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# Adaptive Robust Extended Kalman Filter

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## 1. Introduction

The extended Kalman filter (EKF) is one of the most widely used methods for state estimation with communication and aerospace applications based on its apparent simplicity and tractability (Shi et al., 2002; Bolognani et al., 2003; Wu et al., 2004). However, for an EKF to guarantee satisfactory performance, the system model should be known exactly. Unknown external disturbances may result in the inaccuracy of the state estimate, even cause divergence.

This difficulty has been recognized in the literature (Reif & Unbehauen, 1999; Reif et al., 2000), and several schemes have been developed to overcome it. A traditional approach to improve the performance of the filter is the 'covariance setting' technique, where a positive definite estimation error covariance matrix is chosen by the filter designer (Einicke et al., 2003; Bolognani et al., 2003). As it is difficult to manually tune the covariance matrix for dynamic system, adaptive extended Kalman filter (AEKF) approaches for online estimation of the covariance matrix have been adopted (Kim & ILTIS, 2004; Yu et al., 2005; Ahn & Won, 2006). However, only in some special cases, the optimal estimation of the covariance matrix can be obtained. And inaccurate approximation of the covariance matrix may blur the state estimate.

Recently, the robust  $H_\infty$  filter has received considerable attention (Theodor et al., 1994; Shen & Deng, 1999; Zhang et al., 2005; Tseng & Chen, 2001). The robust filters take different forms depending on what kind of disturbances are accounted for, while the general performance criterion of the filters is to guarantee a bounded energy gain from the worst possible disturbance to the estimation error. Although the robust extended Kalman filter (REKF) has been deeply investigated (Einicke & White, 1999; Reif et al., 1999; Seo et al., 2006), how to prescribe the level of disturbances attenuation is still an open problem. In general, the selection of the attenuation level can be seen as a tradeoff between the optimality and the robustness. In other words, the robustness of the REKF is obtained at the expense of optimality.

This chapter reviews the adaptive robust extended Kalman filter (AREKF), an effective algorithm which will remain stable in the presence of unknown disturbances, and yield accurate estimates in the absence of disturbances (Xiong et al., 2008). The key idea of the AREKF is to design the estimator based on the stability analysis, and determine whether the error covariance matrix should be reset according to the magnitude of the innovation.

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Further analysis shows that the filter can guarantee boundedness of the estimation error despite the unknown disturbances as well as the linearization error. Unlike the AEKF, the calculated innovation is not adopted directly to tune the error covariance matrix, but used as an indicator of the necessity of resetting the covariance matrix, so that the covariance matrix will be reset only when large disturbance occurs. The AREKF can be seen as a REKF with time-varying attenuation level. In comparison with the REKF, the advantage of the AREKF is that the robust behavior is achieved without decreasing accuracy. In addition, no complicated computation procedures are required to implement the AREKF.

The AREKF particularly suits for state estimation in nonlinear stochastic system with large external disturbance. This chapter describes the application of the algorithm to autonomous astronomical navigation for orbit maneuvering spacecraft. The problem is to determine the position vector of the spacecraft according to the spacecraft dynamic model and the measurements from the earth sensor and the star sensor. We consider the case in which the spacecraft performs thrust maneuvers but the unknown acceleration produced by the thruster firings is not known. The unknown acceleration, which may seriously impact the positioning accuracy, is treated as disturbances in the design of the AREKF. Numerical example shows that the AREKF outperforms the usual EKF, the REKF and the AEKF.

The structure of this chapter is as follows. In Section 2, the stability of the REKF is analyzed based on some standard results about the boundedness of stochastic processes. It is specified that the stability of the REKF is not guaranteed. In Section 3, the AREKF is derived to ensure the stability of the filter. The autonomous navigation system that consists of the earth sensor and the star sensor is described in Section 4. In Section 5, the high performance of the AREKF is illustrated through simulations in comparison with the usual EKF, the REKF and the AEKF. Some conclusions are drawn in Section 6.

## 2. Problem statement

### 2.1 REKF algorithm

The considered nonlinear discrete-time system is represented by

$$\mathbf{x}_t = f(\mathbf{x}_{t-1}) + \mathbf{w}_t \quad (1)$$

$$\mathbf{y}_t = h(\mathbf{x}_t) + \mathbf{v}_t \quad (2)$$

where  $\mathbf{x}_t \in \mathbf{R}^l$  and  $\mathbf{y}_t \in \mathbf{R}^m$  denote the state and measurement vectors at time instant  $t$ .  $f(\cdot)$  is a function that describes the dynamics of the state vector.  $h(\cdot)$  is a function that describes the relation between the state vector and the measurement vector. The function  $f(\cdot)$  and  $h(\cdot)$  are assumed to be continuously differentiable.  $\mathbf{w}_t$  and  $\mathbf{v}_t$  are uncorrelated zero mean white noise processes with covariance matrices  $\mathbf{Q}_t$  and  $\mathbf{R}_t$  respectively.

The structure of the standard REKF algorithm is presented as follows (Einicke & White, 1999):

**Prediction:** The one-step prediction of  $\hat{\mathbf{x}}_{t|t-1}$  and its corresponding error covariance matrix  $\Sigma_{t|t-1}$  are

$$\hat{\mathbf{x}}_{t|t-1} = f(\hat{\mathbf{x}}_{t-1}) \quad (3)$$

$$\mathbf{P}_{t|t-1} = \mathbf{F}_t \mathbf{P}_{t-1} \mathbf{F}_t^T + \mathbf{Q}_t \quad (4)$$

$$\boldsymbol{\Sigma}_{t|t-1} = (\mathbf{P}_{t|t-1}^{-1} - \gamma^{-2} \mathbf{L}_t^T \mathbf{L}_t)^{-1} \quad (5)$$

where  $\mathbf{F}_t = \left. \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \right|_{\mathbf{x}=\hat{\mathbf{x}}_{t-1}}$  is the Jacobian matrix of  $f(\mathbf{x}_{t-1})$ ,  $\mathbf{L}_t \in \mathbf{R}^{l \times l}$  is designed as identity matrix  $\mathbf{I}$ , and the tuning parameter  $\gamma$  is found by searching over  $\gamma \neq 0$  such that  $\boldsymbol{\Sigma}_{t|t-1} > 0$ .

**Update:** The estimate of state  $\hat{\mathbf{x}}_t$  and the estimation error covariance matrix  $\mathbf{P}_t$  are

$$\hat{\mathbf{x}}_t = \hat{\mathbf{x}}_{t|t-1} + \mathbf{K}_t [\mathbf{y}_t - h(\hat{\mathbf{x}}_{t|t-1})] \quad (6)$$

$$\mathbf{K}_t = \boldsymbol{\Sigma}_{t|t-1} \mathbf{H}_t^T \mathbf{P}_{y,t}^{-1} \quad (7)$$

$$\mathbf{P}_{y,t} = \mathbf{H}_t \boldsymbol{\Sigma}_{t|t-1} \mathbf{H}_t^T + \mathbf{R}_t \quad (8)$$

$$\mathbf{P}_t = (\boldsymbol{\Sigma}_{t|t-1}^{-1} + \mathbf{H}_t^T \mathbf{R}_t^{-1} \mathbf{H}_t)^{-1} \quad (9)$$

where  $\mathbf{H}_t = \left. \frac{\partial h(\mathbf{x})}{\partial \mathbf{x}} \right|_{\mathbf{x}=\hat{\mathbf{x}}_{t|t-1}}$  is the Jacobian matrix of  $h(\mathbf{x}_t)$ .

Apparently, the structure of the REKF is similar to that of the EKF, and if  $\gamma \rightarrow \infty$ , the REKF reverts to the EKF. The design objective of the REKF is to guarantee the norm of the transfer function between the estimation error and the external disturbances (modeling errors and system noises) to be less than a prescribed attenuation level  $\gamma$

$$\frac{\|\mathbf{L}_t \tilde{\mathbf{x}}_t\|^2}{\|\mathbf{w}_t\|^2 + \|\mathbf{A}_t\|^2 + \|\mathbf{v}_t\|^2} \leq \gamma^2 \quad (10)$$

where  $\|\mathbf{x}\|$  is 2-norm of vector  $\mathbf{x}$ ,  $\tilde{\mathbf{x}}_t = \mathbf{x}_t - \hat{\mathbf{x}}_t$  is the estimation error, and  $\mathbf{A}_t$  represents the model error caused by the linearization error or unknown exogenous inputs.

It can be seen from (10) that the tuning parameter  $\gamma$  in (5) describes the ability of the filter to minimize the effects of the disturbances on the estimation errors. Decreasing  $\gamma$  will enhance robustness of the filter. However, as is pointed out in (Einicke & White, 1999), when a minimum possible  $\gamma$  is adopted, the accuracy of the filter will be decreased. Another limitation of the REKF will be specified in the next sub-section.

## 2.2 Stability analysis

The stability analysis of the REKF is based on the following lemma. The inequalities with random variables in this paper hold with probability one.

