Chapter 14

Maxwell-Fredholm Equations

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Abstract

With the aim to increase the knowledge of the broadcasting properties under circumstances like time reversal, change on refractive index, presence of random obstacles, and so on, we developed new type of hybrid equations named Maxwell-Fredholm equations. These new equations fuse the Maxwell equations' description of the electromagnetic fields with the Fourier transform of the Fredholm integral equations appropriate for a broadcasting process. Now we have a new tool, which resembles the Maxwell equations but including contributions from the Fredholm formulation like the resonant behaviour of the left-hand material conditions. To illustrate the usefulness of this new class of equations, we include an academic example that shows the deflection of an electromagnetic beam traveling among a highly anisotropic and left-handed behaviour media.

Keywords: Maxwell equations, Fredholm equations, left-hand materials conditions, evanescent waves, broadcasting

1. Introduction

In the previous works [1–3], we arrived at the conclusion that if there is a sudden change of the refractive index from positive to negative when we have a broadcasting procedure, a better way to take this phenomenon into account is to formulate the problem through an integral equation. Indeed, we have described the way in which this change is triggered and how the so-called evanescent waves [4–6] are liberated when their confinement is broken. However, the question remains valid about the visualization from the point of view of the traditional formulation of the Maxwell equations. Indeed, there is a qualitative difference between these two points of views because the integral formulation brings their own boundary conditions immersed on their kernel; meanwhile, Maxwell equations need the imposing of the particular boundary conditions directly. Now we have a new goal, that is to create a set of equations with the following properties: first, that preserve the advantage of the integral formulation when studying the changes in the refractive index of the propagation media and second, that we can impose some kind of boundary and initial conditions as in the pure Maxwell equations formulation. Then, we introduce in this chapter Maxwell-Fredholm equations with the aim to increase the knowledge of the broadcasting properties under circumstances like time reversal, change on refractive index, presence of random obstacles, and so on; we do not only obtain new kind of hybrid equations with these properties but we can apply to new kind of electromagnetic problems involving special propagating and broadcasting characteristics which occurs when an electromagnetic beam is strongly deflected in a media with a very high anisotropy and a negative refraction index. The first step to get the hybrid equations is to leave the time domain and transform our generalized Fredholm **integral equations** [7–10] into a system of algebraic equations through a Fourier transform. We must emphasize that the resonant behaviour associated with the transformation of the evanescent waves will be considered when we build the new equations, specifically when we establish the fact that we take a homogeneous equation and we employ the free Green function. In the other hand, we underline the role played of the resonance properties like orthogonally [11–14] and recall how we can overcome the fact that their frequencies cannot be used directly as a mathematical base, so we build the named information packs. After we obtain the Maxwell-Fredholm equations, we apply them to the problem of the beam bending inside a left-hand material. As a part of our procedure, we first show the equivalence of the two formulations that is Maxwell differential equations and generalized Fredholm integral equations. Then, we properly obtain the hybrid equations and apply them to the specific problem mentioned above, a very strong deflection of an electromagnetic beam.

2. Resonances and the Fredholm's eigenvalue

First of all, we recall the generalized homogeneous Fredholm's equations (GHFE) [7–10] taken from a theorem we have proved [7]:

$$f^m(\mathbf{r},\omega) = \eta(\omega)\int_0^\infty K_n^{m(\circ)}\left(\omega;\mathbf{r},\mathbf{r}'\right)f^n\left(\mathbf{r}',\omega\right)dr' \tag{1}$$

Now, we have also proved that when the physical system can be considered as a discrete one, Eq. (1) can be written as:

$$f^m(\mathbf{r}_i,\omega) = \eta(\omega)\mathbf{A}_{i,j}^{m,n}\mathbf{G}_n^{m(\circ)}(\omega;\mathbf{r}_i,\mathbf{r}_j)f^m(\mathbf{r}_i,\omega) \tag{2}$$

