Chapter 9

Quantum Calculus with the Notion $\delta_{\pm}$-Periodicity and Its Applications

Neslihan Nesliye Pelen, Ayşe Feza Güvenilir and Billur Kaymakçalan

Additional information is available at the end of the chapter

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Abstract

The relation between the time scale calculus and quantum calculus and the $\delta_{\pm}$-periodicity in quantum calculus with the notion is considered. As an application, in two-dimensional predator–prey system with Beddington-DeAngelis-type functional response on periodic time scales in shifts is used.

Keywords: predator prey dynamic systems, Beddington-DeAngelis-type functional response, $\delta_{\pm}$-periodic solutions on quantum calculus, periodic time scales in shifts

1. Introduction

The traditional infinitesimal calculus without the limit notion is called calculus without limits or quantum calculus. After the developments in quantum mechanics, q-calculus and h-calculus are defined. In these calculi, h is Planck's constant and q stands for the quantum. These two parameters q and h are related with each other as $q = e^{ih} = e^{2\pi i \tilde{h}}$. This equation $\tilde{h} = \frac{h}{2\pi}$ is the reduced Planck's constant. h-calculus can also be seen as the calculus of the differential equations, and this was first studied by George Boole. Many other scientists also made some studies on h-calculus, and it was shown that it is useful in a number of fields, among them, combinatorics and fluid mechanics. The q-calculus is more useful in quantum mechanics, and it has an intimate connection with commutative relations [1]. In the following, the main notions and its relation to the time scale calculus will be discussed.

In [2], in classical calculus when the equation

$$\frac{f(x) - f(x_0)}{x - x_0}$$

is considered and as x tends to $x_0,$ the differentiation notion is obtained. When the differential equations are considered, the difference of a function is defined as $f(x+1) - f(x)$. In quantum calculus, the q-differential of a function is equal to the following:

$$d_q(f(x)) = f(qx) - f(x)$$

and

$$d_q(x) = qx - x = (q-1)x.$$

Then the q-derivative is defined as follows:

$$\frac{d_q(f(x))}{d_q(x)} = \frac{f(qx) - f(x)}{(q-1)x}.$$

The differentiation in time scale calculus is given in Theorem 1, and if the differentiation notion in this theorem is applied when $\mathbb{T} = q^{\mathbb{N}},$ one can easily see that the same q-derivative is obtained.

As an inverse of q-derivative, one can get q-integral that is also very significant for the structure of this calculus. A function $F(x)$ is a q-antiderivative of $f(x)$ if $D_q F(x) = f(x)$ is satisfied where

$$F(x) = \int f(x) d_q x = (1-q) \sum_{0}^{\infty} x q^j f\left(x q^j\right).$$

This is also called the Jackson integral [3]. When the definition of the antiderivative of a function in time scale calculus is considered, it can be easily seen that when $\mathbb{T} = q^{\mathbb{N}_0},$ these two definitions become equivalent. Therefore, to understand the quantum calculus, it is very important to understand the time scale calculus. In addition to these, the $\delta_{\pm}$-periodicity notion in time scale calculus is defined in Definition 1 in [4] for the application. In this study, by using time scale calculus, the application of $\delta_{\pm}$-periodicity notion of $q^{\mathbb{N}},$ which overlaps with the q-calculus, to a predator–prey system with Beddington-DeAngelis-type functional response is studied.

To understand this application in a much better sense, the following information about the predator–prey dynamic systems is given. Predator–prey equations are also known as the Lotka-Volterra equations. This model was initially proposed by Alfred J. Lotka in the theory of autocatalytic chemical reactions in 1910 [5, 6] which was effectively the logistic Equation [7] and originally derived by Pierre Françis Verhulst [8]. In 1920, Lotka extended this model to “organic systems” by using a plant species and a herbivorous animal species. The findings of this study were published in [9]. In 1925, he obtained the equations to analyze predator–prey interactions in his book on biomathematics [10] arriving at the equations that we know today.

After the development of the equations for predator–prey systems, it becomes important to obtain the type of functional response. The first functional response was proposed by C. S. Holling in [11, 12]. Both the Lotka-Volterra model and Holling's extensions have been used to model the moose and wolf populations in Isle Royale National Park [13]. In addition to these, there are many studies that use the predator–prey dynamic systems with Holling-type functional responses. These studies especially analyze the permanence, stability, periodicity, and such different aspects of these systems. The papers [14], [15, 16] can be some of its examples.

Arditi and Ginzburg made some changes and extension on the functional response of Holling, and this new functional response is known as the ratio-dependent functional response. Also, from this functional response, the semiratio-dependent functional responses are also derived. Again, there are many studies that are about the several structures of the predator–prey dynamic systems such as [14, 17–19], [20, 21].

2. Preliminaries about time scale calculus

The main tool we have used, in this study, is time scale calculus, which was first appeared in 1990 in the thesis of Stephen Hilger [22]. By a time scale, denoted by $\mathbb{T}$, we mean a non-empty closed subset of $\mathbb{R}$. The theory of time scale calculus gives a way to unify continuous and discrete analysis.

The following informations are taken from [14, 23]. The set $\mathbb{T}^{\kappa}$ is defined by $\mathbb{T}^{\kappa} = \mathbb{T}/(\rho(\sup\mathbb{T}), \sup\mathbb{T}]$, and the set $\mathbb{T}_{\kappa}$ is defined by $\mathbb{T}_{\kappa} = \mathbb{T}/[\inf\mathbb{T}, \sigma(\inf\mathbb{T}))$. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) := in(t, \infty)_{\mathbb{T}}, for \;\; t \in \mathbb{T}$. The backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) := \sup(-\infty, t)_{\mathbb{T}}, for \;\; t \in \mathbb{T}$. The forward graininess function $\mu : \mathbb{T} \to \mathbb{R}_0^+$ is defined by $\mu(t) := \sigma(t) - t, for \;\; t \in \mathbb{T}$. The backward graininess function $\nu : \mathbb{T} \to \mathbb{R}_0^+$ is defined by $\nu(t) := t - \rho(t), for \;\; t \in \mathbb{T}$. Here, it is assumed that $\inf 0/ = \sup\mathbb{T}$ and $\sup 0/ = \inf\mathbb{T}$.

For a function $f : \mathbb{T} \to \mathbb{T}$, we define the Δ-derivative of f at $t \in \mathbb{T}^{\kappa}$, denoted by $f^{\Delta}(t)$ for all $\epsilon > 0$. There exists a neighborhood $U \subset \mathbb{T}$ of $t \in \mathbb{T}^{\kappa}$ such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq \epsilon |\sigma(t) - s|$$

for all $s \in U$.

