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Fractal Pyramid: A New Math Tool to Reorient and Accelerate a Spacecraft

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Abstract

An original mathematical instrument matching two different operational procedures aimed to change orientation and velocity of a spacecraft is suggested and described in detail. The tool's basements, quaternion algebra with its square-root (pregeometric) image, and fractal surface are represented in a parenthetical but in a sufficient format, indicating their principle properties providing solution to the operational task. A supplementary notion of vector-quaternion version of relativity theory is introduced since the spacecraft-observer mechanical system appears congenitally relativistic. The new tool is shown to have a simple pregeometric image of a fractal pyramid whose tilt and distortion evoke needed changes in the spacecraft's motion parameters, and the respective math procedures proved to be simplified compared with the traditionally used math methods.

Keywords: spacecraft motion, operation, quaternion, fractal surface

1. Introduction

In classical mechanics, rotation of a rigid body (in particular, a spacecraft) and its translational motion are normally regarded as drastically different actions leading to changes in its position and are respectively described by different groups. Relativistic mechanics, in its turn, deals with these two types of motions "more homogeneously" since rotation and linear motion are described in this case by 4×4 matrices from the Lorentz group $SO(1,3)$. However, it is well known that the special relativity limits itself by inertial motions of the involved frames of reference while use of general relativity comprising any types of motion but demanding math methods of tensor calculus seems unapproved sophisticated. Happily, there exists a simpler vector version of the relativity theory admitting arbitrary accelerated motion of the frames. A brief formulation of the theory is made with the help of quaternion vector units, each set of the

units representing a Cartesian-type frame of reference. In this case, the rotation-and-translation operator is given by 3×3 matrix belonging to the group $SO(3, C)$ known to be 1:1 isomorphic to the group $SO(1, 3)$. However, the calculations of the body's complex motions even within the framework of the vector-quaternion relativity remain prolonged and cumbersome, a simpler method is desired. Such a method is found due to existence of 1:2 isomorphism of the groups $SO(3, C)$ and $SL(2, C)$, the last being a spinor group operating in fractal two-dimensional complex-number valued space (a fractal surface). It is necessary to mention that the subgroup of $SL(2, C)$, rotational group $SU(2)$, is normally used in space-flight practice, providing comparatively simple mathematical computations for a spacecraft reorientation tasks [1, 2]. This method is based upon similarity-type transformations of the initial quaternion triad, in fact assuming nontrivial multiplication of at least three different quaternions, though it straightforwardly gives the data describing the axis of single rotation and value of the respective angle. However, this method provides no translational motion.

In this study, we suggest an essential development of the last (single rotation) method leading, first, to noticeable simplification of computations, and second, to possibility of introduction of additional parameters responsible for the spacecraft acceleration. This development is fully based on fundamental properties of subgeometric dyad forming the fractal space in a way underlying the 3D physical space. Moreover, we suggest subgeometric images (fractal joystick and fractal pyramid) of the math tools realizing the spacecraft's reorientation and acceleration tasks. As well, we give a brief comparative analysis of simplicity (or complexity) of conventional and new methods.

The study is composed as following. In Sections 2–4 we offer a detailed mathematical introduction. In Section 2, we renew our knowledge of quaternion algebra giving traditional (Hamiltonian) and more compact (tensor) notions and correlations. In Section 3, we briefly reproduce the quaternion version of the relativity theory. In Section 4, we consider main notions and properties of the 2D fractal space and show how to build a 3D frame out of a dyad element.

Sections 5–7 are devoted to new math methods making operations of a spacecraft simpler and more functional. Section 5 is devoted to presentation of three methods to reorient a spacecraft with accent on convenience of the single rotation method involving a fractal joystick model. In Section 6, we suggest a very simple way to introduce (apart from space rotation) an acceleration of the spacecraft and demonstrate a subgeometric image of the respective math tool having a shape of fractal pyramid. Finally, in Section 7, we give a sketch of a technological map previewing necessary steps to simultaneously reorient and accelerate the spacecraft followed by a series of relevant pictures.

