Chapter 5

Differential Equations Arising from the 3-Variable Hermite Polynomials and Computation of Their Zeros

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Additional information is available at the end of the chapter

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Abstract

In this paper, we study differential equations arising from the generating functions of the 3-variable Hermite polynomials. We give explicit identities for the 3-variable Hermite polynomials. Finally, we investigate the zeros of the 3-variable Hermite polynomials by using computer.

Keywords: differential equations, heat equation, Hermite polynomials, the 3-variable Hermite polynomials, generating functions, complex zeros

1. Introduction

Many mathematicians have studied in the area of the Bernoulli numbers, Euler numbers, Genocchi numbers, and tangent numbers see [1–15]. The special polynomials of two variables provided new means of analysis for the solution of a wide class of differential equations often encountered in physical problems. Most of the special function of mathematical physics and their generalization have been suggested by physical problems.

In [1], the Hermite polynomials are given by the exponential generating function

$$\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} = e^{2xt-t^2}.$$

We can also have the generating function by using Cauchy's integral formula to write the Hermite polynomials as

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} = \frac{n!}{2\pi i} \oint_C \frac{e^{2tx-t^2}}{t^{n+1}} dt$$

with the contour encircling the origin. It follows that the Hermite polynomials also satisfy the recurrence relation

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x).$$

Further, the two variables Hermite Kampé de Fériet polynomials $H_n(x, y)$ defined by the generating function (see [3])

$$\sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!} = e^{xt+yt^2} \tag{1}$$

are the solution of heat equation

$$\frac{\partial}{\partial y} H_n(x, y) = \frac{\partial^2}{\partial x^2} H_n(x, y), \quad H_n(x, 0) = x^n.$$

We note that

$$H_n(2x, -1) = H_n(x).$$

The 3-variable Hermite polynomials $H_n(x, y, z)$ are introduced [4].

$$H_n(x, y, z) = n! \sum_{k=0}^{[\frac{n}{3}]} \frac{z^k H_{n-3k}(x, y)}{k!(n-3k)!}.$$

The differential equation and he generating function for $H_n(x, y, z)$ are given by

$$\left(3z\frac{\partial^3}{\partial x^3} + 2y\frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial x} - n\right) H_n(x, y, z) = 0$$

and

$$e^{xt+yt^2+zt^3} = \sum_{n=0}^{\infty} H_n(x, y, z) \frac{t^n}{n!}, \tag{2}$$

respectively.

By (2), we get

$$\begin{aligned} \sum_{n=0}^{\infty} H_n(x_1 + x_2, y, z) \frac{t^n}{n!} &= e^{(x_1+x_2)t+yt^2+zt^3} \\ &= \sum_{n=0}^{\infty} x_2^n \frac{t^n}{n!} \sum_{n=0}^{\infty} H_n(x_1, y, z) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} H_l(x_1, y, z) x_2^{n-l} \right) \frac{t^n}{n!}. \end{aligned} \tag{3}$$

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By comparing the coefficients on both sides of (3), we have the following theorem.

Theorem 1. For any positive integer n, we have

$$H_n(x_1 + x_2, y, z) = \sum_{l=0}^{n} \binom{n}{l} H_l(x_1, y, z) x_2^{n-l}.$$

Applying Eq. (2), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} H_n(x, y, z_1 + z_2) \frac{t^n}{n!} &= e^{xt + yt^2 + (z_1 + z_2)t^3} \\ &= \sum_{k=0}^{\infty} z_2^n \frac{t^{3k}}{k!} \sum_{l=0}^{\infty} H_l(x, y, z_1) \frac{t^l}{l!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\left[\frac{n}{3}\right]} \frac{H_{n-3k}(x, y, z_1) z_2^k n!}{k!(n-3k)!}. \right) \frac{t^n}{n!}.\end{aligned}$$

On equating the coefficients of the like power of t in the above, we obtain the following theorem.

Theorem 2. For any positive integer n, we have

$$H_n(x, y, z_1 + z_2) = n! \sum_{k=0}^{\left[\frac{n}{3}\right]} \frac{H_{n-3k}(x, y, z_1) z_2^k}{k!(n-3k)!}.$$

Also, the 3-variable Hermite polynomials $H_n(x, y, z)$ satisfy the following relations

$$\frac{\partial}{\partial y} H_n(x, y, z) = \frac{\partial^2}{\partial x^2} H_n(x, y, z),$$

and

$$\frac{\partial}{\partial z} H_n(x, y, z) = \frac{\partial^3}{\partial x^3} H_n(x, y, z).$$

The following elementary properties of the 3-variable Hermite polynomials $H_n(x, y, z)$ are readily derived form (2). We, therefore, choose to omit the details involved.

Theorem 3. For any positive integer n, we have

1. $H_n(2x, -1, 0) = H_n(x)$.

2. $H_n(x, y_1 + y_2, z) = n! \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{H_{n-2k}(x, y_1, z) y_2^k}{k!(n-2k)!}$.

3. $H_n(x, y, z) = \sum_{l=0}^{n} \binom{n}{l} H_l(x) H_{n-l}(-x, y+1, z)$.

Theorem 4. For any positive integer n, we have

1 $H_n(x_1+x_2, y_1+y_2, z) = \sum_{l=0}^{n} \binom{n}{l} H_l(x_1, y_1, z) H_{n-l}(x_2, y_2)$.

2 $H_n(x_1+x_2, y_1+y_2, z_1+z_2) = \sum_{l=0}^{n} \binom{n}{l} H_l(x_1, y_1, z) H_{n-l}(x_2, y_2, z_2)$.

