\tilde{M}(Mh+1)\Delta<1$, we obtain.

which gives $a_0 + b_0'\Delta + \bar{c}_0 \leq a_0 + b_0\left(\tilde{M}(Mh+1)\Delta + Mh\right) \leq 1$. For appropriate M', we have

$$\begin{aligned} a_1 &\leq a_0 + \frac{M'}{3}(b_0 + c_0)\Delta \\ b_1 &\leq \frac{M'}{3}(b_0 + c_0)\Delta \\ c_1 &\leq \frac{M'}{3}(b_0 + c_0)\Delta \end{aligned}$$

we obtain

$$a_1 + b_1 + c_1 \leq a_0 + M'(b_0 + c_0)\Delta \leq a_0 + b_0 + c_0$$

where Δ is chosen small.

$$U_1 = \exp\left(-(c_1 + \bar{c}_1)\Delta\right) \exp\left(a_1\Delta + b_1\Delta + c_1\Delta\right) \exp\left(-b_1\Delta + \bar{c}_1\Delta\right)$$

$$= \exp\left(-(c_1 + \bar{c}_1)\Delta\right) \exp\left(a_1\Delta + (c_1 + \bar{c}_1)\Delta + b_1'\Delta^2\right)$$

$$= \exp\left(a_1\Delta + b_1'\Delta^2 + c_1'\Delta^2\right)$$

$$= \exp\left(a_2\Delta + b_2\Delta + c_2\Delta\right)$$

where $\bar{c}_1 \in \mathfrak{k}$, such that $\bar{c}_1 \leq Mhb_1$.

where, using bounds derived above $b_1' \leq \tilde{M}(Mh + 1)b_1$, and $c_1' \leq \tilde{M}(Mhb_1 + c_1)$, where using the bound $2\tilde{M}(Mh + 1)\Delta < 1$, we obtain

which gives $a_1 + b_1'\Delta + (c_1 + \bar{c}_1) \leq a_1 + ((1 + Mh)b_1 + c_1) \leq a_0 + b_0 + c_0$.

We can decompose, $(b_1' + c_1')\Delta^2$, into subspaces $\left(a_{1'}^{'} + b_2 + c_2\right)\Delta$, where $a_1'' \leq M(b_1' + c_1')\Delta$, $b_2 \leq M(b_1' + c_1')\Delta$ and $c_2 \leq M(b_1' + c_1')\Delta$, where M as before converts between two different norms.

This gives

$$a_2 \leq a_1 + 4\tilde{M}M^2h(b_1 + c_1)\Delta \; b_2 \leq 4\tilde{M}M^2h(b_1 + c_1)\Delta \; c_2 \leq 4\tilde{M}M^2h(b_1 + c_1)\Delta$$

For $x = 8\tilde{M}M^2h\Delta < \frac{2}{3}$, we have, $a_2 + b_2 + c_2 \leq a_1 + (b_1 + c_1) \leq a_0 + b_0 + c_0$,

Using $(b_k + c_k) \leq x(b_{k-1} + c_{k-1}) \leq x^k(b_0 + c_0)$.

Similarly,

$$|a_k - a_{k-1}|_0 \leq x^k(b_0 + c_0)$$

Note, (a_k, b_k, c_k) is a Cauchy sequences which converges to $(a_\infty, 0, 0)$, where

$$|a_\infty - a_0|_0 \leq x(b_0 + c_0)\sum_{k=0}^{\infty} x^k \leq \frac{x(b_0 + c_0)}{1 - x} \leq C\Delta,$$

where $C = 16\tilde{M}M^2h(b_0 + c_0)$.

