Chapter 12

Stability Conditions for a Class of Nonlinear Systems with Delay

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Additional information is available at the end of the chapter

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Abstract

This chapter presents an extension and offers a more comprehensive overview of our previous paper entitled "Stability conditions for a class of nonlinear time delay systems" published in "Nonlinear Dynamics and Systems Theory" journal. We first introduce a more complete approach of the nonlinear system stability for the single delay case. Then, we show the application of the obtained results to delayed Lur'e Postnikov systems. A state space representation of the class of system under consideration is used and a new transformation is carried out to represent the system, with delay, by an arrow form matrix. Taking advantage of this representation and applying the Kotelyanski lemma in combination with properties of M-matrices, some new sufficient stability conditions are determined. Finally, illustrative example is provided to show the easiness of using the given stability conditions.

Keywords: nonlinear systems, time delay, arrow matrix, M-matrix, Lur'e Postnikov, stability conditions

1. Introduction

Studying stability of dynamical systems with time delay has received the attention of many researchers from the control community in the past decades, see [1–27] and the references therein. Time-varying delay which varies within an interval with nonzero lower bound is encountered in a variety of engineering applications which spreads from recurrent neural networks to chemical reactors and power systems with loss-less transmission lines. It is therefore more appropriate to study stability analysis and control synthesis of these dynamical systems with time-varying delays as these delays are usually time varying in nature. There are mainly two strategies in obtaining stability conditions. We can obtain delay-independent (i.o.d) results [28, 29] and the references therein, which are applicable to delays of arbitrary size or when there is no information about the delay. In general this lack of information about the delay will result in conservative criteria, especially when the delay is relatively small. Whenever it is possible to include information on the size of the delay, we can get delay-dependent (d.d) conditions which are usually less conservative. Most of the systems described above are nonlinear in practical engineering problems. For this reason, the chapter focuses on determining easy to test sufficient stability conditions for nonlinear systems with time-varying delay [30–33].

New delay dependent stability conditions are derived by employing arrow form state space representation [31–34], Kotelyanski lemma and using tools from M-matrix theory and Lyapunov functional method.

The obtained results are exploited to design a state feedback controller that stabilizes Lur'e systems with time-varying delay and sector-bounded nonlinearity [26, 28, 34]. In fact, Lur'e control systems is considered as one the most important classes of nonlinear control systems and continue to be one of the important problems in control theory that has been studied widely because it has many practical applications [32–36].

The chapter is organized as follows: Section 2 presents the notation used throughout the chapter and some facts on M-matrices that will be needed in proving the obtained results. In sections 3 the main results are given. Application of these results to delayed nonlinear nth order all pole plant and the well-known Lur'e systems, is presented in Section 4. Illustrative example is given in Section 5 and some concluding remarks are provided in Section 6.

2. Notation and facts

Let us fix the notation used. Let $C_n = C([-\tau\ 0], R^n)$ be the Banach space of continuous functions mapping the interval $[-\tau\ 0]$ into R^n with the topology of uniform convergence. Let $x_t \in C_n$ be defined by $x_t(\theta) = x(t+\theta), \theta \in [-\tau\ 0]$ where $x(t) = \left(y(t)\ \ \dot{y}(t)\ \ \ldots\ \ y^{(n-1)}(t)\right)'$. For a given $\varphi \in C_n$, we define $\|\varphi\| = \sup_{-\tau \le \theta \le 0} \|\varphi(\theta)\|, \varphi(\theta) \in R^n$. The functions $a_i(.)$, $b_i(.)$, $i = 1, \ldots, n\text{-}1$ are completely continuous mapping the set $J_a \times C_n^H \times S_\varpi$ into R, where $C_n^H = \{\varphi \in C_n, \|\varphi\| < H\}$, $H > 0, J_a = [a\ \ +\infty), a \in R$ and $S_\varpi = \{\varpi, k_1 \le \varpi \le k_2 / k_1 \le k_2 \in R\}$. In the sequel, we denote $(t, x_t, \varpi) = (.)$.

Now we introduce several useful facts, including some definitions of M-matrices and the Kotelyanski lemma that will be used in subsequent parts of the chapter.

Definition 1. The $n \times n$ matrix $A = \left(a_{i,j}\right)_{1 \le i,j \le n}$ is called an M-matrix if the following conditions are satisfied for $i = 1, 2, \ldots, n$ [34]:

1. $a_{i,i} > 0\ \ , a_{i,j} \le 0\ \ (i \ne j, j = 1, 2, \ldots, n)$.
2. Successive principal minors of A are positive, i.e.

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$$\det\begin{pmatrix} a_{1,1} & \dots & a_{1,i} \\ \vdots & \dots & \vdots \\ a_{i,1} & \dots & a_{i,i} \end{pmatrix} > 0$$

Definition 2. The matrix A is the opposite of an M-matrix if $(-A)$ is an M-matrix. There are many equivalent conditions for characterizing an M matrix. In fact, the following definition is the most appropriate for our purposes [34].

Definition 3. The matrix $A = (a_{i,j})_{n\le i,j\le n}$ is called an M-matrix if $a_{i,i} > 0$ $(i = 1, 2, \dots, n)$, $a_{i,j} \le 0$, $i \ne j$, $(i, j = 1, 2, \dots, n)$ and for any vector $\sigma \in R_+^{*n}$, the algebraic equation $A'c = \sigma$ has a solution $c = \left(A'\right)^{-1}\sigma \in R_+^{*n}$ [34].