The 3-variable Hermite polynomials can be determined explicitly. A few of them are

$$
\begin{aligned}
H_0(x,y,z) &= 1,\\
H_1(x,y,z) &= x,\\
H_2(x,y,z) &= x^2+2y,\\
H_3(x,y,z) &= x^3+6xy+6z,\\
H_4(x,y,z) &= x^4+12x^2y+12y^2+24xz,\\
H_5(x,y,z) &= x^5+20x^3y+60xy^2+60x^2z+120yz,\\
H_6(x,y,z) &= x^6+30x^4y+180x^2y^2+120y^3+120x^3z+720xyz+360z^2,\\
H_7(x,y,z) &= x^7+42x^5y+420x^3y^2+840xy^3+210x^4z+2520x^2yz+2520y^2z+2520xz^2,\\
H_8(x,y,z) &= x^8+56x^6y+840x^4y^2+3360x^2y^3+1680y^4+336x^5z+6720x^3yz\\
&\quad +20160xy^2z+10080x^2z^2+20160yz^2.\\
H_9(x,y,z) &= x^9+72x^7y+1512x^5y^2+10080x^3y^3+15120xy^4+504x^6z+15120x^4yz\\
&\quad +90720x^2y^2z+60480y^3z+30240x^3z^2+181440xyz^2+60480z^3,\\
H_{10}(x,y,z) &= x^{10}+90x^8y+2520x^6y^2+25200x^4y^3+75600x^2y^4+30240y^5+720x^7z\\
&\quad +30240x^5yz+302400x^3y^2z+604800xy^3z+75600x^4z^2\\
&\quad +907200x^2yz^2+907200y^2z^2+604800xz^3.
\end{aligned}
$$

Recently, many mathematicians have studied the differential equations arising from the generating functions of special polynomials (see [7, 8, 12, 16–19]). In this paper, we study differential equations arising from the generating functions of the 3-variable Hermite polynomials. We give explicit identities for the 3-variable Hermite polynomials. In addition, we investigate the zeros of the 3-variable Hermite polynomials using numerical methods. Using computer, a realistic study for the zeros of the 3-variable Hermite polynomials is very interesting. Finally, we observe an interesting phenomenon of 'scattering' of the zeros of the 3-variable Hermite polynomials.

2. Differential equations associated with the 3-variable Hermite polynomials

In this section, we study differential equations arising from the generating functions of the 3-variable Hermite polynomials.

Let

$$F = F(t,x,y,z) = e^{xt+yt^2+zt^3} = \sum_{n=0}^{\infty} H_n(x,y,z) \frac{t^n}{n!}, \quad x, y, z, t \in \mathbb{C}. \tag{4}$$

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Then, by (4), we have

$$F^{(1)} = \frac{\partial}{\partial t} F(t,x,y,z) = \frac{\partial}{\partial t}\left(e^{xt+yt^2+zt^3}\right) = e^{xt+yt^2+zt^3}\left(x+2yt+3zt^2\right) \\ = \left(x+2yt+3zt^2\right)F(t,x,y,z), \tag{5}$$

$$F^{(2)} = \frac{\partial}{\partial t} F^{(1)}(t,x,y,z) = (2y+6zt)F(t,x,y,z) + \left(x+2yt+3zt^2\right)F^{(1)}(t,x,y,z) \\ = \left(\left(x^2+2y\right)+(6z+4xy)t+\left(4y^2+6xz\right)t^2+(12yz)t^3+\left(9z^2\right)t^4\right)F(t,x,y,z). \tag{6}$$

Continuing this process, we can guess that

$$F^{(N)} = \left(\frac{\partial}{\partial t}\right)^N F(t,x,y,z) = \sum_{i=0}^{2N} a_i(N,x,y,z)t^i F(t,x,y,z), \, (N=0,1,2,\ldots). \tag{7}$$

Differentiating (7) with respect to t, we have

$$\begin{aligned} F^{(N+1)} = \frac{\partial F^{(N)}}{\partial t} &= \sum_{i=0}^{2N} a_i(N,x,y,z) i t^{i-1} F(t,x,y,z) + \sum_{i=0}^{2N} a_i(N,x,y,z)t^i F^{(1)}(t,x,y,z) \\ &= \sum_{i=0}^{2N} a_i(N,x,y,z) i t^{i-1} F(t,x,y,z) + \sum_{i=0}^{2N} a_i(N,x,y,z)t^i\left(x+2yt+3zt^2\right)F(t,x,y,z) \\ &= \sum_{i=0}^{2N} i a_i(N,x,y,z) t^{i-1} F(t,x,y,z) + \sum_{i=0}^{2N} x a_i(N,x,y,z)t^i F(t,x,y,z) \\ &+ \sum_{i=0}^{2N} 2y a_i(N,x,y,z) t^{i+1} F(t,x,y,z) + \sum_{i=0}^{2N} 3z a_i(N,x,y,z)t^{i+2} F(t,x,y,z) \\ &= \sum_{i=0}^{2N-1} (i+1) a_{i+1}(N,x,y,z) t^i F(t,x,y,z) + \sum_{i=0}^{2N} x a_i(N,x,y,z)t^i F(t,x,y,z) \\ &+ \sum_{i=1}^{2N+1} 2y a_{i-1}(N,x,y,z) t^i F(t,x,y,z) + \sum_{i=2}^{2N+2} 3z a_{i-2}(N,x,y,z)t^i F(t,x,y,z) \end{aligned}$$

Hence we have

$$\begin{aligned} F^{(N+1)} = &\sum_{i=0}^{2N-1} (i+1) a_{i+1}(N,x,y,z) t^i F(t,x,y,z) \\ &+ \sum_{i=0}^{2N} x a_i(N,x,y,z)t^i F(t,x,y,z) \\ &+ \sum_{i=1}^{2N+1} 2y a_{i-1}(N,x,y,z) t^i F(t,x,y,z) \\ &+ \sum_{i=2}^{2N+2} 3z a_{i-2}(N,x,y,z)t^i F(t,x,y,z). \end{aligned} \tag{8}$$

Now replacing N by $N+1$ in (7), we find

$$F^{(N+1)} = \sum_{i=0}^{2N+2} a_i(N+1,x,y,z)t^i F(t,x,y,z). \tag{9}$$

Comparing the coefficients on both sides of (8) and (9), we obtain

$$\begin{aligned}
&a_0(N+1,x,y,z) = a_1(N,x,y,z) + xa_0(N,x,y,z),\\
&a_1(N+1,x,y,z) = 2a_2(N,x,y,z) + xa_1(N,x,y,z) + 2ya_0(N,x,y,z),\\
&a_{2N}(N+1,x,y,z) = xa_{2N}(N,x,y,z) + 2ya_{2N-1}(N,x,y,z) + 3za_{2N-2}(N,x,y,z),\\
&a_{2N+1}(N+1,x,y,z) = 2ya_{2N}(N,x,y,z) + 3za_{2N-1}(N,x,y,z),\\
&a_{2N+2}(N+1,x,y,z) = 3za_{2N}(N,x,y,z),
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
a_i(N+1,x,y,z) = {} & (i+1)a_{i+1}(N,x,y,z) + xa_i(N,x,y,z)\\
& + 2ya_{i-1}(N,x,y,z) + 3za_{i-2}(N,x,y,z), (2 \le i \le 2N-1).
\end{aligned} \tag{11}$$