2. Basic notions and relations of quaternion algebra

Quaternion (Q-) numbers were discovered by Hamilton in 1843 [3]. A quaternion is a math object of the type $q = a1 + bi + cj + dk$ (in Hamilton's notation), where a, b, c, d are real coefficients at the real unit 1 (the symbol is normally omitted in the number) and at three imaginary units i, j, k forming the postulated multiplication table (16 equalities).

$$\begin{aligned} 1^2 = 1, i^2 = j^2 = k^2 = -1, 1i = i1 = i, 1j = j1 = j, 1k = k1 = k, \\ ij = -ji = k, jk = -kj = i, ki = -ik = j, \end{aligned} \quad (1)$$

Q-numbers and the multiplication law (1) can be more compactly rewritten in the vector (and tensor) notations $i, j, k \rightarrow \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3 \rightarrow \mathbf{q}_k, j, k, l, m, n \dots = 1, 2, 3$; then, a quaternion is a sum of scalar (a) and vector ($b_k \mathbf{q}_k$) parts $q \equiv a + b_k \mathbf{q}_k$, where $a, b_k \in \mathbf{R}$, and the multiplication table (1) has the form

$$1\mathbf{q}_k = \mathbf{q}_k 1 = \mathbf{q}_k, \quad \mathbf{q}_k \mathbf{q}_l = -\delta_{kl} + \varepsilon_{klj} \mathbf{q}_j, \quad (2)$$

Summation in repeated indices is implied, and δ_{kl} and ε_{klj} are the 3D Kronecker and Levi-Chivita symbols (see e.g., [4]).

Quaternions admit the same operations as real and complex numbers. Comparison of Q-numbers is reduced to their equality: two Q-numbers are equal if coefficients at respective units are equal. Commutative addition (subtraction) of Q-numbers is made by components. Q-numbers are multiplied as polynomials; the rules (1, 2) state that multiplication is noncommutative (left and right products are defined), but still associative. A quaternion $q \equiv a + b_k \mathbf{q}_k$ has its conjugate $\bar{q} \equiv a - b_k \mathbf{q}_k$, the norm $|q|^2 \equiv q\bar{q} = \bar{q}q$, and the modulus (positive square root from the norm) $|q| \equiv \sqrt{q\bar{q}} = \sqrt{a^2 + b_k b_k}$. Inverse number is $q^{-1} = \bar{q}/|q|^2$; so, for two quaternions q_1 and q_2 , division (left and right) is defined as $(q_1/q_2)_{\text{left}} = q_2 q_1 / |q_2|^2$ and $(q_1/q_2)_{\text{right}} = q_1 q_2 / |q_2|^2$. If q is a product of two multipliers $q_1 = a + b_k \mathbf{q}_k$ and $q_2 = c + d_n \mathbf{q}_n$, then from definition of the norm one finds

$$|q|^2 = |q_1 q_2|^2 = (q_1 q_2)(\overline{q_1 q_2}) = q_1 q_2 \bar{q}_2 \bar{q}_1 = q_1 \bar{q}_1 q_2 \bar{q}_2 = |q_1|^2 |q_2|^2. \quad (3)$$

Written in components, Eq. (3) becomes the famous identity of four squares

$$\begin{aligned} (ac - b_1 d_1 - b_2 d_2 - b_3 d_3)^2 + (ad_1 + cb_1 + b_2 d_2 - b_3 d_3)^2 + (ad_2 + cb_2 + b_3 d_1 - b_1 d_3)^2 + \\ + (ad_3 + cb_3 + b_1 d_2 - b_2 d_1)^2 = (a^2 + b_1^2 + b_2^2 + b_3^2)(c^2 + d_1^2 + d_2^2 + d_3^2). \end{aligned} \quad (4)$$

Identities of the type (4) exist only in four algebras: of real numbers (trivial identity), of complex numbers (two squares), of quaternions (four squares), and of octonions (the last exclusive algebra with one real and seven imaginary units admits identity of eight squares; multiplication in this algebra is no more associative).

