Chapter 14

Weighted Finite-Element Method for Elasticity Problems with Singularity

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Additional information is available at the end of the chapter

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Abstract

In this chapter, the two-dimensional elasticity problem with a singularity caused by the presence of a re-entrant corner on the domain boundary is considered. For this problem, the notion of the R_v-generalized solution is introduced. On the basis of the R_v-generalized solution, a scheme of the weighted finite-element method (FEM) is constructed. The proposed method provides a first-order convergence of the approximate solution to the exact one with respect to the mesh step in the $\mathbf{W}^1_{2,\nu}(\Omega)$-norm. The convergence rate does not depend on the size of the angle and kind of the boundary conditions imposed on its sides. Comparative analysis of the proposed method with a classical finite-element method and with an FEM with geometric mesh refinement to the singular point is carried out.

Keywords: elasticity problem with singularity, corner singularity, R_v-generalized solution, weighted finite-element method, numerical experiments

1. Introduction

The singularity of the solution to a boundary value problem can be caused by the degeneration of the input data (of the coefficients and right-hand sides of the equation and the boundary conditions), by the geometry of the boundary, or by the internal properties of the solution. The classic numerical methods, such as finite-difference method, finite- and boundary-element methods, have insufficient convergence rate due to singularity which has an influence on the regularity of the solution. It results in significant increase of the computational power and time required for calculation of the solution with the given accuracy. For example, the classic finite-element method allows the finding of the solution for the elasticity problem posed in a two-dimensional domain containing a re-entrant corner of on the boundary with convergence rate $O(h^{1/2})$. In this case to compute the solution with the accuracy of 10^{-3} requires a computational power that is one million times greater than in the case of the weighted finite-element method used for the solution of the same problem.

By using meshes refined toward the singularity point, it is possible to construct schemes of the finite-element method with the first order of the rate of convergence of the approximate solution to the exact one [1–3].

In [4, 5], for boundary value problems with strongly singular solutions for which a generalized solution could not be defined and it does not belong to the Sobolev space H^1, it was proposed to define the solution as a R_v-generalized one. The existence and uniqueness of solutions as well as its coercivity and differential properties in the weighted Sobolev spaces and sets were proved [5–10], the weighted finite-element method was built, and its convergence rate was investigated [11–15].

In this chapter, for the Lamé system in domains containing re-entrant corners we will state construction and investigation of the weighted FEM for determination of the R_v-generalized solution [16, 17]. Convergence rate of this method did not depend on the corner size and was equal $O(h)$ (see [18], Theorem 2.1). For the elasticity problems with solutions of two types—with both singular and regular components and with singular component only—a comparative numerical analysis of the weighted finite-element method, the classic FEM, and the FEM with meshes geometrically refined toward the singularity point is performed. For the first two methods, the theoretical convergence rate estimations were confirmed. In addition, it was established that FEM with graded meshes failed on high dimensional meshes but weighted FEM stably found approximate solution with theoretical accuracy under the same computational conditions. The mentioned failure can be explained by a small size of steps of the graded mesh in a neighbourhood of the singular point. As a result, for the majority of nodes, the weighted finite-element method allows to find solution with absolute error which is by one or two orders of magnitude less than that for the FEM with graded meshes.

2. R_v-generalized solution

Let $\Omega = (-1,1) \times (-1,1) \backslash [0,1] \times [-1,0] \subset R^2$ be an L-shaped domain with boundary $\partial\Omega$ containing re-entrant corner of $3\pi/2$ with the vertex located in the point $O(0,0)$, $\overline{\Omega} = \Omega \cup \partial\Omega$.

Denote by $\Omega' = \left\{x \in \Omega : \left(x_1^2 + x_2^2\right)^{1/2} \leq \delta < 1\right\}$ a part of δ-neighbourhood of the point (0,0) laying in the $\overline{\Omega}$. A weight function $\rho(x)$ can be introduced that coincides with the distance to the origin in $\overline{\Omega}'$, and equals δ for $x \in \overline{\Omega}\ \overline{\Omega}'$.

Let $W^1_{2,\alpha}(\Omega, \delta)$be the set of functions satisfying the following conditions:

a. $\left|D^k u(x)\right| \leq c_1 (\delta/\rho(x))^{\alpha+k}$ for $x \in \overline{\Omega}'$, where $k = 0{,}1$ and c_1 is a positive constant independent on k,

b. $\|u\|_{L_{2,\alpha}(\Omega - \Omega')} \geq c_2 > 0$,

with the norm

$$\|u\|_{W^1_{2,\alpha}(\Omega)} = \left(\sum_{|\lambda| \leq 1} \int_\Omega \rho^{2\alpha} \left|D^\lambda u\right|^2 dx\right)^{1/2}, \tag{1}$$

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where $D^\lambda = \partial^{|\lambda|}/\partial x_1^{\lambda_1}\partial x_2^{\lambda_2}$, $\lambda = (\lambda_1,\lambda_2)$, and $|\lambda|=\lambda_1+\lambda_2$; λ_1, λ_2 are nonnegative integers, and α is a nonnegative real number.

Let $L_{2,\alpha}(\Omega,\delta)$ be the set of functions satisfying conditions (a) and (b) with the norm

$$\|u\|_{L_{2,\alpha}(\Omega)} = \left(\int_\Omega \rho^{2\alpha} u^2\, dx\right)^{1/2}.$$

The set $\overset{\circ}{W}{}^1_{2,\alpha}(\Omega,\delta) \subset W^1_{2,\alpha}(\Omega,\delta)$ is defined as the closure in norm (1) of the set $C_0(\Omega,\delta)$ of infinitely differentiable and finite in Ω functions satisfying conditions (a) and (b).