**Lemma 1:** Assume that  $\xi_t$  is the stochastic process and there is a stochastic process  $V(\xi_t)$  as well as the real numbers  $v_{\min} > 0$ ,  $v_{\max} > 0$ ,  $\mu > 0$  and  $0 < \lambda \leq 1$  such that  $\forall t$

$$v_{\min} \|\xi_t\|^2 \leq V(\xi_t) \leq v_{\max} \|\xi_t\|^2 \quad (10)$$

and

$$E[V(\xi_t) | \xi_{t-1}] - V(\xi_{t-1}) \leq \mu - \lambda V(\xi_{t-1}) \quad (11)$$

are fulfilled. Then the stochastic process  $\xi_t$  is bounded in mean square, i.e.

$$E\{\|\xi_t\|^2\} \leq \frac{v_{\max}}{v_{\min}} E\{\|\xi_0\|^2\} (1-\lambda)^t + \frac{\mu}{v_{\min}} \sum_{i=1}^{t-1} (1-\lambda)^i \quad (12)$$

The proof of Lemma 1 is given in (Reif & Unbehauen, 1999; Tarn & Rasis, 1976).

During the stability analysis, the Lyapunov function  $V(\xi_t)$  that represents the energy of  $\xi_t$  should be chosen by the user. Certainly, a properly chosen  $V(\xi_t)$  may facilitate the analysis. The numbers  $v_{\min}$  and  $v_{\max}$  define the low bound and the upper bound of  $V(\xi_t)$  respectively. The lemma specified that if the difference between the conditional expectations of  $V(\xi_t)$  and  $V(\xi_{t-1})$  is not larger than a positive constant  $\mu$  minus the product of  $V(\xi_{t-1})$  and another constant  $\lambda$ , then  $\xi_t$  will be bounded. Intuitively, if the condition (11) is fulfilled, the energy of  $\xi_t$  will not increase arbitrarily.

The prediction error of the REKF is defined as

$$\tilde{\mathbf{x}}_{t|t-1} = \mathbf{x}_t - \hat{\mathbf{x}}_{t|t-1} \quad (13)$$

Substitute (1) and (3) into (13), the prediction error can be written as

$$\tilde{\mathbf{x}}_{t|t-1} = f(\mathbf{x}_{t-1}) + \mathbf{w}_t - f(\hat{\mathbf{x}}_{t-1}) \quad (14)$$

Then the predication error of the REKF is transformed to an equivalent formulation that is easy to handle

$$\tilde{\mathbf{x}}_{t|t-1} = \boldsymbol{\beta}_t \mathbf{F}_t \tilde{\mathbf{x}}_{t-1} + \mathbf{w}_t \quad (15)$$

where  $\boldsymbol{\beta}_t = \text{diag}(\beta_{1,t}, \dots, \beta_{l,t})$  is unknown time-varying matrix used to scale the prediction error caused by the linearization error and the unknown exogenous inputs. Similar formulation has been used in (Boutayeb & Aubry, 1999) and (Xiong et al., 2006) to analyze the stability of the EKF and the UKF respectively. Using (17), the real error covariance matrix of the prediction error can be approximated by

$$\begin{aligned} \bar{\boldsymbol{\Sigma}}_{t|t-1} &= E(\tilde{\mathbf{x}}_{t|t-1} \tilde{\mathbf{x}}_{t|t-1}^T | \tilde{\mathbf{x}}_{t-1}) = E[(\boldsymbol{\beta}_t \mathbf{F}_t \tilde{\mathbf{x}}_{t-1} + \mathbf{w}_t)(\boldsymbol{\beta}_t \mathbf{F}_t \tilde{\mathbf{x}}_{t-1} + \mathbf{w}_t)^T | \tilde{\mathbf{x}}_{t-1}] \\ &= \boldsymbol{\beta}_t \mathbf{F}_t \mathbf{P}_{t-1} \mathbf{F}_t^T \boldsymbol{\beta}_t + \mathbf{Q}_t \end{aligned} \quad (16)$$

With these prerequisites, the sufficient conditions to ensure stability of the REKF are demonstrated in the following theorem.

**Theorem 1.** Consider the nonlinear stochastic system given by (1)-(2) with linear measurement  $h(\mathbf{x}_t) = \mathbf{H}_t \mathbf{x}_t$ , and the REKF as stated by (3)-(9). Let the following assumptions hold for every  $t \geq 0$

1. There are real numbers  $f_{\min}, f_{\max}, \beta_{\min}, \beta_{\max}, h_{\min}, h_{\max}$ , such that:

$$f_{\min}^2 I \leq \mathbf{F}_t \mathbf{F}_t^T \leq f_{\max}^2 I, \beta_{\min}^2 I \leq \boldsymbol{\beta}_t \boldsymbol{\beta}_t^T \leq \beta_{\max}^2 I, h_{\min}^2 I \leq \mathbf{H}_t \mathbf{H}_t^T \leq h_{\max}^2 I \quad (17)$$

2. There are real numbers  $p_{\min}, p_{\max}, r_{\min}, r_{\max}, q_{\max}$ , such that:

$$p_{\min} I \leq \mathbf{P}_t \leq p_{\max} I, r_{\min} I \leq \mathbf{R}_t \leq r_{\max} I, \mathbf{Q}_t \leq q_{\max} I \quad (18)$$

3. The following matrix in-equation is fulfilled

$$\boldsymbol{\Sigma}_{t|t-1} > \bar{\boldsymbol{\Sigma}}_{t|t-1} \quad (19)$$

Then there are real numbers  $\mu_{\max} > 0, 0 < \lambda_{\min} \leq 1$ , such that

$$E\{\|\tilde{\mathbf{x}}_t\|^2\} \leq \frac{p_{\max}}{p_{\min}} E\{\|\tilde{\mathbf{x}}_0\|^2\} (1 - \lambda_{\min})^t + \frac{\mu_{\max}}{p_{\min}} \sum_{i=1}^{t-1} (1 - \lambda_{\min})^i \quad (20)$$

The proof of Theorem 1 can be found in Appendix. The measurement equation is assumed to be linear to simplify the deduction. Nevertheless, the following analysis is expected to remain valid for the system with nonlinear measurement equation if the linearization error of the measurement equation is negligible.

It is clarified in Theorem 1 that under sufficient condition (17)-(19), estimation error of the REKF will remain bounded, and the effect of the initial error  $\tilde{\mathbf{x}}_0$  on the super-bound of  $\tilde{\mathbf{x}}_t$  will diminish as time goes on. In (17), the matrices  $\mathbf{F}_t$ ,  $\boldsymbol{\beta}_t$  and  $\mathbf{H}_t$  are assumed to be bounded. Nevertheless, there are no limitations about the magnitude of the bounds. In general, when the REKF is used in physical processes with finite energy, it is reasonable to assume that  $\boldsymbol{\beta}_t$  is bounded, and the assumption can be verified in practice. In (18), the condition  $p_{\min} I \leq \mathbf{P}_t \leq p_{\max} I$  is related to the observability property of the linearized system and related discussion can be seen in (Reif & Unbehauen, 1999; Boutayeb & Aubry, 1999). The inequalities  $r_{\min} I \leq \mathbf{R}_t \leq r_{\max} I$  and  $\mathbf{Q}_t \leq q_{\max} I$  are trivially true.

Equation (19) is the key condition of Theorem 1. It can be interpreted as follows: to ensure the stability of the filter, the calculated covariance matrix  $\boldsymbol{\Sigma}_{t|t-1}$  should be larger than the real one. This conclusion is similar with the traditional perspective that enlarging the covariance matrices may enhance the filter stability (Einicke et al, 2003; Bolognani et al, 2003; Boutayeb & Aubry, 1999). As unknown matrix  $\boldsymbol{\beta}_t$  may be rather large in the presence of large unknown inputs, from (16), the condition (19) may be violated and stability of the filter can not be guaranteed. A remedy to this potential problem is to tune the parameter  $\gamma$ . From (5), with  $L_t = I$ , in order for  $\boldsymbol{\Sigma}_{t|t-1} - \bar{\boldsymbol{\Sigma}}_{t|t-1}$  to be positive definite, it requires

$$\begin{aligned} (\mathbf{P}_{t|t-1}^{-1} - \gamma^{-2} I)^{-1} - \bar{\boldsymbol{\Sigma}}_{t|t-1} > 0 &\Rightarrow \mathbf{P}_{t|t-1}^{-1} - \bar{\boldsymbol{\Sigma}}_{t|t-1}^{-1} < \gamma^{-2} I \\ &\Rightarrow \gamma^{-2} > \max[\text{eig}(\mathbf{P}_{t|t-1}^{-1} - \bar{\boldsymbol{\Sigma}}_{t|t-1}^{-1})] \end{aligned} \quad (21)$$

where  $\max[\text{eig}(A)]$  denotes the minimum eigenvalue of the matrix  $A$ . If  $\max[\text{eig}(\mathbf{P}_{t|t-1}^{-1} - \bar{\boldsymbol{\Sigma}}_{t|t-1}^{-1})] \leq 0$ , (21) is bound to be fulfilled. Otherwise, the chosen  $\gamma$  should satisfy



$$\gamma < \{\max[\text{eig}(\mathbf{P}_{t|t-1}^{-1} - \bar{\Sigma}_{t|t-1}^{-1})]\}^{-0.5} \quad (22)$$

Hence,  $\gamma$  should be small enough to improve the stability of the REKF. This requirement is consistent with the design criterion of the robust  $H_\infty$  filter that the prescribed attenuation level  $\gamma$  should be as small as possible (Tseng & Chen, 2001). However, in order for the error covariance matrix  $\Sigma_{t|t-1}$  to be positive definite, it requires

$$\mathbf{P}_{t|t-1}^{-1} - \gamma^{-2} \mathbf{I} > 0 \Rightarrow \gamma > \{\max[\text{eig}(\mathbf{P}_{t|t-1})]\}^{0.5} \quad (23)$$

Obviously, if  $\{\max[\text{eig}(\mathbf{P}_{t|t-1})]\}^{0.5} > \{\max[\text{eig}(\mathbf{P}_{t|t-1}^{-1} - \bar{\Sigma}_{t|t-1}^{-1})]\}^{-0.5}$ , it will be difficult to find an appropriate  $\gamma$  to stabilize the filter, i.e., stability of the REKF is not guaranteed.