Geometrically, the imaginary Q-units are associated with three unit vectors initiating a Cartesian coordinate system (Q-triad, Q-frame). This image, in particular, follows from the fact that, according to Eq. (2), each imaginary unit appears as ordered product of the two others: $\mathbf{q}_1 = \mathbf{q}_2 \mathbf{q}_3, \mathbf{q}_2 = \mathbf{q}_3 \mathbf{q}_1, \mathbf{q}_3 = \mathbf{q}_1 \mathbf{q}_2$ (vector products in Gibbs-Heaviside algebra). One can easily construct a set of such units. To demonstrate this, we consider a couple of 2×2 -matrices, $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, B = \begin{pmatrix} d & e \\ f & -d \end{pmatrix}$, traceless: $\text{Tr}A = \text{Tr}B = 0$, and not degenerate: $\det A \neq 0, \det B \neq 0$. We use the matrices to build two different imaginary units as

$$\mathbf{q}_1 = \frac{A}{\sqrt{\det A}}, \quad \mathbf{q}_2 = \frac{B}{\sqrt{\det B}}. \quad (5)$$

We form the product of the two units and demand that its trace vanishes that is given as

$$\mathbf{q}_1 \mathbf{q}_2 = \frac{AB}{\sqrt{\det A \det B}}, \quad \text{Tr}(AB) = 0; \quad (6)$$

then Eq. (6) gives expression for the third imaginary Q-unit $\mathbf{q}_1 \mathbf{q}_2 = \mathbf{q}_3$, and as a whole, we get the Q-triad \mathbf{q}_k , the real unit always remaining the unit matrix $1 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. One readily checks up that the triad given by Eqs. (5) and (6) identically satisfies the multiplication law (2). Built in a similar way, the simplest representation of Q-units $\mathbf{q}^{\tilde{k}}$ is given by the Pauli matrices $\mathbf{p}^{\tilde{k}}$ with factor $-i$: $\mathbf{q}^{\tilde{k}} = -i\mathbf{p}^{\tilde{k}}$

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{q}_1 = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{q}_2 = -i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \mathbf{q}_3 = -i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (7)$$

and the imaginary Q-triad given as Eq. (7) describes a constant Q-vector frame.

However, a Q-frame may be variable, rotating, and moving. There are two types of transformations changing the frame but retaining the form of the multiplication law (2). The first is rotational-type transformation

$$\mathbf{q}_{k'} = O_{k'n} \mathbf{q}_n \quad (8)$$

where $O_{k'n}$ is a 3×3 -matrix (its components are in general complex numbers) having orthogonal properties $O_{k'n} O_{m'n} = \delta_{km}$, hence this matrix belongs to the special orthogonal group of 3D rotations over field of complex numbers $O_{k'n} \in SO(3, \mathbb{C})$. The matrix $O_{n'k}$ can be always represented as a product of plane (or simple) rotations, irreducible representations of $SO(3, \mathbb{C})$. For such matrices, a special notation will be used, e.g., O_n^Θ , where the lower index indicates the rotation axis (the frame's unit vector) and upper index shows the rotation angle. Depending on the math nature of the angle Θ , we distinguish two types of simple rotations. If $\Theta = \alpha \in \mathbb{R}$, then we have a real simple rotation $O_n^\Theta \rightarrow R_n^\alpha$; if the angle is imaginary $\Theta = \eta \in i\mathbb{R}$, then we have a simple hyperbolic rotation $O_n^\Theta \rightarrow H_n^\eta$; for example Eq. (9)

$$R_3^\alpha \equiv \begin{pmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad H_3^\eta \equiv \begin{pmatrix} \cos h\eta & -i \sin h\eta & 0 \\ i \sin h\eta & \cos h\eta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (9)$$

Superposition of any number (N) of real rotations (product of relevant matrices) gives a (nonplane) real rotation

$$\prod_{j=1}^N R_{n_j}^{\alpha_j} = R_{n_1}^{\alpha_1} \dots R_{n_N}^{\alpha_N} \rightarrow R_{k'm} SO(3, R). \quad (10)$$

Product of multiple hyperbolic rotations is physically sensible if accompanied by real rotations in the framework of vector version of theory of relativity (see Section 3); so in general, the matrices of the type

$$\prod_{j=1}^N \prod_{s=1}^M R_{n_j}^{\alpha_j} H_{m_s}^{\eta_s} = R_{n_1}^{\alpha_1} \dots H_{m_1}^{\eta_1} \dots R_{n_N}^{\alpha_N} \dots H_{m_s}^{\eta_s} \rightarrow O_{k'm} \in SO(3, C) \quad (11)$$

are used in applications.

