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# Chaos on Set-Valued Dynamics and Control Sets<sup>1</sup>

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## Abstract

The aim of this chapter is threefold. First, we show some advances in complexity dynamics of set-valued discrete systems in connection with the Devaney's notion of chaos. Secondly, we start to explore some relationships between control sets for the class of linear control systems on Lie groups with chaotic sets. Finally, through several open problems, we invite the readers to give a contribution to this beauty theory.

**Keywords:** chaos, set-valued maps, dynamic, Devaney, control sets

## 1. Introduction

Relevant classes of real problems are modelled by a discrete dynamical system

$$x_{n+1} = f(x_n) \quad , \quad n = 0, 1, 2, \dots \quad (1)$$

where  $(X, d)$  is a metric space and  $f : X \rightarrow X$  is a continuous function. The basic goal of this theory is to understand the nature of the orbit  $O(x, f) = \{f^n(x) / n = 0, 1, 2, \dots\}$  for any state  $x \in X$ , as  $n$  becomes large and, in general this is a hard task. The study of orbits says us how the initial states are moving in the base space  $X$  and, in many cases, these orbits present a chaotic structure. In 1989 in [1], Devaney isolates three main conditions which determine the essential features of chaos.

**Definition 1** Let  $X$  be a metric space and  $f : X \rightarrow X$  a continuous map. Hence,  $f$ .

- a. is *transitive* if for any couple of non-empty open subsets  $U$  and  $V$  of  $X$  there exists a natural number  $k$  such that  $f^k(U) \cap V \neq \emptyset$ .
- b. is *periodically dense* if the set of periodic points of  $f$  is a dense subset of  $X$ .

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- c. has *sensitive dependence on initial conditions* if there is a positive number  $\delta$  (a sensitivity constant) such that for every point  $x \in X$  and every neighbourhood  $N$  of  $x$  there exists a point  $y \in N$  and a non-negative integer number  $n$  such that  $d(f^n(x), f^n(y)) \geq \delta$ .

Next, we mention a remarkable characterisation of transitive maps. In fact, as a consequence of the Birkhoff Transitivity Theorem (see [2] for details), it is possible to prove.

**Proposition 2** *Let  $X$  be a complete metric space which is also perfect (closed and without isolated points). If  $f : X \rightarrow X$  is continuous, then  $f$  is transitive if and only if there exists at least one orbit  $O(x, f)$  dense in  $X$ .*

**Remark 3** *Also, other concepts very useful in this work are the following: i)  $f$  is weakly mixing iff for any non-empty open subsets  $U$  and  $V$  of  $X$  there exists a natural number  $k$  such that  $f^k(U) \cap V \neq \emptyset$  and  $f^k(V) \cap U \neq \emptyset$ . ii)  $f$  is mixing iff given two non-empty open subsets  $U$  and  $V$  of  $X$  there exists a natural number  $k$  such that  $f^n(U) \cap V \neq \emptyset$  for all  $n \geq k$ . iii)  $f$  is exact iff given a non-empty open subsets  $U$  there exists a natural number  $k$  such that  $f^k(U) = X$ . It is clear that  $f$  exact  $\Rightarrow f$  mixing  $\Rightarrow f$  weakly mixing  $\Rightarrow f$  transitive.*

It is worth to point out that sensitivity dependence on initial conditions was widely understood as being the central idea in chaos for many years. However, in a surprising way, Banks et al. has proved that transitivity and periodically density imply sensitivity dependence (for details see [3]). Furthermore, for continuous functions on real intervals, Vellekoop and Berglund in [4] show that transitivity by itself is sufficient to get chaos. This last result is not necessarily true in other type of metric spaces (see Example 4.1 in [5]).

However, sometimes we need to know information about the collective dynamics, i.e. how are moved subsets of  $X$  via iteration or dynamics induced by  $f$ . For example, if  $X$  denotes an ecosystem and  $x \in X$ , then, by using radio telemetry elements, we can obtain information about the movement of  $x$  in the ecosystem  $X$ . In this form, it is possible to build an individual displacement function  $f : X \rightarrow X$ . Of course, this function could be chaotic or not. Eventually, we could also be interested to get information about the collective dynamics induced by  $f$ , means, to follow the dynamics of a group of individuals. Thus, in a natural way the following question appears: what is the relationship between individual and collective dynamics? This is the main topic of this chapter.

Given the system (1), consider the set-valued discrete system associated to  $f$  defined by

$$A_{n+1} = \bar{f}(A_n), \quad n = 0, 1, 2, \dots \quad (2)$$

where  $\bar{f}$  is the natural extension of  $f$  to the metric space  $(\mathcal{K}(X), H)$  of the non-empty compact subsets of  $X$  endowed with the Hausdorff metric  $H$  induced by the original distance  $d$  of  $X$ .

In a more general set up, this work is strictly related with the following fundamental question: what is the relationship between individual and collective chaos?

As a partial response to this question, in this chapter we search the transitivity of a continuous function  $f$  on  $X$  in relation to the transitivity of its extension  $\bar{f}$  to  $\mathcal{K}(X)$ . Our main result here establishes that  $\bar{f}$  transitive implies  $f$  transitive. That is to say, collective chaos implies individual chaos under the dynamics of  $\bar{f}$ .

On the other hand, we propose a new approach to this problem: to study the dynamics induced by  $f$  on the subextension  $\mathcal{K}_c(X)$  of  $\mathcal{K}(X)$ . Precisely, on the class of non-empty compact-convex subsets of  $X$ . We prove that the induced dynamics is less chaotic than the original one!

Finally, we mention that some relevant problems in the theory of control systems can be also approached by the theory of set-valued map. In fact, to any initial state  $x$  of the system, one can associate its reachable set  $\mathcal{A}(x)$ . In other words,  $\mathcal{A}(x)$  contains all the possible states of the manifold that starting from  $x$  you can reach in non-negative time by using the admissible control functions  $\mathcal{U}$  of the system. The aim of this section is twofold. First of all, to apply to the class of linear control systems on Lie groups, the existent relationship between control sets of an affine control system  $\Sigma$  on a Riemannian manifold  $M$  with chaotic sets of the shift flow induced by  $\Sigma$  on  $M \times \mathcal{U}$ , [6]. In particular, we are looking for the consequences of this relation on the controllability property. At the very end, we propose a challenge to the readers to motivate the research on this topic through some open problem relatives to the mentioned relationship.

## 2. Preliminaries

In this section, we mention some notions and fundamental results we use through the chapter.

### 2.1. Extensions

If  $(X, d)$  is a metric space and  $f : X \rightarrow X$  continuous, then we can consider the space  $(\mathcal{K}(X), H)$  of all non-empty and compact subsets of  $X$  endowed with the Hausdorff metric induced by  $d$  and  $\bar{f} : \mathcal{K}(X) \rightarrow \mathcal{K}(X)$ ,  $\bar{f}(A) = f(A)$ , the natural extension of  $f$  to  $\mathcal{K}(X)$ . Also, we denote by  $\mathcal{K}_c(X) = \{A \in \mathcal{K}(X) / A \text{ is convex}\}$ . If  $A \in \mathcal{K}(X)$  we define the “ $\epsilon$ -dilatation of  $A$ ” as the set  $N(A, \epsilon) = \{x \in X / d(x, A) < \epsilon\}$ , where  $d(x, A) = \inf_{a \in A} d(x, a)$ .

The Hausdorff metric on  $\mathcal{K}(X)$  is given by

$$H(A, B) = \inf\{\epsilon > 0 / A \subseteq N(B, \epsilon) \text{ and } B \subseteq N(A, \epsilon)\}.$$

We know that  $(\mathcal{K}(X), H)$  is a complete (separable, compact) metric space if and only if  $(X, d)$  is a complete (separable, compact) metric space, respectively, (see [3, 7, 8]).

Also, if  $A \in \mathcal{K}(X)$ , the set  $B(A, \epsilon) = \{B \in \mathcal{K}(X) / H(A, B) < \epsilon\}$  denotes the ball centred in  $A$  and radius  $\epsilon$  in the space  $(\mathcal{K}(X), H)$ .

Furthermore, given a continuous function  $(I, d) \xrightarrow{f} (I, d)$  on a real interval  $I$ , we also consider the extension  $(\mathcal{K}_c(I), H) \xrightarrow{\bar{f}_c} (\mathcal{K}_c(I), H)$ , where  $\bar{f}_c$  is the restriction  $\bar{f}|_{\mathcal{K}_c(I)}$ .

### 2.2. Baire spaces

In this section, we review some properties of Baire spaces.

**Definition 4** A topological space  $X$  is a Baire space if for any given countable family of closed sets  $\{A_n : n \in \mathbf{N}\}$  covering  $X$ , then  $\text{int}(A_n) \neq \emptyset$  for at least one  $n$ .