According to (10), for a fixed  $\gamma$  which is chosen to ensure  $\Sigma_{t|t-1} > 0$ , the bound of the estimation error  $\|\mathbf{L}_t \tilde{\mathbf{x}}_t\|^2$  will be enlarged in the presence of large linearization error or unknown exogenous inputs. And large deviation of the estimated state from the real one will further augment the linearization error. If this trend is not terminated, the filter will fail to converge. Hence, as can be seen from (23), ability of the REKF to minimize the energy of the estimation error is limited by the maximum eigenvalue of  $\mathbf{P}_{t|t-1}$ .

### 3. The AREKF algorithm

In order to guarantee the stability of the filter, a novel method is proposed to design the REKF. From Theorem 1, the bound of  $\tilde{\mathbf{x}}_t$  can be controlled by enlarging the calculated covariance matrix  $\Sigma_{t|t-1}$ . For the REKF,  $\Sigma_{t|t-1}$  will be enlarged by decreasing  $\gamma$ . However, it may be impossible to choose a suitable  $\gamma$  such that  $\Sigma_{t|t-1}$  is sufficiently large when  $\mathbf{L}_t = \mathbf{I}$ . In fact, this problem can be solved if the matrix  $\mathbf{L}_t$  in (5) is designed as

$$\mathbf{L}_t = \gamma(\mathbf{P}_{t|t-1}^{-1} - \lambda_t^{-1} \mathbf{P}_{t|t-1}^{-1})^{1/2} \quad (24)$$

where  $(\cdot)^{1/2}$  denotes the matrix square root, and  $\lambda_t$  is a tuning parameter which should be large enough such that the following inequation is fulfilled

$$\bar{\Sigma}_{t|t-1} < \lambda_t \mathbf{P}_{t|t-1} \quad (25)$$

With this design, the difficulty of tuning the prescribed level  $\gamma$  is avoid, and  $\lambda_t$  should be tuned instead of  $\gamma$  to obtain better robust behaviour. A practice way to tune the parameter  $\lambda_t$  is given in the next sub-section. Substitute (24) into (5), it is easy to verify that the sufficient condition (19) in Theorem 1 is fulfilled.

Nevertheless, the use of the upper bound  $\lambda_t \mathbf{P}_{t|t-1}$  may be too conservative. Too much emphasis is placed in accommodating the worst case (the largest linearization error) at the expense of optimality. In order to improve stability of the filter without decreasing accuracy, an adaptive scheme to adjust  $\Sigma_{t|t-1}$  in response to the changing environment is given as follows

$$\Sigma_{t|t-1} = \begin{cases} \mathbf{P}_{t|t-1}, & \mathbf{P}_{y,t} > \alpha \bar{\mathbf{P}}_{y,t} \\ (\mathbf{P}_{t|t-1}^{-1} - \gamma^{-2} \mathbf{L}_t^T \mathbf{L}_t)^{-1}, & \text{otherwise} \end{cases} \quad (26)$$

where  $\bar{\mathbf{P}}_{y,t} = E(\tilde{\mathbf{y}}_t \tilde{\mathbf{y}}_t^T | \tilde{\mathbf{x}}_{t-1})$  is the real covariance matrix of the innovation  $\tilde{\mathbf{y}}_t = \mathbf{y}_t - \mathbf{H}_t \hat{\mathbf{x}}_{t|t-1}$ . The parameter  $\alpha > 0$  is introduced to provide an extra degree of freedom to tune the threshold during the implementation process. Although  $\bar{\mathbf{P}}_{y,t}$  is unknown in practice, it can be estimated by

$$\bar{\mathbf{P}}_{y,t} \approx \begin{cases} \tilde{\mathbf{y}}_t \tilde{\mathbf{y}}_t^T, & t = 0 \\ \frac{\rho \bar{\mathbf{P}}_{y,t-1} + \tilde{\mathbf{y}}_t \tilde{\mathbf{y}}_t^T}{\rho + 1}, & t > 0 \end{cases} \quad (27)$$

where  $\rho = 0.98$  is a forgetting factor (Bai, 1999).

The adaptive robust extended Kalman filter (AREKF) has the structure of the REKF, except that the prediction error covariance is calculated by (26) instead of (5). With this particular design, when there is large innovation,  $\Sigma_{t|t-1}$  will be set to  $(\mathbf{P}_{t|t-1}^{-1} - \gamma^{-2} \mathbf{L}_t^T \mathbf{L}_t)^{-1}$  to prevent filter divergence. On the other hand, when the innovation is rather small,  $\Sigma_{t|t-1}$  will be set to  $\mathbf{P}_{t|t-1}$  so that it will not distort the estimation. The stability of the proposed algorithm is analyzed as follows.

**Theorem 2.** Consider the nonlinear stochastic system given by (1)-(2) with linear measurement  $h(\mathbf{x}_t) = \mathbf{H}_t \mathbf{x}_t$ . Assume that  $\text{rank}(\mathbf{H}_t) = l$  and  $\alpha = 1$ , and the real error covariance matrix  $E(\tilde{\mathbf{x}}_{t|t-1} \tilde{\mathbf{x}}_{t|t-1}^T | \tilde{\mathbf{x}}_{t-1})$  is approximated by  $\bar{\Sigma}_{t|t-1}$ . The AREKF is stated by (3), (4), (6)-(9), (24) and (26). Let the following assumptions hold for every  $t \geq 0$

1. There are real numbers  $f_{\min}, f_{\max}, \beta_{\min}, \beta_{\max}, h_{\min}, h_{\max}$ , such that:

$$f_{\min}^2 \mathbf{I} \leq \mathbf{F}_t \mathbf{F}_t^T \leq f_{\max}^2 \mathbf{I}, \beta_{\min}^2 \mathbf{I} \leq \boldsymbol{\beta}_t \boldsymbol{\beta}_t^T \leq \beta_{\max}^2 \mathbf{I}, h_{\min}^2 \mathbf{I} \leq \mathbf{H}_t \mathbf{H}_t^T \leq h_{\max}^2 \mathbf{I} \quad (28)$$

2. There are real numbers  $p_{\min}, p_{\max}, r_{\min}, r_{\max}, q_{\max}$ , such that:

$$p_{\min} \mathbf{I} \leq \mathbf{P}_t \leq p_{\max} \mathbf{I}, r_{\min} \mathbf{I} \leq \mathbf{R}_t \leq r_{\max} \mathbf{I}, \mathbf{Q}_t \leq q_{\max} \mathbf{I} \quad (29)$$

Then there are real numbers  $\mu_{\max} > 0, 0 < \lambda_{\min} \leq 1$ , such that

$$E\{\|\tilde{\mathbf{x}}_t\|^2\} \leq \frac{p_{\max}}{p_{\min}} E\{\|\tilde{\mathbf{x}}_0\|^2\} (1 - \lambda_{\min})^t + \frac{\mu_{\max}}{p_{\min}} \sum_{i=1}^{t-1} (1 - \lambda_{\min})^i \quad (30)$$

*Proof:* The proof of Theorem 2 is similar to that of Theorem 1, except that the sufficient condition shown in (19) is verified according to (24) and (26).

From (2) and (16), the real covariance matrix of the innovation is expressed as

$$\bar{\mathbf{P}}_{y,t} = E[(\mathbf{y}_t - \mathbf{H}_t \hat{\mathbf{x}}_{t|t-1})(\mathbf{y}_t - \mathbf{H}_t \hat{\mathbf{x}}_{t|t-1})^T | \tilde{\mathbf{x}}_{t-1}]$$



$$\begin{aligned}
&= E[(\mathbf{H}_t \tilde{\mathbf{x}}_{t|t-1} + \mathbf{v}_t)(\mathbf{H}_t \tilde{\mathbf{x}}_{t|t-1} + \mathbf{v}_t)^T | \tilde{\mathbf{x}}_{t-1}] \\
&= \mathbf{H}_t \bar{\boldsymbol{\Sigma}}_{t|t-1} \mathbf{H}_t^T + \mathbf{R}_t
\end{aligned} \tag{31}$$

From (8) and (31), we have

$$\mathbf{P}_{y,t} - \bar{\mathbf{P}}_{y,t} = \mathbf{H}_t (\boldsymbol{\Sigma}_{t|t-1} - \bar{\boldsymbol{\Sigma}}_{t|t-1}) \mathbf{H}_t^T \tag{32}$$

As the rank of the measurement matrix  $\mathbf{H}_t$  is assumed to be  $l$  and  $\alpha$  is assumed to be 1, if  $\mathbf{P}_{y,t} > \bar{\mathbf{P}}_{y,t}$ , the condition  $\boldsymbol{\Sigma}_{t|t-1} > \bar{\boldsymbol{\Sigma}}_{t|t-1}$  is fulfilled. Otherwise, if the inequation  $\mathbf{P}_{y,t} > \bar{\mathbf{P}}_{y,t}$  is not fulfilled, inserting (24) into (26), we have

$$\boldsymbol{\Sigma}_{t|t-1} - \bar{\boldsymbol{\Sigma}}_{t|t-1} = \lambda_t \mathbf{P}_{t|t-1} - \bar{\boldsymbol{\Sigma}}_{t|t-1} > 0 \tag{33}$$

Therefore, the condition (19) can always be satisfied. Then the theorem can be proved following the proof process of Theorem 1.