By supposing that we can take a diagonal kernel $K_n^{m(\circ)}\left(\omega;\mathbf{r},\mathbf{r}'\right)$, and that the interaction matrix is:

$$\mathbf{A}_{i,j}^{m,n} = \delta(\mathbf{r}_i - \mathbf{r}_j) \tag{3}$$

Now, we take $f^m(\mathbf{r},\omega) = E^m(\mathbf{r},\omega)$ in Eq. (1) and by applying the differential operator $rot = \nabla\times$ over the non-apostrophe variable $\mathbf{r}$, and we obtain the following equation:

$$\nabla \times \mathbf{E}(\mathbf{r},\omega) = \eta(\omega)\int_0^\infty \nabla \times \mathbf{K}^{(\circ)}\left(\omega;\mathbf{r},\mathbf{r}'\right)\mathbf{E}\left(\mathbf{r}',\omega\right)dr' \tag{4}$$

Now, we use Maxwell equation:

$$rot\mathbf{E}(\mathbf{r},\omega) = -i\omega\boldsymbol{\mu}\mathbf{H}(\mathbf{r},\omega) \tag{5}$$

and, in order to transform Eq. (4), we use the relation:

$$\nabla \times \mathbf{E}(\mathbf{r},\omega)\delta(\mathbf{r}) = \mathbf{D}(\mathbf{r}) \times \mathbf{E}(\mathbf{r},\omega) - i\omega\boldsymbol{\mu}\mathbf{H}(\mathbf{r},\omega) \tag{6}$$

In Eq. (6), the vector $\mathbf{D}(\mathbf{r})$ is defined by:

$$\mathbf{D}(\mathbf{r}) = \delta(y)\delta(z)\widehat{\mathbf{i}} + \delta(x)\delta(z)\widehat{\mathbf{j}} + \delta(x)\delta(y)\widehat{\mathbf{k}} \tag{7}$$

By substituting Eqs. (5)–(7) in Eq. (4), we have after using the derivative properties of the delta function:

$$-i\frac{\eta(\omega)}{\omega}\int_0^\infty \mathbf{D}(\mathbf{r}) \times \mathbf{E}\left(\mathbf{r}',\omega\right)dr' + \eta(\omega)\int_0^\infty \mathbf{K}^{(\circ)}\left(\omega,\mathbf{r}'\right)\boldsymbol{\mu}\mathbf{H}\left(\mathbf{r}',\omega\right)dr' \tag{8}$$

In Eq. (8), the first term seems to be the current of magnetic monopoles, that is, a source term, so must be zero, and the final equation is:

$$\mathbf{H}(\mathbf{r},\omega) = \eta(\omega)\int_0^\infty \mathbf{K}^{(\circ)}\left(\omega,\mathbf{r}'\right)\mathbf{H}\left(\mathbf{r}',\omega\right)dr' \tag{9}$$

So, we can see that it is equivalent to use Maxwell equations or the generalized Fredholm equations. We have shown that the generalized homogeneous fredholm equation (GHFE) can be written in the following compact algebraic form:

$$\mathbf{E}_e^m(\omega) = \left[\eta_e(\omega)\mathbf{G}^{(\circ)}(\omega)\mathbf{A}\right]_n^m \mathbf{E}_e^n(\omega) \tag{10}$$

$$\mathbf{H}_e^m(\omega) = \left[\eta_e(\omega)G^{(\circ)}(\omega)A\right]_n^m \mathbf{H}_e^n(\omega) \tag{11}$$

We can apply operator $rot = \nabla\times$ to Eqs. (10) and (11) and by using Maxwell equations obtaining in terms of the kernels the equations we name Maxwell-Fredholm:

$$rot\mathbf{E}_e(\omega) = -i\omega\boldsymbol{\mu}e^{-ih(\omega_e)}\mathbf{K}^{(\circ)}(\omega)\mathbf{H}_e(\omega) \tag{12}$$

$$rot\mathbf{H}_e(\omega) = i\omega\boldsymbol{\varepsilon}e^{ih(\omega_e)}\mathbf{K}^{(\circ)}(\omega)\mathbf{E}_e(\omega) \tag{13}$$

$$\eta_e(\omega) = e^{ih(\omega_e)} \tag{14}$$

In Eqs. (12) and (13), we must remember that the left-hand side is computed at the final sites, meanwhile, the right-hand term is computed at the initial sites.