For the same function, the ∇-derivative of f at $t \in \mathbb{T}_{\kappa}$, denoted by $f^{\nabla}(t)$, for all $\epsilon > 0$., is defined. There exists a neighborhood $V \subset \mathbb{T}$ of $t \in \mathbb{T}_{\kappa}$ such that

$$|f(s) - f(\rho(t)) - f^{\nabla}(t)(s - \rho(t))| \leq \epsilon |s - \rho(t)|$$

for all $s \in V$.

A function $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous if it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist at left-dense points in $\mathbb{T}$. The class of real rd-continuous functions defined on a time scale $\mathbb{T}$ is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$. If $f \in C_{rd}(\mathbb{T}, \mathbb{R})$, then there exists a function $F(t)$ such that $F^{\Delta}(t) = f(t)$. The delta integral is defined by $\int_a^b f(x)\Delta x = F(b) - F(a)$.

Theorem 1. [23] *Suppose that $f : \mathbb{T} \to \mathbb{R}$ is a function and $t \in \mathbb{T}^{\kappa}$. Then, we have the following*:

1. *If f is delta differentiable at t, then f is continuous at t.*
2. *If f is continuous at a right scattered t, then f is delta differentiable at t with*

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

3. *If t is right dense, then f is delta differentiable at t if and only if the limit*

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case,

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

4. *If f is delta differentiable at t, then*

$$f^{\sigma}(t) = f(t) + \mu(t) f^{\Delta}(t).$$

Theorem 2. [23] *If $a, \ b, \ c, \ d \in \mathbb{T}, \ \alpha \in \mathbb{R}$, and $f, g : \mathbb{T} \to \mathbb{R}$ are rd-continuous, then*

- $\int_a^b [f(t) + g(t)]\Delta t = \int_a^b f(t)\Delta(t) + \int_a^b g(t)\Delta t;$
- $\int_a^b \alpha f(t)\Delta t = \alpha \int_a^b f(t)\Delta t;$
- $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t;$
- $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t;$
- $\int_a^a f(t)\Delta(t) = 0;$
- $\int_a^b f(t) g^{\Delta}(t)\Delta t = fg(b) - fg(a) - \int_a^b f^{\Delta}(t) g(\sigma(t))\Delta t;$
- $\int_a^b f(\sigma(t)) g^{\Delta}(t)\Delta t = fg(b) - fg(a) - \int_a^b f^{\Delta}(t) g(t)\Delta t.$

Theorem 3. [23] *If $a, b \in \mathbb{T}, \ \alpha \in \mathbb{R}$, and $f : \mathbb{T} \to \mathbb{R}$ are rd-continuous, then*

- If $\mathbb{T} = \mathbb{R}$, then

$$\int_a^b f(t)\Delta t = \int_a^b f(t)dt,$$

where the integral on the right is the Riemann integral from calculus.

- *If $\mathbb{T}$ consists of only isolated points and $a < b$, then*

$$\sum_{t\in[a,b)} f(t)\mu(t).$$

Theorem 4. [14] *(Continuation Theorem). Let L be a Fredholm mapping of index zero and C be L-compact on Ω. Assume*

a. *For each $\lambda \in (0,1)$, any y satisfying $Ly = \lambda Cy$ is not on $\delta\Omega$, i.e., $y \notin \delta\Omega$*

b. *For each $y \in \delta\Omega \cap KerL$, $VCy \neq 0$ and the Brouwer degree $\deg\{JVC, \delta\Omega \cap KerL, 0\} \neq 0$. Then, $Ly = Cy$ has at least one solution lying in $DomL \cap \delta\Omega$.*

We will also give the following lemma, which is essential for this chapter.

Definition 1. [4] *Let the time scale $\mathbb{T}$ including a fixed number $t_0 \in \mathbb{T}^*$ where $\mathbb{T}^*$ be a non-empty subset of $\mathbb{T}$, such that there exist operators $\delta\pm : [t_0;\infty)_{\mathbb{T}} \times \mathbb{T}^* \to \mathbb{T}^*$ which satisfy the following properties:*

P.1 With respect to their second arguments, the functions $\delta\pm$ are strictly increasing, i.e., if

$$(S_0, v), (S_0, s) \in D_{\pm} := \{(u,v) \in [t_0, \infty)_{\mathbb{T}} \times \mathbb{T}^* : \delta \pm (u,v) \in \mathbb{T}^*\},$$

then

$S_0 \leq v < s$ implies $\delta \pm (S_0, v) < \delta \pm (S_0, s)$,

P.2 If (S_1, s), $(S_2, s) \in D_-$ with $S_1 < S_2$, then $\delta_-(S_1, s) > \delta_-(S_2, s)$, , and if $(S_1, s), (S_2, s) \in D_+$ with $S1 < S2$, then $\delta_+(S_1, s) < \delta_+(S_2, s)$,

P.3 If $v \in [t_0;\infty)_{\mathbb{T}}$, then $(v, t_0) \in D_+$ and $\delta_+(v, t_0) = s$. Moreover, if $v \in \mathbb{T}^*$, then $(t_0, v) \in D_+$ and $\delta_+(t_0, v) = v$ holds

P.4 If $(u,v) \in D_{\pm}$, then $(u, \delta_{\pm}(u,v)) \in D_{\pm}$ and $\delta_{\mp}(u; \delta_{\pm}(u,v)) = v$, respectively.

P.5 If $(u,v) \in D_{\pm}$ and $(s, \delta_{\pm}(u,v)) \in D_{\pm}$, then $(u, \delta_{\mp}(s,v)) \in D_{\pm}$ and

$$\delta_{\mp}(s, \delta_{\pm}(u, v)) = \delta_{\pm}(u, \delta_{\mp}(s, v)), \textit{respectively}$$

Then the backward operator is δ_-, and the forward operator is δ_+ which are associated with $t_0 \in \mathbb{T}^*$ (called the initial point). Shift size is the variable $u \in [t_0; \infty)_{\mathbb{T}}$ in $\delta_{\pm}(u, v)$. The values $\delta_+(u, v)$ and $\delta_+(u, v)$ in $\mathbb{T}^*$ indicate u unit translation of the term $v \in \mathbb{T}^*$ to the right and left, respectively. The sets $D_{\pm}$ are the domains of the shift operators $\delta_{\pm}$, respectively.

Definition 2. [4] *Let $\mathbb{T}$ be a time scale with the shift operators $\delta\pm$ associated with the initial point $t_0 \in \mathbb{T}^*$. The time scale $\mathbb{T}$ is said to be periodic in shifts $\delta\pm$ if there exists a $q \in (t_0, \infty)_{\mathbb{T}^*}$ such that $(q, t) \in D_{\pm}$ for all $t \in \mathbb{T}^*$. Furthermore, if*

$$Q:=\inf\{q \in (t_0, \infty)_{\mathbb{T}^*} : (q, t) \in D_{\pm} \ \textit{for all} \ \ t \in \mathbb{T}^*\} \neq t_0$$

then P is called the period of the time scale $\mathbb{T}$.