The second type of transformations is performed by an operator U and its inverse U^{-1} is given as

$$\mathbf{q}_{k'} = U \mathbf{q}_k U^{-1}. \quad (12)$$

It is evident that the transformation (12) keeps the form of the basic law (2). The operators U are known to form the (spinor) group $U \in SL(2, C)$ of special linear 2D transformations over field of complex numbers; this group is 2:1 isomorphic to $SO(3, C)$ and similarly to the Lorentz group. A special case of the transformation (12) is a real rotation made by means of the subgroup $SU(2) \in SL(2, C)$, and this spinor subgroup is 2:1 isomorphic to vector group $SO(3, R)$. It is necessary to note that the transformation of the type (12) with $U \in SU(2)$ is most frequently used for solution of a spacecraft orientation problem (see Section 5.2).

As well, in formulation of quaternion relativity (see Section 3), we shall need notion of a biquaternion (BQ-) number. Such a number has the form $b = x + y_k \mathbf{q}_k$, where $x, y_k \in \mathbf{C}$ while $1, \mathbf{q}_k$ are Q-units. BQ-numbers admit addition, multiplication, and conjugation $\bar{b} = x - y_k \mathbf{q}_k$. But the norm is not well defined since the product $b\bar{b} = x^2 + y_k y_k$ in general is not a real (and positive) number. A real number "norm" exists in the subset of vector biquaternions

$$b = (w_k + i z_k) \mathbf{q}_k \quad (13)$$

whose real and imaginary parts are mutually orthogonal

$$w_k z_k = 0 \rightarrow \|b\|^2 = b\bar{b} = w_k w_k - z_k z_k. \quad (14)$$

There are evidently zero dividers in Eq. (14), hence division is not well defined, but the subset (13 and 14) comprises basic formulas describing relative motion of arbitrary accelerated frames of reference.

3. Vector-quaternion version of the relativity theory

According to Eqs. (13) and (14), the interval of Einstein's relativity theory¹

$$ds^2 = dx_0^2 - dx_k dx_k = dt^2 - dr^2 \quad (15)$$

admits a BQ-square root

$$ds = (ie_k dt + dx_k) \mathbf{q}_k \quad (16)$$

where displacement of observed object dx_k is orthogonal to a unit vector e_k directing change in time dt : $e_k dx_k = 0$. Under these conditions, square of Eq. (16) yields Eq. (15) $ds ds = ds^2$. It is convenient to explicitly relate displacement dx_k to a plane orthogonal to time-directing vector e_k with the help of metric-projector $b_{kn} \equiv \delta_{kn} - e_k e_n$ $dx_k = dx_n b_{nk}$,

then the orthogonality condition is fulfilled automatically $e_k dx_k = e_k dx_n b_{kn} = 0$.

The interval (15) is invariant under Lorentz transformations of coordinate system $dx^{\alpha'} = L_{\lambda}^{\alpha'} dx^{\lambda}$, $L_{\lambda}^{\alpha'} \in SO(1, 3)$, while the Q-frame can be subject to $SO(3, C)$ rotations $\mathbf{q}_{k'} = O_{k'l} \mathbf{q}_l$; simultaneous application of the transformations, together with demand that the BQ-vector (16) form be conserved, leads to correlation between components of matrices $O_{k'l}$ and $L_{\alpha'\lambda}$ ² [5, 6]

$$ie_k O_{s'k} = ie_{s'} L_{0'0} + L_{m'0} b_{m's'}, \quad (17)$$

$$b_{nk} O_{s'k} = -ie_{s'} L_{0'm} - L_{m'k} b_{m's'}. \quad (18)$$

Eqs. (17) and (18) in particular mean that within the group $SO(3, C)$ a set of ordered simple rotations of the type (11) are distinguished, real and hyperbolic, each performed about one-unit vector of Q-triad. If for instance, direction No. 1 of $L_{\alpha'\lambda}$ is not involved in the transformation ($e_k = e_{k'} = \delta_{1k}$), then Eqs. (17) and (18) represent the matrix O as function of components of Lorentz matrix L

$$O_{k'm} = \begin{pmatrix} L_{0'}^{0'} & -iL_{2'}^{0'} & -iL_{3'}^{0'} \\ iL_{0'}^{2'} & L_{2'}^{2'} & L_{3'}^{2'} \\ iL_{0'}^{3'} & L_{2'}^{3'} & L_{3'}^{3'} \end{pmatrix}. \quad (19)$$