*Remarks:*

- 1) This theorem shows that the stability of the proposed algorithm can be ensured with the assumption  $\text{rank}(\mathbf{H}_t) = l$ . Nevertheless, the AREKF can be used even if  $\text{rank}(\mathbf{H}_t) < l$ . According to (28), if the hypothesis  $\mathbf{P}_{y,t} > \bar{\mathbf{P}}_{y,t}$  is not satisfied, then the condition  $\boldsymbol{\Sigma}_{t|t-1} > \bar{\boldsymbol{\Sigma}}_{t|t-1}$  is bound to be violated. In other words, it indicates that the stability of the filter is not guaranteed. Thus it is proper to adopt the robust filtering technique to adjust the covariance matrix  $\boldsymbol{\Sigma}_{t|t-1}$  when  $\mathbf{P}_{y,t} > \bar{\mathbf{P}}_{y,t}$  is not satisfied.
- 2) The small positive number  $\alpha$  in (26) is introduced to avoid resetting the covariance matrix  $\boldsymbol{\Sigma}_{t|t-1}$  frequently. Note that the condition  $\boldsymbol{\Sigma}_{t|t-1} > \bar{\boldsymbol{\Sigma}}_{t|t-1}$  is sufficient but not necessary to ensure stability of the filter. Numerical simulations have shown that even if the condition is not globally satisfied as  $\alpha < 1$ , the AREKF can tolerate much higher prediction error than the REKF and the traditional EKF.
- 3) The key idea of the AREKF is to design the estimator based on the stability analysis, and determine whether the error covariance matrix  $\boldsymbol{\Sigma}_{t|t-1}$  should be reset according to the hypothesis test. Unlike the REKF, the proposed algorithm is designed based on the tuning parameter  $\lambda_t$  instead of the prescribed attenuation level  $\gamma$ . Sufficient large  $\lambda_t$  can generally be found for practical systems with finite energy. In contrast, as discussed earlier, it may be impossible to obtain appropriate  $\gamma$  for the REKF. Thus, the proposed technique is more efficient to prevent instability in the case of large predication error. In addition, as the AREKF switches between the REKF and the traditional EKF under the control of the innovation covariance  $\bar{\mathbf{P}}_{y,t}$ , and it will not work in the REKF mode unless the estimated covariance  $\bar{\mathbf{P}}_{y,t}$  exceeds the threshold  $\alpha \mathbf{P}_{y,t}$ , it is expected to be more accurate than the traditional REKF in the case of negligible predication error.
- 4) Note that the estimate of  $\bar{\mathbf{P}}_{y,t}$  is not adopted directly in the algorithm, but used as an indicator of the filter instability. Even if the covariance matrix  $\boldsymbol{\Sigma}_{t|t-1}$  is reset inappropriately

due to the inaccurate approximation in (27), as  $(\mathbf{P}_{t|t-1}^{-1} - \gamma^{-2} \mathbf{L}_t^T \mathbf{L}_t)^{-1} > \mathbf{P}_{t|t-1}$ , from (19) in Theorem 1, it will not affect the stability of the filter. Certainly, in order to avoid decreasing accuracy of the filter, the parameter  $\alpha$  should be fine tuned so that  $\Sigma_{t|t-1}$  will not be reset frequently.

5) In fact, the estimation errors of the EKF and the UKF can be written as a uniform formulation which is similar to (A.3) (See (Xiong, 2006) for more explanations about the reformulation). Hence, conclusions as shown in Theorem 1 and Theorem 2 can also be drawn for the UKF, and the proof goes roughly the same as those shown in this paper.

6) A robust adaptive Kalman filter for linear systems with stochastic uncertainties has been proposed in (Wang & Balakrishnan, 1999). The key idea of the algorithm is to minimize the mean square estimation error, and a convex optimization problem should be solved according to the time-varying model and the measurements at each step of the algorithm. In contrast, the purpose of this paper is to provide a technique to stabilize the filter for nonlinear systems with external disturbances, and the hypothesis test is adopted to control the error covariance matrix of the estimator.

7) This section provides a technique to enhance the ability of the EKF to handle large external disturbances. In the presence of both model-plant mismatch and disturbances, other methods should be used together with the proposed technique to design the estimator. The most popular method to account for the model mismatch is to estimate the error terms adaptively and compensate the mismatch effect according to the estimates (Kwon, 2006). Another effective method is the extended robust  $H_\infty$  filter technique developed in (Seo et al., 2006). When this technique is adopted, several terms that scale the magnitude of the model error should be added in the error covariance matrix of the filter.

8) Although the paper focus on the external disturbance in state equation, the result in this paper can be extended to the systems with disturbance in measurement equations, and the innovation covariance matrix  $\mathbf{P}_{y,t}$  may play a similar role to the error covariance matrix  $\Sigma_{t|t-1}$  in ensuring the filter stability. In other words, enlarging the covariance matrix of the innovation could enhance ability of the filter to handle large prediction errors in the innovation. However, if both the state equation and the measurement equation are nonlinear, the stability analysis will be more complicated. Further works are required to give some principles for the design of  $\mathbf{P}_{y,t}$ .

#### 4. The autonomous navigation system

Currently, the orbit of a spacecraft is determined by earth stations, which require expensive equipments and extensive ground operations. As the number of the satellites increases, the number of the earth stations and system maintenance cost may increase significantly. This problem can be partly solved by using the autonomous navigation technique (Wiegand, 1996; Ma & Zhai, 2004; Huang et al., 2004).

The usefulness of autonomy has been recognized in the literature and several studies have been done to realize the autonomous navigation. The EKF forms the basis of many spacecraft navigation algorithm today (Vasile et al., 2002; Psiaki & Hinks, 2007; Nolet, 2007). Here we study the astronomical navigation technique method that employs the earth sensor and the star sensor. The basic measurement of the navigation system is illustrated in Fig.1.

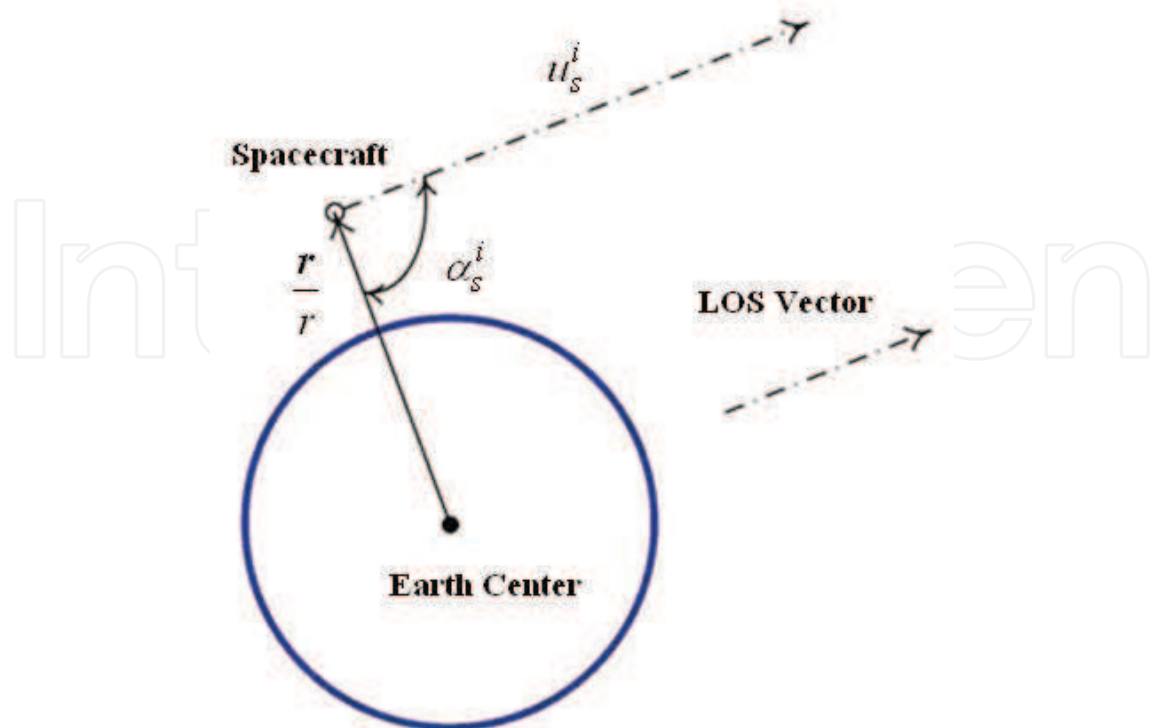


Fig. 1. The angle between the satellite position vector and the LOS vector

In Fig.1,  $\mathbf{r}/r$  is the unit position vector that points from the center of Earth to the spacecraft. It can be obtained from the earth sensor.  $\mathbf{u}_s^i$  is the line of sight (LOS) vector that points from the spacecraft to the star. It can be obtained from the star sensor.  $\alpha_s$  is the angle between the unit position vector  $\mathbf{r}/r$  and the LOS vector  $\mathbf{u}_s^i$ . The measurement  $\alpha_s$  does not depend on the estimate of the spacecraft attitude.