Kotelyanski Lemma

The real parts of the eigenvalues of a matrix A, with non-negative off diagonal elements, are less than a real number μ if and only if all those of the matrix M, $M = I_n - \mu A$, are positive, with I_n the $n \times n$ identity matrix [34, 35].

3. Sufficient stability conditions

Our work consists of determining stability conditions for systems described by the following equation:

$$\begin{cases} y^{(n)}(t) + \sum_{i=0}^{n-1} a_i(t, x_t, \varpi) y^{(i)}(t) + \sum_{j=0}^{n} b_j(t, x_t, \varpi) y^{(j)}(t\text{-}\tau) = u(t) \\ y^{(i)}(t) = \varphi_i(t), t \in [\text{-}\tau\ 0], i = 0, \dots, n\text{-}1, \end{cases} \tag{1}$$

where τ is a constant delay and $a_i(.)$, $b_i(.)$, $i = 1, \dots, n\text{-}1$ are nonlinear functions.

We start by representing the system (1), under another form. Using the following notation:

$$x_{i+1}(t) = y^{(i)}(t), i = 0, \dots, n\text{-}1 \tag{2}$$

we get:

$$\begin{cases} \dot{x}_i(t) = x_{i+1}(t) \ \ i = 1, \dots, n\text{-}1 \\ \dot{x}_n(t) = -\sum_{i=0}^{n-1} a_i(.) x_i(t) - \sum_{i=0}^{n-1} b_i(.) x_i(t\text{-}\tau) \end{cases} \tag{3}$$

or under matrix form:

$$\dot{x}(t) = A(.)x(t) + B(.)x(t\text{-}\tau) \tag{4}$$

A(.) and B(.) are $n \times n$ matrices given by:

$$A(.) = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ -a_0(.) & -a_1(.) & \dots & -a_{n-1}(.) \end{bmatrix}, B(.) = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \dots & \vdots \\ 0 & \dots & 0 \\ -b_0(.) & \dots & -b_{n-1}(.) \end{bmatrix} \tag{5}$$

The regular basis change P transforms the original system to the new one defined by:

$$x(t) = Pz(t), \tag{6}$$

with:

$$P = \begin{bmatrix} 1 & 1 & \dots & 1 & 0 \\ \alpha_1 & \alpha_2 & \dots & \alpha_{n-1} & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \dots & \alpha_{n-1}^{n-1} & 1 \end{bmatrix} \tag{7}$$

The new state space representation is:

$$\dot{z}(t) = F(.)z(t) + D(.)z(t\text{-}\tau) \tag{8}$$

with:

$$F(.) = P^{-1}A(.)P = \begin{bmatrix} \alpha_1 & & & & \beta_1 \\ & \alpha_2 & & & \beta_2 \\ & & \ddots & & \vdots \\ & & & \alpha_{n-1} & \beta_{n-1} \\ \gamma_1(.) & \gamma_2(.) & \dots & \gamma_{n-1}(.) & \gamma_n(.) \end{bmatrix} \tag{9}$$

Elements of the matrix F(.) are defined in [33] by:

$$\begin{cases} \gamma_i(.) = -p_A(\alpha_i, .) \text{ for } i = 1, \dots, n\text{-}1, \\ \gamma_n(.) = -a_{n-1}(.) - \sum_{i=1}^{n-1} \alpha_i \end{cases} \tag{10}$$

where

$$p_A(s, .) = s^n + \sum_{i=0}^{n-1} a_i(.) s^i \tag{11}$$

and

$$\beta_i = \left. \frac{\lambda\text{-}\alpha_i}{Q(\lambda)} \right|_{\lambda = \alpha_i} \text{for } \; i = 1, \dots, n\text{-}1 \tag{12}$$

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where

$$Q(\lambda) = \prod_{j=1}^{n-1} (\lambda\text{-}\alpha_j) \tag{13}$$

and the matrix D(.) is given by:

$$D(.) = P^{-1}B(.)P = \begin{pmatrix} O_{n-1,\,n-1} & & & O_{n-1,\,1} \\ \delta_1(.) & \dots & \delta_{n-1}(.) & \delta_n(.) \end{pmatrix} \tag{14}$$

Elements of the matrix D(.) are defined in [18] by:

$$\begin{cases} \delta_i(.) = -p_B(\alpha_i, .),\ i = 1, \dots, n\text{-}1 \\ \delta_n(.) = -b_{n-1}(.) \end{cases} \tag{15}$$

Based on this transformation and the arbitrary choice of parameters α_i, $i = 1, \dots, n-1$ which play an important role in simplifying the use of aggregate techniques, we give now the main result. Let us start by writing our system in another form. By using the Newton-Leibniz formula

$$x(t\text{-}\tau) = \int_{t-\tau}^{t} \dot{x}(u)du \tag{16}$$

Equation (Eq. 8) becomes

$$\dot{z}(t) = (F(.) + D(.))z(t)\text{-}D(.)\int_{t-\tau}^{t} \dot{x}(\theta)d\theta \tag{17}$$

Let Ω be a domain of R^n, containing a neighborhood of the origin, and $\sup_{J_\tau, \Omega, S_\varpi}$ the suprema calculated for $t \in J_\tau (i.e\ t \geq \tau)$, for functions x with values in Ω, and for ϖ in S_ϖ.