The matrix (19) may describe a series of simple rotations, but real rotations should be always performed about vector \mathbf{q}_1 (initial or transformed), while hyperbolic rotations are allowed about

¹Standard interval of special relativity is regarded for simplicity; similarly, interval of general relativity can be considered in tangent space $ds^2 = \theta_0^2 - \theta_k \theta_k$ with $\theta_{(\alpha)} = g_{(\alpha)\lambda} dy^{\lambda}$ being basic one-form and Greek indices in brackets enumerating tangent space tetrad, and those without brackets are related to curved manifold holonomic coordinates

$$\eta_{\alpha\beta} = \text{diag}(1, -1, -1, -1)$$

² $ds^2 = \theta_0^2 - \theta_k \theta_k \theta_{(\alpha)} = g_{(\alpha)\lambda} dy^{\lambda}$: four-dimensional indices are raised and lowered by Minkowski metric $\eta_{\alpha\beta} = \text{diag}(1, -1, -1, -1)$.

vectors \mathbf{q}_2 and \mathbf{q}_3 . It is easily checked up that all matrices O of the type (19) constitute a subgroup $SO(1, 2) \subset SO(3, C)$ of the ordered rotations of Q -triads.

Main idea of Q -version of relativity is to replace line element of Einstein's relativity (15) and its invariance under Lorentz group by adequate BQ-vector (16) invariant under rotational group represented by matrices $O \in SO(1, 2)$. Then, instead of quadratic form of four-dimensional coordinates, an observer has at his disposal a movable Q -triad with time and distances measured along its unit vectors and dealt with the vector basement as with the Newtonian mechanics or general relativity in tetrad formulation. However, on this way, an essential peculiarity arises. Eq. (16) implies that the constructed space-time model has six dimensions, and it is a symmetric sum of two three-dimensional (3D) spaces $Q_6 = R_3 \oplus T_3$, where R_3 is the usual 3D space where coordinate and velocity change, whereas T_3 is also a 3D space but imaginary with respect to R_3 . In this model, the observer works only with some sections of the 6D space; but since the objects of the observations are found in real 3D space, and imaginary time axis is distinguished, an illusion of four dimensions emerges.

Physical measurements in the Q -model are made with the help of three spatial rulers \mathbf{q}_k and built-in geometric clock represented by "imaginary time rulers" (Pauli-type matrices) $\mathbf{p}_k \equiv i\mathbf{q}_k$, the two triads being obviously co-aligned. The tool-set $\Sigma \equiv \{\mathbf{p}_k, \mathbf{q}_k\}$ with an observer in the initial point represents full physical frame of reference, Eq. (16) can be rewritten as

$$ds = e_k dt \mathbf{p}_k + dx_k \mathbf{q}_k. \quad (20)$$

Now, the principal statement of the Q -version of relativity follows: all physically sustainable frames of reference are interconnected by "rotational equations"

$$\Sigma' = O\Sigma, \quad O \in SO(1, 2). \quad (21)$$

The sustainability means form-invariance of BQ-vector (16) or (20) under transformations (21). Kinematic effects of special relativity are straightforwardly found in the Q -version; here, we demonstrate only one effect important for fractal pyramid technology accelerating a spacecraft (see Section 6).

Boost. Σ -observer always can align one of his spatial vectors (e.g., \mathbf{q}_2) with velocity of moving body, so basic BQ-vector can be written in the form

$$ds = dt \mathbf{p}_1 + dr \mathbf{q}_2. \quad (22)$$

Let the frame Σ' be a result of a hyperbolic rotation of a constant frame Σ

$$\Sigma' = H_3^\eta \Sigma, \quad (23)$$

with the matrix H_3^η from Eq. (9b) (rotation about \mathbf{q}_3 by angle η). This simple rotation, physically a boost, obviously keeping BQ-vector (20) form-invariant

$$dt\mathbf{p}_1 + dr\mathbf{q}_2 = dt'\mathbf{p}_{1'} + dr'\mathbf{q}_{2'} \quad (24)$$

yields familiar coordinate transformations

$$dt' = dt \cosh \eta + dr \sinh \eta, \quad dr' = dt \sinh \eta + dr \cosh \eta \quad (25)$$

with respective effects of length and time segments contraction. If observed particle is the body of reference of the frame Σ' , then $dr' = 0$, and one finds that the frame Σ' is moving with the velocity

$$V = \frac{dr}{dt} = \tanh \psi. \quad (26)$$

Specific features of the Q-vector version of relativity will be effectively used below in the fractal-pyramid math method to operate a spacecraft. Now, we turn to notions of a fractal space.