One can say that $\varphi \in W^{1/2}_{2,\alpha}(\partial\Omega,\delta)$ if there exists a function Φ from $W^1_{2,\alpha}(\Omega,\delta)$ such that $\Phi(x)|_{\partial\Omega} = \varphi(x)$ and

$$\|\varphi\|_{W^{1/2}_{2,\alpha}(\partial\Omega,\delta)} = \inf_{\Phi|_{\partial\Omega}=\phi} \|\Phi\|_{W^1_{2,\alpha}(\Omega,\delta)}.$$

For the corresponding spaces and sets of vector-functions are used notations$\mathbf{W}^1_{2,\alpha}(\Omega,\delta)$, $\mathbf{L}_{2,\alpha}(\Omega,\delta)$, $\overset{\circ}{\mathbf{W}}{}^1_{2,\alpha}(\Omega,\delta)$.

Let $\mathbf{u} = (u_1,u_2)$ be a vector-function of displacements. Assume that $\overline{\Omega}$ is a homogeneous isotropic body and the strains are small. Consider a boundary value problem for the displacement field $\mathbf{u}$ for the Lamé system with constant coefficients λ and μ:

$$-\left(2\mathbf{div}\left(\mu\varepsilon(\mathbf{u})\right) + \nabla(\lambda \mathrm{div}\mathbf{u})\right) = \mathbf{f}, \quad x \in \Omega, \tag{2}$$

$$u_i = q_i, \quad x \in \partial\Omega, \tag{3}$$

Here, $\varepsilon(\mathbf{u})$ is a strain tensor with components $\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$.

Assume that the right-hand sides of (2), (3) satisfy the conditions

$$\mathbf{f} \in \mathbf{L}_{2,\beta}(\Omega,\delta), \quad q_i \in W^{1/2}_{2,\beta}(\partial\Omega,\delta), \quad i = 1,2, \quad \beta > 0. \tag{4}$$

Denoted by

$$a_1(\mathbf{u},\mathbf{v}) = \int_\Omega \left[(\lambda+2\mu)\frac{\partial u_1}{\partial x_1}\frac{\partial\left(\rho^{2\nu}v_1\right)}{\partial x_1} + \mu\frac{\partial u_1}{\partial x_2}\frac{\partial\left(\rho^{2\nu}v_1\right)}{\partial x_2} + \lambda\frac{\partial u_2}{\partial x_2}\frac{\partial\left(\rho^{2\nu}v_1\right)}{\partial x_1} + \mu\frac{\partial u_2}{\partial x_1}\frac{\partial\left(\rho^{2\nu}v_1\right)}{\partial x_2}\right]dx,$$

$$a_2(\mathbf{u},\mathbf{v}) = \int_\Omega \left[\lambda\frac{\partial u_1}{\partial x_1}\frac{\partial\left(\rho^{2\nu}v_2\right)}{\partial x_2} + \mu\frac{\partial u_1}{\partial x_2}\frac{\partial\left(\rho^{2\nu}v_2\right)}{\partial x_1} + (\lambda+2\mu)\frac{\partial u_2}{\partial x_2}\frac{\partial\left(\rho^{2\nu}v_2\right)}{\partial x_2} + \mu\frac{\partial u_2}{\partial x_1}\frac{\partial\left(\rho^{2\nu}v_2\right)}{\partial x_1}\right]dx,$$

$$l_1(\mathbf{v}) = \int_\Omega \rho^{2\nu} f_1 v_1 dx, \quad l_2(\mathbf{v}) = \int_\Omega \rho^{2\nu} f_2 v_2 dx$$

the bilinear and linear forms and $a(\mathbf{u},\mathbf{v}) = (a_1(\mathbf{u},\mathbf{v}), a_2(\mathbf{u},\mathbf{v}))$, $l(\mathbf{v}) = (l_1(\mathbf{v}), l_2(\mathbf{v}))$.

Definition 1

A function $\mathbf{u}_v$ *from the set* $\mathbf{W}^1_{2,\nu}(\Omega, \delta)$ *is called an* R_v*-generalized solution to the problem (2), (3) if it satisfies boundary condition (3) almost everywhere on* $\partial\Omega$ *and for every* $\mathbf{v}$ *from* $\mathbf{W}^1_{2,\nu}(\Omega, \delta)$ *the integral identity*

$$a(\mathbf{u}_\nu, \mathbf{v}) = l(\mathbf{v}) \tag{5}$$

holds for any fixed value of ν *satisfying the inequality*

$$\nu \geq \beta. \tag{6}$$

In [17], for the boundary value problem (2)–(3) with homogeneous boundary conditions, existence and uniqueness of its R_v-generalized solution were established.

Theorem 1

Let condition (4) be satisfied. Then for any $\nu > \beta$ *there always exists parameter* δ *such that the problem (2)–(3) with homogeneous boundary conditions has a unique* R_v*-generalized solution* $\mathbf{u}_v$ *in the set* $\mathbf{W}^{\circ 1}_{2,\alpha}(\Omega, \delta)$*. In this case*

$$\|\mathbf{u}_\nu\|_{\mathbf{W}^1_{2,\nu}(\Omega)} \leq c_3 \|\mathbf{f}\|_{\mathbf{L}_{2,\beta}(\Omega)}, \tag{7}$$

where c_3 *is a positive constant independent of* $\mathbf{f}$.

Then for any $\nu > \beta$*, there always exists parameter* δ *such that the problem (2)–(3) with homogeneous boundary conditions has a unique* R_v*-generalized solution* $\mathbf{u}_v$ *in the set* $\mathbf{W}^{\circ 1}_{2,\alpha}(\Omega, \delta)$.

Comment 1

At present, there exists a complete theory of classical solutions to boundary value problems with smooth initial data (equation coefficients, right hands of solution and boundary conditions) and with smooth enough domain boundary [19–22].

On the basis of the generalized solution-wide investigations of boundary value problems with discontinuous initial data and not smooth domain boundary were performed in Sobolev and different weighted spaces [23–26]. On the basis of the Galerkin method, theories of difference schemes, finite volumes, and finite-element method were developed to find approximate generalized solution [27].