Then we define the state vector, the dynamic model and the measurement model that are used in the spacecraft navigation filter. The spacecraft navigation filter uses the following 6-dimensional state vector

$$\mathbf{x}_t = [\mathbf{r} \quad \dot{\mathbf{r}}]^T \quad (34)$$

where  $\mathbf{r} = [r_x \quad r_y \quad r_z]^T$  is the position vector of the spacecraft in Earth-centered inertial (ECI) coordinates,  $\dot{\mathbf{r}} = [v_x \quad v_y \quad v_z]^T$  is the velocity vectors of the spacecraft in ECI coordinates.

The orbital dynamic model is written in the form of (1) with the function  $f(\mathbf{x}_{t-1})$  as follows (Zhang & Fang, 2003)

$$f(\mathbf{x}_{t-1}) = \mathbf{x}_{t-1} + \varphi(\mathbf{x}_{t-1})\Delta t \quad (35)$$

where the propagation period  $\Delta t$  is defined as the time between two predictions, and

$$\varphi(\mathbf{x}_{t-1}) = \begin{bmatrix} v_x \\ v_y \\ v_z \\ -\frac{\mu r_x}{r^3} \left[ 1 + \frac{3}{2} J_2 \left( \frac{R_e}{r} \right)^2 \left( 1 - 5 \frac{r_z^2}{r^2} \right) \right] \\ -\frac{\mu r_y}{r^3} \left[ 1 + \frac{3}{2} J_2 \left( \frac{R_e}{r} \right)^2 \left( 1 - 5 \frac{r_z^2}{r^2} \right) \right] \\ -\frac{\mu r_z}{r^3} \left[ 1 + \frac{3}{2} J_2 \left( \frac{R_e}{r} \right)^2 \left( 3 - 5 \frac{r_z^2}{r^2} \right) \right] \end{bmatrix} \quad (36)$$

$\mu$  is the Earth's gravitational constant,  $R_e$  is the radius of Earth,  $r_i = (r_x^2 + r_y^2 + r_z^2)^{0.5}$  is the distance of the satellite from the center of Earth. Although the motion of the spacecraft is affected by the Earth's non-spherical mass distribution, atmospheric drag, solar radiation pressure and the forces caused by the thruster firings, the dynamic model in the algorithm is limited to the two-body equations of motion augmented by J2 perturbations. For most Earth-orbiting satellites that are placed in a set orbit with no mission operations deviating from that orbit, the unmodeled terms are relatively small and their effects are represented as the process noise  $w_t$ .

The measurement equation is written in the form of (2) with the function  $h(\mathbf{x}_t)$  as follows (Wei, 2004)

$$h(\mathbf{x}_t) = \begin{bmatrix} \alpha_s^1 \\ \alpha_s^2 \end{bmatrix} \quad (37)$$

where

$$\alpha_s^i = \arccos\left(-\frac{\mathbf{u}_s^i \cdot \mathbf{r}}{|\mathbf{r}|}\right), i = 1, 2 \quad (38)$$

The superscript  $i$  is used to distinguish the measurements obtained from different star sensors.

Based on the previous state equation and measurement equation, the standard EKF can be used to estimate the position and velocity vector of the spacecraft. Note that the unknown acceleration produced by the thruster firings is not taken into consideration in the orbital dynamic model. Hence, the standard algorithm is only valid for one specific orbit. If a spacecraft's mission requires it to maneuver at some point, such as merely changing its location along the orbit track or possibly altering its entire orbit shape, the unknown acceleration, which can be seen as the external disturbances of the navigation system, will degrade the performance of the EKF.

In order to achieve better accuracy, the AREKF is adopted in this Section. For a maneuvering spacecraft, the filter innovation  $\tilde{\mathbf{y}}_t$  will be enlarged due to the unknown acceleration. From (26), if the covariance matrix  $\bar{\mathbf{P}}_{y,t}$  calculated according to  $\tilde{\mathbf{y}}_t$  exceeds the

specified threshold  $\frac{1}{\alpha} \bar{\mathbf{P}}_{y,t}$ , the matrix  $\Sigma_{t|t-1}$  will be reset as  $(\mathbf{P}_{t|t-1}^{-1} - \gamma^{-2} \mathbf{L}_t^T \mathbf{L}_t)^{-1}$  such that the effect of the measurements to correct the state estimate will be reinforced. On the other hand, when the spacecraft stops maneuvering, the innovation will decrease as the filter goes on, and from (26), the original  $\mathbf{P}_{t|t-1}$  will be used to calculate the gain  $\mathbf{K}_t$ , such that the information contained in the orbital dynamic model will be fully used to handle the unfavorable effect of the measurement noise.

## 5. Simulation

The autonomous navigation system described in Section 4 is considered in this sub-section to demonstrate the improvement of the AREKF over the EKF, the REKF and the AEKF. The simulated truth orbital states are obtained by using a high accurate numerical orbit propagator, and the simulated measurements are obtained according to the simulated states and the measurement model. The simulated measurements are processed by different filters to obtain the state estimates. The estimates are compared with the simulated states to evaluate the performance of the filter.

The numerical simulation is implemented under the following assumptions. An Earth-orbiting satellite performs maneuver from 7293s to 8373s and from 11279s to 12068s. The height of the perigee of the orbit is raised from 500km to 2000km. The measurement precisions are  $0.02^\circ$  for earth sensor and  $5''$  for star sensor. The measurements are used to correct the predicted state  $\hat{\mathbf{x}}_{t|t-1}$  with a sampling interval of 100s. The initial position error and the corresponding error covariance matrix of the filters are listed below

$$\tilde{\mathbf{x}}_0 = [p_r \quad p_r \quad p_r \quad p_v \quad p_v \quad p_v]^T \quad (39)$$

$$\mathbf{P}_0 = \text{diag}([p_r^2 \quad p_r^2 \quad p_r^2 \quad p_v^2 \quad p_v^2 \quad p_v^2]) \quad (40)$$

where  $p_r = 5000\text{m}$ ,  $p_v = 10\text{m/s}$ . The process noise covariance matrix is set to

$$\mathbf{Q}_k = \text{diag}([q_r^2 \quad q_r^2 \quad q_r^2 \quad q_v^2 \quad q_v^2 \quad q_v^2]) \quad (41)$$

where  $q_r = 2 \times 10^{-5}\text{m}$ ,  $q_v = 2 \times 10^{-4}\text{m/s}$ .

The proposed AREKF can cope with the large linearization error caused by the initial error or the external disturbances. As the effect of the initial estimation error on the performance of the nonlinear filter has been analyzed in (Xiong et al., 2006), here emphasis is on the impact of the disturbances, i.e., the unknown acceleration produced by the thruster firings.

First, the usual EKF is performed to estimate the position and the velocity of the maneuvering spacecraft according to (3), (4), (6)-(9) with the covariance matrix  $\Sigma_{t|t-1} = \mathbf{P}_{t|t-1}$ . The result obtained from the EKF serves as a baseline reference for the comparison. The estimation errors are plotted versus time in Fig.2. Obviously, the performance of the EKF is seriously degraded due to the unknown acceleration in dynamic model. It indicates that this algorithm is sensitive to the disturbances. In addition, the root mean square (RMS) error of the EKF is computed from the data collected in the last 2000s of the simulation. The equation to compute RMS error can be expressed as

$$\sigma_i = \sqrt{\frac{1}{n-1} \sum_{t=T-n}^T (r_{i,t} - \hat{r}_{i,t})^2}, i = x, y, z, n = 2000 \quad (42)$$

where  $T$  is the time of the simulation. In order to facilitate the comparison, the square root of  $\sigma_x$ ,  $\sigma_y$  and  $\sigma_z$  are computed by using the following equation

$$\sigma_p = \sqrt{\sigma_x^2 + \sigma_y^2 + \sigma_z^2} \quad (43)$$

The RMS errors of the filters are summarized in Table 1.