Definition 3. [4] *(Periodic function in shifts δ_+ and δ_-). Let $\mathbb{T}$ be a time scale that is periodic in shifts δ_+ and δ_- with the period Q. We say that a real valued function g defined on $\mathbb{T}^*$ is periodic in shifts if there exists a $\tilde{T} \in [Q, \infty)_{\mathbb{T}^*}$ such that*

$$g(\delta_{\pm}(\tilde{T}, t)) = g(t).$$

The smallest number $\tilde{T} \in [Q, \infty)_{\mathbb{T}^*}$ such that is called the period of f.

Definition 1, Definition 2, and Definition 3 are from [4].

[24]

Notation 1 $\delta_+^2(T, \kappa) = \delta_+(T, \delta_+(T, \kappa))$,

$$\delta_+^3(T, \kappa) = \delta_+(T, \delta_+(T, \delta_+(T, \kappa))), \ldots$$

$$\delta_+^n(T, \kappa) = \delta_+(, \delta_+(T, \delta_+(T, \delta_+(\ldots.)))).$$

Lemma 1. [24] *Let our time scale $\mathbb{T}$ be periodic in shifts, and for each $t \in \mathbb{T}^*$, $\left(\delta_+^n(T, t)\right)^{\Delta}$ is constant. Then, $\frac{\int_\kappa^{\delta_+(T,\kappa)} u(t)\Delta t}{mes(\delta_+(T,\kappa))}$ is also constant $\forall \kappa \in \mathbb{T}$,*

where $\kappa = \delta_{\pm}^m(T, t_0)$ for $m \in \mathbb{N}$ and $mes(\delta_+(T, \kappa))$ = $\int_\kappa^{\delta_+(T,\kappa)} 1\Delta t$. Here, $u(t)$ is a periodic function in shifts.

Proof. We get the desired result, if we can be able to show that for any $\kappa_1 \neq \kappa_2$ ($\kappa_1, \kappa_2 \in \mathbb{T}$).

$$\frac{\int_{\kappa_1}^{\delta_+(T,\kappa_1)} u(t)\Delta t}{mes(\delta_+(T,\kappa_1))} = \frac{\int_{\kappa_2}^{\delta_+(T,\kappa_2)} u(t)\Delta t}{mes(\delta_+(T,\kappa_2))}.$$

Since $\mathbb{T}$ is a periodic time scale in shifts (WLOG $\kappa_2 > \kappa_1$), there exits $n \in \mathbb{N}$ such that $\kappa_2 = \delta_+^n(T,\kappa_1)$. Hence, it is also enough to show that

$$\frac{\int_{\kappa_1}^{\delta_+(T,\kappa_1)} u(t)\Delta t}{mes(\delta_+(T,\kappa_1))} = \frac{\int_{\delta_+^n(T,\kappa_1)}^{\delta_+\left(T,\delta_+^n(T,\kappa_1)\right)} u(t)\Delta t}{mes(\delta_+(T,\delta_+^n(T,\kappa_1)))}.$$

Because of the definition of the time scale and u, $u(\kappa_1) = u\left(\delta_+^n(T,\kappa_1)\right)$, $u(\delta_+(T,\kappa_1)) = u\left(\delta_+^{n+1}(T,\kappa_1)\right)$, and for each $t \in [\kappa_1, \delta_+(T,\kappa_1)]$, $u(t) = u\left(\delta_+^n(T,t)\right)$. By using change of variables, we get the result. If $s = \delta_+^n(T,t)$, then by the assumption of the lemma $\Delta s = \tilde{c}\Delta t$. When $s = \delta_+^n(T,\kappa_1)$, then $t = \delta_-^n(T,s) = \kappa_1$, and when $s = \delta_+^{n+1}(T,\kappa_1)$, then $t = \delta_-^n(T,s) = \delta_+(T,\kappa_1)$.

$$\int_{\delta_+^n(T,\kappa_1)}^{\delta_+^{n+1}(T,\kappa_1)} u(s)\Delta s = \tilde{c}\int_{\kappa_1}^{\delta_+(T,\kappa_1)} u(t)\Delta t,$$

$$\int_{\delta_+^n(T,\kappa_1)}^{\delta_+^{n+1}(T,\kappa_1)} 1\Delta t = \tilde{c}\int_{\kappa_1}^{\delta_+(T,\kappa_1)} 1\Delta t,$$

and

$$\frac{\int_{\kappa_1}^{\delta_+(T,\kappa_1)} u(t)\Delta t}{mes(\delta_+(T,\kappa_1))} = \frac{\tilde{c}\int_{\kappa_1}^{\delta_+(T,\kappa_1)} u(t)\Delta t}{\tilde{c}\ \ mes(\delta_+(T,\kappa_1)).}$$

Hence, proof follows. □

Remark 1. [24] *It is obvious that if* $\mathbb{T} = \{0\} \cup q^{\mathbb{Z}}$, *then* $mes(\delta_+(T,t))$ *is equal for each* t *in* $\{0\} \cup q^{\mathbb{Z}}$.

The equation that we investigate is

$$\begin{aligned} x^{\Delta}(t) &= a(t) - b(t)\exp(x(t)) - \frac{c(t)\exp(y(t))}{\alpha(t) + \beta(t)\exp(x(t)) + m(t)\exp(y(t))}, \\ y^{\Delta}(t) &= -d(t) + \frac{f(t)\exp(x(t))}{\alpha(t) + \beta(t)\exp(x(t)) + m(t)\exp(y(t))}, \end{aligned} \tag{2.1}$$

In Eq. (2.1), let $a(t) = a(\delta_{\pm}(T,t))$, $b(\delta_{\pm}(T,t)) = b(t)$, $c(\delta_{\pm}(T,t)) = c(t)$, $d(\delta_{\pm}(T,t)) = d(t)$, $f(\delta_{\pm}(T,t)) = f(t)$, $\alpha(\delta_{\pm}(T,t)) = \alpha(t)$, $\beta(\delta_{\pm}(T,t)) = \beta(t)$, and $m(\delta_{\pm}(T,t)) = m(t)$, and $\int_{\kappa}^{\delta_+(T,\kappa)} a(t)\Delta t$, $\int_{\kappa}^{\delta_+(T,\kappa)} b(t)\Delta t$, $\int_{\kappa}^{\delta_+(T,\kappa)} d(t)\Delta t > 0$. $\beta^l = \min_{t\in[\kappa,\delta_+(T,\kappa)]}\beta(t)$, $m^l = \min_{t\in[\kappa,\delta_+(T,\kappa)]}$

$m(t)$, $\beta^u = \max_{t\in[\kappa,\delta_+(T,\kappa)]}\beta(t)$, and $m^u = \max_{t\in[\kappa,\delta_+(T,\kappa)]}m(t)$, such that $\kappa = \delta_{\pm}^m(T, t_0)$ for $m\in\mathbb{N}$. $m(t) > 0$ and $c(t), f(t), b(t) > 0$ $\alpha(t)\geq 0$, $\beta(t) > 0$. Each function is from $C_{rd}(\mathbb{T}, \mathbb{R})$.