**Definition 5** In any Baire space  $X$ ,

1.  $D \subset X$  is called nowhere dense if  $\text{int}(cl(D)) = \emptyset$ .
2. Any countable union of nowhere dense sets is called a set of first category.
3. Any set not of first category is said to be of second category.
4. The complement of a set of first category is called a residual set.

**Remark 6** It is important to note that:

- a. Any complete metric space is a Baire space.
- b. Every residual set is of second category in  $X$ .
- c. Every residual set is dense in  $X$ .
- d. The complement of a residual set is of first category.
- e. If  $B$  is of first category and  $A \subseteq B$ , then  $A$  is of first category.

(For details, see [8–10])

In particular, if  $X = I$  is an interval, then  $\mathcal{C}(X)$  and  $\mathcal{C}(X, \mathbb{R})$ , endowed with the respective supremum metrics, are Baire spaces.

In a Baire space  $X$ , we say that “most elements of  $X$ ” verify the property (P) if the set of all  $x \in X$  that do not verify property (P) is of first category in  $X$ . In this form, sets of second category can be regarded as “big” sets. A relevant area of the real analysis is to estimate the “size” of some sets associated to a continuous interval function  $f$  such as the set  $\mathcal{P}(f)$  of periodic points of  $f$ , or the set  $\mathcal{F}(f)$  of fixed points of  $f$ . Typically, continuous interval functions have a first category set of periodic points (see [11]) and, in particular, a first category set of fixed points. It has also been recently proved that a typical continuously differentiable interval function has a finite set of fixed points and a countable set of periodic points (see [12] and references therein). It is also well-known that the class of nowhere differentiable functions  $\mathcal{ND}(I)$  is a residual set in  $\mathcal{C}(I)$  (see [13, 14]). Also, a special class of functions in  $\mathcal{C}(I)$  is the class  $\mathcal{CNL}(I)$  of all continuous functions whose graphs “cross no lines” defined in a negative way as follows (see [10]):

**Definition 7** Let  $f : [a, b] \rightarrow [a, b]$  a continuous map and  $L : \mathbb{R} \rightarrow \mathbb{R}$  a function whose graph is a straight line. We say that  $L$  crosses  $f$  (or  $f$  crosses  $L$ ) if there exists  $x_0 \in [a, b]$  and  $\delta > 0$  such that  $f(x_0) = L(x_0)$  and either.

- (a)  $L(x) \leq f(x)$  for all  $x \in [x_0 - \delta, x_0] \cap [a, b]$  and  $L(x) \geq f(x)$  for all  $x \in [x_0, x_0 + \delta] \cap [a, b]$ ; or.
- (b)  $L(x) \geq f(x)$  for all  $x \in [x_0 - \delta, x_0] \cap [a, b]$  and  $L(x) \leq f(x)$  for all  $x \in [x_0, x_0 + \delta] \cap [a, b]$ .

The following result can be found in [10]:

**Theorem 8** ([10]) The set  $\mathcal{CNL}(I) = \{f \in \mathcal{C}(I) / f \text{ crosses no lines}\}$  is residual in  $\mathcal{C}(I)$ .

The set  $\mathcal{CNL}(I)$  will play an important role in the next sections.

### 2.3. The dynamics of control theory

In Section 7, we propose some challenges through the relationship between the notion of chaotic sets in the Devaney sense and control sets for the class of Linear Control Systems on Lie Groups, [15]. In particular, we explicitly show some results concerning the controllability property in terms of chaotic dynamics.

In the sequel, we follow the relevant book *The Dynamics of Control* by Colonius and Kliemann, [6]. Let  $M$  be a  $d$  dimensional smooth manifold. By an *affine control system*  $\Sigma$  in  $M$ , we understand the family of ordinary differential equations:

$$\Sigma : \dot{x}(t) = X(x(t)) + \sum_{j=1}^m u_j(t) Y^j(x(t)), \quad u = (u_1, \dots, u_m) \in \mathcal{U} \quad (3)$$

where  $X, Y^j, j = 0, 1, \dots, m$  are arbitrary  $C^\infty$  vector fields on  $M$ . The set  $\mathcal{U} \subset L^\infty(\mathbb{R}, \Omega \subset \mathbb{R}^m)$  is the class of restricted admissible control functions where  $\Omega \subset \mathbb{R}^m$  with  $0 \in \text{int}\Omega$ , is a compact and convex set.

Assume  $\Sigma$  satisfy the Lie algebra rank condition, i.e.

$$\text{for any } x \in M \Rightarrow \text{Span}_{\mathcal{LA}} \{X, Y^1, \dots, Y^m\}(x) = d.$$

Of course,  $\mathcal{LA}$  means the Lie algebra generated by the vector fields through the usual notion of Lie bracket. Furthermore, the *ad*-rank condition for  $\Sigma$  is defined as follows:

$$\text{for any } x \in M \Rightarrow \text{Span} \{ad^i(Y^j) : j = 1, \dots, m \text{ and } i = 0, 1, \dots\}(x) = d.$$

For each  $u \in \mathcal{U}$  and each initial value  $x \in M$ , there exists an unique solution  $\varphi(t, x, u)$  defined on an open interval containing  $t = 0$ , satisfying  $\varphi(0, x, u) = x$ . Since we are concerned with dynamics on Lie Groups, without loss of generality we assume that the vector fields  $X, Y^1, \dots, Y^m$  are complete. Then, we obtain a mapping  $\Phi$  satisfying the *cocycle property*

$$\Phi : \mathbb{R} \times M \times \mathcal{U} \rightarrow M, \quad (t, x, u) \mapsto \Phi(t, x, u) \text{ and } \Phi(t + s, x, u) = \Phi(t, \Phi(s, x, u), \Theta_s u)$$

for all  $t, s \in \mathbb{R}, x \in M, u \in \mathcal{U}$ . Where, for any  $t \in \mathbb{R}$ , the map  $\Theta_t$  is the *shift flow* on  $\mathcal{U}$  defined by  $(\Theta_s u)(t) := u(t + s)$ . Hence,  $\Phi$  is a skew-product flow. The topology here is given by the product topology between the topology of the manifold and the weak\* topology on  $\mathcal{U}$ .

It turns out the following results.

**Lemma 9** [6] *Consider the set  $\mathcal{U}$  equipped with the weak\* topology associated to  $L^\infty(\mathbb{R}, \mathbb{R}^m) = (L^1(\mathbb{R}, \mathbb{R}^m))^*$  as a dual vector space. Therefore,*

1.  $(\mathcal{U}, d)$  is a compact, complete and separable metric space with the distance given by

$$d(u_1, u_2) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|\int_{\mathbb{R}} \langle u_1(t) - u_2(t), v_n(t) \rangle dt|}{1 + |\int_{\mathbb{R}} \langle u_1(t) - u_2(t), v_n(t) \rangle dt|}.$$

Here,  $\{v_n : n \in \mathbb{N}\} \subset L^1(\mathbb{R}, \mathbb{R}^m)$  is a dense set of Lebesgue integrable functions.



2. The map  $\Theta : \mathbb{R} \times \mathcal{U} \rightarrow \mathcal{U}$  defines a continuous dynamical systems on  $\mathcal{U}$ . Its periodic points are dense and the shift is topologically mixing (and then topologically transitive).
3. The map  $\Phi$  defines a continuous dynamical system on  $M \times \mathcal{U}$ .

On the other hand, the completely controllable property of  $\Sigma$ , i.e. the possibility to connect any two arbitrary points of  $M$  through a  $\Sigma$ -trajectory in positive time, is one of the most relevant issue for any control system. But, few systems have this property. A more realistic approach comes from a Kliemann notion introduced in [16].

**Definition 10** A non-empty set  $C \subset M$  is called a control set of (3) if.

- i. for every  $x \in M$  there exists  $u \in \mathcal{U}$  such that  $\{\varphi(t, x, u) : t \geq 0\} \subset C$
- ii. for every  $x \in C$ ,  $C \subset \text{cl}(\mathcal{A}(x))$
- iii.  $C$  is maximal with respect to the properties (i) and (ii).

$\mathcal{A}(x)$  denotes the states that can be reached from  $x$  by  $\Sigma$  in positive time and  $\text{cl}$  its closure

$$\mathcal{A}(x) = \{y \in M : \exists u \in \mathcal{U} \text{ and } t > 0 \text{ with } y = \varphi(t, x, u)\}.$$

Moreover, for an element  $x \in M$ , the set of points that can be steered to  $x$  through a  $\Sigma$ -trajectory in positive time is denoted by

$$\mathcal{A}^*(x) = \bigcup_{\tau > 0} \{y \in M : \exists u \in \mathcal{U}, \quad e = \varphi_{\tau, u}(x)\}.$$

Finally, we mention that the Lie algebra rank condition warranty that the system is locally accessible, which means that for every  $\tau > 0$ ,

$$\text{int}(\mathcal{A}_{\leq \tau}(x)) \text{ and } \text{int}(\mathcal{A}_{\leq \tau}^*(x)) \text{ are non empty, for any } x \in M.$$

### 3. $\bar{f}$ transitive implies $f$ transitive

As we explain, in terms of the original dynamics and its extensions a natural question arises: what are the relations between individual and collective chaos? As a partial response to this question, in the sequel, we show that the transitivity of the extension  $\bar{f}$  implies the transitivity of  $f$ . For that, we need to describe some previous results.