Also, we remember that Eq. (1) can be written for the electric field $\mathbf{E}(\mathbf{r}, \omega)$ as:

$$\mathbf{E}(\mathbf{r}, \omega) = \eta(\omega)\int_0^\infty \mathbf{K}^{(\circ)}\left(\omega, \mathbf{r}'\right)\mathbf{E}\left(\mathbf{r}', \omega\right)dr' \tag{15}$$

which is a form identical to Eq. (9).

At this point, it is important to emphasize that Eqs. (9) and (15) are homogeneous generalized integral equations that properly allow us to follow the behaviour of a left-hand material media, that is a media with a negative refractive index; and the Maxwell-Fredholm Eqs. (12) and (13) also have a structure guided for the same purpose.

3. The role of orthogonality properties

There is a very important property of the resonant solutions for the generalized Fredholm equations, that is, the orthogonality between different resonances [7]. Indeed, we are giving an alternative point of view as the established in the work of Li et al. [5] or by Kong et al. [6], concerning the physical interpretation of a resonance. If the resonances would constitute a band of resonant states, we could use these properties directly as a mathematical base to represent any kind of desired broadcasting signal, but the set of resonant solutions is made of punctual frequencies that only permit a pedestrian kind of information transmission, perhaps like a telegraph mode in which even a single frequency can be used as a succession of signal, non-signal intervals. This last kind of information is very far from the goals of an efficient broadcasting. Nevertheless, if we use some results we have obtained previously like the definition of information packs, we can reach our desired results. Suppose that we want to send a signal represented by the function $S(t)$ and that we know that the propagating media bring us a set of resonances one of which we can name ρ so that the associated resonant frequency will be known as ω_ρ. Now we can project the original signal over a sub-space generated with the aid of Communication theory [15–20] by the rule:

$$S_\rho(t) = \sum_{-\infty}^{\infty} P_{m,\rho}\frac{\sin\left[\pi\left(2\omega_\rho t - m\right)\right]}{\pi\left(2\omega_\rho t - m\right)} \tag{16}$$

In expression (16) the span coefficients are:

$$P_{m,\rho} = S\left(\frac{m}{2\omega_\rho}\right) \tag{17}$$

So, we have a collection of signals that are projections of the original $S(t)$. Then, we can emit simultaneously the different projections $S_\rho(t)$ and when they arrive to their destination, we can decode and rebuild the original $S(t)$. There is a limitation that comes also from communication theory about the frequencies appeared in every pack, that is, these frequencies cannot be major than the respective resonant frequency ω_ρ. During the broadcasting, the orthogonality properties of the resonances and the structure of the information packs guarantee that there is no interference between the different projections.

In this chapter, we do not show how we can apply Eq. (16) explicitly, but we suppose that the signal we enter through the initial electric and magnetic fields comes from the building of information packs. In this manner, we are using resonances in two different ways, first by using the Maxwell-Fredholm equations created for an explicitly homogeneous situation and second, by the projection of the original signal over the sub-spaces generated with the rules (16) and (17).

4. Academic example

We have obtained a new type of algebraic equations named the Maxwell-Fredholm equations as our principal goal, in which we incorporate the resonant behaviour and we can apply them in the following academic special case, in which we suppose the media has very large left-handed material properties also with a very large inhomogeneity that force a light beam to follows a circular trajectory and we describe the phenomena with the aid of parabolic coordinates. We also suppose that we know the tensor $\boldsymbol{\varepsilon}$ [21, 22] in an appropriate form that directly operates over a space of parabolic coordinates (ξ, η, φ). The specific relation between the new and old systems is depicted in the following equations:

$$\widehat{\mathbf{i}} = \frac{\xi}{\sqrt{2}|\xi|}\left(\widehat{\boldsymbol{\xi}}_0 + \widehat{\boldsymbol{\eta}}_0\right) \tag{18}$$

$$\widehat{\mathbf{j}} = \frac{\xi}{|\xi|\sqrt{4+\pi^2}}\left(\sqrt{2}\widehat{\boldsymbol{\xi}}_0 + \sqrt{2}\widehat{\boldsymbol{\eta}}_0 + \pi\widehat{\boldsymbol{\varphi}}_0\right) \tag{19}$$

$$\widehat{\mathbf{k}} = 2\left(\xi\widehat{\boldsymbol{\xi}}_0 - \sqrt{\frac{1}{4} - \xi^2}\,\widehat{\boldsymbol{\eta}}_0\right) \tag{20}$$

We underline that the vectors $\widehat{\boldsymbol{\xi}}_0$, $\widehat{\boldsymbol{\eta}}_0$ and $\widehat{\boldsymbol{\varphi}}_0$ are functions of the coordinates (ξ, η, φ).