In addition, by (7), we have

$$F(t,x,y,z) = F^{(0)}(t,x,y,z) = a_0(0,x,y,z)F(t,x,y,z), \tag{12}$$

which gives

$$a_0(0,x,y,z) = 1. \tag{13}$$

It is not difficult to show that

$$\begin{aligned}
& xF(t,x,y) + 2ytF(t,x,y,z) + 3zt^2F(t,x,y,z)\\
& = F^{(1)}(t,x,y,z)\\
& = \sum_{i=0}^{2} a_i(1,x,y,z)F(t,x,y,z)\\
& = \big(a_0(1,x,y,z) + a_1(1,x,y,z)t + a_2(1,x,y,z)t^2\big)F(t,x,y,z).
\end{aligned} \tag{14}$$

Thus, by (14), we also find

$$a_0(1,x,y,z) = x, \quad a_1(1,x,y,z) = 2y, \quad a_2(1,x,y,z) = 3z. \tag{15}$$

From (10), we note that

$$\begin{aligned}
& a_0(N+1,x,y,z) = a_1(N,x,y,z) + xa_0(N,x,y,z),\\
& a_0(N,x,y,z) = a_1(N-1,x,y,z) + xa_0(N-1,x,y,z), \ldots\\
& a_0(N+1,x,y,z) = \sum_{i=0}^{N} x^i a_1(N-i,x,y,z) + x^{N+1},
\end{aligned} \tag{16}$$

and

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$$
\begin{aligned}
&a_{2N+2}(N+1,x,y,z) = 3za_{2N}(N,x,y,z),\\
&a_{2N}(N,x,y,z) = 3za_{2N-2}(N-1,x,y,z), \dots \\
&a_{2N+2}(N+1,x,y,z) = (3z)^{N+1}.
\end{aligned}
\tag{17}
$$

Note that, here the matrix $a_i(j,x,y)_{0\le i\le 2N+2,\,0\le j\le N+1}$ is given by

$$
\begin{pmatrix}
1 & x & 2y+x^2 & \cdot & \cdots & \cdot \\
0 & 2y & 4xy+6z & \cdot & \cdots & \cdot \\
0 & 3z & 6xz+4y^2 & \cdot & \cdots & \cdot \\
0 & 0 & 12yz & \cdot & \cdots & \cdot \\
0 & 0 & (3z)^2 & \cdot & \cdots & \cdot \\
0 & 0 & 0 & \cdot & \cdots & \cdot \\
0 & 0 & 0 & (3z)^3 & \cdots & \cdot \\
\vdots & \vdots & \vdots & \vdots & \ddots & \cdot \\
0 & 0 & 0 & 0 & \cdots & (3z)^{N+1}
\end{pmatrix}
$$

Therefore, we obtain the following theorem.

Theorem 5. For $N = 0, 1, 2, \dots,$ the differential equation

$$
F^{(N)} = \left(\frac{\partial}{\partial t}\right)^N F(t,x,y,z) = \left(\sum_{i=0}^{N} a_i(N,x,y,z)t^i\right) F(t,x,y,z)
$$

has a solution

$$
F = F(t,x,y,z) = e^{xt+yt^2+zt^3},
$$

where

$$
\begin{aligned}
&a_0(N+1,x,y,z) = \sum_{i=0}^{N} x^i a_1(N-i,x,y,z) + x^{N+1},\\
&a_1(N+1,x,y,z) = 2a_2(N,x,y,z) + xa_1(N,x,y,z) + 2ya_0(N,x,y,z),\\
&a_{2N}(N+1,x,y,z) = xa_{2N}(N,x,y,z) + 2ya_{2N-1}(N,x,y,z) + 3za_{2N-2}(N,x,y,z),\\
&a_{2N+1}(N+1,x,y,z) = 2ya_{2N}(N,x,y,z) + 3za_{2N-1}(N,x,y,z),\\
&a_{2N+2}(N+1,x,y,z) = (3z)^{N+1},
\end{aligned}
$$

and

$$a_i(N+1,x,y,z) = (i+1)a_{i+1}(N,x,y,z) + xa_i(N,x,y,z) \\ + 2ya_{i-1}(N,x,y,z) + 3za_{i-2}(N,x,y,z), (2 \le i \le 2N-1).$$

From (4), we note that

$$F^{(N)} = \left(\frac{\partial}{\partial t}\right)^N F(t,x,y,z) = \sum_{k=0}^{\infty} H_{k+N}(x,y,z)\frac{t^k}{k!}. \tag{18}$$

By (4) and (18), we get

$$\begin{aligned} e^{-nt}\left(\frac{\partial}{\partial t}\right)^N F(t,x,y,z) &= \left(\sum_{m=0}^{\infty}(-n)^m\frac{t^m}{m!}\right)\left(\sum_{m=0}^{\infty}H_{m+N}(x,y,z)\frac{t^m}{m!}\right) \\ &= \sum_{m=0}^{\infty}\left(\sum_{k=0}^{m}\binom{m}{k}(-n)^{m-k}H_{N+k}(x,y,z)\right)\frac{t^m}{m!}. \end{aligned} \tag{19}$$

By the Leibniz rule and the inverse relation, we have

$$\begin{aligned} e^{-nt}\left(\frac{\partial}{\partial t}\right)^N F(t,x,y,z) &= \sum_{k=0}^{N}\binom{N}{k}n^{N-k}\left(\frac{\partial}{\partial t}\right)^k\left(e^{-nt}F(t,x,y,z)\right) \\ &= \sum_{m=0}^{\infty}\left(\sum_{k=0}^{N}\binom{N}{k}n^{N-k}H_{m+k}(x-n,y,z)\right)\frac{t^m}{m!}. \end{aligned} \tag{20}$$

Hence, by (19) and (20), and comparing the coefficients of $\frac{t^m}{m!}$ gives the following theorem.

Theorem 6. Let m, n, N be nonnegative integers. Then

$$\sum_{k=0}^{m}\binom{m}{k}(-n)^{m-k}H_{N+k}(x,y,z) = \sum_{k=0}^{N}\binom{N}{k}n^{N-k}H_{m+k}(x-n,y,z). \tag{21}$$

If we take $m = 0$ in (21), then we have the following corollary.