**Lemma 11** [5] Let  $A$  be a non-empty open subset of  $X$ . If  $K \in \mathcal{K}(X)$  and  $K \subset A$ , then there exists  $\epsilon > 0$  such that  $N(K, \epsilon) \subset A$ .

**Definition 12** Let  $A \subset X$  be. Then the extension of  $A$  to  $\mathcal{K}(X)$  is given by  $e(A) = \{K \in \mathcal{K}(X) / K \subset A\}$ .

**Remark 13**  $e(A) = \emptyset \Leftrightarrow A = \emptyset$ .

**Lemma 14** [5] Let  $A \subset X$  be,  $A \neq \emptyset$ , an open subset of  $X$ . Then,  $e(A)$  is a non-empty open subset of  $\mathcal{K}(X)$ .

**Lemma 15** [5] *If  $A, B \subset X$ , then: i)  $e(A \cap B) = e(A) \cap e(B)$ , ii)  $\bar{f}(e(A)) \subseteq e(f(A))$ , and iii)  $\bar{f}^p = \overline{f^p}$ , for every  $p \in \mathbb{N}$ .*

Now, we are in a position to prove the following results

**Theorem 16** *Let  $f : X \rightarrow X$  be a continuous function. Then,  $\bar{f}$  transitive implies  $f$  transitive.*

**Proof:** Let  $A, B$  be two non-empty open sets in  $X$ . Due to Lemma 13,  $e(A)$  and  $e(B)$  are non-empty open sets in  $\mathcal{K}(X)$ . Thus, by transitivity of  $\bar{f}$ , there exists some  $k \in \mathbb{N}$  such that

$$\bar{f}^k(e(A)) \cap e(B) = \overline{f^k}(e(A)) \cap e(B) \neq \emptyset$$

and, from Lemma 14, we obtain

$$e(f^k(A)) \cap e(B) = e(f^k(A) \cap B) \neq \emptyset$$

which implies  $f^k(A) \cap B \neq \emptyset$  and, consequently,  $f$  is a transitive function.

## 4. Two examples

Now we show that, in general, the converse of Theorem 15 is not true.

**Example 4.1** (Translations of the circle). If  $\lambda \in \mathbb{R}$  is an irrational number and we define  $T_\lambda : S^1 \rightarrow S^1$  by  $T_\lambda(e^{i\theta}) = e^{i(\theta+2\pi\lambda)}$ , then it was shown by Devaney [1] that each orbit  $\{T_\lambda^n(e^{i\theta})/n \in \mathbb{N}\}$  is dense in  $S^1$  and, due Proposition 2,  $T_\lambda$  is transitive. Nevertheless,  $T_\lambda$  has no periodic points and, because  $T_\lambda$  is isometric, it does not exhibit sensitive dependence on initial conditions either.

If  $K \in \mathcal{K}(S^1)$ , because  $\overline{T_\lambda}$  preserves diameter, then  $\text{diam}(K) = \text{diam}(\overline{T_\lambda}^n(K))$ , for all  $n \in \mathbb{N}$ .

Now, let  $K \in \mathcal{K}(S^1)$  such that  $\text{diam}(K) = 1$ , and let  $\epsilon > 0$  sufficiently small. Then

$$\begin{aligned} F \in U = B(K, \epsilon) &\Rightarrow \text{diam}(F) \approx 1 \\ G \in V = B(\{1\}, \epsilon) &\Rightarrow \text{diam}(G) \approx 0. \end{aligned}$$

Thus,  $\text{diam}(\overline{T_\lambda}^n(F)) \approx 1 \quad \forall n \in \mathbb{N}$  and, consequently,  $\overline{T_\lambda}^n(U) \cap V = \emptyset$  for all  $n \in \mathbb{N}$ , which implies that  $\overline{T_\lambda}$  is not transitive on  $\mathcal{K}(S^1)$ .

**Example 4.2** Define the “tent” function  $f : [0, 1] \rightarrow [0, 1]$  as  $f(x) = 2x$  if  $0 \leq x \leq 1/2$  and  $f(x) = 2(1 - x)$  if  $1/2 \leq x \leq 1$ .

It is not difficult to show that  $f$  is an exact function on  $[0, 1]$ . In fact, intuitively we can see that, after each iteration, the number of tent in the graphics is increasing, whereas the base of each tent is decreasing and they are uniformly distributed over the interval  $[0, 1]$ .



Thus, if  $U$  is an arbitrary non-empty open subset of  $[0, 1]$ , then  $U$  contains an open interval  $J$  and, after certain number of iterations, there exists a tent, with height equal to one, whose base is contained in  $J$ , which implies that  $f(U) = [0, 1]$  and, according to Remark 3,  $f$  is an exact mapping and, consequently,  $f$  is a mixing function.

The conclusions in Examples 4.1 and 4.2 come from the next result, Banks [17] in 2005.

**Theorem 17** *If  $f : X \rightarrow X$  is continuous, then the following conditions are equivalent:*

i)  $f$  is weakly mixing, ii)  $\bar{f}$  is weakly mixing, iii)  $\bar{f}$  is transitive.

Hitherto, we have used the strong topology induced by the  $H$ -metric on  $\mathcal{K}(X)$ . However, considering the  $w^e$ -topology on  $\mathcal{K}(X)$  generated by the sets  $e(A)$  with  $A$  an open set in  $X$ , we obtain the following complementary result, see [5]:

**Theorem 18** *For a continuous map  $f : X \rightarrow X$  the following conditions are equivalent:*

i)  $f$  is transitive in  $(X, d)$ , ii)  $\bar{f}$  is transitive in the  $w^e$ -topology.

## 5. Sensitivity and periodic density of $\bar{f}$

Let  $f : X \rightarrow X$  be a continuous function and let  $\bar{f}$  be its corresponding extension to the hyper-space  $\mathcal{K}(X)$ . Then, the study of sensitivity of  $f$  in the base space in relation to the sensitivity of  $\bar{f}$  on  $\mathcal{K}(X)$  has been very exhaustively analysed in the last years. Román and Chalco published the first result in this direction [18] in 2005, where the authors prove

**Theorem 19**  *$\bar{f}$  sensitively dependent implies  $f$  sensitively dependent.*

**Proof:** If  $\bar{f}$  has sensitive dependence, then there exists a constant  $\delta > 0$  such that for every  $K \in \mathcal{K}(X)$  and every  $\epsilon > 0$  there exists  $G \in B(K, \epsilon)$  and  $n \in \mathbb{N}$  such that  $H(f^n(K), f^n(G)) \geq \delta$ .

Now, let  $x \in X$  be and  $\epsilon > 0$ . Then, taking  $K = \{x\} \in \mathcal{K}(X)$ , we have that there exists  $G \in B(\{x\}, \epsilon)$  and  $n \in \mathbb{N}$  such that  $H(f^n(\{x\}), f^n(G)) = H(f^n(x), f^n(G)) \geq \delta$ .

Thus,  $H(f^n(x), f^n(G)) = \sup_{y \in G} d(f^n(x), f^n(y)) \geq \delta$  and, due to the compactness of  $G$  and the continuity of  $f$ , there exists  $y_0 \in G$  such that  $H(f^n(x), f^n(G)) = d(f^n(x), f^n(y_0)) \geq \delta$ .

But,  $G \in B(x, \epsilon)$  implies  $G \subset B(x, \epsilon)$  and, consequently,  $y_0 \in B(x, \epsilon)$ . This proves that  $f$  is sensitively dependent (with constant  $\delta$ ).

The reverse of this theorem is not true. In fact, recently Sharma and Nagar [19] show an example where  $(X, d)$  is sensitive but  $(\mathcal{K}(X), H)$  is not. Now, in order to overcome that shortcoming, the authors in [19] introduce the following notion of sensitivity:

**Definition 20** (Stronger sensitivity [19]). *Let  $f : X \rightarrow X$  be a continuous function. Then  $f$  is strongly sensitive if there exists  $\delta > 0$  such that for each  $x \in X$  and each  $\epsilon > 0$ , there exists  $n_0 \in \mathbb{N}$  such that for every  $n \geq n_0$ , there is a  $y \in X$  with  $d(x, y) < \epsilon$  and  $d(f^n(x), f^n(y)) > \delta$ .*

Obviously, the notion of stronger sensitivity is more restrictive than sensitivity, and the authors in [19] obtain the following results:

**Theorem 21** *If  $f : X \rightarrow X$  is a continuous function and  $(\mathcal{K}(X), H, \bar{f})$  is strongly sensitive then  $(X, d, f)$  is strongly sensitive.*

In the compact case, it is possible to obtain a characterization as follows.