Also we have:

$$x = \sqrt{\xi\eta}\cos(\varphi) \tag{21}$$

$$y = \sqrt{\xi\eta}\sin(\varphi) \tag{22}$$

$$z = \frac{1}{2}(\xi - \eta) \tag{23}$$

And, the rotational of a vector $\mathbf{V}$ in parabolic coordinates is:

$$\nabla \times \mathbf{V}(\xi,\eta,\varphi) =$$

$$\frac{1}{(\xi^2+\eta^2)\xi\eta}\left\{(\xi^2+\eta^2)^{1/2}\widehat{\boldsymbol{\xi}}_0\left[\xi V_\varphi + \xi\eta\frac{\partial V_\varphi}{\partial\eta} - (\xi^2+\eta^2)^{1/2}\frac{\partial V_\eta}{\partial\varphi}\right]\right.$$

$$-(\xi^2+\eta^2)^{1/2}\widehat{\boldsymbol{\eta}}_0\left[\eta V_\varphi + \xi\eta\frac{\partial V_\varphi}{\partial\xi} - (\xi^2+\eta^2)^{1/2}\frac{\partial V_\xi}{\partial\varphi}\right)]$$

$$\left.+\xi\eta\widehat{\boldsymbol{\varphi}}_0\left[\xi(\xi^2+\eta^2)^{-1/2}V_\eta + (\xi^2+\eta^2)^{1/2}\frac{\partial V_\eta}{\partial\xi} - \eta(\xi^2+\eta^2)^{-1/2}V_\xi - (\xi^2+\eta^2)^{1/2}\frac{\partial V_\xi}{\partial\eta}\right]\right\} \tag{24}$$

Let us take Eq. (13) and make in the left-hand term:

$$rot\mathbf{H}_e^{'}(\omega) = i\omega\boldsymbol{\varepsilon}^{'}\mathbf{E}_e^{'}(\omega) \tag{25}$$

So we arrive to the equation:

$$i\omega\mathbf{E}_e^{'}(\omega) = i\omega\boldsymbol{\varepsilon}e^{ih(\omega_e)}\mathbf{K}^{(\circ)}(\omega)\mathbf{E}_e(\omega) \tag{26}$$

Now, we suppose that the electric field points towards the unitary vector $\widehat{\boldsymbol{\xi}}_0$ that implies that Eq. (26) becomes (see **Figure 1**):

$$\mathbf{E}_e^{'}(\omega) = \boldsymbol{\varepsilon}e^{ih(\omega_e)}\mathbf{K}^{(\circ)}(\omega)\widehat{\boldsymbol{\xi}}_0 E_\xi(\omega) \tag{27}$$