Lemma 2. [24] *Let $t_1, t_2\in[\kappa, \delta_+(T, \kappa)]$ and $t\in\{0\}\cup q^{\mathbb{Z}}$. κ is defined as in Lemma 1. If $g : \{0\}\cup q^{\mathbb{Z}} \to R$ is periodic function in shifts, then*

$$g(t)\leq g(t_1) + \int_\kappa^{\delta_+(T,\kappa)} |g^\Delta(s)|\Delta s \quad \text{and} \quad g(t)\geq g(t_2) - \int_\kappa^{\delta_+(T,\kappa)} |g^\Delta(s)|\Delta s.$$

Proof. We only show the first inequality as the proof of the second inequality is similar to the proof of the other one. Since g is a periodic function in shifts, without loss of generality, it suffices to show that the inequality is valid for $t\in[\kappa, \delta_+(T, \kappa)]$. If $t = t_1$ then the first inequality is obviously true. If $t > t_1$

$$g(t) - g(t_1)\leq|g(t) - g(t_1)| = \left|\int_{t_1}^t g^\Delta(s)\Delta s\right|\leq \int_{t_1}^t g^\Delta(s)|\Delta s\leq \int_\kappa^{\delta_+(T,\kappa)} |g^\Delta(s)|\Delta s.$$

Therefore,

$$g(t)\leq g(t_1) + \int_\kappa^{\delta_+(T,\kappa)} |g^\Delta(s)|\Delta s.$$

If

$$t < t_1$$

$$g(t_1) - g(t)\geq - |g(t_1) - g(t)| = -\left|\int_t^{t_1} g^\Delta(s)\Delta s\right|\geq - \int_t^{t_1} |g^\Delta(s)|\Delta s\leq - \int_\kappa^{\delta_+(T,\kappa)} |g^\Delta(s)|\Delta s,$$

that gives $g(t)\leq g(t_1) + \int_\kappa^{\delta_+(T,\kappa)} |g^\Delta(s)|\Delta s$.

The proof is complete. □

Remark 2. [14] *Consider the following equation:*

$$\begin{aligned}\tilde{x}^{'}(t) &= a(t)\tilde{x}(t) - b(t)\tilde{x}^2(t) - \frac{c(t)\tilde{y}(t)\tilde{x}(t)}{\alpha(t) + \beta(t)\tilde{x}(t) + m(t)\tilde{y}(t)},\\ \tilde{y}^{'}(t) &= -d(t)\tilde{y}(t) + \frac{f(t)\tilde{x}(t)\tilde{y}(t)}{\alpha(t) + \beta(t)\tilde{x}(t) + m(t)\tilde{y}(t)}.\end{aligned} \tag{2.2}$$

This is the predator–prey dynamic system that is obtained from ordinary differential equations. Let $\mathbb{T} = \mathbb{R}$. In (2.1), *by taking $\exp(x(t)) = \tilde{x}(t)$ and $\exp(y(t)) = \tilde{y}(t)$, we obtain the equality* (2.2), *which is the standard predator–prey system with Beddington-DeAngelis functional response.*

Let $\mathbb{T} = \mathbb{Z}$. By using equality (2.1), *we obtain*

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$$x(t+1) - x(t) = a(t) - b(t)\exp(x(t)) - \frac{c(t)\exp(y(t))}{\alpha(t) + \beta(t)\exp(x(t)) + m(t)\exp(y(t))},$$
$$y(t+1) - y(t) = -d(t) + \frac{f(t)\exp(x(t))}{\alpha(t) + \beta(t)\exp(x(t)) + m(t)\exp(y(t))}$$

Here, again by taking $\exp(x(t)) = \tilde{x}(t)$ *and* $\exp(y(t)) = \tilde{y}(t)$, *we obtain*

$$\begin{aligned} \tilde{x}(t+1) &= \tilde{x}(t)\exp\left[a(t) - b(t)\tilde{x}(t) - \frac{c(t)\tilde{y}(t)}{\alpha(t) + \beta(t)\tilde{x}(t) + m(t)\tilde{y}(t)}\right], \\ \tilde{y}(t+1) &= \tilde{y}(t)\exp\left[-d(t) + \frac{f(t)\tilde{x}(t)}{\alpha(t) + \beta(t)\tilde{x}(t) + m(t)\tilde{y}(t)}\right], \end{aligned} \tag{2.3}$$

which is the discrete time predator–prey system with Beddington-DeAngelis-type functional response and also the discrete analogue of Eq. (2.2). *This system was studied in* [25, 26]. *Since Eq.* (2.1) *incorporates Eqs.* (2.2) and (2.3) *as special cases, we call Eq.* (2.1) *the predator–prey dynamic system with Beddington-DeAngelis functional response on time scales.*

For Eq. (2.1), $\exp(x(t))$ *and* $\exp(y(t))$ *denote the density of prey and the predator. Therefore,* $x(t)$ *and* $y(t)$ *could be negative. By taking the exponential of* $x(t)$ *and* $y(t)$, *we obtain the number of preys and predators that are living per unit of an area. In other words, for the general time scale case, our equation is based on the natural logarithm of the density of the predator and prey. Hence,* $x(t)$ *and* $y(t)$ *could be negative.*

For Eqs. (2.2) and (2.3), *since* $exp(x(t)) = \tilde{x}(t)$ *and* $exp(y(t)) = \tilde{y}(t)$, *the given dynamic systems directly depend on the density of the prey and predator.*

3. Application of $\delta_{\pm}$-periodicity of Q-calculus

The following theorem is the modified version of Theorem 8 from [24].

Theorem 5. *Assume that for the given time scale* $\mathbb{T} = \{0\} \cup q^{\mathbb{Z}}$, *while* $T \in q^{\mathbb{Z}}$, $mes(\delta_+(T,t))$ *is equal for each* $t \in \mathbb{T}$. *In addition to conditions on coefficient functions and*

Lemma 1 if $\int_\kappa^{\delta_+(T,\kappa)} a(t)\Delta t - \int_\kappa^{\delta_+(T,\kappa)} \frac{c(t)}{m(t)}\Delta t > 0$ *and*

$$\left(\frac{\int_\kappa^{\delta_+(T,\kappa)} a(t)\Delta t - \int_\kappa^{\delta_+(T,\kappa)} \dfrac{c(t)}{m(t)}\Delta t}{\int_\kappa^{\delta_+(T,\kappa)} b(t)\Delta t}\right) exp\left[-\left(\int_\kappa^{\delta_+(T,\kappa)} |a(t)|\Delta t + \int_\kappa^{\delta_+(T,\kappa)} a(t)\Delta t\right)\right]$$
$$\cdot \int_\kappa^{\delta_+(T,\kappa)} f(t)\Delta t - \beta^u\left(\int_\kappa^{\delta_+(T,\kappa)} d(t)\Delta t\right) - \alpha^u\left(\int_\kappa^{\delta_+(T,\kappa)} d(t)\right)\Delta t > 0$$

are satisfied, then there exist at least one $\delta_{\pm}$-periodic solution.