Let us call boundary value problem a problem with strong singularity if its generalized solution could not be defined. This solution does not belong to the Sobolev space W^1_2 (H^1), or, in other words, the Dirichlet integral of the solution diverges. In [4, 5], we suggested to define a solution to the boundary value problems with strong singularity as an R_v-generalized one in the weighted Sobolev space. The essence of this approach is in introducing weight function into the integral equality. The weight function coincides with the distance to the singular points in their neighbourhoods. The role (sense, mission) of this function is in suppressing of the solution singularity caused by the problem features and is in assuring convergence of

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integrals in both parts of the integral equality. Taking into account the local character of the singularity, we define weight function as the distance to each singularity point inside the disk of radius δ centered in that points, and outside these disks the weight function equals δ. An exponent of the weight function in the definition of the R_ν-generalized solution as well as weighted space containing this solution depend on the spaces to which problem initial data belongs, on geometrical features of the boundary (re-entrant corners), and on changing of the boundary condition type.

In [13, 14], for the transformed system of Maxwell equations in the domain with re-entrant corner in which the solution does not depend on the space W_2^1, the weighted edge-based finite-element method was developed on the basis of introducing the R_ν-generalized solution. Convergence rate of this method is $O(h)$, and it does not depend on the size of singularity as opposed to other methods [28, 29].

The proposed approach of introducing R_ν-generalized solution allows to effectively find solutions not only to the boundary value problems with divergent Dirichlet integral but also to problems with weak singularity when the solution belongs to the W_2^1 and does not belong to the space W_2^2.

3. The weighted finite-element method

A finite-element scheme for problems (2)–(3) is constructed relying on the definition of an R_ν-generalized solution. For this purpose, a quasi-uniform triangulation T^h of $\overline{\Omega}$ and introduction of special basis functions are constructed.

The domain $\overline{\Omega}$ is divided into a finite number of triangles K (called finite elements) with vertices P_k (k = 1,…,N), which are triangulation nodes. Denoted by $\Omega^h = \cup_{K \in T^h} K$—the union of all elements; here, h is the longest of their side lengths. It is required that the partition satisfies the conventional constraints imposed on triangulations [10]. Denote by $P = \{P_k\}_{k=1}^{k=n}$, the set of triangulation internal nodes; by $P = \{P_k\}_{k=n+1}^{k=N}$, the set of nodes belonging to the $\partial\Omega$.

Each node $P_k \in P$ is associated with a function Ψ_k of the form

$$\Psi_k(x) = \rho^{\nu^*}(x)\phi_k(x), \quad k = 1, \dots, n,$$

where $\phi_k(x)$ is linear on each finite element, $\phi_k(P_j) = \delta_{kj}$, $k, j = 1, \dots, n$ δ_{kj} is the Kronecker delta, and ν^* is a real number.

The set V^h is defined as the linear span of the system of basis functions $\{\Psi_k\}_{k=1}^{k=n}$. Denote the corresponding vector set by $\mathbf{V}^h = \left[V^h\right]^2$. In set $\mathbf{V}^h$, one singled out the subset $\mathbf{V}^{\circ h} = \left\{\mathbf{v} \in \mathbf{V}^h, v_i (P_k)|_{P_k \in \partial\Omega} = 0, i = 1, 2\right\}$.

Associated with the constructed triangulation, the finite-element approximation of the displacement vector components has the form

$$u^h_{\nu,1} = \sum_{k=1}^{n} d_{2k-1}\Psi_k, \quad u^h_{\nu,2} = \sum_{k=1}^{n} d_{2k}\Psi_k, \quad d_j = \rho^{-\nu^*}\left(P_{\left[\frac{j+1}{2}\right]}\right)c_j, \quad j = 1, \dots, 2n.$$

Definition 2

An approximate R_ν-generalized solution to the problems (2)–(3) by the weighted finite-element method is a function $\mathbf{u}^h_\nu \in \mathbf{V}^h$ such that it satisfies the boundary condition (3) in the nodes of the boundary $\partial\Omega$ and for arbitrary $\mathbf{v}^h(x) \in \mathbf{V}^h$ and $\nu > \beta$ the integral identity

$$a\left(\mathbf{u}^h_\nu, \mathbf{v}^h\right) = l\left(\mathbf{v}^h\right),$$

holds, where $\mathbf{u}^h_\nu = \left(u^h_{\nu,1}, u^h_{\nu,2}\right)$.

In [18], it was shown that convergence rate of the approximate solution to the exact one does not depend on size of the re-entrant corner and is always equal to $O(h)$ when weighted finite-element method is used for finding an R_ν-generalized solution to elasticity problem. The next section explains results of comparative numerical analysis for the model problems (2)–(3) of the weighted FEM using the classical finite-element method and the FEM with geometrically graded meshes of two kinds.

4. Results of numerical experiments

In the domain, Ω is considered a Dirichlet problem for the Lamé system (2), (3) with constant coefficients $\lambda = 3$ and $\mu = 5$. Two kinds of vector-function $\mathbf{u} = (u_1, u_2)$ were used as a solution to the problem.

Problem A

Components of the solution $\mathbf{u}$ of the model problem (2), (3) contain only a singular component

$$u_1 = \cos(x_1)\cos^2(x_2)\left(x_1^2 + x_2^2\right)^{0.3051},$$

$$u_2 = \cos^2(x_1)\cos(x_2)\left(x_1^2 + x_2^2\right)^{0.3051}.$$

Singularity order of u_1, u_2 corresponds to the size of the re-entrant corner $\gamma = 3\pi/2$ on the domain boundary [30].