**Theorem 22** *Let  $(X, d)$  be a compact metric space and  $f : X \rightarrow X$  a continuous function. Then  $(\mathcal{K}(X), H, \bar{f})$  is strongly sensitive if and only if  $(X, d, f)$  is strongly sensitive.*

In connection with these results, recently Subrahmonian ([20], 2007) has been shown that most of the important sensitive dynamical systems are all strongly sensitive (the author here calls them cofinitely sensitive). Hence, we can say that for most cases, sensitivity is equivalent in both cases  $(X, d)$  and  $(\mathcal{K}(X), H)$ . It turns out that, strongly sensitivity and sensitivity are equivalent on the class of interval functions, which implies that

**Theorem 23** *If  $f : I \rightarrow I$  is a continuous function, the following conditions are equivalent.*

a)  $(I, d, f)$  is sensitive, b)  $(\mathcal{K}(I), H, \bar{f})$  is sensitive.

We finish this section assuming the existence of a dense set of periodic points for  $\bar{f}$ , we have

**Theorem 24** *Let  $(X, d)$  be a compact metric space and  $f : X \rightarrow X$  a continuous function. If  $f : X \rightarrow X$  has a dense set of periodic points then  $\bar{f} : \mathcal{K}(X) \rightarrow \mathcal{K}(X)$  has the same property.*

**Proof:** Let  $K \in \mathcal{K}(X)$  and  $\epsilon > 0$ . Then there exists a  $\epsilon/2$ -net covering  $K$ , That is to say, there are  $x_1, \dots, x_p$  in  $K$  such that  $K \subset B(x_1, \epsilon/2) \cup \dots \cup B(x_p, \epsilon/2)$ . Because  $f$  has periodic density, there are  $y_i \in X$  and  $n_i \in \mathbb{N}$  such that:

$$y_i \in B(x_i, \epsilon/2), \forall i = 1, \dots, p \text{ and } f^{n_i}(y_i) = y_i, \forall i = 1, \dots, p.$$

Now, take  $G = \{y_1, \dots, y_p\}$ . By construction, we have  $H(K, G) < \epsilon$  and, moreover,  $f^{n_1 n_2 \dots n_p}(y_i) = y_i$ , for all  $i = 1, \dots, p$ . Therefore,  $f^{n_1 n_2 \dots n_p}(G) = G$ , which implies that  $\bar{f}$  has periodic density.

The converse of this theorem is no longer true (for a counterexample, see Banks [17]). However, to find conditions on  $\bar{f}$  warranting the existence of a dense set of periodic points for  $f$  is a very hard problem which still remains open.

## 6. The dynamics on the $(\mathcal{K}_c(I), H)$ extension

In the previous sections, we have studied the diagram

$$\begin{array}{ccc} (\mathcal{K}(X), H) & \xrightarrow{\bar{f}} & (\mathcal{K}(X), H) \\ \uparrow & & \uparrow \\ (X, d) & \xrightarrow{f(X, d)} & \end{array} \quad (4)$$

and the chaotic relationships between  $f$  and  $\bar{f}$ . However, in the setting of mathematical modeling of many real-world applications, it is necessary to take into account additional considerations such as vagueness or uncertainty on the variables. This implies the use of interval parameters and, consequently, to deal with interval systems. That is, it is necessary to consider an interval  $X = I$  and to study the following new diagram:

$$\begin{array}{ccc} (\mathcal{K}_c(I), H) & \xrightarrow{\bar{f}_c} & (\mathcal{K}_c(I), H) \\ \uparrow & & \uparrow \\ (I, d) & \xrightarrow{f(I, d)} & (I, d) \end{array} \quad (5)$$

along with the analysis of the connection between their respective dynamical relationships. Here  $\bar{f}_c$  denotes the restriction of  $\bar{f}$  to  $\mathcal{K}_c(I)$ , the class of all compact subintervals of  $I$ . For  $A = [a, b], B = [c, d] \in \mathcal{K}_c(I)$ , the Hausdorff metric can be explicitly computed as

$$H(A, B) = \max\{|a - c|, |b - d|\}. \quad (6)$$

The aim of this section is to show that the Devaney complexity of the extension  $\bar{f}_c$  on  $\mathcal{K}_c(I)$  is less or equal than the complexity of  $f$  on the base space  $I$ . More precisely,  $\bar{f}_c$  is never transitive for any continuous function  $f \in \mathcal{C}(I)$ . Also, we will show that  $\bar{f}_c$  has no dense set of periodic points for most functions  $f \in \mathcal{C}(I)$ . Finally, we prove that  $\bar{f}_c$  has no sensitive dependence for most functions  $f \in \mathcal{C}(I)$ .

As a motivation, we present the following examples.

**Example 6.1** Consider the “tent” function  $f : [0, 1] \rightarrow [0, 1]$  defined by

$$f(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq \frac{1}{2} \\ 2(1 - x) & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Then it is well known that  $f$  is D-chaotic on  $[0, 1]$  (see [1]). Moreover, because  $f$  is a mixing function on  $[0, 1]$ , then  $\bar{f}$  is transitive on  $\mathcal{K}([0, 1])$  (see [17]). Also, we observe that  $x = \frac{2}{3}$  is a fixed point of  $f$ . On the other hand, it is clear that if  $K$  is a compact and convex subset of  $X = [0, 1]$ , then  $\bar{f}(K)$  is also a compact and convex subset of  $X$ . Consequently, if we let  $\mathcal{K}_c([0, 1])$  denote the class of all closed subintervals of  $[0, 1]$ , then we can consider  $\bar{f}_c$  as a mapping  $\bar{f}_c : \mathcal{K}_c([0, 1]) \rightarrow \mathcal{K}_c([0, 1])$ . We recall that  $\mathcal{K}_c([0, 1])$  is a closed subspace of  $\mathcal{K}([0, 1])$  (see [21]). Now, considering the open balls  $B([0, 1], \frac{1}{10})$  and  $B(\{0\}, \frac{1}{10})$  in  $\mathcal{K}_c([0, 1])$ , we have.

$$K \in B([0, 1], \frac{1}{10}) \Rightarrow \frac{2}{3} \in K \text{ which implies } \frac{2}{3} \in \bar{f}_c^p(K), \quad \forall p \in \mathbb{N}.$$

On the other hand, if  $F \in B(\{0\}, \frac{1}{10})$ , then  $F \subset [0, 1/10]$ . Consequently,  $H(\bar{f}_c^p(K), F) \geq \frac{17}{30}$  for every  $K \in B([0, 1], \frac{1}{10})$  and  $F \in B(\{0\}, \frac{1}{10})$ .

Therefore,

$$\bar{f}_c^p \left( B \left( [0, 1], \frac{1}{10} \right) \right) \cap B \left( \{0\}, \frac{1}{10} \right) = \emptyset, \quad \forall p \in \mathbb{N}.$$

Thus,  $\bar{f}_c$  is not transitive on  $\mathcal{K}_c([0, 1])$ .

Example 6.1 shows a function  $f$  which is transitive on the base space  $X = [0, 1]$  and  $\bar{f}$  is also transitive on the total extension  $\mathcal{K}([0, 1])$ , but  $\bar{f}_c$  is not transitive on the subextension  $\mathcal{K}_c([0, 1])$ .

The following example shows a function  $f : [0, 1] \rightarrow [0, 1]$  with a dense set of periodic points, and where the total extension of  $f$  to  $\mathcal{K}([0, 1])$  also has a dense set of periodic points, whereas  $\bar{f}_c$  does not have a dense set of periodic points on  $\mathcal{K}_c([0, 1])$ .

**Example 6.2.** Let  $X = [0, 1]$  and consider the “logistic” function  $f : [0, 1] \rightarrow [0, 1]$  defined by  $f(x) = 4x(1 - x)$ . It is well known that  $f$  is D-chaotic on  $[0, 1]$  (see [1]). Moreover,  $f$  is a mixing function. Thus, in particular,  $f$  has a dense set of periodic points and, therefore,  $\bar{f}$  also has a dense set of periodic points on the total extension  $\mathcal{K}([0, 1])$  (see Theorem 24).

However,  $\bar{f}_c$  has no a dense set of periodic points on  $\mathcal{K}_c(X)$ .

In order to see this, we claim that the open ball  $B\left(\left[\frac{1}{8}, \frac{3}{8}\right], \frac{1}{8}\right)$  in  $(\mathcal{K}_c([0, 1]), H)$  does not contain periodic points of  $\bar{f}_c$ .