Proof. $X := \left\{ \begin{bmatrix} u \\ v \end{bmatrix} \in C_{rd}\left(\{0\} \cup q^{\mathbb{Z}}, \mathbb{R}^2\right) : u(\delta_\pm(T,t)) = u(t), v(\delta_\pm(T,t)) = v(t) \right\}$ with the norm:

$$\left\| \begin{bmatrix} u \\ v \end{bmatrix} \right\| = \max_{t \in [t_0, \delta_+(T,t_0)]_{\mathbb{T}}} (|u(t)|, |v(t)|)$$

$Y := \left\{ \begin{bmatrix} u \\ v \end{bmatrix} \in C_{rd}\left(\{0\} \cup q^{\mathbb{Z}}, \mathbb{R}^2\right) : u(\delta_\pm(T,t)) = u(t), v(\delta_\pm(T,t)) = v(t) \right\}$ with the norm:

$$\left\| \begin{bmatrix} u \\ v \end{bmatrix} \right\| = \max_{t \in [t_0, \delta_+(T,t_0)]_{\mathbb{T}}} (|u(t)|, |v(t)|)$$

Let us define the mappings L and C by $L : DomL \subset X \to Y$ such that

$$L\left(\begin{bmatrix} u \\ v \end{bmatrix} \right) = \begin{bmatrix} u^\Delta \\ v^\Delta \end{bmatrix}$$

and $C : X \to Y$ such that

$$C\left(\begin{bmatrix} u \\ v \end{bmatrix} \right) = \begin{bmatrix} a(t) - b(t)\, exp\,(u(t)) - \dfrac{c(t)\, exp\,(v(t))}{\alpha(t) + \beta(t)\, exp\,(u(t)) + m(t)\, exp\,(v(t))} \\ -d(t) + \dfrac{f(t)\, exp\,(u(t))}{\alpha(t) + \beta(t)\, exp\,(u(t)) + m(t)\, exp\,(v(t))} \end{bmatrix}$$

Then, $KerL = \left\{ \begin{bmatrix} u \\ v \end{bmatrix} : \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \right\}$, c_1 and c_2 are constants.

$$ImL = \left\{ \begin{bmatrix} u \\ v \end{bmatrix} : \begin{bmatrix} \int_\kappa^{\delta_+(T,\kappa)} u(t)\Delta t \\ \int_\kappa^{\delta_+(T,\kappa)} v(t)\Delta t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}.$$

ImL is closed in Y. Its obvious that $dimKerL = 2$. To show $dimKerL = codimImL = 2$, we have to prove that $KerL \oplus ImL = Y$. It is obvious that when we take an element from Ker L, an element from Im L, we find an element of Y by summing these two elements. If we take an element $\begin{bmatrix} u \\ v \end{bmatrix} \in Y$, and WLOG taking $u(t)$, we have $\int_\kappa^{\delta_+(T,\kappa)} u(t)\Delta t = I$ where I is a constant. Let us define a new function $g = u - \frac{I}{mes(\delta_+(T,\kappa))}$. Since $\frac{I}{mes(\delta_+(T,\kappa))}$ is constant by Lemma 1, if we take the integral of g from κ to $\delta_+(T,\kappa)$, we get

$$\int_\kappa^{\delta_+(T,\kappa)} g(t)\Delta t = \int_\kappa^{\delta_+(T,\kappa)} u(t)\Delta t - I = 0.$$

Similar steps are used for v. $\begin{bmatrix} u \\ v \end{bmatrix} \in Y$ can be written as the summation of an element from Im L and an element from Ker L. Also, it is easy to show that any element in Y is uniquely expressed as the summation of an element Ker L and an element from Im L. So, *codimImL* is also 2, we get the desired result. Hence, L is a Fredholm mapping of index zero. There exist continuous projectors $U : X \to X$ and $V : Y \to Y$ such that

$$U\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \frac{1}{mes(\delta_+(T,\kappa))}\begin{bmatrix} \int_\kappa^{\delta_+(T,\kappa)} u(t)\Delta t \\ \int_\kappa^{\delta_+(T,\kappa)} v(t)\Delta t \end{bmatrix}$$

and

$$V\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \frac{1}{mes(\delta_+(T,\kappa))}\left(\begin{bmatrix} \int_\kappa^{\delta_+(T,\kappa)} u(t)\Delta t \\ \int_\kappa^{\delta_+(T,\kappa)} v(t)\Delta t \end{bmatrix}\right).$$

The generalized inverse $K_U = ImL \to DomL \cap KerU$ is given:

$$K_U\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} \int_\kappa^t u(s)\Delta s - \frac{1}{mes(\delta_+(T,\kappa))}\int_\kappa^{\delta_+(T,\kappa)}\int_\kappa^t u(s)\Delta s \\ \int_\kappa^t v(s)\Delta s - \frac{1}{mes(\delta_+(T,\kappa))}\int_\kappa^{\delta_+(T,\kappa)}\int_\kappa^t v(s)\Delta s \end{bmatrix}.$$

$$VC\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) =$$

$$\frac{1}{mes(\delta_+(T,\kappa))}\left(\begin{bmatrix} \int_\kappa^{\delta_+(T,\kappa)} a(s) - b(s)\,exp\,(u(s)) - \frac{c(s)\,exp\,(v(s))}{\alpha(s) + \beta(s)\,exp\,(u(s)) + m(s)\,exp\,(v(s))}\Delta s \\ \int_\kappa^{\delta_+(T,\kappa)} -d(s) + \frac{f(s)\,exp\,(u(s))}{\alpha(s) + \beta(s)\,exp\,(u(s)) + m(s)\,exp\,(v(s))}\Delta s \end{bmatrix}\right)$$