The above iterative procedure generates k_1' and k_2'' in Eq. (47), such that

$$\exp\left((K_1'Ad_K(X_d)K_1)\Delta\right) = \exp\left(-k_1''\right) \exp\left(a_0\Delta + C\Delta^2\right) \exp\left(-Ak_2''A'\right).$$

where $a_0\Delta + C\Delta^2 \in \mathfrak{f}$. By using a stabilizer H_1, H_2, we can rotate them to $\mathfrak{a}$ such that

$$\exp\left(Ad_K(X_d)\Delta\right)K_1AK_2 = K_aH_1 \exp\left(a_0'\Delta + C'\Delta^2\right)AH_2K_b$$

such that $H_1^{-1}\left(a_0\Delta + C\Delta^2\right)H_1 = a_0'\Delta + C'\Delta^2$ is in $\mathfrak{a}$ and $a_0' = P\left(H_1^{-1}a_0H_1\right)$ is projection onto $\mathfrak{a}$ such that

$$P\left(H_1^{-1}a_0H_1\right) = \sum_k \alpha_k \mathcal{W}_k(X_d).$$

This follows because the orthogonal part of $Ad_{\overline{K}}(X_d)$ to $\mathfrak{f}$ written as $Ad_{\overline{K}}(X_d)^{\perp}$ remains orthogonal of $\mathfrak{f}$

$$\left\langle H^{-1}Ad_K(X_d)^{\perp}H, \mathfrak{a}\right\rangle = \left\langle Ad_K(X_d)^{\perp}, H\mathfrak{a}H^{-1}\right\rangle = \left\langle Ad_K(X_d)^{\perp}, \mathfrak{a}''\right\rangle = 0$$

($a'' \in \mathfrak{f}$), remains orthogonal to $\mathfrak{a}$. Therefore
$P\left(H_1^{-1}a_0H_1\right) = P\left(H_1^{-1}Ad_{\overline{K}}(X_d)H_1\right) = \sum_k \alpha_k \mathcal{W}_k(X_d)$.

$$\exp\left(Ad_K(X_d)\Delta\right)K_1AK_2 = K_a \exp\left(a + a_{0'}\Delta + C'\Delta^2\right)K_b.$$

Lemma 1 Given $P = K_1 \underbrace{\exp\left(a + a_1\Delta\right)}_{A_1} K_2 = K_3 \underbrace{\exp\left(b - b_1\Delta\right)}_{A_2} K_4$, where $a, b, a_1, b_1 \in \mathfrak{a}$. We can express

$$\exp(b) = K_a \exp\left(a + a_1\Delta + \mathcal{W}(b_1)\Delta\right)K_b,$$

where $\mathcal{W}(b_1)$ is Weyl element of b_1. Furthermore

$$\exp\left(b + b_2\Delta\right) = K_{a''} \exp\left(a + a_1\Delta + \mathcal{W}(b_1)\Delta + \mathcal{W}(b_2)\Delta\right)K_{b''}.$$

Proof. Note, $A_2 = K_3^{-1}PK_4^{-1}$, commutes with b_1. This implies
$A_2 = \tilde{K}\exp\left(a + a_1\Delta\right)K$ commutes with b_1. This implies $A_2b_1A_2^{-1} = b_1$, i.e., $\tilde{K}\exp\left(a + a_1\Delta\right)Ad_K(b_1)\exp\left(-(a + a_1\Delta)\right)\tilde{K}' = b_1$, which implies that $Ad_K(b_1) \in \mathfrak{f}$. Recall, from Remark 5,

$$\exp\left(a + a_1\Delta\right)Ad_K(b_1)\exp\left(-(a + a_1\Delta)\right) = \sum_k c_k\left(Y_k\cos(\lambda_k) + X_k\sin(\lambda_k)\right).$$

This implies $\sum_k c_k \sin(\lambda_k)X_k = 0$, implying $\lambda_k = n\pi$. Therefore,

$$\exp\left(2(a + a_1\Delta)\right)Ad_K(b_1)\exp\left(-2(a + a_1\Delta)\right) = Ad_K(b_1).$$

We have shown existence of H_1 such that $H_1Ad_K(b_1)H_1^{-1} \in \mathfrak{a}$, using H_1, H_2 as before,

$$\begin{aligned}\tilde{K}\exp\left(a + a_1\Delta\right)K\exp\left(b_1\Delta\right) &= \tilde{K}H_2\exp\left(a + a_1\Delta\right)H_1\exp\left(Ad_K(b_1)\Delta\right)K \\ &= K_a\exp\left(a + a_1\Delta + \mathcal{W}(b_1)\Delta\right)K_b.\end{aligned}$$