In fact, if  $K = [c, d] \in B\left(\left[\frac{1}{8}, \frac{3}{8}\right], \frac{1}{8}\right)$ , then  $|c - \frac{1}{8}| < \frac{1}{8}$  and  $|d - \frac{3}{8}| < \frac{1}{8}$ , which implies that  $0 < c < \frac{1}{4}$  and  $\frac{1}{4} < d < \frac{1}{2}$ .

Thus, we obtain that  $\frac{1}{4} \in K \Rightarrow \frac{3}{4} \in f(K) \Rightarrow f(K) \neq K$ .

On the other hand,

$$\frac{3}{4} \in f(K) \Rightarrow \frac{3}{4} \in f^n(K), \forall n \geq 2 \Rightarrow f^n(K) \neq K, \forall n \geq 1$$

and, consequently,  $\bar{f}_c$  has no periodic points in the ball  $B\left(\left[\frac{1}{8}, \frac{3}{8}\right], \frac{1}{4}\right) \subseteq (\mathcal{K}_c([0, 1]), H)$ , which implies that  $\bar{f}_c$  has no dense set of periodic points on  $(\mathcal{K}_c([0, 1]), H)$ .

**Lemma 25**  $\bar{f}_c$  transitive on  $\mathcal{K}_c([a, b])$  implies  $f$  transitive on  $[a, b]$ .

**Proof.** Let  $U, V$  non-empty open subsets of  $X = [a, b]$ . We can choose  $x \in U, y \in V$  and  $\epsilon > 0$  such that  $B(x, \epsilon) \subset U$  and  $B(y, \epsilon) \subset V$ . Now, in  $\mathcal{K}_c([a, b])$  consider the open balls  $B(\{x\}, \epsilon)$  and  $B(\{y\}, \epsilon)$  with respect to the  $H$ -metric. Due to the transitivity of  $\bar{f}_c$  on  $\mathcal{K}_c([a, b])$ , there exists  $n \in \mathbb{N}$  such that  $\bar{f}_c^n(B(\{x\}, \epsilon)) \cap B(\{y\}, \epsilon) \neq \emptyset$ .

Therefore, there exists an interval  $J \in B(\{x\}, \epsilon)$  such that  $\bar{f}_c^n(J) = f^n(J) \in B(\{y\}, \epsilon)$ . However,  $J \subset B(x, \epsilon)$  and, analogously,  $f^n(J) \subset B(y, \epsilon)$ , which implies that  $f^n(B(x, \epsilon)) \cap B(y, \epsilon) \neq \emptyset$  and, consequently,  $f^n(U) \cap V \neq \emptyset$ . And  $f$  is a transitive function on  $[a, b]$ .

It is well-known that if  $X = I$  is an interval, then most functions  $f \in \mathcal{C}(I)$  has no dense orbits, that is to say, there exists a residual set  $D \subset \mathcal{C}(I)$  such that every function  $f \in D$  has no point whose orbit is dense in  $I$  (see [22]) and, consequently, most functions  $f \in \mathcal{C}(I)$  are not transitive. From Lemma 24, we can conclude that  $\bar{f}_c$  is not transitive for most functions  $f \in \mathcal{C}(I)$ .

The next theorem provides a stronger result.

**Theorem 26** *Let  $f : [a, b] \rightarrow [a, b]$  be continuous. Then  $\bar{f}_c$  is not transitive on  $\mathcal{K}_c([a, b])$ .*

**Proof.** By Schauder Theorem,  $f$  has at least one fixed point  $p \in [a, b]$ .

Case 1. Suppose that  $p \in (a, b)$  and let  $r = \max\{p - a, b - p\}$ . Without loss of generality, we can suppose that  $r = p - a$  and, because  $a < b$ , it is clear that  $r > 0$ .

Now, let  $r' = b - p > 0$  and let  $\epsilon = \frac{r'}{2}$ . If we consider the open balls  $B([a, b], \epsilon)$ ,  $B(\{a\}, \epsilon) \in \mathcal{K}_c([a, b])$ , it follows that  $K \in B([a, b], \epsilon) \Rightarrow p \in K \Rightarrow p \in \bar{f}^n(K)$  for any  $n \in \mathbb{N}$ .

On the other hand,

$$F \in B(\{a\}, \epsilon) \Rightarrow H(F, \{a\}) < \epsilon \Rightarrow F \subset [a, a + \epsilon] .$$

Because  $r' < r$  we get

$$H(\bar{f}^n(K), F) \geq p - a - \epsilon = r - \frac{r'}{2} > 0$$

for each  $K \in B([a, b], \epsilon)$ ,  $F \in B(a, \epsilon)$  and for any  $n \in \mathbb{N}$ . Thus,

$$\bar{f}^n(B([a, b], \epsilon)) \cap B(a, \epsilon) = \emptyset , \quad \forall n \in \mathbb{N}.$$

Consequently,  $\bar{f}$  is not transitive on  $\mathcal{K}_c([a, b])$ .

Case 2. Suppose that  $f$  has no fixed points in  $(a, b)$ . From the continuity of  $f$ , we have that  $f(x) > x$  for all  $x \in (a, b)$  or  $f(x) < x$  for all  $x \in (a, b)$ . This clearly implies that  $f$  is not a transitive function, and consequently, due to Lemma 24,  $\bar{f}_c$  is not transitive on  $\mathcal{K}_c([a, b])$ .

An important question to answer is what about the size of the set of periodic points of  $\bar{f}_c$ . It is clear that there are some functions  $f \in \mathcal{C}(I)$  with a dense set of periodic points on  $I$ , and such that their extensions  $\bar{f}_c$  also has a dense set of periodic points on  $\mathcal{K}_c(I)$  (for instance,  $f(x) = x$ ). Therefore, an analogous result to Theorem 26, but for periodic density of  $\bar{f}_c$ , cannot be obtained. However, as we will see, most functions  $f \in \mathcal{C}(I)$  do not have an extension  $\bar{f}_c$  with a dense set of periodic points on  $\mathcal{K}_c(I)$ . To prove it, we need the following lemma.

**Lemma 27** *Let  $I$  be a compact interval in  $\mathbb{R}$ , and  $f : I \rightarrow I$  be a continuous function. If we suppose that  $\bar{f}_c$  has periodic density on  $\mathcal{K}_c(I)$ , then  $f$  has periodic density on  $I$ .*

**Proof.** If  $x_0 \in I$  and  $\epsilon > 0$  then  $\{x_0\} \in \mathcal{K}_c(I)$  and, consequently, there exists  $K \in \mathcal{K}_c(I)$  and  $n \in \mathbb{N}$  such that

a.  $H(\{x_0\}, K) < \epsilon$

b.  $\bar{f}_c^n(K) = K$ .

Combining a. and b. we get

$$d(x_0, f^n(x)) < \epsilon, \text{ for all } x \in K. \quad (7)$$

Because  $\bar{f}^n(K) = \bar{f}^n(K) = f^n(K) = K$  and  $f^n$  is continuous on  $K$  then, by the Schauder's Fixed Point Theorem, there exists  $x_p \in K$  such that  $f^n(x_p) = x_p$ . Thus,  $x_p$  is a periodic point of  $f$  and, due to (7), we obtain  $d(x_0, x_p) < \epsilon$ . Hence,  $f$  has periodic density on  $I$ .  $\square$

**Theorem 28** Let  $I = [a, b]$  be a compact interval in  $\mathbb{R}$ . Then  $\bar{f}_c$  does not have a dense set of periodic points in  $\mathcal{K}_c(I)$ , for most functions  $f \in \mathcal{C}(I)$ .

**Proof.** The proof is based on an exhaustive analysis of the behaviour of the fixed points of  $f$ . We connect this analysis with an adequate residual set in  $\mathcal{C}(I)$ . The analysis of each fixed point of  $f$  is fundamental to decide whether the function  $f$  allows or not an extension  $\bar{f}_c$  that has a dense set of periodic points. More precisely, the behaviour of each fixed point will imply only two (mutually exclusive) options:

- A.  $\bar{f}_c$  does not have a dense set of periodic points, or.
- B.  $f \in [\mathcal{CNL}(I)]^c$ , which is a set of first category in  $\mathcal{C}(I)$ .

Towards this end, let  $f : [a, b] \rightarrow [a, b]$  be a continuous function. By the Schauder's Fixed Point Theorem,  $f$  has at least one fixed point  $p \in [a, b]$ . The proof is divided in.

Case 1.  $f$  has no fixed points in  $(a, b)$ .

In this case, we have the following three subcases:

1i)  $p = a$  is the unique fixed point of  $f$ .