Let

$$a(t) - b(t)\,exp\,(u(t)) - \frac{c(t)\,exp\,(v(t))}{\alpha(t) + \beta(t)\,exp\,(u(t)) + m(t)\,exp\,(v(t))} = C_1$$

$$-d(t) + \frac{f(t)\,exp\,(u(t))}{\alpha(t) + \beta(t)\,exp\,(u(t)) + m(t)\,exp\,(v(t))} = C_2$$

$$\frac{1}{mes(\delta_+(T,\kappa))}\int_\kappa^{\delta_+(T,\kappa)} a(s) - b(s)\,exp\,(u(s)) - \frac{c(s)\,exp\,(v(s))}{\alpha(s) + \beta(s)\,exp\,(u(s)) + m(s)\,exp\,(v(s))}\Delta s = \overline{C}_1$$

and

$$\frac{1}{mes(\delta_+(T,\kappa))}\int_\kappa^{\delta_+(T,\kappa)} -d(s) + \frac{f(s)\,exp\,(u(s))}{\alpha(s)+\beta(s)\,exp\,(u(s))+m(s)\,exp\,(v(s))}\Delta s = \overline{C}_2$$

$$K_U(I-V)C\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = K_U\left(\begin{bmatrix} C_1-\overline{C}_1 \\ C_2-\overline{C}_2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} \int_\kappa^t C_1(s)-\overline{C}_1(s)\Delta s - \frac{1}{mes(\delta_+(T,\kappa))}\int_\kappa^{\delta_+(T,\kappa)}\int_\kappa^t C_1(s)-\overline{C}_1(s)\Delta s \\ \int_\kappa^t C_2(s)-\overline{C}_2(s)\Delta s - \frac{1}{mes(\delta_+(T,\kappa))}\int_\kappa^{\delta_+(T,\kappa)}\int_\kappa^t C_2(s)-\overline{C}_2(s)\Delta s \end{bmatrix}.$$

Clearly, VC and $K_U(I-V)C$ are continuous. Here, X and Y are Banach spaces. Since for the given time scale $\mathbb{T}$ while T is constant, $mes(\delta_+(T,t))$ is equal for each $t\in\mathbb{T}$; then, we can apply Arzela-Ascoli theorem, and by using Arzela-Ascoli theorem, we can find that $\overline{K}_U(I-V)C(\overline{\Omega})$ is compact for any open bounded set $\Omega\subset X$. Additionally, $VC(\overline{\Omega})$ is bounded. Thus, C is L-compact on $\overline{\Omega}$ with any open bounded set $\Omega\subset X$.

To apply the continuation theorem, we investigate the below operator equation:

$$\begin{aligned} x^\Delta(t) &= \lambda\left[a(t)-b(t)\,exp\,(x(t)) - \frac{c(t)\,exp\,(y(t))}{\alpha(t)+\beta(t)\,exp\,(x(t))+m(t)\,exp\,(y(t))}\right] \\ y^\Delta(t) &= \lambda\left[-d(t)+\frac{f(t)\,exp\,(x(t))}{\alpha(t)+\beta(t)\,exp\,(x(t))+m(t)\,exp\,(y(t))}\right] \end{aligned} \tag{3.1}$$

Let $\begin{bmatrix} x \\ y \end{bmatrix}\in X$ be any solution of system (3.1). Integrating both sides of system (3.1) over the interval $[0,w]$, we obtain

$$\begin{cases} \int_\kappa^{\delta_+(T,\kappa)} a(t)\Delta t = \int_\kappa^{\delta_+(T,\kappa)} b(t)\,exp\,(x(t)) + \frac{c(t)\,exp\,(y(t))}{\alpha(t)+\beta(t)\,exp\,(x(t))+m(t)\,exp\,(y(t))}\Delta t & , \\ \int_\kappa^{\delta_+(T,\kappa)} d(t)\Delta t = \int_\kappa^{\delta_+(T,\kappa)} \frac{f(t)\,exp\,(x(t))}{\alpha(t)+\beta(t)\,exp\,(x(t))+m(t)\,exp\,(y(t))}\Delta t & , \end{cases} \tag{3.2}$$

From (3.1) and (3.2), we get

$$\begin{aligned} \int_\kappa^{\delta_+(T,\kappa)} |x^\Delta(t)|\Delta t &\le \lambda\left[\int_\kappa^{\delta_+(T,\kappa)} |a(t)|\Delta t + \int_\kappa^{\delta_+(T,\kappa)} b(t)\,exp\,(x(t)) + \frac{c(t)\,exp\,(y(t))}{\alpha(t)+\beta(t)\,exp\,(x(t))+m(t)\,exp\,(y(t))}\Delta t\right], \\ &\le \lambda\left[\int_\kappa^{\delta_+(T,\kappa)} |a(t)|\Delta t + \int_\kappa^{\delta_+(T,\kappa)} a(t)\Delta t\right] \\ &\le \int_\kappa^{\delta_+(T,\kappa)} |a(t)|\Delta t + \int_\kappa^{\delta_+(T,\kappa)} a(t)\Delta t{:=}M_1 \end{aligned} \tag{3.3}$$

$$\begin{aligned} \int_\kappa^{\delta_+(T,\kappa)} |y^\Delta(t)|\Delta t &\le \lambda\left[\int_\kappa^{\delta_+(T,\kappa)} |d(t)|\Delta t + \int_\kappa^{\delta_+(T,\kappa)} \frac{f(t)\,exp\,(x(t))}{\alpha(t)+\beta(t)\,exp\,(x(t))+m(t)\,exp\,(y(t))}\Delta t\right] \\ &\le \lambda\left[\int_\kappa^{\delta_+(T,\kappa)} |d(t)|\Delta t + \int_\kappa^{\delta_+(T,\kappa)} d(t)\Delta t\right] \\ &\le \int_\kappa^{\delta_+(T,\kappa)} |d(t)|\Delta t + \int_\kappa^{\delta_+(T,\kappa)} d(t)\Delta t{:=}M_2 \end{aligned} \tag{3.4}$$

Since $\begin{bmatrix} x \\ y \end{bmatrix} \in X$, then there exist η_i, ξ_i and $i = 1, 2$ such that

$$\begin{aligned} x(\xi_1) &= \min_{t \in t \in [\kappa, \delta_+(T,\kappa)]} x(t), x(\eta_1) = \max_{t \in t \in [\kappa, \delta_+(T,\kappa)]} x(t), \\ y(\xi_2) &= \min_{t \in t \in [\kappa, \delta_+(T,\kappa)]} y(t), y(\eta_2) = \max_{t \in t \in [\kappa, \delta_+(T,\kappa)]} y(t) \end{aligned} \tag{3.5}$$

If ξ_1 is the minimum point of $x(t)$ on the interval $[\kappa, \delta_+(T,\kappa)]$ because $x(t)$ is a function that is periodic in shifts for any $n \in \mathbb{N}$ on the interval $\left[\delta_+^n(T,\kappa_1), \delta_+^{n+1}(T,\kappa_1)\right]$, the minimum point of $x(t)$ is $\delta_+^n(T,\xi_1)$ and $x(\xi_1) = x\left(\delta_+^n(T,\xi_1)\right)$. We have similar results for the other points for ξ_2, η_1, and η_2.