Problem B

Solution $\mathbf{u}$ of the model problems (2, 3) contains both singular and regular components—regular part belongs to the $\mathbf{W}^2_2(\Omega)$

$$u_1 = \cos(x_1)\cos^2(x_2)\left(x_1^2 + x_2^2\right)^{0.3051} + \left(x_1^2 + x_2^2\right),$$

$$u_2 = \cos^2(x_1)\cos(x_2)\left(x_1^2 + x_2^2\right)^{0.3051} + \left(x_1^2 + x_2^2\right).$$

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4.1. Comparative analysis of the generalized and R_ν-generalized solutions

Results of numerical experiments presented in this subsection were obtained using the code "Proba-IV" [31] with regular meshes which were built by the following scheme:

Domain Ω was divided into squares by lines parallel to coordinate axis, with distance equal to $1/N$ between them, where N is a half of number of partitioning segments along the greater side;

Each square was subdivided into two triangles by the diagonal.

In this case, size of the mesh-step h could be computed by $h = \sqrt{2}/N$. Example of the regular mesh for $N = 4$ is presented in **Figure 1**.

Calculations were performed for different values of N. Optimal parameters δ, ν, and ν^* were obtained by the program complex [32]. Generalized solution was determined by the integral equality (5) for $\nu = 0$.

One calculated the errors $\mathbf{e} = (e_1, e_2) = \left(u_1 - u_1^h, u_2 - u_2^h\right)$ and $\mathbf{e}_\nu = (e_{\nu,1}, e_{\nu,2}) = \left(u_1 - u_{\nu,1}^h, u_2 - u_{\nu,2}^h\right)$ of numerical approximation to the generalized $\mathbf{u}^h = \left(u_1^h, u_2^h\right)$ and R_ν-generalized $\mathbf{u}_\nu^h = \left(u_{1,\nu}^h, u_{2,\nu}^h\right)$ solutions, respectively. Problems A and B in **Tables 1** and **4**, respectively, present values of relative errors of the generalized solution in the norm of the Sobolev space $\mathbf{W}_2^1\left(\eta = \frac{\|\mathbf{e}\|_{W_2^1}}{\|\mathbf{u}\|_{W_2^1}}\right)$ and the R_ν-generalized one in the norm of the weighted Sobolev space $\mathbf{W}_{2,\nu}^1$ $\left(\eta_\nu = \frac{\|\mathbf{e}_\nu\|_{W_{2,\nu}^1}}{\|\mathbf{u}\|_{W_{2,\nu}^1}}\right)$ with different values of h. In addition, these tables contain ratios between error norms, obtained on meshes with step reducing twice. **Figures 2** and **3** show the convergence rates of the generalized and R_ν-generalized solutions to the corresponding problems with the logarithmic scale. The dashed line in the figures corresponds to convergence with the rate $O(h)$. **Tables 2** and **3** (Problem A) and **Tables 5** and **6** (Problem B) give limit values: number of nodes where $|e_1|$, $|e_2|$, $|e_{v,1}|$, and $|e_{v,2}|$ belong to the giving range, this number in percentage to the total number of nodes, and pictures of the absolute error distribution in the domain Ω.

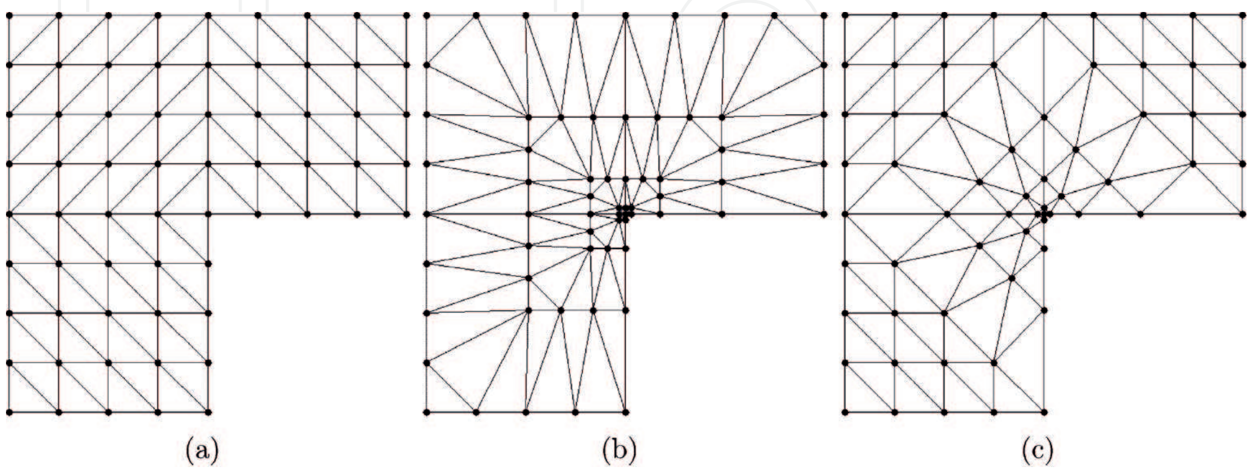


Figure 1. Example of regular mesh (**a**), and graded meshes I (**b**) and II (**c**) ($N = 4$, $\kappa = 0.4$).

4.1.1. Problem A

$2N$	128		256		512		1024		2048		4096
h	1.105e-2		5.524e-3		2.762e-3		1.381e-3		6.905e-4		3.453e-4
η	6.963e-2	1.52	4.579e-2	1.52	3.007e-2	1.52	1.972e-2	1.53	1.293e-2	1.53	8.476e-3
η_ν	7.011e-2	1.55	4.522e-2	1.64	2.756e-2	2.17	1.272e-2	2.21	5.745e-3	1.98	2.902e-3

Table 1. Dependence of relative errors of the generalized (η) and R_ν-generalized (η_ν) ($\delta = 0.0029$, $\nu = 1.2$, $\nu^* = 0.16$) solution to problem A on mesh step.