We have, either

$$\begin{aligned} f(x) > x, \forall x \in (a, b) & \quad (\Rightarrow x < f(x) < f^2(x) < \dots < f^n(x) < \dots), \text{ or} \\ f(x) < x, \forall x \in (a, b) & \quad (\Rightarrow x > f(x) > f^2(x) > \dots > f^n(x) > \dots). \end{aligned}$$

In both cases it follows that  $f$  has no periodic points in  $(a, b)$ .

1ii)  $p = b$  is the unique fixed point of  $f$ .

This case is analogous to the case 1i).

1iii)  $p = a$  and  $p = b$  are the unique fixed points of  $f$ .

This case is also analogous to the cases 1i) and 1ii).

Therefore, in case 1 the function  $f$  does not have a dense set of periodic points in  $[a, b]$ . Due to Lemma 24,  $\bar{f}_c$  does not have a dense set of periodic points in  $\mathcal{K}_c([a, b])$ .



Case 2.  $f$  has at least one fixed point  $p \in (a, b)$ .

We have the following subcases:

2i)  $\exists q \in (a, b)$ ,  $q \neq p$  such that  $f(q) = p$ .

Without loss of generality, suppose that  $q \in (a, p)$ . Then, taking  $0 < \epsilon < \min\{\frac{q-a}{2}, \frac{p-q}{2}\}$ , we can consider the open ball  $B([q - \epsilon, q + \epsilon], \epsilon)$  in the space  $\mathcal{K}_c([a, b])$ . If  $J = [c, d] \in B([q - \epsilon, q + \epsilon], \epsilon)$ , from (6) we have

$$|c - (q - \epsilon)| < \epsilon \quad \text{and} \quad |d - (q + \epsilon)| < \epsilon$$

which implies that  $a < c < q$  and  $q < d < p$  and, consequently,  $q \in J$  whereas  $p \notin J$ . Thus,

$$q \in J \Rightarrow f(q) = p \in f(J) \Rightarrow f(J) \neq J. \quad (8)$$

On the other hand,  $p \in f(J)$  implies that

$$p \in f^n(J), \forall n \geq 2 \Rightarrow f^n(J) \neq J, \forall n \geq 2, \quad (9)$$

and, consequently,  $\bar{f}_c$  has no periodic points in the ball  $B([q - \epsilon, q + \epsilon], \epsilon) \subseteq (\mathcal{K}_c([a, b]), H)$ , which implies that  $\bar{f}_c$  does not have a dense set of periodic points on  $(\mathcal{K}_c([a, b]), H)$ .

2ii)  $q = a$ ,  $q \neq p$ , is the unique point such that  $f(a) = p$ .

Without loss of generality, we can suppose that  $f(x) > p$ , for all  $x \in (a, p)$ .

Now, in addition to hypothesis 2ii), we have two subcases:

2iia<sub>1</sub>)  $f$  does not cross the line  $y = p$  and  $f(x) > p$  for all  $x \in (a, p)$ .

In this situation,  $f(x) \geq p$  for all  $x \in [a, b]$ . Thus, choosing  $q \in (a, p)$  and  $0 < \epsilon < \max\{\frac{q-a}{2}, \frac{p-q}{2}\}$ , we can consider the open ball  $B(\{q\}, \epsilon)$  to have

$$K = [c, d] \in B(\{q\}, \epsilon) \Rightarrow K \subset (a, p). \quad (10)$$

From our hypothesis, we obtain

$$f^n(z) > p, \quad \forall z \in K, \quad \forall n \in \mathbb{N}, \quad (11)$$

which implies that  $f^n(K) \neq K$ ,  $\forall n \in \mathbb{N}$ . Consequently,  $\bar{f}_c$  has no periodic points in the ball  $B(\{q\}, \epsilon)$ . In other words,  $\bar{f}_c$  does not have a dense set of periodic points in  $\mathcal{K}_c(I)$ .

2iia<sub>2</sub>)  $f$  does not cross the line  $y = p$  and  $f(x) < p$  for all  $x \in (a, p)$ .

In this case,  $f(x) \geq p$  for all  $x \in [a, b]$ . Thus, choosing  $q \in (p, b)$  and  $0 < \epsilon < \max\{\frac{q-p}{2}, \frac{b-q}{2}\}$ , we can consider the open ball  $B(\{q\}, \epsilon)$  to obtain

$$K = [c, d] \in B(\{q\}, \epsilon) \Rightarrow K \subset (p, b). \quad (12)$$

Again, from our hypothesis, we get

$$f^n(z) < p, \quad \forall z \in K, \quad \forall n \in \mathbb{N}, \quad (13)$$

which implies that  $f^n(K) \neq K, \forall n \in \mathbb{N}$  and, consequently,  $\bar{f}_c$  has no periodic points in the ball  $B(\{q\}, \epsilon)$ . In other words,  $\bar{f}_c$  does not have a dense set of periodic points in  $\mathcal{K}_c(I)$ .

2iib)  $f$  crosses the line  $y = p$ .

It is clear that, in this case,  $f \in [\mathcal{CNL}(I)]^c$  which, due to Theorem 8 and Remark 6, is a set of first category in  $\mathcal{C}([a, b])$ .

2iii)  $q = b, q \neq p$ , is the unique point such that  $f(b) = p$ .

This case is analogous to case 2ii) and, consequently, if  $f$  does not cross the line  $y = p$  then  $\bar{f}_c$  does not have a dense set of periodic points in  $\mathcal{K}_c(I)$ , whereas if  $f$  crosses the line  $y = p$ , then  $f \in [\mathcal{CNL}(I)]^c$ .

2iv)  $q_1 = a$  and  $q_2 = b, q_1, q_2 \neq p$ , are the unique points such that  $f(a) = f(b) = p$ .

In this case, we have the following subcases:

2iva<sub>1</sub>)  $f$  does not cross the line  $y = p$  and  $f(x) > p$  and  $f(x) > p$  for all  $x \in (a, b) \setminus \{p\}$ .

This case is analogous to the case 2iia<sub>1</sub>) and the same is true for 2iva<sub>2</sub>) when  $f$  does not cross the line  $y = p$  and  $f(x) < p$  for all  $x \in (a, b) \setminus \{p\}$  which is analogous to the case 2iia<sub>2</sub>) Finally, there only remains two subcases:

2ivb<sub>1</sub>)  $f$  crosses the line  $y = p$  and  $f(x) > p$  in  $(a, p)$  and  $f(x) < p$  in  $(p, b)$ , and.

2ivb<sub>2</sub>)  $f$  crosses the line  $y = p$  and  $f(x) < p$  in  $(a, p)$  and  $f(x) > p$  in  $(p, b)$ .

It is clear that in both cases  $f \in [\mathcal{CNL}(I)]^c$ .

Thus, as a direct consequence of the analysis of the behaviour of the set of fixed points of  $f$ , it turns out that the unique cases in which  $f$  could have an extension  $\bar{f}_c$  with a dense set of periodic points on  $\mathcal{K}_c(I)$  are when there exists a fixed point  $p$  of  $f$  such that  $f$  crosses the line  $y = p$  at  $x = p$ . In other words, we obtain

$$\mathcal{HDS}(I) = \{f \in \mathcal{C}(I) / \bar{f}_c \text{ has a dense set of periodic points in } \mathcal{K}_c(I)\} \Rightarrow \mathcal{HDS}(I) \subseteq [\mathcal{CNL}(I)]^c,$$

But,  $\mathcal{CNL}(I)$  is a residual set in  $\mathcal{C}(I)$ , therefore from Remark 6, we conclude that  $\mathcal{HDS}(I)$  is of first category in  $\mathcal{C}(I)$ . Equivalently,  $\bar{f}_c$  does not have a dense set of periodic points, for most functions  $f \in \mathcal{C}(I)$ , which ends the proof.

Finally based on the following result,

**Theorem 29** ([23]) *For most functions  $f \in \mathcal{C}(I)$ , the set of all points where  $f$  is sensitive is dense in the set of all periodic points of  $f$ .*

we show an analogous result for the sensitivity property, as follows.

**Theorem 30** *For most functions  $f \in \mathcal{C}(I)$ , the extension  $\bar{f}_c \in \mathcal{C}(\mathcal{K}_c(I))$  is not sensitive.*

**Proof.** This is a direct consequence of Theorem 28 and Theorem 29.

## 7. Control sets of linear systems and chaotic dynamics

The aim of this section is twofold. First of all, to start to apply to the class of linear control systems on Lie groups, the existent relationship between control sets of an affine control system  $\Sigma$  on a Riemannian manifold  $M$  with chaotic sets of the shift flow induced by  $\Sigma$  on  $M \times \mathcal{U}$ , [6]. In particular, we are looking for the consequences of this relation on the controllability property. The second part is intended to motivate the research on this topic to writing down some open problems relatives to this relationship.