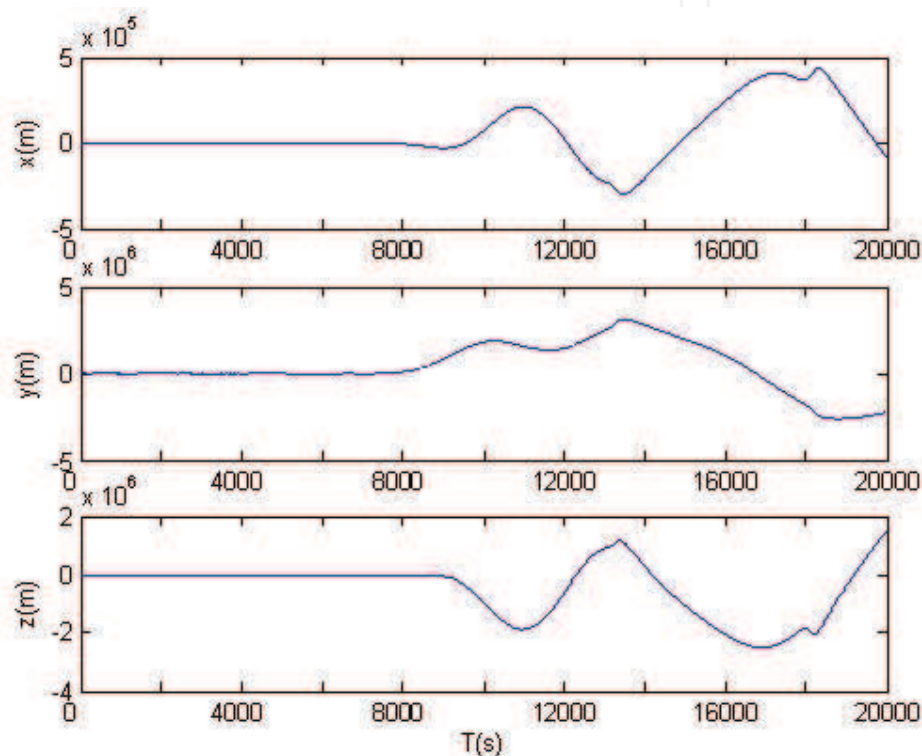


Fig. 2. Estimation error of the EKF

Second, the REKF is implemented according to (3)-(9). It was found that  $\gamma = 8000$  is sufficient to satisfy the condition  $\gamma > \{\max[\text{eig}(\mathbf{P}_{t|t-1})]\}^{0.5}$  for the entire run. Fig.3 shows the graph of the position error of the REKF, and the RMS error of the REKF is listed in Table1 1. Although the RMS error is slightly reduced by using the REKF instead of the EKF, performance of the REKF is also severely degraded by the disturbance. As the ability of the REKF to tolerant large disturbances is limited by the scaling parameter  $\gamma$ , which is chosen to ensure  $\Sigma_{t|t-1} > 0$ , there is a potential problem of instability in the method. From (22) and (23), it is difficult to choose an appropriate  $\gamma$  such that  $\Sigma_{t|t-1} > \bar{\Sigma}_{t|t-1}$  is fulfilled when the magnitude of the disturbances is rather large. Surely, the problem can be partly solved by choosing time-vary scaling parameter  $\gamma$  (Fu et al, 2001). However, additional computation is required to implement this technique. In addition, the performance of the REKF may be improved by tuning the noise covariance matrices  $\mathbf{Q}_t$  and  $\mathbf{R}_t$ . However, in general, it is difficult to obtain appropriate  $\mathbf{Q}_t$  and  $\mathbf{R}_t$ .



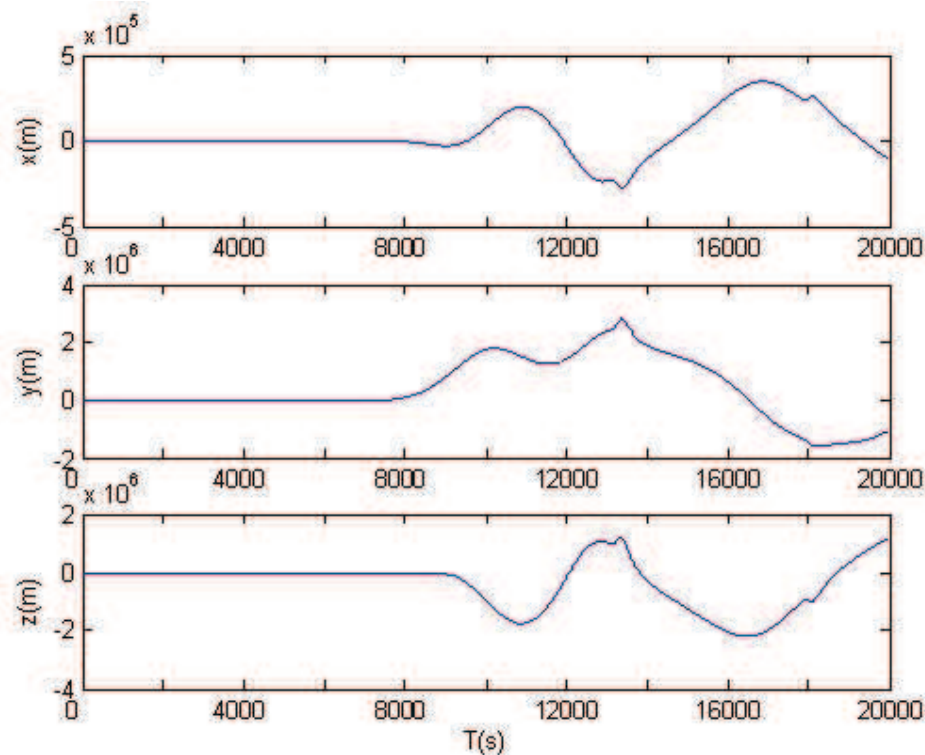


Fig. 3. Estimation error of the REKF

In order to clarify the superiority of the proposed algorithm, the AREKF is also compared with the AEKF proposed in (Moghaddamjoo & Kirilin, 1989) and developed in (Ashokaraj et al, 2002). A brief description of the AEKF is collected in the Appendix. It is shown in (Ashokaraj et al, 2002) that the performance of the AEKF is superior to that of the EKF. The AEKF algorithm can be implemented by using (3), (A.16), (A.17) and (6)-(9). For estimation of  $\hat{\mathbf{Q}}_t$ , the matrix  $\mathbf{A}$  and  $\mathbf{B}$  in (A17) are set to be

$$\mathbf{A} = a \times \begin{bmatrix} 0 & 0 & 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T, \quad \mathbf{B} = \mathbf{A}^T \quad (44)$$

where  $a = 1.5$ . The parameter  $k$  in (A.16) is set to be 2, i.e.,  $\hat{\mathbf{Q}}_t$  is determined by

$$\hat{\mathbf{Q}}_t = \hat{\mathbf{Q}}_{t-1} \exp[\mathbf{H}_t (\mathbf{C}_{y,1} + \mathbf{C}_{y,2}) \mathbf{C}_{y,0}^{-1} \mathbf{H}_t^T] \quad (45)$$

$\mathbf{C}_{y,i}$  ( $i = 1, 2$ ) is estimated by

$$\mathbf{C}_{y,i} = \frac{1}{N-1} \sum_{j=t-N+1}^t \tilde{\mathbf{y}}_j \tilde{\mathbf{y}}_{j-i}^T \quad (46)$$

The window size  $N$  is set to be 15 for this simulation. The performance of the AEKF is shown in Fig.4 and Table 1. The estimation result of the AEKF is better than those of the EKF and the REKF. It shows that the AEKF is more effective to eliminate the unfavorable effect of the considered disturbances in comparison with the EKF and the REKF. However, the error

curves also fluctuate widely during the maneuver. On the other hand, it is specified in (Xiong et al., 2008) the AEKF is somewhat less accurate than the standard EKF when there are no disturbances in dynamic model, for the noise covariance matrix is reset to the estimated value  $\hat{\mathbf{Q}}_t$  in every step of the filter, and  $\hat{\mathbf{Q}}_t$  which may deviate from its true value due to the inaccuracy of the state estimate. In addition, to implement the adaptive EKF, many tuning parameters (such  $k$ ,  $N$  and the elements in matrices  $\mathbf{A}$  and  $\mathbf{B}$ ) have to be designed. In contrast, for the AREKF, only parameters  $\lambda_t$  and  $\alpha$  should be tuned. And from (26), the covariance matrix  $\Sigma_{t|t-1}$  will not be reset unless  $\bar{\mathbf{P}}_{y,t}$  is enlarged sufficiently due to the disturbances.

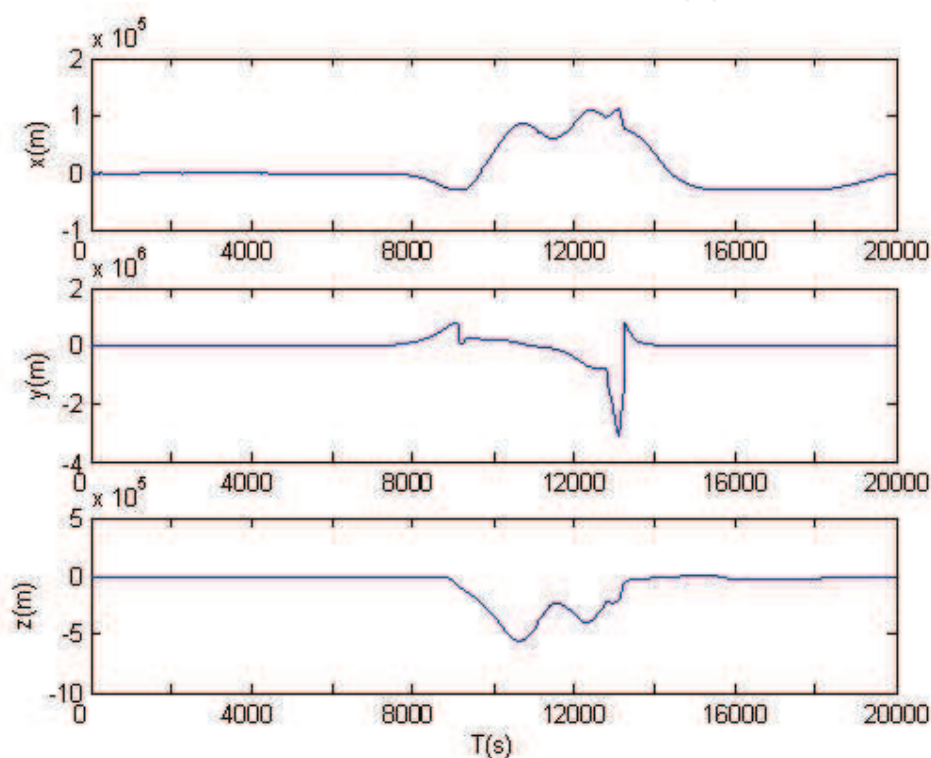


Fig. 4. Estimation error of the AEKF

Finally, we illustrate the application of the AREKF. The only difference from the REKF is that the prediction error covariance matrix is calculated by (26) instead of (5). As explained in Section 3, the design of the parameter  $\lambda_t$  is crucial to control the accuracy and stability of the filter. In order to obtain appropriate  $\lambda_t$ ,  $\lambda_t$  is tuned according to trace of the covariance matrix  $\bar{\mathbf{P}}_{y,t}$  calculated from the innovation  $\tilde{\mathbf{y}}_t$ . In this scenario,  $\lambda_t$  is set to be