Corollary 7. For $N = 0, 1, 2, \ldots,$ we have

$$H_N(x,y,z) = \sum_{k=0}^{N}\binom{N}{k}n^{N-k}H_k(x-n,y,z).$$

For $N = 0, 1, 2, \ldots,$ the differential equation

$$F^{(N)} = \left(\frac{\partial}{\partial t}\right)^N F(t,x,y,z) = \left(\sum_{i=0}^{N}a_i(N,x,y,z)t^i\right)F(t,x,y,z)$$

has a solution

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$$F = F(t, x, y, z) = e^{xt+yt^2+zt^3}.$$

Here is a plot of the surface for this solution. In **Figure 1(left)**, we choose $-2 \le z \le 2$, $-1 \le t \le 1$, $x = 2$, and $y = -4$. In **Figure 1(right)**, we choose $-5 \le x \le 5$, $-1 \le t \le 1$, $y = -3$, and $z = -1$.

3. Distribution of zeros of the 3-variable Hermite polynomials

This section aims to demonstrate the benefit of using numerical investigation to support theoretical prediction and to discover new interesting pattern of the zeros of the 3-variable Hermite polynomials $H_n(x, y, z)$. By using computer, the 3-variable Hermite polynomials $H_n(x, y, z)$ can be determined explicitly. We display the shapes of the 3-variable Hermite polynomials $H_n(x, y, z)$ and investigate the zeros of the 3-variable Hermite polynomials $H_n(x, y, z)$. We investigate the beautiful zeros of the 3-variable Hermite polynomials $H_n(x, y, z)$ by using a computer. We plot the zeros of the $H_n(x, y, z)$ for $n = 20$, $y = 1, -1, 1+i, -1-i$, $z = 3, -3, 3+i, -3-i$ and $x \in \mathbb{C}$ (**Figure 2**). In **Figure 2(top-left)**, we choose $n = 20$, $y = 1$, and $z = 3$. In **Figure 2(top-right)**, we choose $n = 20$, $y = -1$, and $z = -3$. In **Figure 2(bottom-left)**, we choose $n = 20$, $y = 1 + i$, and $z = 3 + i$. In **Figure 2(bottom-right)**, we choose $n = 20$, $y = -1 - i$, and $z = -3 - i$.

In **Figure 3(top-left)**, we choose $n = 20$, $x = 1$, and $y = 1$. In **Figure 3(top-right)**, we choose $n = 20$, $x = -1$, and $y = -1$. In **Figure 3(bottom-left)**, we choose $n = 20$, $x = 1 + i$, and $y = 1 + i$. In **Figure 3(bottom-right)**, we choose $n = 20$, $x = -1 - i$, and $y = -1 - i$.

Stacks of zeros of the 3-variable Hermite polynomials $H_n(x, y, z)$ for $1 \le n \le 20$ from a 3-D structure are presented (**Figure 3**). In **Figure 4(top-left)**, we choose $n = 20$, $y = 1$, and $z = 3$. In **Figure 4 (top-right)**, we choose $n = 20$, $y = -1$, and $z = -3$. In **Figure 4(bottom-left)**, we choose $n = 20$, $y = 1 + i$, and $z = 3 + i$. In **Figure 4(bottom-right)**, we choose $n = 20$, $y = -1 - i$, and $z = -3 - i$.

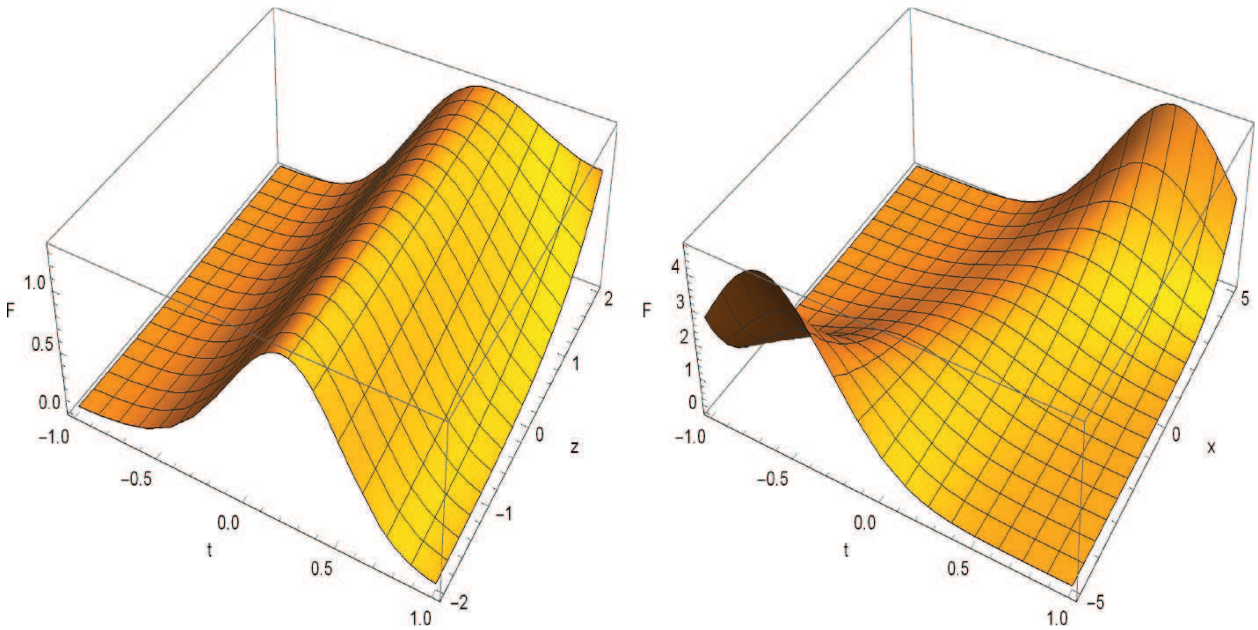

Figure 1. The surface for the solution $F(t, x, y, z)$.