By the first equation of systems (3.2) and (3.5)

$$\begin{aligned} \int_\kappa^{\delta_+(T,\kappa)} a(t)\Delta t &\leq \int_\kappa^{\delta_+(T,\kappa)} \left[b(t) \, exp\left(x(\eta_1)\right) + \frac{c(t)}{m(t)} \Delta t \right] \\ &= exp\left(x(\eta_1)\right) \int_\kappa^{\delta_+(T,\kappa)} b(t)\Delta t + \int_\kappa^{\delta_+(T,\kappa)} \frac{c(t)}{m(t)} \Delta t. \end{aligned}$$

Since $\int_\kappa^{\delta_+(T,\kappa)} b(t)\Delta t > 0$, so we get

$$x(\eta_1) \geq \ln\left(\frac{\int_\kappa^{\delta_+(T,\kappa)} a(t)\Delta t - \int_\kappa^{\delta_+(T,\kappa)} \frac{c(t)}{m(t)} \Delta t}{\int_\kappa^{\delta_+(T,\kappa)} b(t)\Delta t} \right) := l_1$$

Using the second inequality in Lemma 2, we have

$$\begin{aligned} x(t) &\geq x(\eta_1) - \int_\kappa^{\delta_+(T,\kappa)} |x^\Delta(t)|\Delta t \\ &\geq x(\eta_1) - \left(\int_\kappa^{\delta_+(T,\kappa)} |a(t)|\Delta t + \int_\kappa^{\delta_+(T,\kappa)} a(t)\Delta t \right) \\ &= l_1 - M_1 := H_1 \end{aligned} \tag{3.6}$$

By the first equation of systems (3.2) and (3.5)

$$\begin{aligned} \int_\kappa^{\delta_+(T,\kappa)} a(t)\Delta t &\geq \int_\kappa^{\delta_+(T,\kappa)} b(t) \, exp\left(x(\xi_1)\right)\Delta t \\ &= exp\left(x(\xi_1)\right) \int_\kappa^{\delta_+(T,\kappa)} b(t)\Delta t. \end{aligned}$$

Then, we get

$$x(\xi_1) \leq \ln\left(\frac{\int_\kappa^{\delta_+(T,\kappa)} a(t)\Delta t}{\int_\kappa^{\delta_+(T,\kappa)} b(t)\Delta t} \right) := l_2$$

Using the first inequality in Lemma 2, we have

$$\begin{aligned} x(t) &\le x(\xi_1) + \int_\kappa^{\delta_+(T,\kappa)} |x^\Delta(t)|\Delta t \\ &\le x(\xi_1) + \left(\int_\kappa^{\delta_+(T,\kappa)} |a(t)|\Delta t + \int_\kappa^{\delta_+(T,\kappa)} a(t)\Delta t\right) \\ &= l_2 + M_1 := H_2 \end{aligned} \tag{3.7}$$

By Eq. (3.6) and (3.7), $\max_{t\in[\kappa,\delta_+(T,\kappa)]}|x(t)| \le \max\{|H_1|,|H_2|\} := B_1$. From the second equation of system (3.2) and the second equation of system (3.6), we can derive that

$$\begin{aligned} \int_\kappa^{\delta_+(T,\kappa)} d(t)\Delta t &\le \int_\kappa^{\delta_+(T,\kappa)} \frac{f(t)\, exp\,(x(t))}{\beta^l\, exp\,(x(t)) + m^l\, exp\,(y(t))}\Delta t \\ &\le \int_\kappa^{\delta_+(T,\kappa)} \frac{f(t)e^{H_2}}{\beta^l e^{H_2} + m^l\, exp\,(y(\xi_2))}\Delta t \\ &= \frac{e^{H_2}}{\beta^l e^{H_2} + m^l\, exp\,(y(\xi_2))}\int_\kappa^{\delta_+(T,\kappa)} f(t)\Delta t. \end{aligned}$$

Therefore,

$$exp\,(y(\xi_2)) \le \frac{1}{m^l}\left(\frac{e^{H_2}\int_\kappa^{\delta_+(T,\kappa)} f(t)\Delta t}{\int_\kappa^{\delta_+(T,\kappa)} d(t)\Delta t} - \beta^l e^{H_2}\right)$$

By the assumption of the Theorem 5, we get,

$$\int_\kappa^{\delta_+(T,\kappa)} f(t)\Delta t - \beta^l\left(\int_\kappa^{\delta_+(T,\kappa)} d(t)\right)\Delta t > 0 \text{ and}$$

$$y(\xi_2) \le \ln\left(\frac{1}{m^l}\left(\frac{e^{H_2}\int_\kappa^{\delta_+(T,\kappa)} f(t)\Delta t}{\int_\kappa^{\delta_+(T,\kappa)} d(t)\Delta t} - \beta^l e^{H_2}\right)\right) := L_1$$

Hence, by using the first inequality in Lemma 2 and the second equation of system (3.2)

$$\begin{aligned} y(t) &\le y(\xi_2) + \int_\kappa^{\delta_+(T,\kappa)} |y^\Delta(t)|\Delta t \\ &\le y(\xi_2) + \left(\int_\kappa^{\delta_+(T,\kappa)} |d(t)|\Delta t + \int_\kappa^{\delta_+(T,\kappa)} d(t)\Delta t\right) \\ &\le L_1 + M_2 := H_3. \end{aligned} \tag{3.8}$$

Again, using the second equation of system (3.2), we obtain

$$\begin{aligned} \int_\kappa^{\delta_+(T,\kappa)} d(t)\Delta t &\ge \int_\kappa^{\delta_+(T,\kappa)} \frac{f(t)\, exp\,(x(t))}{\alpha^u + \beta^u\, exp\,(x(t)) + m^u\, exp\,(y(t))}\Delta t \\ &\ge \int_\kappa^{\delta_+(T,\kappa)} \frac{f(t)e^{H_1}}{\alpha^u + \beta^u e^{H_1} + m^u\, exp\,(y(\eta_2))}\Delta t \\ &= \frac{e^{H_1}}{\alpha^u + \beta^u e^{H_1} + m^u\, exp\,(y(\eta_2))}\int_\kappa^{\delta_+(T,\kappa)} f(t)\Delta t, \end{aligned}$$

$$exp\left(y(\eta_2)\right) \geq \frac{1}{m^u}\left(\frac{e^{H_1}\int_\kappa^{\delta_+(T,\kappa)} f(t)\Delta t}{\int_\kappa^{\delta_+(T,\kappa)} d(t)\Delta t} - \beta^u e^{H_1} - \alpha^u\right).$$

Using the assumption of the Theorem 5, we obtain

$$e^{H_1}\left(\int_\kappa^{\delta_+(T,\kappa)} f(t)\Delta t - \beta^u\left(\int_\kappa^{\delta_+(T,\kappa)} d(t)\Delta t\right)\right) - \alpha^u\left(\int_\kappa^{\delta_+(T,\kappa)} d(t)\Delta t\right) > 0$$

and

$$y(\eta_2) \geq \ln\left(\frac{1}{m^u}\left(\frac{e^{H_1}\int_\kappa^{\delta_+(T,\kappa)} f(t)\Delta t}{\int_\kappa^{\delta_+(T,\kappa)} d(t)\Delta t} - \beta^u e^{H_1} - \alpha^u\right)\right) {:=} L_2.$$