$$\lambda_t = \frac{\text{trace}(\bar{\mathbf{P}}_{y,t})}{\text{trace}(\mathbf{P}_{y,t})} \quad (47)$$

and  $\alpha$  is set to be 0.2 to avoid resetting  $\Sigma_{t|t-1}$  frequently when the disturbances are rather small. The estimation errors of the AREKF are plotted in Fig.5.

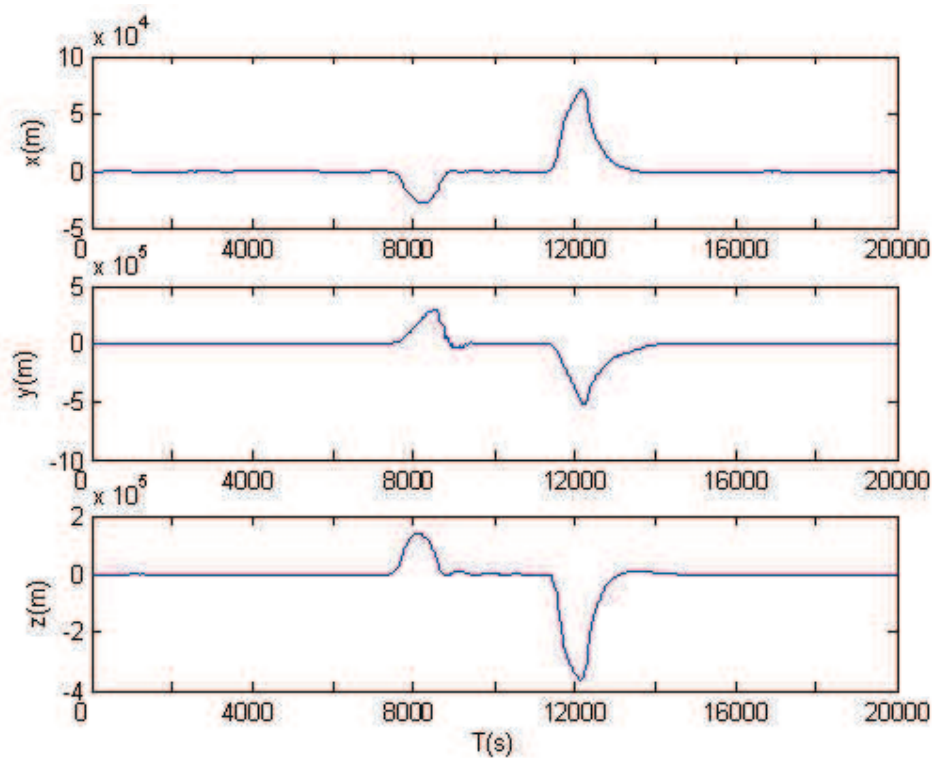


Fig. 5. Estimation error of the AREKF

As we expected, the proposed algorithm is robust enough to cope with the disturbances. Unlike the AEKF, the covariance matrix of the AREKF is reset only when the disturbances are large enough. Since the impact of the disturbances is partly eliminated, it is not surprising that the accuracy of the autonomous navigation system is further improved by using the AREKF instead of the AEKF. From Fig. 2-5 and Table 1, it is evident that the proposed method outperforms the usual EKF, the REKF and the AEKF in the presence of large external disturbances. The RMS error of the AREKF for the considered navigation system is on the order of 200m.

Filtering Algorithm	RMS error (18000s < t < 20000s)			
	$\sigma_x$ (m)	$\sigma_y$ (m)	$\sigma_z$ (m)	$\sigma_p$ (m)
EKF	$0.2771 \times 10^6$	$2.4096 \times 10^6$	$1.2367 \times 10^6$	$2.7175 \times 10^6$
REKF	$0.1382 \times 10^6$	$1.4027 \times 10^6$	$0.7055 \times 10^6$	$1.5762 \times 10^6$
AEKF	$1.9091 \times 10^4$	$1.2786 \times 10^4$	$0.6441 \times 10^4$	$2.3863 \times 10^4$
AREKF	126.1731	143.1503	69.5699	203.1050

Table 1. Performance Comparison of the Filtering Algorithms

6. Conclusion

The AREKF is proposed here as a modification of the REKF, that switches between the REKF mode and the normal EKF mode under the control of the innovation. In the presence of large external disturbances, the proposed algorithm is more effective than the REKF to ensure boundedness of the estimation error. On the other hand, in the absence of disturbances, it can yield more accurate estimates. In comparison with the adaptive EKF, the main advantage of the AREKF is its ease of application, as few parameters need to be tuned

for the design of the estimator. The proposed method is successfully applied to determine the position and velocity of the orbit maneuvering spacecraft based on the information obtain from the earth sensor and the star sensor. Numerical simulation shows that the AREKF is more effective than the EKF, the REKF and the AEKF to eliminate the unfavorable effect of the unknown acceleration produced by the thruster firings.

In addition, the adaptive robust filtering technique is expected to be effective to improve the stability of other nonlinear filters, such as the unscented Kalman filter (UKF) and the particle filter.

## 7. Appendix

### 7.1 Proof of Theorem 1

*Proof:* First, choose the Lyapunov function

$$V_t(\tilde{\mathbf{x}}_t) = \tilde{\mathbf{x}}_t^T \mathbf{P}_t^{-1} \tilde{\mathbf{x}}_t \quad (\text{A.1})$$

Because of (16), we have the bounds for the function  $V_t(\tilde{\mathbf{x}}_t)$

$$\frac{1}{p_{\max}} \|\tilde{\mathbf{x}}_t\|^2 \leq V_t(\tilde{\mathbf{x}}_t) \leq \frac{1}{p_{\min}} \|\tilde{\mathbf{x}}_t\|^2 \quad (\text{A.2})$$

i.e., (10) hold with  $v_{\min} = \frac{1}{p_{\max}}$  and  $v_{\max} = \frac{1}{p_{\min}}$ . To satisfy the requirement (11) for the application of Lemma 1, it needs an upper bound on  $E[V_t(\tilde{\mathbf{x}}_t) | \tilde{\mathbf{x}}_{t-1}] - V_{t-1}(\tilde{\mathbf{x}}_{t-1})$ .

From (5), (13) and (2), estimation error of the REKF can be written as

$$\tilde{\mathbf{x}}_t = \boldsymbol{\beta}_t \mathbf{F}_t \tilde{\mathbf{x}}_{t-1} - \mathbf{K}_t \mathbf{H}_t \boldsymbol{\beta}_t \mathbf{F}_t \tilde{\mathbf{x}}_{t-1} + (\mathbf{I} - \mathbf{K}_t \mathbf{H}_t) \mathbf{w}_t - \mathbf{K}_t \mathbf{v}_t \quad (\text{A.3})$$

Substitute (A.3) into (A.1), and taking the conditional expectation yields:

$$\begin{aligned} E[V_t(\tilde{\mathbf{x}}_t) | \tilde{\mathbf{x}}_{t-1}] &= E[(\boldsymbol{\beta}_t \mathbf{F}_t \tilde{\mathbf{x}}_{t-1})^T \mathbf{P}_t^{-1} (\boldsymbol{\beta}_t \mathbf{F}_t \tilde{\mathbf{x}}_{t-1}) - (\mathbf{K}_t \mathbf{H}_t \boldsymbol{\beta}_t \mathbf{F}_t \tilde{\mathbf{x}}_{t-1})^T \mathbf{P}_t^{-1} (\boldsymbol{\beta}_t \mathbf{F}_t \tilde{\mathbf{x}}_{t-1}) - \\ &\quad (\boldsymbol{\beta}_t \mathbf{F}_t \tilde{\mathbf{x}}_{t-1})^T \mathbf{P}_t^{-1} (\mathbf{K}_t \mathbf{H}_t \boldsymbol{\beta}_t \mathbf{F}_t \tilde{\mathbf{x}}_{t-1}) + (\mathbf{K}_t \mathbf{H}_t \boldsymbol{\beta}_t \mathbf{F}_t \tilde{\mathbf{x}}_{t-1})^T \mathbf{P}_t^{-1} (\mathbf{K}_t \mathbf{H}_t \boldsymbol{\beta}_t \mathbf{F}_t \tilde{\mathbf{x}}_{t-1}) + \\ &\quad \mathbf{w}_t^T (\mathbf{I} - \mathbf{K}_t \mathbf{H}_t)^T \mathbf{P}_t^{-1} (\mathbf{I} - \mathbf{K}_t \mathbf{H}_t) \mathbf{w}_t + \mathbf{v}_t^T \mathbf{K}_t^T \mathbf{P}_t^{-1} \mathbf{K}_t \mathbf{v}_t | \tilde{\mathbf{x}}_{t-1}] \end{aligned} \quad (\text{A.4})$$