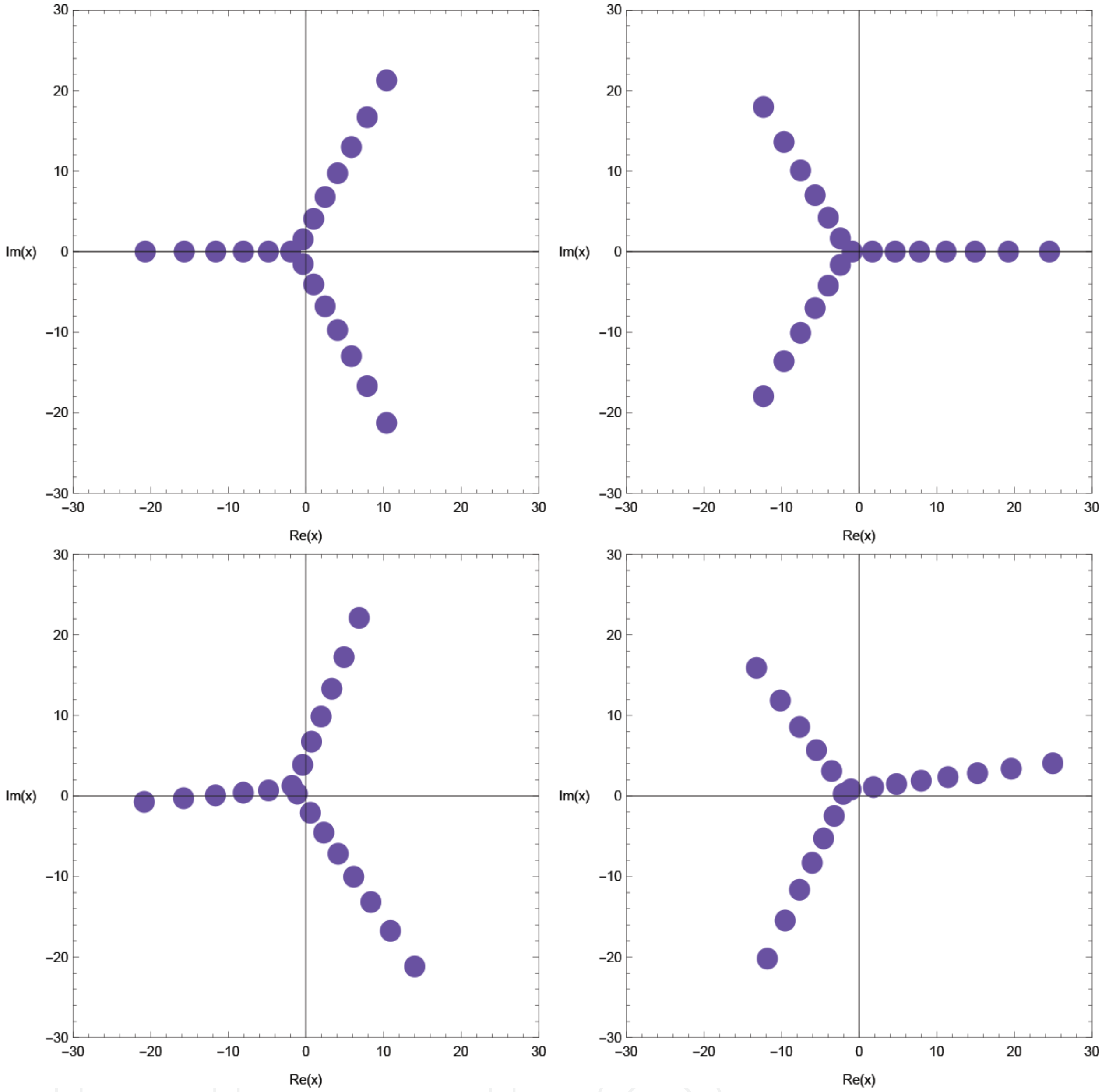

Figure 2. Zeros of $H_n(x, y, z)$.

Our numerical results for approximate solutions of real zeros of the 3-variable Hermite polynomials $H_n(x, y, z)$ are displayed (**Tables 1–3**).

The plot of real zeros of the 3-variable Hermite polynomials $H_n(x, y, z)$ for $1 \leq n \leq 20$ structure are presented (**Figure 5**).

In **Figure 5(left)**, we choose $y = 1$ and $z = 3$. In **Figure 5(right)**, we choose $y = -1$ and $z = -3$.

Stacks of zeros of $H_n(x, -2, 4)$ for $1 \leq n \leq 40$, forming a 3D structure are presented (**Figure 6**). In **Figure 6(top-left)**, we plot stacks of zeros of $H_n(x, -2, 4)$ for $1 \leq n \leq 20$. In **Figure 6(top-right)**, we draw x and y axes but no z axis in three dimensions. In **Figure 6(bottom-left)**, we draw y