$\lvert e_1 \rvert$	$\lvert e_2 \rvert$	**Limit values**	$\lvert e_1 \rvert$		$\lvert e_2 \rvert$	
Distribution				**Number**	**%**	**Number**
		$\geq 5e-6$	48.077	6045579	48.077	6045579
		$\geq 1e-6$	29.387	3695290	29.387	3695290
		$\geq 5e-7$	6.724	845468	6.724	845468
		$\geq 1e-7$	9.624	1210192	9.624	1210192
		$\geq 5e-8$	2.564	322449	2.564	322449
		≥ 0	3.624	455743	3.624	455743

Table 2. Number, percentage equivalence, and distribution of nodes where absolute errors $|e_i|$ $(i = 1, 2)$ of finding components of the approximate generalized solution to problem A are not less than given limit values, $2N = 4096$.

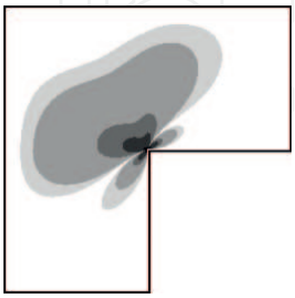
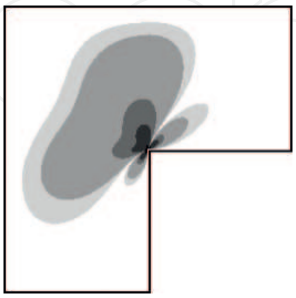
$\lvert e_1 \rvert$	$\lvert e_2 \rvert$	**Limit values**	$\lvert e_1 \rvert$		$\lvert e_2 \rvert$	
Distribution			**%**	**Number**	**%**	**Number**
		$\geq 5e-6$	0.033	4102	0.033	4102
		$\geq 1e-6$	0.764	96075	0.764	96075
		$\geq 5e-7$	2.457	308985	2.457	308985
		$\geq 1e-7$	21.704	2729186	21.704	2729186
		$\geq 5e-8$	12.589	1582976	12.589	1582974
		≥ 0	62.454	7853397	62.454	7853399

Table 3. Number, percentage equivalence, and distribution of nodes where absolute errors $|e_{\nu,i}|$ $(i = 1, 2)$ of finding components of the approximate R_ν-generalized solution to problem A ($\delta = 0.0029$, $\nu = 1.2$, $\nu^* = 0.16$) are not less than given limit values, $2N = 4096$.

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4.1.2. Problem B

$2N$	128		256		512		1024		2048		4096
h	1.105e-2		5.524e-3		2.762e-3		1.381e-3		6.905e-4		3.453e-4
η	2.849e-2	1.54	1.850e-2	1.53	1.205e-2	1.53	7.870e-3	1.53	5.146e-3	1.53	3.367e-3
η_ν	2.868e-2	1.57	1.827e-2	1.65	1.107e-2	2.16	5.117e-3	2.21	2.319e-3	1.98	1.171e-3

Table 4. Dependence of relative errors of the generalized (η) and R_ν-generalized (η_ν) ($\delta = 0.0029$, $\nu = 1.2$, $\nu^* = 0.16$ solution of the problem B on the mesh step.

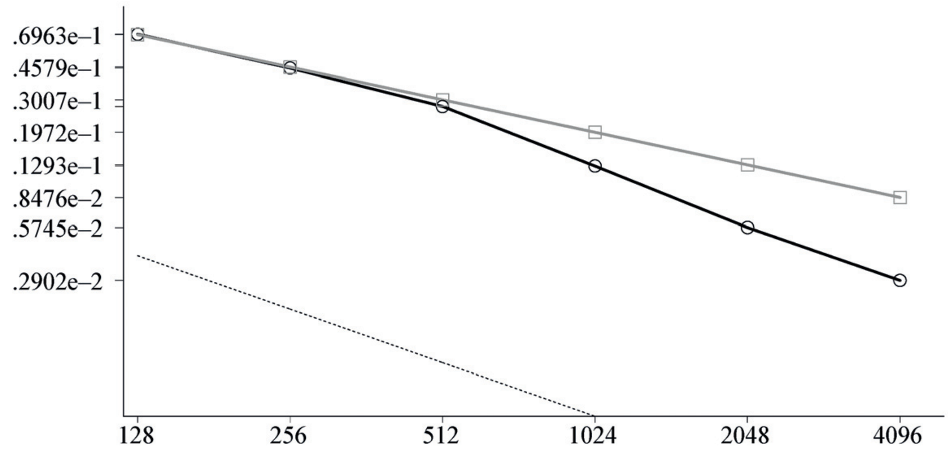


Figure 2. Chart of η for the generalized (squared line) and of η_ν for R_ν-generalized (circled line) (δ=0.0029, ν=1.2, ν^*=0.16) solutions to the problem A in dependence on the number of subdivisions $2N$.

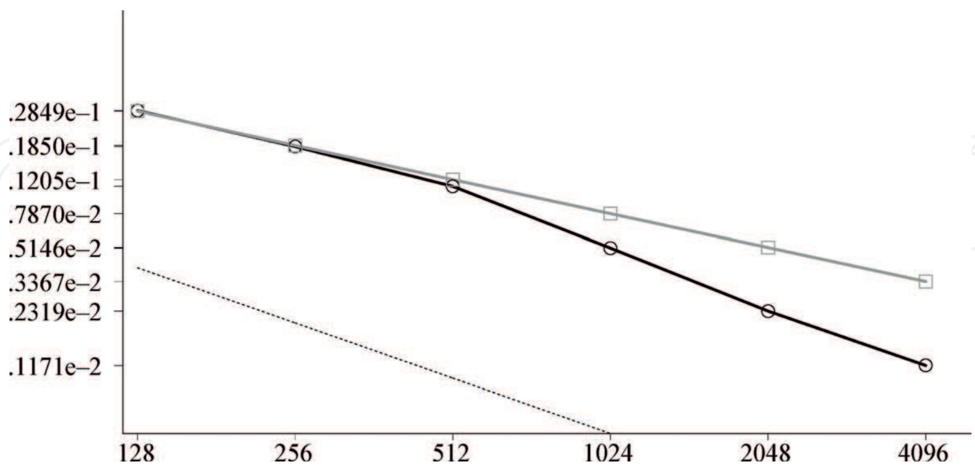


Figure 3. Chart of η for the generalized (squared line) and of η_ν for R_ν-generalized (circled line) ($\delta = 0.0029$, $\nu = 1.2$, $\nu^* = 0.16$) solutions to the problem B in dependence on the number of subdivisions $2N$.