4. Fractal space underlying physical space

In this section, we show that a 3D space (e.g., physical space) may be endowed with a pregeometry [7] mathematically described by a complex-numbered surface, a 2D fractal space, each its vector having dimensionality half compared to that of the 3D space. We start with 2D space and construct out of its basic elements a basis of 3D space.

Let there exist a smooth 2D space (surface) endowed with a metric g_{AB} (and inverse: $g^{BC}g^{BC} \rightarrow g_{AB}g^{BC} = \delta_A^C$) and with a system of coordinates $x^A = \{x^1, x^2\}$; here $A, B, C = 1, 2$, δ_A^C is a 2D Kronecker symbol, summation in repeated indices is also implied. The line element of the surface is

$$ds^2 = g_{AB}dx^A dx^B; \quad (27)$$

the surface may be curved, so covariant and contravariant metric components differ. In a point, we choose a couple of unit orthogonal vectors a^A, b^B (a dyad)

$$g_{AB}a^A a^B = 1, \quad (28)$$

$$g_{AB}a^A b^B = 0. \quad (29)$$

A domain of the surface in vicinity of the dyad's initial point (together with respective part of tangent plane having the metric $\delta_{MN} = \delta_m^N = \delta_m^N$) will be called a "2D-cell."

Considering direct (tensor) products of the dyad vectors with mixed components [8], we can construct only four such products (2×2 matrices): two idempotent matrices

$$G_B^A \equiv a^A a_B, \quad H_B^A \equiv b^A b_B \rightarrow G_B^A G_C^B = G_C^A, \quad H_B^A H_C^B = H_C^A, \quad (30a)$$

and two nilpotent matrices

$$D^{AB} \equiv a^A b_B, \quad F^{AB} \equiv b^A a_B \rightarrow D^{AB} D^{BC} = 0, \quad F^{AB} F^{BC} = 0. \quad (30b)$$

Next, we built sum and difference of the idempotent matrices

$$E \equiv E_B^A \equiv G_B^A + H_B^A = a^A a_B + b^A b_B, \quad E^2 = E, \quad (31a)$$

$$\tilde{K} \equiv \tilde{K}_B^A \equiv G_B^A - H_B^A = a^A a_B - b^A b_B, \quad \tilde{K}^2 = E, \quad (31b)$$

and sum and difference of the nilpotent matrices

$$\tilde{I} \equiv \tilde{I}_B^A = D^{AB} + F^{AB} = a^A b_B + b^A a_B, \quad \tilde{I}^2 = E, \quad (31c)$$

$$J \equiv J_B^A = D^{AB} - F^{AB} = a^A b_B - b^A a_B, \quad J^2 = -E. \quad (31d)$$

If the units Eqs. (31b) and (31c) are slightly corrected so that their product is the third unit (31d), then we obtain the basis of quaternion (and biquaternion) numbers

$$1 \equiv E, \quad \mathbf{q}_1 = -i \tilde{I}, \quad \mathbf{q}_2 = -i J, \quad \mathbf{q}_3 = -i \tilde{K}. \quad (32)$$

Now, we recall the spectral theorem (of the matrix theory) stating that any invertible matrix with distinct eigenvalues can be represented as a sum of idempotent projectors with the eigenvalues as coefficients, the projectors being direct products of vectors of a biorthogonal basis. The unit \mathbf{q}_3 defined in Eqs. (32), (31b) is the characteristic example

$$\mathbf{q}_3|_A^B = ia^A a_B - ib^A b_B = iG_B^A - iH_B^A. \quad (33)$$

Right and left eigenfunctions of \mathbf{q}_3 are vectors a^A, b^B and covectors a_A, b_B of the dyad, respectively; the eigenvalues are $+i$ (for a) and $-i$ (for b), and G_B^A, H_B^A are the projectors.