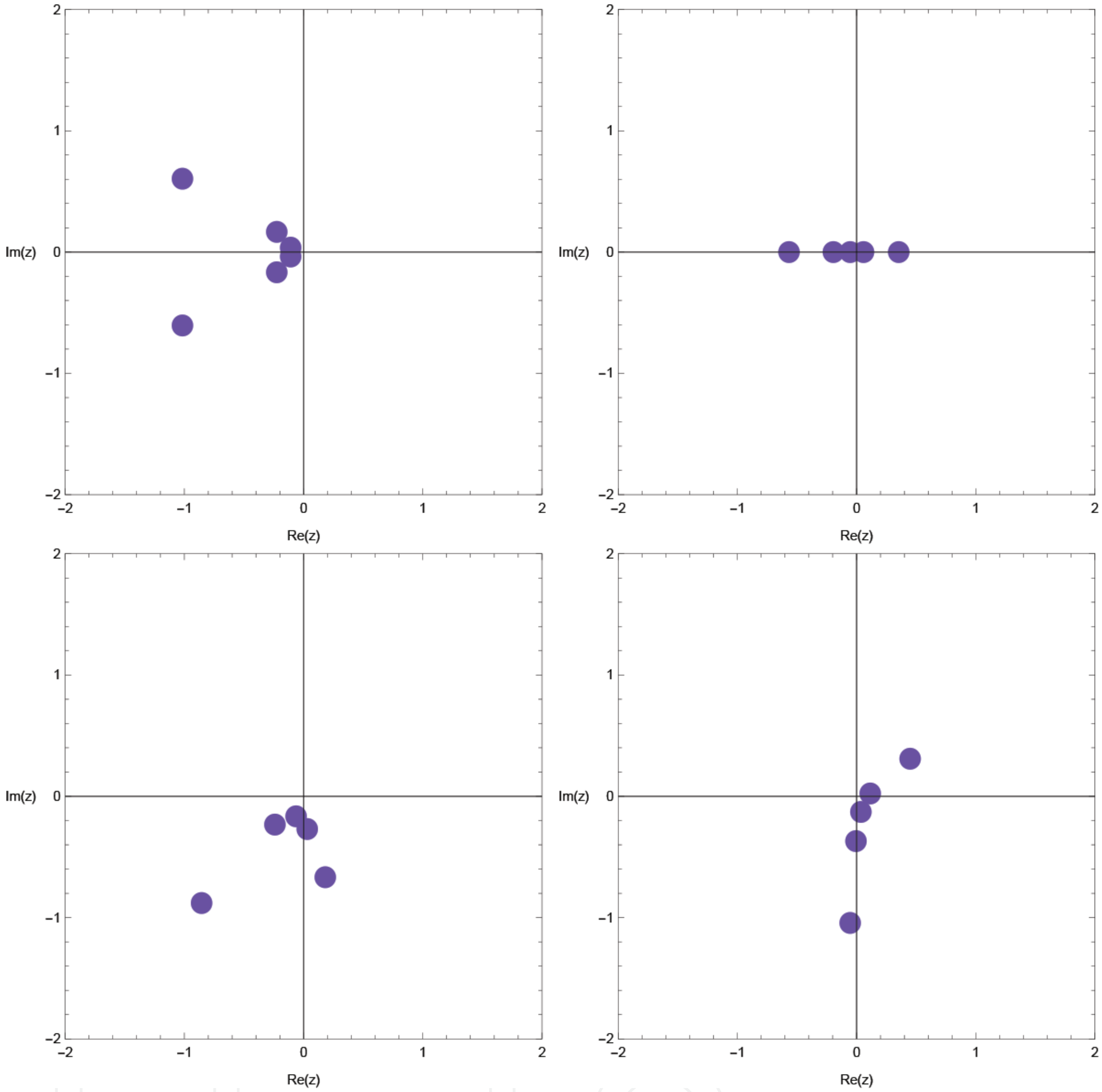

Figure 3. Zeros of $H_n(x, y, z)$.

and z axes but no x axis in three dimensions. In **Figure 6(bottom-right)**, we draw x and z axes but no y axis in three dimensions.

It is expected that $H_n(x, y, z), x \in \mathbb{C}, y, z \in \mathbb{R}$, has $\mathrm{Im}(x) = 0$ reflection symmetry analytic complex functions (see **Figures 2–7**). We observe a remarkable regular structure of the complex roots of the 3-variable Hermite polynomials $H_n(x, y, z)$ for $y, z \in \mathbb{R}$. We also hope to verify a remarkable regular structure of the complex roots of the 3-variable Hermite polynomials $H_n(x, y, z)$ for $y, z \in \mathbb{R}$ (**Tables 1** and **2**). Next, we calculated an approximate solution satisfying $H_n(x, y, z) = 0, x \in \mathbb{C}$. The results are given in **Tables 3** and **4**.

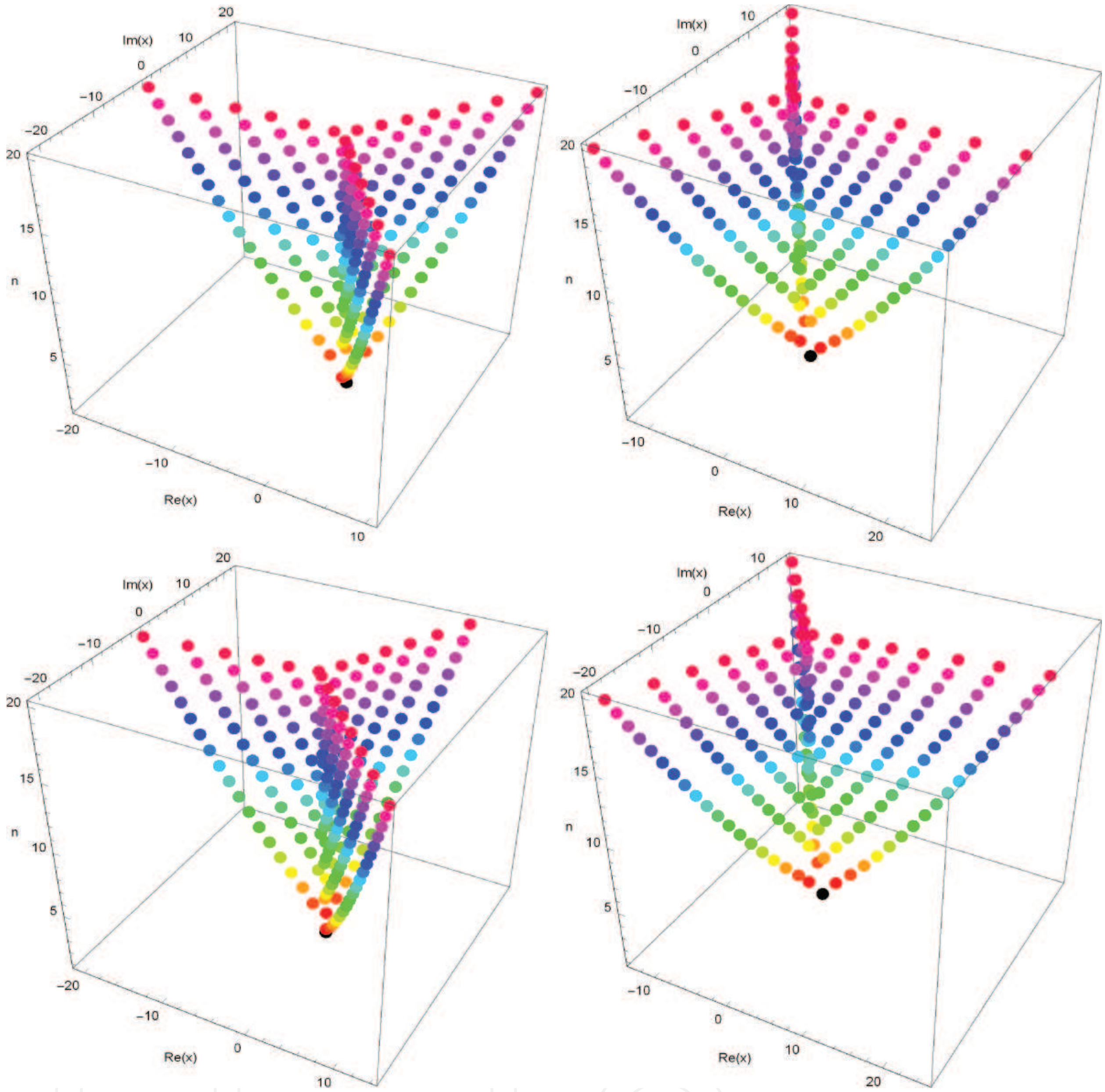

Figure 4. Stacks of zeros of $H_n(x, y, z)$, $1 \le n \le 20$.