$\|e_1\|$	$\|e_2\|$	Limit values	$\|e_1\|$		$\|e_2\|$	
Distribution				Number	%	Number
		$\geq 5e-6$	48.078	6045622	48.078	6045622
		$\geq 1e-6$	29.387	3695278	29.387	3695278
		$\geq 5e-7$	6.724	845466	6.724	845466
		$\geq 1e-7$	9.624	1210158	9.624	1210159
		$\geq 5e-8$	2.564	322439	2.564	322438
		≥ 0	3.624	455758	3.624	455758

Table 5. Number, percentage equivalence, and distribution of nodes where absolute errors $|e_i|$ $(i = 1, 2)$ of finding components of the approximate generalized solution to problem B are not less than given limit values, $2N = 4096$.

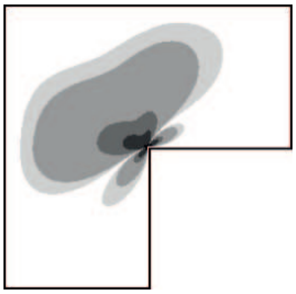
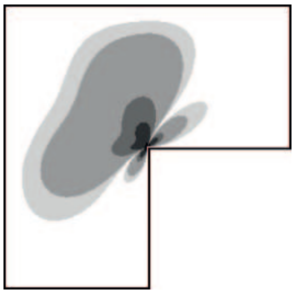
$\|e_1\|$	$\|e_2\|$	Limit values	$\|e_1\|$		$\|e_2\|$	
Distribution			%	Number	%	Number
		$\geq 5e-6$	0.033	4108	0.033	4108
		$\geq 1e-6$	0.771	96899	0.771	96899
		$\geq 5e-7$	2.481	311996	2.481	311996
		$\geq 1e-7$	21.789	2739862	21.789	2739863
		$\geq 5e-8$	12.588	1582876	12.588	1582876
		≥ 0	62.339	7838980	62.339	7838979

Table 6. Number, percentage equivalence, and distribution of nodes where absolute errors $|e_{\nu,i}|$ $(i = 1, 2))$ of finding components of the approximate R_ν-generalized solution to problem B $((\delta = 0.0029,\ \nu = 1.2,\ \nu^* = 0.16)$ are not less than given limit values, $2N = 4096$.

4.2. FEM with graded mesh: comparative analysis

This subsection presents results of error analysis for finding generalized solution to the problems A and B by the FEM with graded meshes of two kinds (for detailed information about graded meshes, see [2, 33, 34]).

Mesh I. This partitioning was built by the following scheme

1. In the domain Ω, for a given N, regular mesh was constructed as described in section 4.1.
2. Level $l = \max_{i=1,2} (|N - [(x_i + 1)N]|)$ was determined for each node. Here, x_i $(i = 1, 2)$ are initial node coordinates on the regular mesh, $[\cdot]$ means integer part.
3. New coordinates of nodes of the graded mesh are calculated by the formula $([(x_i + 1)N] - N)l^{-1}(l/N)^{1/\kappa}$ $(i = 1, 2)$.

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Mesh II. Constructing process for this mesh differs from the one described earlier in the level-calculating mode. Here, $l = \sum_{i=1}^{2} |N - [(x_i + 1)N]|$. In this case, new coordinates are determined only for nodes with $l \leq N$.

Examples of meshes I and II are shown in **Figure 1(b)** and **(c)**, respectively.

The FEM solution obtained on described graded meshes converges with the first rate on the mesh step when the value of the parameter κ is less than the order of singularity [2, 33].

Calculations were performed for different values of N and κ. For each node, one calculated the errors $\mathbf{e}_I = \mathbf{u} - \mathbf{u}_I^h$ and $\mathbf{e}_{II} = \mathbf{u} - \mathbf{u}_{II}^h$ of the approximate generalized solutions $\mathbf{u}_I^h$, $\mathbf{u}_{II}^h$ obtained on meshes I and II, respectively. The values of relative errors of the generalized solution to the problems A and B in the norm of the Sobolev space $\mathbf{W}_2^1$ for different values of h and κ for mesh I $\left(\eta_I = \frac{\|\mathbf{e}_I\|_{W_2^1}}{\|\mathbf{u}\|_{W_2^1}}\right)$ are presented in **Tables 7** and **10**, respectively, and for mesh II $\left(\eta_{II} = \frac{\|\mathbf{e}_{II}\|_{W_2^1}}{\|\mathbf{u}\|_{W_2^1}}\right)$ are presented in **Tables 8** and **11,** respectively. In addition, these tables contain ratios between error norms and between mesh steps obtained with nodes number increasing four times. **Figures 4** and **5** show the convergence rates of the generalized solutions to the corresponding problems for meshes I and II with the logarithmic scale. Dashed line in the figures corresponds to convergence with the rate $O(h)$ as in paragraph 1. Besides, for the problems A and B, **Tables 9** and **12**, respectively, contain limit values for the following data: number of nodes where $|e_{1,II}|$, $|e_{2,II}|$ belong to the giving range, this number in percentage to the total number of nodes, and pictures of the absolute error distribution in the domain Ω.