Next, using the special form of system (Eq. (1)) and applying the notation $\sup_{J_\tau, \Omega, S_\varpi} = \sup_{[.]}$, we can announce the following theorem.

Theorem 2.1. The system (Eq. (1)) is asymptotically stable, if there exist distinct parameters $\alpha_i < 0, i = 1, \dots, n\text{-}1,$ such that the matrix $\tilde{F}(.)$ is the opposite of an M-matrix, where $\tilde{F}(.)$ is given by

$$\tilde{F}(.) = \begin{bmatrix} \alpha_1 & & & & |\beta_1| \\ & \alpha_2 & & & |\beta_2| \\ & & \ddots & & \vdots \\ & & & \alpha_{n-1} & |\beta_{n-1}| \\ \tilde{\gamma}_1(.) & \tilde{\gamma}_2(.) & \dots & \tilde{\gamma}_{n-1}(.) & \tilde{\gamma}_n(.) \end{bmatrix} \tag{18}$$

and the elements $\tilde{\gamma}_i(.), i = 1, \dots, n$, are given by

$$\begin{cases} \tilde{\gamma}_i(.) = \dfrac{\left|\gamma_i(.) + \delta_i(.)\right| + \tau|\alpha_i\ |\sup_{[.]}\ |\delta_i(.)|}{1 - \tau \sup_{[.]}\ |\delta_n(.)|}, \quad i = 1, \ldots, n\text{-}1 \\ \tilde{\gamma}_n(.) = \gamma_n(.) + \delta_n(.) + \dfrac{\tau \sup_{[.]}\ |\delta_n(.)|\left|\gamma_n(.) + \delta_n(.)\right|}{1 - \tau \sup_{[.]}\ |\delta_n(.)|} + \displaystyle\sum_{i=1}^{n} \dfrac{\tau\ \left|\beta_i\ \right|\sup_{[.]}\ |\delta_i(.)|}{1 - \tau \sup_{[.]}\ |\delta_n(.)|} \end{cases} \tag{19}$$

Proof:

We use the following vector norm $p(z) = \left(\ p_1(z)\ p_2(z)\ p_3(z) \ldots p_n(z)\ \right)'$, where

$$\begin{cases} p_i(z) = |z_i\ |, i = 1, \ldots, n\text{-}1 \\ p_n(z) = |z_n| + \dfrac{\displaystyle\sum_{i=1}^{n} \sup_{[.]}\ |\delta_i(.)|}{1 - \tau \sup_{[.]}\ |\delta_n(.)|} \displaystyle\int_{-\tau}^{0} \int_{t+\theta}^{t} |\dot{z}_i(\vartheta)|\ d\vartheta d\theta \end{cases} \tag{20}$$

with the condition

$$\tau \sup_{[.]}\ |\delta_n(.)| < 1 \tag{21}$$

Let $V(t)$ be a radially unbounded Lyapunov function given by (Eq. (22)).

$$V\ (t) = \left\langle\ \left(p(z(t))\right)', w \right\rangle = \sum_{i=1}^{n} w_i p_i(z(t)) \tag{22}$$

where $w \in R_+^n$, $w_i > 0, i = 1, \ldots, n$. First, note that

$$V(t_0) \le \sum_{i=1}^{n} w_i\ |z_i(t_0)| + w_n \left(|z_n(t_0)| + \frac{\sup_{[.]}\ (|\delta_n(.)|)}{1\text{-}\tau \sup_{[.]}\ (|\delta_n(.)|)} \sup_{[-\tau,0]}\ \ |\dot{\varphi}_n| \frac{\tau^2}{2} \right) := r < +\infty$$

and

$$V\ (t) \ge \sum_{i=1}^{n} w_i\ |z_i(t)|$$

The right Dini derivative of $V(t)$, along the solution of (Eq. (22)), gives

$$D^+V(t) = \sum_{i=1}^{n} w_i \frac{d^+ p_i(z(t))}{dt^+} \tag{23}$$

For clarification reasons, each element of $\frac{d^+ p_i(z(t))}{dt^+}$, i = 1, …, n is calculated separately. Let us begin with the first (n-1) elements. Because $|z_i\ | = z_i \mathrm{sign}(z_i)$, we can write, for $i = 1, \ldots, n\text{-}1$,

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$$\begin{aligned}\frac{d^+p_i(z(t))}{dt^+}=\frac{d^+|z_i(t)|}{dt^+} &= \frac{d^+z_i(t)}{dt^+}\text{sign}(z_i(t))\\ &= \alpha_i|z_i(t)|+\beta_i z_n(t)\text{sign}(z_i(t))\\ &\le \alpha_i|z_i(t)|+|\beta_i||z_n(t)|\end{aligned} \tag{24}$$

and

$$\frac{d^+p_n(z)}{dt^+}=\frac{d^+|z_n|}{dt^+}+\frac{\sum_{i=1}^{n}\sup_{[.]}|\delta_i(.)|}{1\text{-}\tau\sup_{[.]}|\delta_n(.)|}\frac{d^+}{dt^+}\int_{-\tau}^{0}\int_{t+\theta}^{t}|\dot z_i(\upsilon)|\,d\upsilon d\theta \tag{25}$$