The plot of real zeros of the 3-variable Hermite polynomials $H_n(x, y, z)$ for $1 \le n \le 20$ structure are presented (**Figure 7**).

In **Figure 7(left)**, we choose $x = 1$ and $y = 2$. In **Figure 7(right)**, we choose $x = -1$ and $y = -2$.

Finally, we consider the more general problems. How many zeros does $H_n(x, y, z)$ have? We are not able to decide if $H_n(x, y, z) = 0$ has n distinct solutions. We would also like to know the number of complex zeros $C_{H_n(x,y,z)}$ of $H_n(x, y, z)$, $\mathrm{Im}(x) \neq 0$. Since n is the degree of the polynomial $H_n(x, y, z)$, the number of real zeros $R_{H_n(x,y,z)}$ lying on the real line $\mathrm{Im}(x) = 0$ is then $R_{H_n(x,y,z)} = n - C_{H_n(x,y,z)}$, where $C_{H_n(x,y,z)}$ denotes complex zeros. See **Tables 1** and **2** for

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Degree n	Real zeros	Complex zeros
1	1	0
2	0	2
3	1	2
4	2	2
5	1	4
6	2	4
7	3	4
8	2	6
9	3	6
10	4	6
11	3	8
12	4	8
13	3	10
14	4	10

Table 1. Numbers of real and complex zeros of $H_n(x, 1, 3)$.

Degree n	Real zeros	Complex zeros
1	1	0
2	2	0
3	1	2
4	2	2
5	3	2
6	2	4
7	3	4
8	4	4
9	3	6
10	4	6
11	5	6
12	6	6
13	5	8
14	6	8

Table 2. Numbers of real and complex zeros of $H_n(x, -1, -3)$.

Degree n	x
1	0
2	—
3	-1.8845
4	$3.1286,\quad -0.17159$
5	-4.5385
6	$-5.8490,\quad -1.3476$
7	$-7.1098,\quad -2.1887,\quad -0.36350$
8	$-8.3241,\quad -3.4645$
9	$-9.4984,\quad -4.6021,\quad -1.1118$
10	$-10.637,\quad -5.7212,\quad -1.5785,\quad -0.61919$
11	$-11.745,\quad -6.8105,\quad -2.8680$
12	$-12.824,\quad -7.8743,\quad -3.8894,\quad -0.99513$

Table 3. Approximate solutions of $H_n(x,1,3)=0, x\in \mathrm{R}$.

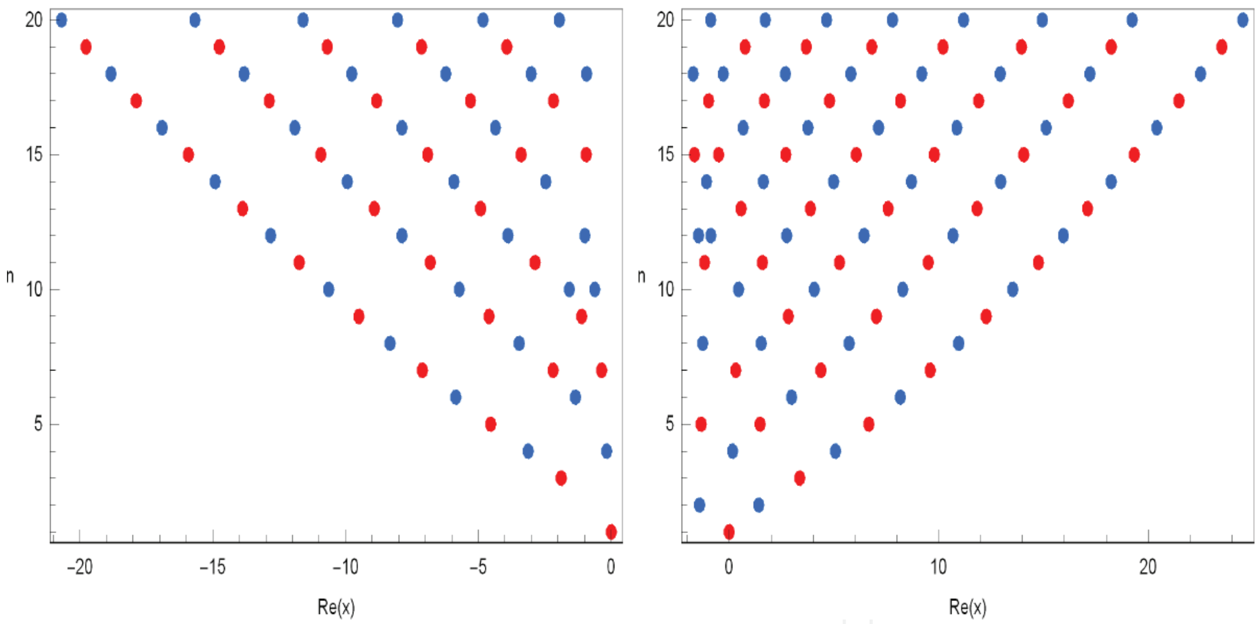

Figure 5. Real zeros of $H_n(x,y,z), 1\le n\le 20$.

tabulated values of $R_{H_n(x,y,z)}$ and $C_{H_n(x,y,z)}$. The author has no doubt that investigations along these lines will lead to a new approach employing numerical method in the research field of the 3-variable Hermite polynomials $H_n(x,y,z)$ which appear in mathematics and physics. The reader may refer to [2, 11, 13, 20] for the details.