4.2.1. Problem A

$2N$	128		256		512		1024		2048		4096
$\kappa = 0.3$											
η_I	2.659e-2	2.00	1.332e-2	2.00	6.675e-3	1.91	3.501e-3	0.75	4.650e-3	0.27	1.741e-2
h	0.062263	1.979	0.031459	1.99	0.015812	1.995	0.007926	1.997	0.003968	1.999	0.001985
$\kappa = 0.4$											
η_I	2.111e-2	2.00	1.057e-2	1.99	5.302e-3	1.78	2.971e-3	0.53	5.559e-3	0.26	2.154e-2
h	0.044928	1.986	0.02262	1.993	0.011349	1.997	0.005684	1.998	0.002845	1.999	0.001423
$\kappa = 0.5$											
η_I	1.990e-2	1.99	1.001e-2	1.99	5.038e-3	1.71	2.940e-3	0.46	6.401e-3	0.25	2.513e-2
h	0.034611	1.99	0.017387	1.995	0.008714	1.998	0.004362	1.999	0.0021823	1.999	0.001092
$\kappa = 0.6$											
η_I	2.315e-2	1.92	1.204e-2	1.93	6.254e-3	1.70	3.678e-3	0.50	7.292e-3	0.26	2.818e-2
h	0.030169	1.993	0.015135	1.997	0.007580	1.998	0.003793	1.999	0.0018973	1.9996	0.0009489

Table 7. Dependence of relative errors of the generalized solution to problem A with mesh I on the mesh step for different κ.

$2N$	128		256		512		1024		2048		4096
$\kappa = 0.3$											
η_{II}	2.392e-2	2.00	1.196e-2	2.00	5.982e-3	1.99	3.012e-3	1.46	2.059e-3	0.36	5.687e-3
h	0.05114	1.982	0.025805	1.99	0.012962	1.995	0.006496	1.998	0.003252	1.999	0.001627
$\kappa = 0.4$											
η_{II}	1.974e-2	2.00	9.879e-3	2.00	4.942e-3	1.97	2.511e-3	1.16	2.167e-3	0.30	7.154e-3
h	0.038606	1.988	0.019417	1.994	0.009737	1.997	0.004876	1.999	0.00244	1.999	0.001220
$\kappa = 0.5$											
η_{II}	1.954e-2	1.98	9.857e-3	1.99	4.963e-3	1.93	2.565e-3	0.94	2.726e-3	0.28	9.725e-3
h	0.031006	1.99	0.015564	1.996	0.007797	1.998	0.003902	1.999	0.001952	1.9995	0.000976
$\kappa = 0.6$											
η_{II}	2.339e-2	1.91	1.225e-2	1.92	6.386e-3	1.90	3.368e-3	1.14	2.966e-3	0.31	9.712e-3
h	0.025906	1.995	0.012987	1.997	0.006502	1.999	0.003253	1.999	0.001627	1.9997	0.000814

Table 8. Dependence of relative errors of the generalized solution to problem A with mesh II on the mesh step for different κ.

$\|e_1\|$	$\|e_2\|$	**Limit values**	$\|e_1\|$		$\|e_2\|$	
Distribution			**%**	**Number**	**%**	**Number**
		$\geq 5e-6$	0.001	6	0.001	6
		$\geq 1e-6$	35.524	278645	35.479	278292
		$\geq 5e-7$	13.631	106920	13.770	108011
		$\geq 1e-7$	33.363	261697	33.377	261808
		$\geq 5e-8$	7.020	55066	6.984	54782
		≥ 0	10.461	82051	10.389	81486

Table 9. Number, percentage equivalence, and distribution of nodes where absolute errors $|e_{i,II}|$ $(i = 1, 2)$ of finding components of the approximate generalized solution to problem A obtained with mesh II ($\kappa = 0.5$) are not less than given limit values, $2N = 1024$.

4.2.2. Problem B

$2N$	128		256		512		1024		2048		4096
$\kappa = 0.3$											
η_I	9.851e-3	1.99	4.955e-3	1.97	2.510e-3	1.36	1.845e-3	0.33	5.639e-3	0.25	2.247e-2
h	0.062263	1.979	0.031459	1.99	0.015812	1.995	0.007926	1.997	0.003968	1.999	0.001985
$\kappa = 0.4$											
η_I	7.712e-3	1.99	3.870e-3	1.95	1.988e-3	0.98	2.034e-3	0.28	7.218e-3	0.25	2.866e-2
h	0.044928	1.986	0.02262	1.993	0.011349	1.997	0.005684	1.998	0.002845	1.999	0.001423

$\kappa = 0.5$											
η_I	7.625e-3	1.99	3.839e-3	1.92	1.995e-3	0.87	2.301e-3	0.27	8.676e-3	0.25	3.454e-2
h	0.034611	1.99	0.017387	1.995	0.008714	1.998	0.004362	1.999	0.002182	1.999	0.001091
$\kappa = 0.6$											
η_I	9.330e-3	1.92	4.849e-3	1.88	2.584e-3	0.91	2.847e-3	0.28	1.016e-2	0.25	4.001e-2
h	0.034611	1.99	0.017387	1.995	0.008714	1.998	0.004362	1.999	0.002182	1.999	0.001091

Table 10. Dependence of relative errors of the generalized solution to problem B with mesh I on the mesh step for different κ.

$2N$	128		256		512		1024		2048		4096
$\kappa = 0.3$											
η_{II}	5.963e-3	2.00	2.982e-3	2.00	1.492e-3	1.91	7.819e-4	0.77	1.013e-3	0.27	3.757e-3
h	0.05114	1.982	0.025805	1.99	0.012962	1.995	0.006496	1.998	0.003252	1.999	0.0016267
$\kappa = 0.4$											
η_{II}	6.349e-3	2.00	3.178e-3	2.00	1.591e-3	1.87	8.490e-4	0.67	1.263e-3	0.26	4.805e-3
h	0.038606	1.988	0.019417	1.994	0.009737	1.997	0.004876	1.999	0.00244	1.999	0.0012203
$\kappa = 0.5$											
η_{II}	7.441e-3	1.98	3.756e-3	1.98	1.894e-3	1.83	1.037e-3	0.60	1.717e-3	0.26	6.606e-3
h	0.031006	1.99	0.015564	1.996	0.007797	1.998	0.003902	1.999	0.001952	1.9995	0.0009763
$\kappa = 0.6$											
η_{II}	9.574e-3	1.91	5.000e-3	1.92	2.602e-3	1.85	1.409e-3	0.78	1.804e-3	0.27	6.660e-3
h	0.025906	1.995	0.012987	1.997	0.006502	1.999	0.003253	1.999	0.001627	1.9997	0.00081366

Table 11. Dependence of relative errors of the generalized solution to problem B with mesh II on the mesh step for different κ.