because

$$\frac{\sum_{i=1}^{n}\sup_{[.]}|\delta_i(.)|}{1-\tau\sup_{[.]}|\delta_n(.)|}\frac{d^+}{dt^+}\int_{-\tau}^{0}\int_{t+\theta}^{t}|\dot z_i(\vartheta)|\,d\vartheta d\theta=\frac{\sum_{i=1}^{n}\sup_{[.]}|\delta_i(.)|}{1\text{-}\tau\sup_{[.]}(|\delta_n(.)|)}\left(\tau|\dot z_i(t)|-\int_{t-\tau}^{t}|\dot z_i(\vartheta)|d\vartheta\right)$$

and

$$\frac{d^+|z_n(t)|}{dt^+}\le\left(\gamma_n(.)+\delta_n(.)\right)|z_n(t)|+\sum_{i=1}^{n\text{-}1}|\gamma_i(.)+\delta_i(.)||z_i(t)|+\sum_{i=1}^{n}\sup_{[.]}|\delta_i(.)|\int_{t\text{-}\tau}^{t}|\dot z_i(\theta)|d\theta$$

Finally, it is easy to see that equation (Eq. (25)) can be overvalued by the following one

$$\frac{d^+p_n(z)}{dt^+}\le\sum_{i=1}^{n}\tilde\gamma_i(.)|z_i|$$

Then we obtain the following inequality

$$D^+V(t)<\langle\tilde F(.)|z(t)|,w\rangle \tag{26}$$

where $|z(t)|=(\,|z_1(t)|\quad\ldots\quad|z_n(t)|\,)'$, and

$$\tilde F(.)=\begin{bmatrix}\alpha_1 & & & & |\beta_1|\\ & \alpha_2 & & & |\beta_2|\\ & & \ddots & & \vdots\\ & & & \alpha_{n\text{-}1} & |\beta_{n\text{-}1}|\\ \tilde\gamma_1(.) & \tilde\gamma_2(.) & \ldots & \tilde\gamma_{n\text{-}1}(.) & \tilde\gamma_n(.)\end{bmatrix} \tag{27}$$

Because the nonlinear elements of $\tilde F(.)$ are isolated in the last row, the eigenvector $v(t,x_t,\varpi)$ relative to the eigenvalue λ_m is constant [34, 35], where λ_m is such that $\text{Re}(\lambda_m)=\max_i\{\text{Re}(\lambda_i),\lambda_i\in\lambda(\tilde F(.))\}$. Then, in order to have $D^+V(t)<0$, it is sufficient to have $\tilde F(.)$ as the opposite of an M-matrix. Indeed, according to properties of M-matrices, we have $\forall\sigma \in R_{+}^{*n}, \exists w \in R_{+}^{*n}$ such that $-\left(\tilde{F}'(.)\right)^{-1}\sigma = w$. This enables us to write the following equation

$$D^{+}V(t) < \left\langle \left(\tilde{F}(.)|z(t)|\right)', w \right\rangle = \left\langle |z(t)|', \tilde{F}'(.)w \right\rangle = \left\langle |z(t)|', -\sigma \right\rangle = -\sum_{i=1}^{n} \sigma_i |z_i(t)| < 0 \tag{28}$$

This completes the proof of theorem.

Corollary 2.1. The system (Eq. (1)) is asymptotically stable, if there exist distinct parameters $\alpha_i < 0, i = 1, \ldots, n\text{-}1,$ such that the following condition:

$$\mu(.) + 2\tau\nu(.) - \xi(.) < 0 \tag{29}$$

is satisfied.

where:

$$\begin{cases} \mu(.) = \gamma_n(.) + \delta_n(.) + \tau \sup_{[.]} |\delta_n(.)|\left(\left|\gamma_n(.) + \delta_n(.)\right| - \left(\gamma_n(.) + \delta_n(.)\right)\right) \\ \nu(.) = \sum_{i=1}^{n-1} \left|\beta_i\right| \sup_{[.]} |\delta_i(.)| \\ \xi(.) = \sum_{i=1}^{n-1} \dfrac{\left|\gamma_i(.) + \delta_i(.)\right| \left|\beta_i\right|}{\alpha_i} + \end{cases} \tag{30}$$

Proof:

Basing on definition 1 and definition 2, the choice of $\alpha_k < 0$, $k = 1, \ldots, n\text{-}1$, $\alpha_i \neq \alpha_j$ for $i \neq j$, the condition of signs on the principal minors is as follows

$$\det\begin{pmatrix} -\alpha_1 & & 0 \\ & \ddots & \\ 0 & & -\alpha_i \end{pmatrix} > 0 \;, \quad (i = 1, 2, 3, \ldots, n\text{-}1) \tag{31}$$

and

$$\det\left(-\tilde{F}(.)\right) = -\left(\tilde{\gamma}_n(.) - \sum_{i=1}^{n-1} \frac{\tilde{\gamma}_i(.)\left|\beta_i\right|}{\alpha_i}\right) \prod_{i=1}^{n-1} (-\alpha_i) > 0 \tag{32}$$

which yields to the following condition

$$\tilde{\gamma}_n(.) - \sum_{i=1}^{n-1} \frac{\tilde{\gamma}_i(.)\left|\beta_i\right|}{\alpha_i} < 0 \tag{33}$$