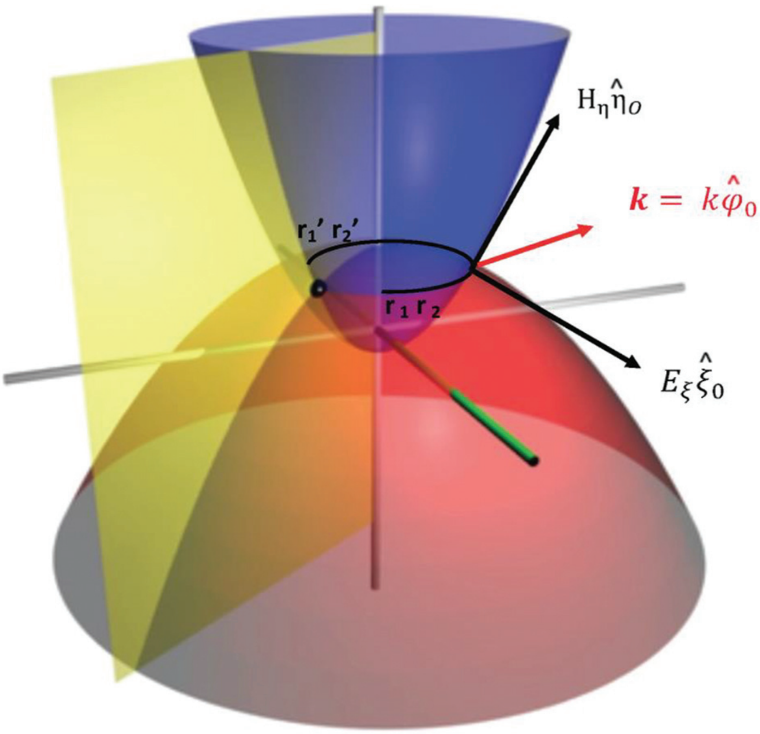

Figure 1. Electric and magnetic fields in parabolic coordinates and the beam trajectory.

Defining the permittivity tensor:

$$\boldsymbol{\varepsilon} \tag{28}$$

In principle, there is a dependence on the frequency ω but, for convenience, we bequeath this to the kernel, in order to easy look the contribution of the tensor $\boldsymbol{\varepsilon}$, which operates on the column vectors in the (ξ, η, φ) space bending the beam trajectory (see **Figure 1**):

$$\boldsymbol{\varepsilon} = \varepsilon \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{29}$$

In terms of this last tensor, Eq. (26) can be written as:

$$\mathbf{E}_e^{'}(\omega) = e^{ih(\omega_e)} \boldsymbol{\varepsilon} \mathbf{K}^{(\circ)}(\omega) \widehat{\boldsymbol{\xi}}_0 E_\xi(\omega) \tag{30}$$

For simplicity, we propose that we have only two punctual emitters with the kernel given by:

$$\mathbf{K}^{(\circ)} = \begin{bmatrix} {}^1\mathbf{K}(\circ) & \mathbf{0} \\ \mathbf{0} & {}^2\mathbf{K}(\circ) \end{bmatrix} \tag{31}$$

On matrix (23), the elements are:

$${}^{1,2}\mathbf{K}^{(\circ)} = \begin{bmatrix} \dfrac{\sin\left[(\omega - \omega_p)\delta\right]}{(\omega - \omega_p)\delta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{32}$$

As we have said, we suppose that the electric fields at the two initial points only have a ξ component, for example:

$$E_\xi(\mathbf{r}_1) = E_\xi(\mathbf{r}_2) = E_0 \cos(\omega_0) \tag{33}$$

In Eq. (25), we impose the condition that the $rot\mathbf{H}_e^{'}(\omega)$ does not have φ or η components. Also:

$$H_\xi{}' = 0 \tag{34}$$

and

$$H_\varphi{}' = 0 \tag{35}$$

which means that $H_\eta{}'$ satisfy the partial differential equation:

$$\frac{\partial H_\eta{}'}{\partial \xi} = -\frac{\xi}{\xi^2 + \eta^2} H_\eta{}' \tag{36}$$

and then

$$H_{\eta}{}' = C_0 \frac{\varphi}{\xi\eta} e^{-\frac{1}{2}\ln\left(\xi^2+\eta^2\right)} \tag{37}$$

In (29), C_0 is a constant determined by the field at the starting point in Eq. (25).

The field $\mathbf{H}_e{}'$ has the components:

$$H_{\eta}{}' = C_0 \frac{\varphi}{\xi\eta} e^{-\frac{1}{2}\ln\left(\xi^2+\eta^2\right)} \tag{38}$$

$$H_{\xi}{}' = 0 \tag{39}$$

$$H_{\varphi}{}' = 0 \tag{40}$$

Now, because of Eqs. (31)–(33):

$$\mathbf{E}_e'(\omega) = \boldsymbol{\varepsilon} e^{ih(\omega_e)} \begin{bmatrix} {}^1\mathbf{K}(\circ) & 0 \\ 0 & {}^2\mathbf{K}(\circ) \end{bmatrix} \widehat{\boldsymbol{\xi}}_0 E_{\xi}(\omega) \tag{41}$$