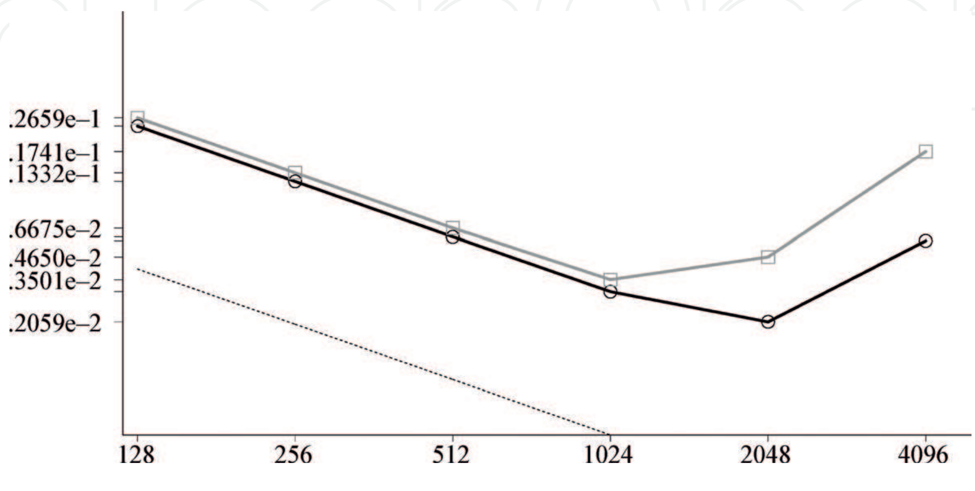


Figure 4. Chart of η_I for mesh I (squared line) and of η_{II} for mesh II (circled line) for problem A depending on the number of subdivisions $2N$; $\kappa = 0.3$.

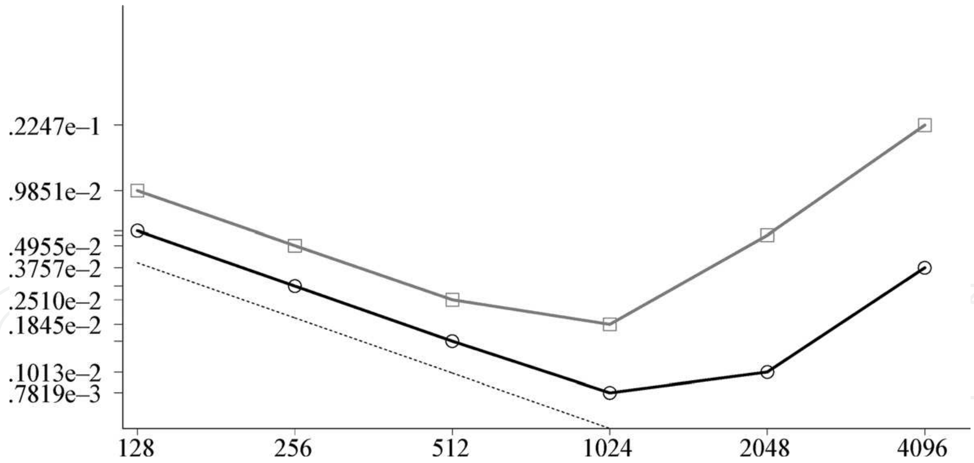


Figure 5. Chart of η_I for mesh I (squared line) and of η_{II} for mesh II (circled line) for problem B depending on the number of subdivisions $2N$; $\kappa = 0.3$.

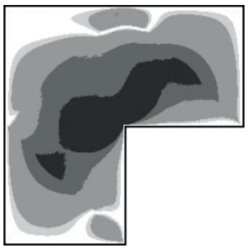
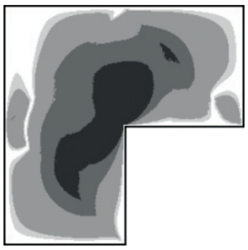
$\lvert e_1 \rvert$	$\lvert e_2 \rvert$	**Limit values**	$\lvert e_1 \rvert$		$\lvert e_2 \rvert$	
Distribution			%	**Number**	%	**Number**
		$\geq 5e-6$	0.001	6	0.001	6
		$\geq 1e-6$	23.718	186038	23.282	182623
		$\geq 5e-7$	18.518	145255	19.047	149398
		$\geq 1e-7$	34.864	273467	35.327	277097
		$\geq 5e-8$	8.084	63407	7.899	61956
		≥ 0	14.816	116212	14.445	113305

Table 12. Number, percentage equivalence, and distribution of nodes where absolute errors $|e_{i,II}|$ $(i = 1, 2)$ of finding components of the approximate generalized solution to problem B obtained with mesh II ($\kappa = 0.5$) are not less than given limit values, $2N = 1024$.

5. Conclusions

Presented numerical results have demonstrated that:

1. An approximate R_ν-generalized solution to the problem (2)–(4) converges to the exact one with the rate $O(h)$ in the norm of the set $\mathbf{W}^1_{2,\nu}(\Omega, \delta)$ in contrast with the generalized solution, which converges with the rate $O(h^{0.61})$ for the classical FEM;

2. FEM with graded meshes fails on high-dimensional grids because of the small mesh size near the singular point, but the weighted FEM stably allows to find approximate solution with the accuracy $O(h)$ under the same computational conditions;

For the approximate R_ν-generalized solution obtained by the weighted finite-element method, an absolute error value is by one or two orders of magnitude less than the approximate generalized one obtained by the FEM or by the FEM with graded meshes; this holds for the overwhelming majority of nodes.

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