Replacing each term in (Eq. (33)) of by its expression we get

$$\tilde{\gamma}_n(.)-\sum_{i=1}^{n-1}\frac{\tilde{\gamma}_i(.)|\beta_i|}{\alpha_i}:=\gamma_n(.)+\delta_n(.)+\frac{\tau\sup_{[.]}|\delta_n(.)||\gamma_n(.)+\delta_n(.)|}{1-\tau\sup_{[.]}|\delta_n(.)|}+\frac{\tau\sum_{i=1}^{n-1}|\beta_i|\sup_{[.]}|\delta_i(.)|}{1-\tau\sup_{[.]}|\delta_n(.)|}$$

$$-\sum_{i=1}^{n-1}\frac{\left(|\gamma_i(.)+\delta_i(.)|+\tau|\alpha_i|\sup_{[.]}|\delta_i(.)|\right)|\beta_i|}{\left(1-\tau\sup_{[.]}|\delta_n(.)|\right)\alpha_i}$$

$$=\left(1-2\sup_{[.]}|\delta_n(.)|\right)\left(\gamma_n(.)+\delta_n(.)\right)+\tau\sum_{i=1}^{n-1}|\beta_i|\sup_{[.]}|\delta_i(.)|$$

$$-\sum_{i=1}^{n-1}\frac{\left(|\gamma_i(.)+\delta_i(.)|-\tau\alpha_i\sup_{[.]}|\delta_i(.)|\right)|\beta_i|}{\alpha_i}$$

which can be re-written as:

$$\begin{aligned}&\mu(.)+\tau\ \nu(.)-\sum_{i=1}^{n-1}\frac{|\gamma_i(.)+\delta_i(.)||\beta_i|}{\alpha_i}-\sum_{i=1}^{n-1}\frac{-\tau\alpha_i\sup_{[.]}|\delta_i(.)||\beta_i|}{\alpha_i}\\&=\mu(.)+\tau\ \nu(.)-\xi(.)+\tau\ \nu(.)\\&=\mu(.)+2\tau\ \nu(.)-\xi(.)\end{aligned}$$

where:

$$\begin{cases}\mu(.)=(1-2\tau\sup_{[.]}(\delta_n(.))\left(\gamma_n(.)+\delta_n(.)\right)\\ \nu(.)=\sum_{i=1}^{n-1}|\beta_i|\sup_{[.]}|\delta_i(.)|\\ \xi(.)=\sum_{i=1}^{n-1}\frac{|\gamma_i(.)+\delta_i(.)||\beta_i|}{\alpha_i}\end{cases}$$

which completes the proof.

Remark 2.1. If the couple $\left(p_A(s,.)+p_B(s,.),Q(s)\right)$ forms a positive pair, then there exist distinct negative parameters α_i, i = 1, …, n-1, verifying the condition $\left(\gamma_i(.)+\delta_i(.)\right)\ \beta_i>0$ for i = 1, …, n-1.

Using Theorem 2.1 and Remark 2.1, the obtained supremum of time delay is a function of α_i values, i = 1, …, n-1. As a result, a sufficient condition for asymptotic stability of our system is when values of the time delay are less than this supremum.

Corollary 2.1. If the couple $(D(s,.)+N(s,.),Q(s))$ forms a positive pair and there exist distinct negative parameters α_i, i = 1, …, n-1, such that:

$$2\tau\left(\left(\gamma_n(.)+\delta_n(.)\right)\sup\left[.\right]\left|\delta_n(.)\right|\text{-}\nu(.)\right)+\frac{D(0,.)+N(0,.)}{Q(0)}>0 \tag{34}$$

then the system (Eq. (1)) is asymptotically stable.

Proof.

According to Remark 2.1, we find that

$$\begin{aligned}\gamma_n(.)+\delta_n(.)\text{-}\sum_{j=1}^{n\text{-}1}\frac{\left|\gamma_j(.)+\delta_j(.)\right|\left|\beta_j\right|}{\alpha_j}&=\gamma_n(.)+\delta_n(.)\text{-}\sum_{j=1}^{n\text{-}1}\frac{\left(\gamma_j(.)+\delta_j(.)\right)\beta_j}{\alpha_j}\\&=\text{-}\frac{D(0,.)+N(0,.)}{Q(0)}\end{aligned}$$

The result of Theorem 2.1 becomes

$$2\tau\left(\left(\gamma_n(.)+\delta_n(.)\right)\sup\left[.\right]\left|\delta_n(.)\right|\text{-}\nu(.)\right)+\frac{D(0,.)+N(0,.)}{Q(0)}>0$$

This completes the proof of corollary.

Remark 2.2

- Theorem 2.1 depends on the new basis change, where parameters α_i of the matrix P are arbitrary chosen such that matrix $T(.)$ is the opposite of an M-matrix. The appropriate choice of the set of free parameters α_i makes the given stability conditions satisfied.

- The theorem takes into account the fact that delayed terms may stabilize our system. Theorem 2.1 can hold even if $p_A(s,.)$ is unstable. This is another advantage as the majority of previously published results assume that $p_A(s,.)$ is linear and stable.

4. Application to delayed nonlinear nth order all pole plant

Consider the complex system S given in **Figure 1**.