Applying the theorem again to

$$K_a\exp\left(a + a_1\Delta + \mathcal{W}(b_1)\Delta\right)K_b\exp\left(b_2\Delta\right) = K_{a''}\exp\left(a + a_1\Delta + \mathcal{W}(b_1)\Delta + \mathcal{W}(b_2)\Delta\right)K_{b''}.$$

Lemma 2 Given $P_i = K_1^iA^iK_2^i = K_1^i\exp\left(a^i\right)K_2^i$, we have $P_{i,i+1} = \exp\left(H_i^+\Delta_i^+\right)P_i$, and $P_{i,i+1} = \exp\left(-H_{i+1}^-\Delta_{i+1}^-\right)P_{i+1}$, where $H_i^+ = Ad_{K_i}(X_d)$. From above we can express

$$P_{i,i+1} = K_a^{i+}\exp\left(a^i + a_1^{i+}\Delta_+^i + a_2^{i+}\left(\Delta_+^i\right)^2\right)K_b^{i+}.$$

where a_1^{i+} and a_2^{i+} are first and second order increments to a_i in the positive direction. The remaining notation is self-explanatory.

$$P_{i,i+1} = K_a^{(i+1)-} \exp\left(a^{i+1} - a_1^{(i+1)-}\Delta_-^{i+1} - a_2^{(i+1)-}\left(\Delta_-^{i+1}\right)^2\right)K_b^{(i+1)-}.$$

$$\exp\left(a^{i+1}\right) = K_1 \exp\left(a^i + a_1^{i+}\Delta_+^i + a_2^{i+}\left(\Delta_+^i\right)^2 + \mathcal{W}\left(a_1^{(i+1)-}\Delta_-^{i+1} + a_2^{(i+1)-}\left(\Delta_-^{i+1}\right)^2\right)\right)K_2.$$

$$\begin{aligned}\mathcal{W}\left(a_1^{(i+1)-}\Delta_-^{i+1} + a_2^{(i+1)-}\left(\Delta_-^{i+1}\right)^2\right) &= \mathcal{P}\left(\mathcal{W}\left(a_1^{(i+1)-}\right)\right)\Delta_-^{i+1} + \mathcal{P}\left(\mathcal{W}\left(a_2^{(i+1)-}\right)\right)\left(\Delta_-^{i+1}\right)^2 \\ &= \sum_k \alpha_k \mathcal{W}_k(X_d)\Delta_-^{i+1} + o\left(\left(\Delta_-^{i+1}\right)^2\right)\end{aligned}$$

where, $a^i, a_1^i, a_2^i \in \mathfrak{a}$.

Using Lemma 1 and 2, we can express

$$P_n(T) = K_1 \exp\left(a_n\right) \exp K_2 = K_1 \exp\left(\sum_i \mathcal{W}\left(a_i^+\right)\Delta_i^+ + \mathcal{W}\left(a_{i+1}^-\right)\Delta_{i+1}^-\right) \exp\left(\underbrace{\sum o\left(\Delta^2\right)}_{\leq \varepsilon T}\right)K_2$$

Letting ε go to 0, we have

$$P_n(T) = K_1 \exp\left(T\sum_i \alpha_i \mathcal{W}_i(X_d)\right)K_2.$$

Hence the proof of theorem. **q.e.d.**

4. Conclusion

In this chapter, we studied some control problems that derive from time optimal control of coupled spin dynamics in NMR spectroscopy and quantum information and computation. We saw how dynamics was decomposed into fast generators $\mathfrak{k}$ (local Hamiltonians) and slow generators $\mathfrak{p}$ (couplings) as a Cartan decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$. Using this decomposition, we used some convexity ideas to completely characterize the reachable set and time optimal control for these problems.