Using (6) and (7), it can be verified that

$$\mathbf{K}_t = \mathbf{P}_t \mathbf{H}_t^T \mathbf{R}_t^{-1}$$

And then (A.4) becomes

$$E[V_t(\tilde{\mathbf{x}}_t) | \tilde{\mathbf{x}}_{t-1}] = \tilde{\mathbf{x}}_{t-1}^T \mathbf{F}_t^T \boldsymbol{\beta}_t [\boldsymbol{\Sigma}_{t|t-1}^{-1} - \mathbf{H}_t^T (\mathbf{R}_t^{-1} - \mathbf{R}_t^{-1} \mathbf{H}_t \mathbf{P}_t \mathbf{H}_t^T \mathbf{R}_t^{-1}) \mathbf{H}_t] \boldsymbol{\beta}_t \mathbf{F}_t \tilde{\mathbf{x}}_{t-1} + \boldsymbol{\mu}_t \quad (\text{A.5})$$

where

$$\boldsymbol{\mu}_t = E[\mathbf{w}_t^T (\mathbf{I} - \mathbf{K}_t \mathbf{H}_t)^T \mathbf{P}_t^{-1} (\mathbf{I} - \mathbf{K}_t \mathbf{H}_t) \mathbf{w}_t + \mathbf{v}_t^T \mathbf{K}_t^T \mathbf{P}_t^{-1} \mathbf{K}_t \mathbf{v}_t] \quad (\text{A.6})$$

It can be seen from (14) and condition (17) that

$$\Sigma_{t|t-1} \geq \beta_t F_t P_{t-1} F_t^T \beta_t + Q_t \geq \beta_t F_t P_{t-1} F_t^T \beta_t \quad (\text{A.7})$$

In addition, it is easy to verified that

$$R_t^{-1} - R_t^{-1} H_t P_t H_t^T R_t^{-1} = (H_t \Sigma_{t|t-1} H_t^T + R_t)^{-1} \quad (\text{A.8})$$

Using (A.7) and (A.8), we have

$$E[V_t(\tilde{\mathbf{x}}_t) | \tilde{\mathbf{x}}_{t-1}] \leq \tilde{\mathbf{x}}_{t-1}^T P_{t-1}^{-1} \tilde{\mathbf{x}}_{t-1} - \tilde{\mathbf{x}}_{t-1}^T F_t^T \beta_t H_t^T (H_t \Sigma_{t|t-1} H_t^T + R_t)^{-1} H_t \beta_t F_t \tilde{\mathbf{x}}_{t-1} + \mu_t \quad (\text{A.9})$$

Subtracting  $V_{t-1}(\tilde{\mathbf{x}}_{t-1})$  from both sides of (A.9), we establish that

$$E[V_t(\tilde{\mathbf{x}}_t) | \tilde{\mathbf{x}}_{t-1}] - V_{t-1}(\tilde{\mathbf{x}}_{t-1}) \leq \mu_t - \lambda_t V_{t-1}(\tilde{\mathbf{x}}_{t-1}) \quad (\text{A.10})$$

where

$$\lambda_t = [\tilde{\mathbf{x}}_{t-1}^T F_t^T \beta_t H_t^T (H_t \Sigma_{t|t-1} H_t^T + R_t)^{-1} H_t \beta_t F_t \tilde{\mathbf{x}}_{t-1}] / V_{t-1}(\tilde{\mathbf{x}}_{t-1}) \quad (\text{A.11})$$

The aim of the following deduction is to determine domains of  $\mu_t$  and  $\lambda_t$ . Because both sides of (A.6) are scalars, taking the trace of the equation will not change its value. Under assumption (15) and (16), we obtain

$$\begin{aligned} \mu_t &\leq \text{tr}[(P_t^{-1} + H_t^T K_t^T P_t^{-1} K_t H_t) Q_t] + \text{tr}(K_t^T P_t^{-1} K_t R_t) \\ &= \text{tr}[(P_t^{-1} + H_t^T R_t^{-1} H_t P_t H_t^T R_t^{-1} H_t) Q_t] + \text{tr}(R_t^{-1} H_t P_t H_t^T) \\ &\leq (p_{\min}^{-1} + h_{\max}^4 r_{\min}^{-2} p_{\max}) q_{\max} l + h_{\max}^2 p_{\max} r_{\min}^{-1} m \\ &\stackrel{\Delta}{=} \mu_{\max} \end{aligned} \quad (\text{A.12})$$

Obviously,  $\mu_{\max} > 0$ . Applying the matrix inversion lemma (see, e.g., (Lewis, 1986), Appendix A2, p.347) yields:

$$\begin{aligned} &V_{t-1}(\tilde{\mathbf{x}}_{t-1}) - \lambda_t V_{t-1}(\tilde{\mathbf{x}}_{t-1}) \\ &= \tilde{\mathbf{x}}_{t-1}^T [P_{t-1}^{-1} - F_t^T \beta_t H_t^T (H_t \Sigma_{t|t-1} H_t^T + R_t)^{-1} H_t \beta_t F_t] \tilde{\mathbf{x}}_{t-1} \\ &> \tilde{\mathbf{x}}_{t-1}^T [P_{t-1}^{-1} - F_t^T \beta_t H_t^T (H_t P_{t|t-1} H_t^T + R_t)^{-1} H_t \beta_t F_t] \tilde{\mathbf{x}}_{t-1} \\ &= \tilde{\mathbf{x}}_{t-1}^T [P_{t-1} + P_{t-1} F_t^T \beta_t H_t^T (H_t Q_t H_t^T + R_t)^{-1} H_t \beta_t F_t P_{t-1}]^{-1} \tilde{\mathbf{x}}_{t-1} > 0 \end{aligned} \quad (\text{A.13})$$

Hence,  $\lambda_t < 1$ . Under assumption (15) and (16),

$$\lambda_t \geq p_{\min} (h_{\min} \beta_{\min} f_{\min})^2 [p_{\max} (h_{\max} \beta_{\max} f_{\max})^2 + q_{\max} h_{\max}^2 + r_{\max}]^{-1}$$

$$\overset{\Delta}{=} \lambda_{\min} > 0 \quad (\text{A.14})$$

Then we obtain the following inequality

$$E[V_t(\tilde{\mathbf{x}}_t) | \tilde{\mathbf{x}}_{t-1}] - V_{t-1}(\tilde{\mathbf{x}}_{t-1}) \leq \mu_{\max} - \lambda_{\min} V_{t-1}(\tilde{\mathbf{x}}_{t-1}) \quad (\text{A.15})$$

Where  $\mu_{\max} > 0$ ,  $0 < \lambda_{\min} < 1$ . Finally, applying Lemma 1, we can draw a conclusion that the estimation error  $\tilde{\mathbf{x}}_t$  is bounded in mean square.

## 7.2 The AEKF algorithm

The key idea of the AEKF is to stabilize the filter under any unknown external disturbances by adjusting the noise covariance matrix  $\mathbf{Q}_t$ . In other words,  $\mathbf{Q}_t$  is modified to insert a negative feedback in the estimation process such that the property of the filter will be optimized.

The difference between the AEKF and the traditional EKF is that the prediction error covariance matrix of the AEKF is calculated as

$$\Sigma_{t|t-1} = \mathbf{F}_t \mathbf{P}_{t-1} \mathbf{F}_t^T + \hat{\mathbf{Q}}_t \quad (\text{A.16})$$

And  $\hat{\mathbf{Q}}_t$  is adjusted according to the innovation of the filter

$$\hat{\mathbf{Q}}_t = \hat{\mathbf{Q}}_{t-1} \exp \left[ \mathbf{A} \left( \sum_{i=1}^k \mathbf{C}_{y,i} \right) \mathbf{C}_{y,0}^{-1} \mathbf{B}^T \right] \quad (\text{A.17})$$

where  $\mathbf{C}_{y,i} = E(\tilde{\mathbf{y}}_t \tilde{\mathbf{y}}_{t-i}^T)$  is the autocorrelation of the innovation  $\tilde{\mathbf{y}}_t$ ,  $\mathbf{A}$  and  $\mathbf{B}$  are coefficient matrices which should be found experimentally, and their dimensions are  $(l \times m)$ . It is suggested in (Psiaki & Hinks, 2007) to choose a small number of autocorrelation  $k$ .

In this method,  $\hat{\mathbf{Q}}_t$  is used to control the Kalman gain  $\mathbf{K}_t$  adaptively. From (A.16), when the estimate of  $\mathbf{C}_{y,i}$  increases,  $\hat{\mathbf{Q}}_t$  will be increased. On the other hand, when  $\mathbf{C}_{y,i}$  decreases,  $\hat{\mathbf{Q}}_t$  will be decreased. This behaviour can be seen as a negative feedback which has a stabilizing role in the performance of the algorithm (See (Psiaki & Hinks, 2007) for more explanations).

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