### 7.1. Linear control systems on lie groups

Let  $G$  be a connected  $d$  dimensional Lie group with Lie algebra  $\mathfrak{g}$ . A linear control system  $\Sigma_L$  on  $G$  is an affine system determined by

$$\Sigma_L : \dot{x}(t) = \mathcal{X}(x(t)) + \sum_{j=1}^m u_j(t) Y^j(x(t)), \quad u = (u_1, \dots, u_m) \in \mathcal{U} \quad (14)$$

where  $\mathcal{X}$  is linear, that is, its flow  $(\mathcal{X}_t)_{t \in \mathbb{R}}$  is a one-parameter group of  $G$ -automorphism, the control vectors  $Y^j$ ,  $j = 1, \dots, m$  are invariant vector fields, as elements of  $\mathfrak{g}$ . The restricted class of admissible control  $\mathcal{U}$  is the same as before.

Certainly, the drift vector field  $\mathcal{X}$  is complete and the same is true for every invariant vector field  $Y^j$ ,  $j = 1, \dots, m$ . As usual, we assume that  $\Sigma_L$  satisfy the Lie algebra rank condition, i.e.

$$\text{for any } x \in M \Rightarrow \text{Span}_{\mathcal{LA}} \{ \mathcal{X}, Y^1, \dots, Y^m \}(x) = d.$$

The system is said to be controllable if  $\mathcal{A}(e) = \mathcal{A}$  is  $G$ .

The class of systems  $\Sigma_L$  is huge and contains many relevant algebraic systems as the classical linear and bilinear systems on Euclidean spaces [6], and the class of invariant systems on Lie groups, [24]. Furthermore, according to the Jouan Equivalence Theorem [25],  $\Sigma_L$  is also relevant in applications. It approaches globally any affine non-linear control system  $\Sigma$  on a Riemannian manifold when the Lie algebra of the dynamics of  $\Sigma$  is finite dimensional.

One can associate to  $\mathcal{X}$  a derivation  $\mathcal{D}$  of  $\mathfrak{g}$  defined by  $\mathcal{D}Y = -[\mathcal{X}, Y](e)$ ,  $Y \in \mathfrak{g}$ . Indeed, the Jacobi identity shows  $\mathcal{D}[X, Y] = [\mathcal{D}X, Y] + [X, \mathcal{D}Y]$  is in fact a derivation. The relation between  $\varphi_t$  and  $\mathcal{D}$  is given by the formula

$$\varphi_t(\exp Y) = \exp(e^{t\mathcal{D}}Y), \quad \text{for all } t \in \mathbb{R}, Y \in \mathfrak{g}.$$

Consider the generalised eigenspaces of  $\mathcal{D}$  defined by

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : (\mathcal{D} - \alpha)^n X = 0 \text{ for some } n \geq 1\}$$

where  $\alpha \in \text{Spec}(\mathcal{D})$ . Then,  $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$  when  $\alpha + \beta$  is an eigenvalue of  $\mathcal{D}$  and zero otherwise.

Therefore, it is possible to decompose  $\mathfrak{g}$  as  $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^-$ , where

$$\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^-, \text{ where}$$

$$\mathfrak{g}^+ = \bigoplus_{\alpha: \operatorname{Re}(\alpha) > 0} \mathfrak{g}_\alpha, \quad \mathfrak{g}^0 = \bigoplus_{\alpha: \operatorname{Re}(\alpha) = 0} \mathfrak{g}_\alpha \text{ and } \mathfrak{g}^- = \bigoplus_{\alpha: \operatorname{Re}(\alpha) < 0} \mathfrak{g}_\alpha.$$

Actually,  $\mathfrak{g}^+, \mathfrak{g}^0, \mathfrak{g}^-$  are Lie algebras and  $\mathfrak{g}^+, \mathfrak{g}^-$  are nilpotent. Denote by  $G^+, G^-$  and  $G^0$  the connected and closed Lie subgroups of  $G$  with Lie algebras  $\mathfrak{g}^+, \mathfrak{g}^-$  and  $\mathfrak{g}^0$  respectively.

Despite the fact that for an invariant system the global controllability property is local, this class has been studied for more than 50 years, see [24] and the references there in. The important point to note here is: for an invariant system the reachable set from the identity is a semigroup. However, in [26] the authors show that this is not the case for a linear system which turns the problem more complicated. Therefore, we would like to explore the mentioned connection between control sets and the Devaney and Colonius-Kliemann ideas. This section is the starting point for the  $\Sigma_L$  class. We begin with a fundamental result.

**Theorem 31** *Assume the system  $\Sigma_L$  satisfy the Lie algebra rank condition. Therefore, there exists a control set*

$$C_e = \operatorname{cl}(\mathcal{A}(e)) \cap \mathcal{A}^*(e)$$

which contains the identity element  $e$  in its interior. Here,  $\mathcal{A}^*(e)$  is the set of states of  $G$  that can be sent by  $\Sigma_L$  to  $e$  in positive time.

For a proof in a more general set up, see [6].

Recently, we were able to establish some algebraic, topological, and dynamical conditions on  $\Sigma_L$  to study uniqueness and boundness of control sets and its consequences on controllability. But, the state of arts is really far from being complete. In order to approach this problem for  $\Sigma_L$ , as in [27] we assume here that  $G$  has finite semisimple centre, i.e. all semisimple Lie subgroups of  $G$  have finite center. We notice that any nilpotent and solvable Lie group, and any semisimple Lie group with finite centre has the finite semisimple centre property. But also, the product between groups with finite semisimple centre have the same property. We also assume that  $\mathcal{A}$  is open. This is true if for example, the system satisfy the *ad*-rank condition. About the uniqueness and boundness of control sets of a linear systems, we know few things [27].

**Theorem 32** *Let  $\Sigma_L$  a linear control system on the Lie group  $G$ .*

1. If  $G = G^- G^0 G^+$  is decomposable,  $C_e$  is the only control set with non-empty interior. In particular, this is true for any solvable Lie group.

2. Suppose that  $G$  is semisimple or nilpotent, it turns out that

$$\text{if } \operatorname{cl}(\mathcal{A}_{G^-}), \operatorname{cl}(\mathcal{A}_{G^+}^*) \text{ and } G^0 \text{ are compact sets } C \text{ is bounded.}$$

3. If  $G$  is a nilpotent simply connected Lie group, it follows that

$$C \text{ is bounded} \Leftrightarrow \operatorname{cl}(\mathcal{A}_{G^-}) \text{ and } \operatorname{cl}(\mathcal{A}_{G^+}^*) \text{ are compact sets and } D \text{ is hyperbolic.}$$

Furthermore, it is possible to determine algebraic sufficient conditions to decide when  $C$  is bounded. Actually, in a forthcoming paper we show that

**Theorem 33** Let  $\Sigma_L$  be a linear control system on the Lie group  $G$ . Assume that  $G$  is decomposable and  $G^{+,0}$  is a normal subgroup of  $G$ . Hence,  $\text{cl}(G^- \cap \mathcal{A})$  is compact.

An analogous result is obtained for  $G^+ \cap \mathcal{A}$  assuming that  $G^{-,0}$  is normal. Of course,  $G^{+,0}$  is a normal subgroup of  $G$  if and only if  $\mathfrak{g}^+ \oplus \mathfrak{g}^0$  is an ideal of  $\mathfrak{g}$ . On the other hand,

$$\mathfrak{g}^+ \oplus \mathfrak{g}^0 \text{ and } \mathfrak{g}^+ \oplus \mathfrak{g}^0 \text{ are ideals of } \mathfrak{g} \Leftrightarrow [\mathfrak{g}^+, \mathfrak{g}^0] = 0 \text{ and } [\mathfrak{g}^+, \mathfrak{g}^-] \subset \mathfrak{g}^0.$$

## 7.2. Chaos and control sets

We start with an explicitly relationship between chaotic subsets of  $M \times \mathcal{U}$  and the  $\Sigma$ -control sets.

**Theorem 34** Let  $\mathfrak{C} \subset M \times \mathcal{U}$  and the canonical projection  $\pi_M : M \times \mathcal{U} \rightarrow M$ . Hence,

$$\pi_M(\mathfrak{C}) = \{x \in M : \text{there exists } u \in \mathcal{U} \text{ with } (x, u) \in \mathfrak{C}\}$$

is compact and its non-void interior consists of locally accessible points. Then,

1.  $\mathfrak{C}$  is a maximal topologically mixing set if and only if there exists a control  $\mathcal{C}$  such that

$$\mathfrak{C} = \text{cl}\{(x, u) \in M \times \mathcal{U} : \varphi(t, x, u) \in \text{int}(\mathcal{C}) \text{ for every } t \in \mathbb{R}\}$$

In this case,  $\mathcal{C}$  is unique and  $\text{int}(\mathcal{C}) = \text{int}(\pi_M(\mathfrak{C}))$ ,  $\text{cl}(\mathcal{C}) = \text{cl}(\pi_M(\mathfrak{C}))$ .