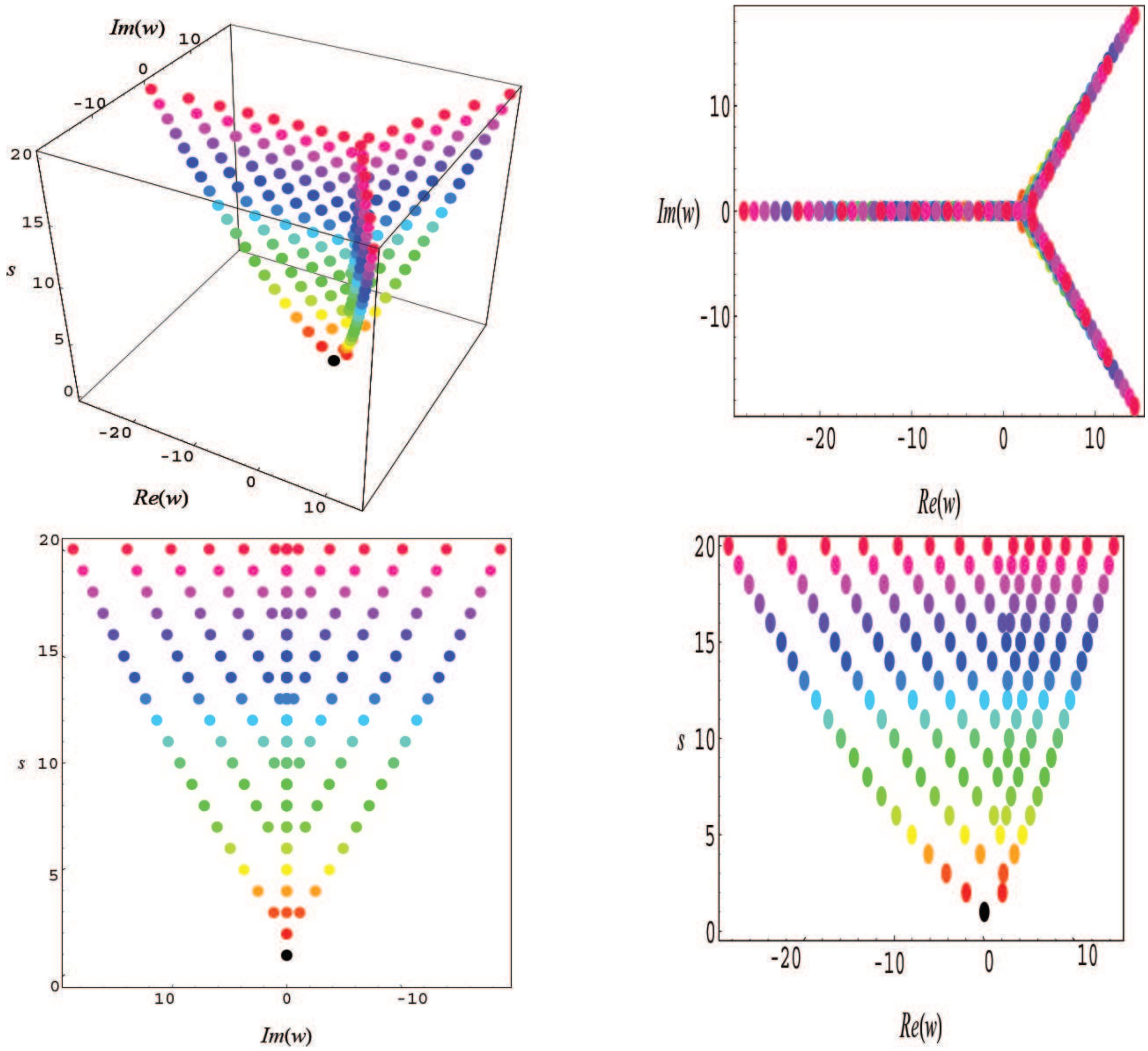

Figure 6. Stacks of zeros of $H_n(x, -2, 4)$ *for* $1 \le n \le 20$.

degree n	x
1	0
2	−1.4142, 1.4142
3	3.3681
4	0.16229, 5.0723
5	−1.3404, 1.4745, 6.6661
6	2.9754, 8.1678
7	0.31213, 4.3783, 9.5946

degree n	x
8	$-1.2604, 1.5304, 5.7274, 10.959$
9	$2.8224, 7.0271, 12.270$
10	$0.44594, 4.0615, 8.2834, 13.535$
11	$-1.1740, 1.5825, 5.2667, 9.5013, 14.760$
12	$-1.4659, -0.87728, 2.7469, 6.4398, 10.685, 15.949$

Table 4. Approximate solutions of $H_n(x, -1, -3) = 0, x \in \mathbb{R}$.

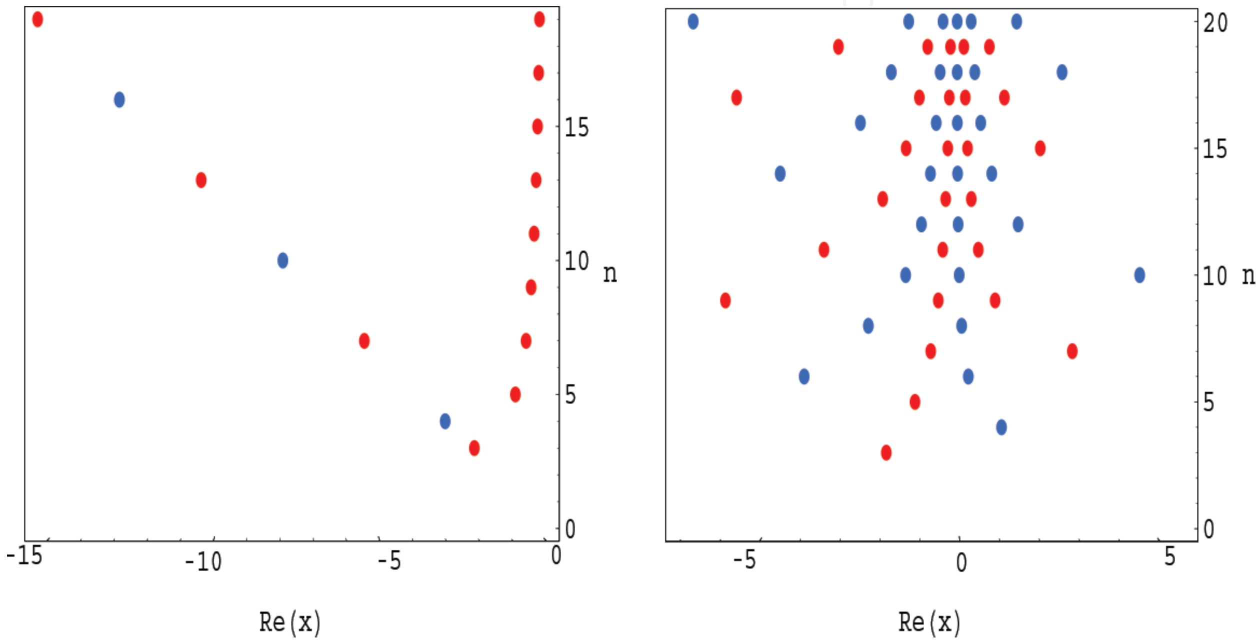

Figure 7. Real zeros of $H_n(x, y, z), 1 \le n \le 20$.

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