Author details

Navin Khaneja
Systems and Control Engineering, IIT Bombay, India

*Address all correspondence to: navinkhaneja@gmail.com

References

[1] Nielsen M, Chuang I. Quantum Information and Computation. New York: Cambridge University Press; 2000

[2] Ernst RR, Bodenhausen G, Wokaun A. Principles of Nuclear Magnetic Resonance in One and Two Dimensions. Oxford: Clarendon Press; 1987

[3] Cavanagh J, Fairbrother WJ, Palmer AG, Skelton NJ. Protein NMR spectroscopy. In: Principles and Practice. New York: Academic Press; 1996

[4] Khaneja N, Brockett RW, Glaser SJ. Time optimal control of spin systems. Physical Review A. 2001;**63**:032308

[5] Khaneja N, Glaser SJ. Cartan decomposition of $SU(2^n)$ and control of spin systems. Chemical Physics. 2001; **267**:11-23

[6] D'Alessandro D. Constructive controllability of one and two spin 1/2 particles. Proceedings 2001 American Control Conference; Arlington, Virginia; June 2001

[7] Kraus B, Cirac JI. Optimal creation of entanglement using a two qubit gate. Physical Review A. 2001;**63**:062309

[8] Bennett CH, Cirac JI, Leifer MS, Leung DW, Linden N, Popescu S, et al. Optimal simulation of two-qubit hamiltonians using general local operations. Physical Review A. 2002;**66**: 012305

[9] Khaneja N, Glaser SJ, Brockett RW. Sub-Riemannian geometry and optimal control of three spin systems. Physical Review A. 2002;**65**:032301

[10] Vidal G, Hammerer K, Cirac JI. Interaction cost of nonlocal gates. Physical Review Letters. 2002;**88**:237902

[11] Hammerer K, Vidal G, Cirac JI. Characterization of nonlocal gates. Physical Review A. 2002;**66**:062321

[12] Yuan H, Khaneja N. Time optimal control of coupled qubits under non-stationary interactions. Physical Review A. 2005;**72**:040301(R)

[13] Yuan H, Khaneja N. Reachable set of bilinear control systems under time varying drift. System and Control Letters. 2006;**55**:501

[14] Zeier R, Yuan H, Khaneja N. Time optimal synthesis of unitary transformations in fast and slow qubit system. Physical Review A. 2008;**77**: 032332

[15] Yuan H, Zeier R, Khaneja N, Lloyd S. Constructing two qubit gates with minimal couplings. Physical Review A. 2009;**79**:042309

[16] Reiss T, Khaneja N, Glaser S. Broadband geodesic pulses for three spin systems: Time-optimal realization of effective trilinear coupling terms and indirect SWAP gates. Journal of Magnetic Resonance. 2003;**165**:95

[17] Khaneja N, Glaser S. Efficient transfer of coherence through Ising spin chains. Physical Review A. 2002;**66**: 060301

[18] Khaneja N, Heitmann B, Spörl A, Yuan H, Schulte-Herbrüggen T, Glaser SJ. Shortest paths for efficient control of indirectly coupled qubits. Physical Review A. 2007;**75**:012322

[19] Yuan H, Zeier R, Khaneja N. Elliptic functions and efficient control of Ising spin chains with unequal couplings. Physical Review A;**77**:032340

[20] Yuan H, Khaneja N. Efficient synthesis of quantum gates on a three-

spin system with triangle topology. Physical Review A;**84**:062301

[21] Yuan H, Wei D, Zhang Y, Glaser S, Khaneja N. Efficient synthesis of quantum gates on indirectly coupled spins. Physical Review A;**89**:042315

[22] Redfield AG. The theory of relaxation processes. Advances in Magnetic and Optical Resonance. 1965; **1**:1-32

[23] Lindblad G. On the generators of quantum dynamical semigroups. Communications in Mathematical Physics. 1976;**48**:199

[24] Helgason S. Differential Geometry, Lie Groups, and Symmetric Spaces. Cambridge: Academic Press; 1978

[25] Brockett RW. System theory on group manifolds and Coset spaces. SIAM Journal of Control. 1972;**10**: 265-284

[26] Jurdjevic V, Sussmann H. Control systems on lie groups. Journal of Differential Equations. 1972;**12**:313-329

[27] Jurdjevic V. Geometric Control Theorey. New York: Cambridge University Press; 1997

[28] Kostant B. On convexity, the Weyl group and the Iwasawa decomposition. Annales scientifiques de l’cole Normale SupÃ©rieure. 1973;**6**(4):413-455