$D(s)=p_A(s)$ defined by (Eq. (11)) and $p_B(s)=1$, respectively. In this case $\tilde{f}_i(.)$ are constants and g is a function satisfying the finite sector condition.

Let $\widehat{g}$ be a function defined as follows

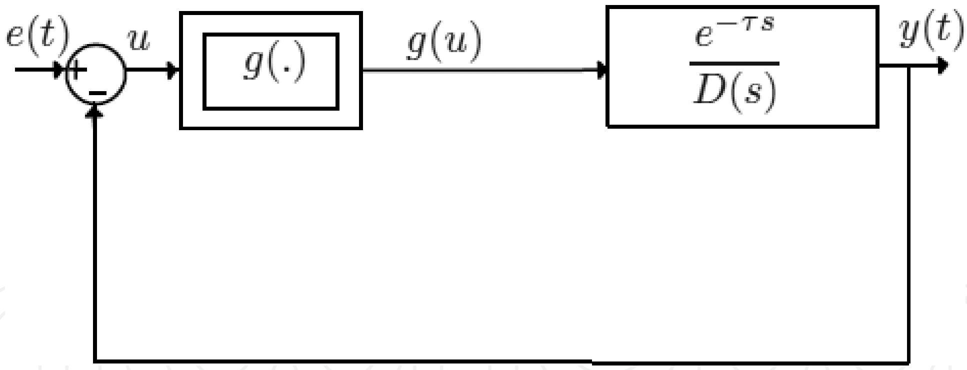

Figure 1. Block diagram of studied system.

$$\widehat{g}(e(\theta), y(\theta)) = \frac{g(e(\theta) - y(\theta))}{e(\theta) - y(\theta)}, e(\theta) \neq y(\theta) \ \ \forall \theta \in [-\tau + \infty[\tag{35}$$

$$\sup_{[.]} |\widehat{g}(e(t), y(t))| = \overline{g} \in R_{+}^{*}.$$

The presence of delay in the system of **Figure 1** makes stability study difficult. The following steps show how to represent this system in the form of system (Eq. (1)). Then we can write

$$y^{(n)}(t) + \sum_{i=0}^{n-1} a_i \frac{d^i y(t)}{dt^i} = -\widehat{g}\,(e(t-\tau), y(t-\tau))y(t-\tau) + \widehat{g}(e(t-\tau), y(t-\tau))e(t-\tau).$$

Using the following notation $\widehat{g}(.) = \widehat{g}(e(t-\tau), bx(t-\tau))$, therefore

$$y^{(n)}(t) + \sum_{i=0}^{n-1} a_i y^{(i)}(t) + \widehat{g}\,(.)y(t-\tau) = \widehat{g}(.)e(t-\tau). \tag{36}$$

It is clear that system (Eq. (36)) is equivalent to system (Eq. (1)) in the special cases $e(\theta) = 0$ and $e(\theta) = -Kx(\theta)$, $x(t) = \left(y(t), \dot{y}(t), \dots, y^{(n)}(t)\right)'$, $\forall\ \theta \in [-\tau + \infty[$. We will now consider each case separately.

4.1. Case $e(t) = 0$

In case, $e(t) = 0 \ \ \forall t \in [-\tau + \infty[$, the description of the system becomes

$$y^{(n)}(t) + \sum_{i=0}^{n-1} a_i y^{(i)}(t) + \widehat{g}(.)y(t-\tau) = 0.$$

This is a special representation of system (Eq. (1)) where $\tilde{f}_i(.) = a_i$, $\tilde{g}_1(.) = \widehat{g}(.)$ $\tilde{g}_i(.) = 0$ $\forall$ $i = 2, \dots, n-1$, $D(s,.) = D(s)$, $N(s,.) = \widehat{g}(.)$, $\gamma_n(.) = \gamma_n = -a_{n-1} - \sum_{i=1}^{n-1} \alpha_i$ and $\delta_n(.) = 0$.

A sufficient stability condition for this system is given in the following proposition.

Proposition 4.1. If there exist distinct $\alpha_i < 0$ $i = 1, \ldots, n-1$, such that the following conditions

$$\begin{cases} \gamma_n < 0 \\ \mu_1(.) + 2\tau\nu_1(.) - \xi_1(.) < 0 \end{cases} \tag{37}$$

where

$$\begin{cases} \mu_1(.) = \gamma_n \\ \nu_1(.) = \overline{g} \\ \xi_1(.) = \dfrac{|D(\alpha_1) + \widehat{g}(.)||\beta_1|}{\alpha_1} + \displaystyle\sum_{i=2}^{n-1} \dfrac{|D(\alpha_i)||\beta_i|}{\alpha_i} \end{cases} \tag{38}$$

are satisfied. Then the system S is asymptotically stable.

Suppose that $D(s)$ admits n distinct real roots p_i, $i = 1, \ldots, n$ among which there are $n-1$ negative ones. By using the fact that $a_{n-1} = -\sum_{i=1}^{n} p_i$, then the choice $\alpha_i = p_i$, $\forall i = 1, .., n-2$ and $\alpha_{n-1} = p_{n-1} + \varepsilon$ permit us to write $\gamma_n = -a_{n-1} - \sum_{i=1}^{n-1} p_i = p_n - \varepsilon$. In this case the last proposition becomes.