As mentioned above, the similarity transformation of the units

$$\mathbf{q}_{k'} = U \mathbf{q}_k U^{-1}, \quad U \in SL(2, C) \quad (34)$$

preserves the form of algebras' multiplication law (2). Therefore, vector units from Eq. (32) can be obtained from a single unit, say, \mathbf{q}_3 by a transformation (34). Then, all vector units have same eigenvalues $\pm i$, and the eigenfunctions of the derived units are linear combinations of the eigenfunctions of the initial unit [9]. This also means that the mapping (34) is a secondary one, but the primary one is $SL(2, C)$ transformation of dyad vectors, thus forming a set of spinors from the viewpoint of the 3D space described by the triad vectors \mathbf{q}_k .

Hereinafter, we introduce shorter 2D-index-free matrix notations for the dyad: a vector is a column, a co-vector is a row, and a parity indicator $+$ or $-$ marks the sign of the eigenvalue $\pm i$

$$a^A \rightarrow \psi^+, \quad a_A \rightarrow \varphi^+, \quad b^A \rightarrow \psi^-, \quad b_A \rightarrow \varphi^-; \quad (35)$$

this helps to rewrite the above expressions more compactly. The dyad orthonormality conditions (28, 29) acquire the form

$$\varphi^\pm \psi^\pm = 1, \quad \varphi^\mp \psi^\pm = \varphi^\pm \psi^\mp = 0, \quad (36)$$

the idempotent projectors are denoted as $C^+ \equiv G = \psi^+ \varphi^+$, $C^- \equiv H = \psi^- \varphi^-$,

and the units (32) are expressed through the single dyad vectors (co-vectors) as

$$1 = \psi^+ \varphi^+ + \psi^- \varphi^-, \quad (37a)$$

$$\mathbf{q}_1 = -i (\psi^+ \varphi^- + \psi^- \varphi^+), \quad (37b)$$

$$\mathbf{q}_2 = \psi^+ \varphi^- - \psi^- \varphi^+, \quad (37c)$$

$$\mathbf{q}_3 = i (\psi^+ \varphi^+ - \psi^- \varphi^-). \quad (37d)$$

Eq. (37) obviously demonstrates that the dyad elements are in a way “square roots” from 3D vector units. So, if we put dimensionality of any 3D line to be a unity, then dimensionality of a line on the 2D space (e.g., dimensionality of a dyad vector) must be $\frac{1}{2}$; hence from the viewpoint of the 3D space, the surface determined by a dyad is fractal. The next important observation concerns transformations. The transformation (34) clearly results from the $SL(2, C)$ transformations of the dyad vectors (covectors)

$$\psi'^\pm = U \psi^\pm, \quad \varphi' = \varphi'^\pm U^{-1}. \quad (38)$$

So, apart from vector-type (8) and spinor-type (12) transformations of a Q-triad (an element of 3D space), there exists a possibility to deal with more fundamental math elements, vectors, and covectors describing “pregeometric” 2D cell of a fractal surface. These simpler math objects are subject to evidently simpler mapping (38); moreover, in the following sections, we will show that the operators of the transformations, being themselves BQ-numbers, suggest simpler and less numerous equations to solve, thus reducing degree of math load and probability of mistakes.

5. Three methods to reorient a spacecraft and fractal joystick

The orientation tasks are relevant with computations over 3D flat space modeling a local domain of the physical space. Two types of the orientation problem solutions are traditional:

(i) a series of subsequent several angles rotation and (ii) a one-angle rotation about an instant axis. Mixed variants exist, but are less productive, and they are not normally considered.

If magnitudes involved in calculations are generically measured in real numbers, then both techniques (i) and (ii) are based on the vector rotation group $SO(3, R)$. Math content of the technique (i) implies a multiple set of plane rotations [of type of Eq. (9a)] by Euler (or Krylov, or others) angles about selected axes. The technique (ii) in its turn represents a nontrivial problem of determining the instant axis of a single rotation.