So, the electric field at the final points is:

$$E_{\xi}{}'(\mathbf{r}_{1,2}) = \frac{1}{\varepsilon'} C_0 \frac{\varphi'}{\xi_{1,2}'\eta_{1,2}'} e^{-\frac{1}{2}\ln\left(\xi_{1,2}'{}^2+\eta_{1,2}'{}^2\right)} = \varepsilon e^{ih(\omega_e)} \frac{\sin(\omega-\omega_p)\delta}{(\omega-\omega_p)\delta} \cos(\omega_0) E_0 \tag{42}$$

The value of E_0 is really a function of (ξ, η, φ) that is, we know that:

$$E_0\left(\xi_{1,2}, \eta_{1,2}, \varphi_{1,2}\right) = D_0 \frac{\varphi e^{-\frac{1}{2}\ln\left(\xi_{1,2}{}^2+\eta_{1,2}{}^2\right)}}{\xi_{1,2}\eta_{1,2}} \tag{43}$$

and we obtain from (34):

$$\frac{1}{\varepsilon'} C_0 \frac{\varphi'}{\xi_{1,2}'\eta_{1,2}'} e^{-\frac{1}{2}\ln\left(\xi_{1,2}'{}^2+\eta_{1,2}'{}^2\right)} = \varepsilon e^{ih(\omega_e)} \frac{\sin(\omega-\omega_p)\delta}{(\omega-\omega_p)\delta} \cos(\omega_0) D_0 \frac{e^{-\frac{1}{2}\ln\left(\xi_{1,2}{}^2+\eta_{1,2}{}^2\right)}}{\xi\eta} \tag{44}$$

We can see that in parabolic coordinates there is a strong dependence on the specific values of the vectors (ξ, η, φ) and $\left(\xi', \eta', \varphi'\right)$. But we can say that all the initial conditions depend on the non-primed variables and put in a named constant factor E_0:

That is

$$\frac{1}{\varepsilon'} C_0 \frac{\varphi'}{\xi_{1,2}'\eta_{1,2}'} = \varepsilon e^{ih(\omega_e)} \frac{\sin(\omega-\omega_p)\delta}{(\omega-\omega_p)\delta} \cos(\omega_0) D_0 E_0 \tag{45}$$

From (37), we can see that we must calculate C_0 for every selected $\left(\xi', \eta', \varphi'\right)$.

Then the fields can be calculated through the Eqs. (38)–(43) and (45).

5. Conclusions

We have seen how it is possible to use the hybrid Maxwell-Fredholm equations to understand some kind of problems like the bending of a light beam inside a left-hand material described with an extremely deflective tensor $\boldsymbol{\varepsilon}$. So a light beam begin his trajectory measured at the two points $\mathbf{r}_1$, $\mathbf{r}_2$ and we determine with the aid of the Maxwell-Fredholm equations the electromagnetic fields at the final points $\mathbf{r}_1{}'$ and $\mathbf{r}_2{}'$ given by the Eqs. (33)–(45). Explicitly, we have shown how we can point towards the permittivity tensor as responsible for the beam deflection because in the kernel of the Maxwell-Fredholm equations only appear the free Green function $\mathbf{G}^{(\circ)}$. In this chapter, we have showed how we can add several tools to observe the behaviour of general devices and we have obtained that the Maxwell-Fredholm equations results in an appropriate tool in some interesting physical situations as the academic example illustrates. It is shown how we can select an appropriate system of coordinates as in the specific case of parabolic coordinates that we used in the example. Even if we do not establish an explicit link between the building of the named information packs and proper use of the Maxwell-Fredholm equations, we suppose that the initial signal is indeed a mix of different information packs and then it is possible to conclude that implementation of both cause a better achievement of signal transmission. In addition, we can say in advance that in a near future, the Maxwell-Fredholm equations could be used in extremely different problems like the nanofluid flux [23] providing we can establish a left-hand materials propagation condition for the electromagnetic field inside the nanofluid and others like a system of split ring resonators [24].

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