By using the second inequality in Lemma 2

$$\begin{aligned} y(t) &\geq y(\eta_2) - \int_\kappa^{\delta_+(T,\kappa)} |y^\Delta(t)|\Delta t \\ &\geq y(\eta_2) - \left(\int_\kappa^{\delta_+(T,\kappa)} |d(t)|\Delta t + \int_\kappa^{\delta_+(T,\kappa)} d(t)\Delta t\right) \\ &= L_2 - M_2 {:=} H_4. \end{aligned} \tag{3.9}$$

By Eq. (3.8) and (3.9), we have $\max_{t\in[t_0,\delta_+(T,t_0)]}|y(t)| \leq \max\{|H_3|, |H_4|\} {:=} B_2$. Obviously, B_1 and B_2 are both independent of λ. Let $M = B_1 + B_2 + 1$. Then, $\max_{t\in[t_0,\delta_+(T,t_0)]}\left\|\begin{bmatrix} x \\ y \end{bmatrix}\right\| < M$. Let $\Omega = \left\{\left\|\begin{bmatrix} x \\ y \end{bmatrix}\right\| \in X : \left\|\begin{bmatrix} x \\ y \end{bmatrix}\right\| < M\right\}$; then, Ω verifies the requirement (a) in Theorem 4. When $\begin{bmatrix} x \\ y \end{bmatrix} \in KerL \cap \partial\Omega$, $\begin{bmatrix} x \\ y \end{bmatrix}$ is a constant with $\left\|\begin{bmatrix} x \\ y \end{bmatrix}\right\| = M$, ; then,

$$\begin{aligned} VC\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= \left(\begin{bmatrix} \int_\kappa^{\delta_+(T,\kappa)} a(s) - b(s)\,exp\,(x) - \dfrac{c(s)\,exp\,(y)}{\alpha(s) + \beta(s)\,exp\,(x) + m(s)\,exp\,(y)}\Delta t \\ \int_\kappa^{\delta_+(T,\kappa)} -d(s) + \dfrac{f(s)\,exp\,(x)}{\alpha(s) + \beta(s)\,exp\,(x) + m(s)\,exp\,(y)}\Delta t \end{bmatrix}\right) \\ &\neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

$$JVC\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = VC\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

where $J : ImV \to KerL$ is the identity operator.

Let us define the homotopy such that $H_\nu = \nu(JVC) + (1-\nu)G$ where

$$G\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \int_{\kappa}^{\delta_+(T,\kappa)} a(s) - b(s)\, exp\,(x)\Delta t \\ \int_{\kappa}^{\delta_+(T,\kappa)} d(s) - \dfrac{f(s)\, exp\,(x)}{\alpha(s) + \beta(s)\, exp\,(x) + m(s)\, exp\,(y)} \Delta t \end{bmatrix}$$

Take DJ_G as the determinant of the Jacobian of G. Since $\begin{bmatrix} x \\ y \end{bmatrix} \in KerL$, then Jacobian of G is

$$\begin{bmatrix} -e^x \int_{\kappa}^{\delta_+(T,\kappa)} b(s)\Delta t & 0 \\ \int_{\kappa}^{\delta_+(T,\kappa)} \dfrac{-e^x f(s)}{\alpha(s) + \beta(s)e^x + m(s)e^y} \Delta t + \int_{\kappa}^{\delta_+(T,\kappa)} \dfrac{(e^x)^2 f(s)\beta(s)}{(\alpha(s) + \beta(s)e^x + m(s)e^y)^2} \Delta t & -\int_{\kappa}^{\delta_+(T,\kappa)} \dfrac{e^x e^y f(s) m(s)}{(\alpha(s) + \beta(s)e^x + m(s)e^y)^2} \Delta t \end{bmatrix}$$

All the functions in Jacobian of G is positive; then, $signDJ_G$ is always positive. Hence,

$$\deg(JVC, \Omega \cap KerL, 0) = \deg(G, \Omega \cap KerL, 0) = \sum_{\begin{bmatrix} x \\ y \end{bmatrix} \in G^{-1}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)} signDJ_G\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) \neq 0.$$

Thus, all the conditions of Theorem 4 are satisfied. Therefore, system (2.1) has at least a positive $\delta\pm$-periodic solution. □

Example 1 *Let* $\mathbb{T} = \{0\} \cup q^{\mathbb{Z}}$. $\delta_{\pm}(q,t)$ *is the shift operator and* $t_0 = 1$.

$$\begin{aligned} x^{\Delta}(t) &= \left((-1)^{\frac{\ln|t|}{\ln(q)}} + 4\right) - \left((-1)^{\frac{\ln|t|}{\ln(q)}} + 0.5\right) exp\,(x(t)) - \frac{exp\,(y(t))}{exp\,(x(t)) + 2\, exp\,(y(t))}, \\ y^{\Delta}(t) &= -0.3 + \frac{\left((-1)^{\frac{\ln|t|}{\ln(q)}} + 7\right) \; exp\,(x(t))}{exp\,(x(t)) + 2\, exp\,(y(t))}, \end{aligned} \tag{3.10}$$

Each function in system (12) is $\delta_{\pm}(q^2, t)$ *periodic and satisfies Theorem 1; then, the system has at least one* $\delta_{\pm}(q^2, t)$ *periodic solution. Here,* $mes(\delta_{+}(q^2, t)) = 2$.

4. Conclusion

The important results of this study are:

1. The definition of $\delta_{\pm}$-periodicity notion is adapted to the quantum calculus.
2. The importance of time scale calculus is pointed out for the analysis of quantum calculus.
3. As an application, the $\delta_{\pm}$-periodicity notion for quantum calculus is used for the predator–prey dynamic system whose coefficient functions are $\delta_{\pm}$ periodic.

As a result, it is seen that one can define a periodicity notion that is applicable to the structure of the quantum calculus. Additionally, it is shown that this notion is useful for different applications. One of its applications is analyzed in this study with an example.

5. Discussion

There are many studies about the predator–prey dynamic systems on time scale calculus such as [14, 19, 27, 28]. All of these cited studies are about the periodic solutions of the considered system on a periodic time scale. However, in the world, there are many different species. While investigating the periodicity notion of the different life cycle of the species, the w-periodic time scales could be a little bit restricted. Therefore, if the life cycle of this kind of species is appropriate to the Beddington-DeAngelis functional response, then the results that we have found in that study are becoming more useful and important.

In addition to these, the $\delta_{\pm}$-periodic solutions for predator–prey dynamic systems with Holling-type functional response, semiratio-dependent functional response, and monotype functional response can be also taken into account for future studies. In that dynamic systems, delay conditions and impulsive conditions can also be added for the new investigations.

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Author details

Neslihan Nesliye Pelen*, Ayşe Feza Güvenilir and Billur Kaymakçalan

*Address all correspondence to: nesliyeaykir@gmail.com

Faculty of Science, Department of Mathematics, Ondokuz Mayis University, Samsun, Turkey

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