Proposition 4.2. If $D(s)$ admits $n-1$ distinct real negative roots such that the following conditions

$$\begin{cases} p_n - \varepsilon < 0 \\ \mu_2(.) + 2\tau\nu_2(.) - \xi_2(.) < 0 \end{cases} \tag{39}$$

are satisfied, where

$$\begin{cases} \mu_2(.) = p_n - \varepsilon \\ \nu_2(.) = \overline{g} \\ \xi_2(.) = \dfrac{|\widehat{g}(.)||\beta_1|}{\alpha_1} + \dfrac{|D(\alpha_{n-1})||\beta_{n-1}|}{\alpha_{n-1}} \end{cases} \tag{40}$$

then the system S is asymptotically stable.

4.2. Case $e(t) = -Kx(t)$

In this case, take $e(t) = -Kx(t)$ with $K = (k_0, k_1, \ldots, k_{n-1})$, then the obtained system has the same form as (Eq. (1)), with $\widehat{g}_1^K(.) = \widehat{g}^K(.)(k_0 + 1)$ and $\widehat{g}_i^K(.) = \widehat{g}^K(.)k_{i-1}$, $i = 2, \ldots, n$.

The stabilizing values of K can be obtained by making the following changes:

$$\gamma_n = -a_{n-1} - \sum_{i=1}^{n-1} \alpha_i, \delta_n^K(.) = -\widehat{g}^K(.)k_{n-1}, \nu_1^K(.) = \overline{g}^K \sum_{i=1}^{n-1} \left| \tilde{N}(\alpha_i) \right| \text{ where } \overline{g}^K = \sup_{[.]} |\widehat{g}^K(.)|$$

and $\tilde{N}(\alpha) = (1 + k_0) + \sum_{i=1}^{n-1} (b_i + k_i)\alpha^i$.

Proposition 4.3. If there exist distinct $\alpha_i < 0$, $i = 1, \dots, n-1$, such that the following conditions

$$\begin{cases} \gamma_n - \widehat{g}K(.)k_{n-1} < 0 \\ \tau < \dfrac{1}{2\overline{g}^K|k_{n-1}|} \\ \mu_1^K(.) + 2\tau\nu_1^K(.) - \xi_1^k(.) < 0 \end{cases} \tag{41}$$

where

$$\begin{cases} \mu_1^K(.) = \left(1 - 2\overline{g}^K\tau|k_{n-1}|\right)\left(\gamma_n + \delta_n^K(.)\right) \\ \nu_1^K(.) = \overline{g}^K \displaystyle\sum_{i=1}^{n-1} |\beta_i||\tilde{N}(\alpha_i)| \\ \xi_1^K(.) = \displaystyle\sum_{i=1}^{n-1} |D(\alpha_i) + \widehat{g}^K(.)\dfrac{\tilde{N}(\alpha_i)||\beta_i|}{\alpha_i} \end{cases} \tag{42}$$

are satisfied. Then the system S is asymptotically stable.

By a special choice of K the result of proposition 3.3 can be simplified. In fact, if the conditions of this proposition are verified we can choose the vector K such that $D(p_i) = \tilde{N}(p_i)$. In this case we obtain $D(p_i) = \tilde{N}(p_i) = 0$, $\forall\, i = 1, \dots, n-1$ and $\nu_1(.) = \xi_1(.) = 0$ which yields the following new proposition.

Proposition 4.4. If $D(s)$ admits $n - 1$ distinct real negative roots p_i such that the following conditions are satisfied.

$$\begin{cases} \gamma_n - \widehat{g}^K(.)k_{n-1} < 0 \\ \tau < \dfrac{1}{2\overline{g}^K|k_{n-1}|} \\ \mu_1^K(.) < 0 \end{cases} \tag{43}$$

Then the system S is asymptotically stable.

5. Illustrative example

Let us study the same example in [34] defined by **Figure 2** which refer to the dynamics of a time-delayed DC motor speed control system with nonlinear gain, Block diagram of time-delayed DC motor speed control system with nonlinear gain.

where:

- $p_1 = \frac{1}{T_e}$ and $p_2 = \frac{1}{T_m}$ where T_e and T_m are, respectively, electrical constant and mechanical constant.
- τ_f presents the feedback delay between the output and the controller. This delay represents the measurement and communication delays (sensor-to-controller delay).
- τ_c the controller processing and communication delay (controller-to-actuator delay) is placed in the feedforward part between the controller and the DC motor.
- $g(.) : R \to R$ is a function that represents a nonlinear gain.

The process of **Figure 2** can also be modeled by **Figure 1**, where $\tau = \tau_f + \tau_c$.