2. The periodic points of  $\Phi$  are dense in  $\mathfrak{C}$ .
3.  $\Phi$  restrict to  $\mathfrak{C}$  is topologically mixing, topologically transitive and has sensitive dependence on initial conditions.

In order to apply this fundamental result for a non-controllable linear control system, the boundness property of its control set is crucial. Let us assume that  $\mathcal{C}$  is a bounded control set with non-empty interior of  $\Sigma_L$  and define  $\mathfrak{C} = \pi_M^{-1}(\mathcal{C}) = \text{cl}(\mathcal{C} \times \mathcal{U}_\mathcal{C})$  where

$$\mathcal{U}_\mathcal{C} = \{u \in \mathcal{U} : \text{exist } x \in \mathcal{C} \text{ with } \varphi(t, x, u) \in \text{int}(\mathcal{C}) \text{ for every } t \in \mathbb{R}\}.$$

The Lie group  $G$  is finite dimensional and  $\mathcal{U}_\mathcal{C}$  is a closed subset of the compact class of admissible control  $\mathcal{U} \subset L^\infty(\mathbb{R}, \Omega \subset \mathbb{R}^m)$  with the weak\* topology. Since the projection is a continuous map, it turns out that  $\pi_M(\mathfrak{C})$  is compact and  $\mathfrak{C}$ ,  $\mathcal{C}$  are uniquely defined.

On the other hand, we are assuming that  $\Sigma_L$  satisfy the Lie algebra rank condition, hence the system is locally accessible at any point of the state space. Therefore, we are in a position to apply Theorem 32, first, for some classes of controllable linear systems, as follows.

**Theorem 35** Let  $\Sigma_L$  be a linear control system on a Lie group  $G$ . Any condition.

1.  $G$  is compact, or
2.  $G$  is Abelian, or

3.  $G$  has the finite semisimple centre property and the Lyapunov spectrum of  $\mathcal{D}$  is  $\{0\}$  implies that the skew flow  $\Phi$  is chaotic in  $G \times \mathcal{U}$ .

**Proof.** Under the hypothesis in (1), any control set is bounded. Furthermore, if  $G$  is compact, the Lie algebra rank condition assures that the linear control system  $\Sigma_L$  is controllable on  $G$ , see [15]. Hence,  $\Phi$  is topologically mixing, topologically transitive and the periodic points of  $\Phi$  are dense in  $G \times \mathcal{U}$ , which give us the desired conclusion.

It is well known that any Abelian Lie group is a product  $G = \mathbb{R}^m \times T^n$  between the Euclidean space  $\mathbb{R}^m$  and the torus  $T^n = S^1 \times \dots \times S^1$  ( $n$  times), for some  $m, n \in \mathbb{N}$ . In this case,  $\Sigma_L$  is also controllable [15]. Indeed, since the automorphism group of  $T^n$  is discrete, any linear vector field on the torus is trivial. But, we are assuming the Lie algebra condition on  $G$  which coincides with the Kalman rank condition in  $\mathbb{R}^m$ . And, on the compact part, we apply (1). Hence, the skew flow  $\Phi$  is chaotic in  $G \times \mathcal{U}$ . In fact,  $\pi_M^{-1}(\mathcal{C}) = G \times \mathcal{U}$  and the hypothesis of the compactness on the projection in Theorem 32 is not necessary for the lifting, see Proposition 4.3.3 in [6]. The same is true for (3). Actually, for this more general set up, we recently prove that the system is also controllable, [28, 29].

In the sequel, we use some topological properties of  $\mathcal{C}_e$  to translate these properties to its associated chaotic set  $\mathfrak{C}$ , as follows.

**Theorem 36** *Let  $\Sigma_L$  be a linear control system on a Lie group  $G$ . It holds.*

1. If  $G = G^- G^0 G^+$  there exists one and only one chaotic set  $\mathfrak{C} = \pi_M^{-1}(\mathcal{C}_e)$  in  $G \times \mathcal{U}$  given by

$$\mathfrak{C} = cl\{(x, u) : \varphi(t, x, u) \in \text{int}(\mathcal{C}_e) \text{ for every } t \in \mathbb{R}\} \subset M \times \mathcal{U}$$

2. If  $G$  is nilpotent and  $\mathcal{D}$  has only eigenvalues with non-positive real parts, then the only chaotic set  $\mathfrak{C} = \pi_M^{-1}(\mathcal{C})$  in  $G \times \mathcal{U}$  is closed
3. If  $G$  is nilpotent and  $\mathcal{D}$  has only eigenvalues with non-negative real parts then the only chaotic set  $\mathfrak{C} = \pi_M^{-1}(\mathcal{C})$  in  $G \times \mathcal{U}$  is open

**Proof.** If  $G$  is decomposable, we know that there exists just one control set: the one which contains the identity element. Hence,  $\mathfrak{C} = \pi_M^{-1}(\mathcal{C}_e)$  is the only chaotic set of  $\Phi$  on  $G \times \mathcal{U}$  which proves (1). To prove (2) and (3), we observe that the Lyapunov spectrum condition on the derivation  $\mathcal{D}$  associated to the drift vector field  $\mathcal{X}$  is equivalent to the control set  $\mathcal{C}_e$  be closed or open, respectively. Since the projection  $\pi_G : G \times \mathcal{U} \rightarrow G$  is a continuous map with the weak\* topology on  $\mathcal{U}$ , the lifting  $\pi_G^{-1}(\mathcal{C}_e)$  is both closed and open, respectively.

### 7.3. Challenge

In this very short section, we would like to invite the readers to work on the relationship between chaotic and control sets. We suggest to go further in this research through some specific examples on low-dimensional Lie groups. For that, we give some relevant information



about two groups of dimension three: the simply connected nilpotent Heisenberg Lie group  $H$  and the special linear group  $SL(2, \mathbb{R})$ . We finish by computing an example on  $H$ .

1. The nilpotent Lie algebra  $\mathfrak{h} = (\mathbb{R}^3, +, \cdot)$ , has the basis  $\{E_{12}, E_{23}, E_{13}\}$  with  $[E_{12}, E_{23}] = E_{13}$ . Here,  $E_{ij}$  denotes the real matrix of order 3 with zero everywhere except 1 in the position  $ij$ . The associated Heisenberg Lie group has the matrix representation

$$G = \left\{ g = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\} \xrightarrow{\varphi: g \rightarrow (x, y, z)} \mathbb{R}^3.$$

As invariant vector fields, the basis elements of  $\mathfrak{g}$  has the following description

$$E_{12} = \frac{\partial}{\partial x}, E_{23} = \frac{\partial}{\partial y} + x \frac{\partial}{\partial z} \text{ and } E_{13} = \frac{\partial}{\partial z}.$$

The canonical form of any  $\mathfrak{g}$ -derivation is given by

$$\mathcal{D} = \begin{pmatrix} a & d & 0 \\ b & e & 0 \\ c & f & a+e \end{pmatrix} : a, b, c, d, e, f \in \mathbb{R}.$$

Any linear vector field  $\mathcal{X}$  reads as

$$\mathcal{X}(x, y, z) = (ax + dy) \frac{\partial}{\partial x} + (bx + ey) \frac{\partial}{\partial y} + \left( \frac{b}{2}x^2 + \frac{d}{2}y^2 + cx + fy + (a+e)z \right) \frac{\partial}{\partial z}.$$

2. The vector space  $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$  of all real matrices of order three and trace zero is the Lie algebra of the Lie group  $G = SL(2, \mathbb{R}) = \det^{-1}(1)$ . Let us consider the following generators of  $\mathfrak{g}$ :  $Y^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ ,  $Y^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$  and  $Y^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ . The Lie group  $G$  is semisimple, then any  $\mathfrak{g}$  derivation is inner which means that there exists an invariant vector field  $Y$  such that  $ad(Y)$  represents . Thus, a general form of a derivation reads as

$$\alpha \, ad(Y^1) + \beta \, ad(Y^2) + \gamma \, ad(Y^3).$$

**Example 7.1** On the Heisenberg Lie group, consider the system

$$\Sigma_L : \dot{g}(t) = \mathcal{X}(g(t)) + u_1(t)E_{12}(g(t)) + u_2(t)E_{23}(g(t)), \quad u = (u_1, u_2) \in \mathcal{U} \quad (15)$$

where  $\mathcal{X}$  is determined by the derivation  $\mathcal{D} = ad(E_{12}) = E_{32}$ . Since the group is nilpotent, it has the semisimple finite centre property. The Lyapunov spectrum of  $\mathcal{D}$  reduces to zero. Finally, the reachable set from the identity  $\mathcal{A}$  is open. In fact, the  $ad$ -rank condition is obviously true because  $\mathcal{D}(E_{12}) = E_{13}$ . It turns out that the skew flow  $\Phi$  is chaotic in  $H \times \mathcal{U}$ .

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