It is clear that model of **Figure 2** is a particular form of delayed Lurie system in the case where $D(s) = s(s+p_1)(s+p_2) = s^3 + (p_1+p_2)s^2 + p_1p_2s$ and $N(s) = 1$. Thereafter, applying the result of Theorem 2.1, a stability condition of the system is that the matrix $T(.)$ given by:

$$T(.) = \begin{pmatrix} \alpha_1 & 0 & |(\alpha_1 - \alpha_2)^{-1}| \\ 0 & \alpha_2 & |(\alpha_2 - \alpha_1)^{-1}| \\ t_1(.) & t_2(.) & t_3(.) \end{pmatrix}$$

where:

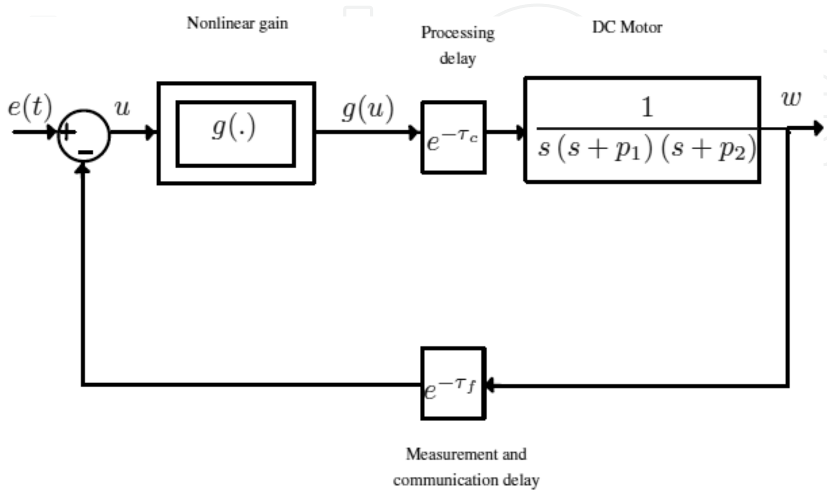

Figure 2. Delayed nonlinear model of DC motor speed control.

$$t_1(.) = |\gamma_1 + \widehat{g}(.)| + \tau|\alpha_1|\overline{g}, \quad t_2(.) = |\gamma_2|, \quad t_3(.) = \gamma_3 + \tau|\beta_1|\overline{g}$$

must be the opposite of an M-matrix. By choosing α_i, $i = 1, 2$, negative real and distinct, we get the following stability condition:

$$\gamma_3 + 2\tau|\beta_1|\overline{g} - \frac{|\beta_1||\gamma_1 + \widehat{g}(.)|}{\alpha_1} - \frac{|\beta_2||\gamma_2|}{\alpha_2} < 0$$

For the particular choice of $\alpha_1 = -p_1$ and $\alpha_2 = -p_2 + \varepsilon$, $\varepsilon > 0$.

yields $|\beta_1| = |\beta_2| = |(\varepsilon + p_1 - p_2)^{-1}|$ and we obtain the new stability condition:

$$2\tau\overline{g} + |p_1|^{-1}|\widehat{g}(.)| + |\alpha_2|^{-1}|D(\alpha_2)| < \varepsilon|\varepsilon + p_1 - p_2|$$

Assume that we have this inequality $\overline{g} < |D(\alpha_2)|$, we can find from \ref.{ops} the stabilizing delay given by the following condition:

$$\tau < \frac{1}{2}\left(\frac{\varepsilon|\varepsilon + p_1 - p_2|}{|D(\alpha_2)|} - |p_1|^{-1}| - |\alpha_2|^{-1}\right) \tag{44}$$

By applying the control $e(t) = -Kx(t)$ with $K = (k_0, k_1, k_2)$, we can determine the stabilizing values of K can be obtained by making the following changes:

$$\gamma_3 = -(p_1 + p_2) - \sum_{i=1}^{2}\alpha_i, \delta_1^K(.) = -\widehat{g}^K(.)(k_0 + 1), \delta_i^K(.) = -\widehat{g}^K(.)k_{i-1}, \quad i = 2, 3$$

$$v_1^K(.) = \overline{g}^K \sum_{i=1}^{2} |\beta_i||\tilde{N}(\alpha_i)| \text{ where } \overline{g}^K = \sup_{[.]} |\widehat{g}^K(.)| \text{ and } \tilde{N}(\alpha) = 1 + k_0 + \sum_{i=1}^{2} k_i\alpha^i$$

If we choose $\alpha_i < 0$, $i = 1, 2$, such that the following conditions

$$D(\alpha_i) = \tilde{N}(\alpha_i) = 0, \forall, i = 1, 2$$

we get

$$\frac{1 + k_0}{k_2} = p_1 + p_2, \quad \frac{k_1}{k_2} = p_1p_2$$

and from proposition 3.3 the stabilizing gain values satisfying the following relations:

$$\begin{cases} 0 - \overline{g}^K(.)k_2 < 0 \\ |k_2| < \dfrac{1}{2\tau\overline{g}^K} \end{cases} \tag{45}$$

Finally we find the domain of stabilizing k_0, k_1, k_2 as follows:

$$\begin{cases} 0 < k_2 < \dfrac{1}{2\tau\overline{g}^K} \\ k_1 = p_1 p_2 k_2 \\ \text{and} \\ k_0 = (p_1 + p_2)k_2 - 1 \end{cases} \tag{46}$$

6. Conclusion

In this chapter, a joined and structured procedure for the analysis of delayed nonlinear systems is proven. A complete structured analysis formulation based on the comparison principle and vector norms for the asymptotic stability is presented. Based on the arrow form matrices, and by taking into account for the system parameters, a new stability conditions are synthesized, leading to a practical estimation of the stability domain. In order to highlight the feasibility and the main capabilities of the proposed approach, the case of nonlinear nth order all pole plant and delayed Lur'e Postnikov systems are presented and discussed. In addition, the simplicity of the application of these criteria is demonstrated on model of time-delayed DC motor speed control.

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