Quaternions are widely known to fit better than real numbers for the orientation tasks mostly due to the fact that three vector units represent models of three mutually orthogonal gyroscope axes. As well, use of the Q-algebra formalism essentially simplifies calculations, especially for the technique (ii), since both the vector rotation group $SO(3, R)$ and its spinor “equivalent” $SU(2)$ reflection group can be used whatever enigmatic were formulas describing spinor rotations. However, the quaternion algebra reveals its unique property to split axial 3D vectors into dyad sets belonging to a fractal subspace as in Eq. (37), see also the basic work [10]. The above-described fractalization procedure, mathematically nontrivial and much less known, on the one hand clarifies “mysterious” two-side $SU(2)$ quaternion vector multiplication and on the other hand endows all algebraic objects and actions with distinct geometric sense; moreover, the calculations become most primitive. Solution of a spacecraft reorientation task as transformation of a fractal dyad represents the third math method (iii) suggested here. However, all three math methods are described in detail in this section.

5.1. Quaternion $SO(3, R)$ approach to the reorientation problem: Technique (i)

Orientation of a spacecraft in 3D space is determined by three angles between axes of some global coordinate system and unit vectors of a frame attached to the moving body taking into account its physical symmetry. The global coordinates, e.g., are represented by a spherical system, and its local initiating vectors pointing: \mathbf{q}_1 to the north along the Earth’s meridian), \mathbf{q}_2 along a parallel, and \mathbf{q}_3 to zenith direction. The directing vectors \mathbf{q}_k are considered constant. Then, the orientation of a spacecraft bearing a frame $\mathbf{q}_{k'}$, (with $\mathbf{q}_{1'}$ along the body, $\mathbf{q}_{2'}$ a transverse one, and $\mathbf{q}_{3'}$ along gravity) is determined by three angles: “yaw” ψ , the angle between \mathbf{q}_1 and $\mathbf{q}_{1'}$ (rotation about \mathbf{q}_3); “roll” φ , angle $\mathbf{q}_2 - \mathbf{q}_{2'}$ (rotation about \mathbf{q}_1); and “pitch” θ , angle $\mathbf{q}_3 - \mathbf{q}_{3'}$ (rotation about \mathbf{q}_2). Within these notations, the spacecraft’s orientation in the space is described by the matrix equation

$$\mathbf{q}_{n'} = R_{n'k} \mathbf{q}_k, R \in SO(3, R). \quad (39)$$

Outlined above technique (i) demands that the matrix $R_{n'k}$ be represented as a product of simple rotations, irreducible representations of $SO(3, R)$ [a special notation for such matrix is $R_{n'}^\alpha$, see Section 2, Eqs. (9, 10)], each performed about a frame’s unit vector. Simple rotations with the above parameters of the probe’s orientations are given by the matrices

$$R_3^\psi \equiv \begin{pmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix}, R_2^\varphi \equiv \begin{pmatrix} \cos \varphi & 0 & -\sin \varphi \\ 0 & 1 & 0 \\ \sin \varphi & 0 & \cos \varphi \end{pmatrix}, R_1^\theta \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix}. \quad (40)$$

Direct reorientation problem, i.e., reaching object's assigned orientation, can be solved by a sequence of plane rotations mathematically described by a sequent multiplication of matrices [see Eq. (10)]. This problem has no unique solution since the group $SO(3, R)$ is not commutative; i.e., different multiplication order of the matrices (40) with the same parameters (angles) generally gives different result; e.g., the products $R = R_3^\psi R_2^\varphi R_1^\theta$ and $R' = R_1^\theta R_2^\varphi R_3^\psi$ are, in general, different $R \neq R'$. Vice versa, different orders of the matrix product with other parameters may yield the same result, e.g., products $R = R_3^\psi R_2^\varphi R_1^\theta$ and $R' = R_1^{\theta'} R_2^{\varphi'} R_3^{\psi'}$ may represent equivalent rotational result $R = R'$. The possibility to represent an arbitrary $SO(3, R)$ matrix as a product of its irreducible representations given in different order in particular entails uncertainty in solution of the inverse problem when one has to determine values of angles securing an assigned reorientation of the spacecraft. Therefore, the technique (i) does not provide single-valued results.

Even with more difficult, we meet trying to use matrices from the group $SO(3, R)$ in the technique (ii). As is known from the theory of matrices (see e.g., [11]) in this case, we have to solve the characteristic equation $RX = X$ searching for the matrix operator R an eigenvector X with unit eigenvalue, the vector X pointing direction of the instant rotation axis. This tough algebraic task then followed by sophisticated calculations aimed to find the instant rotation angle. The use of hypercomplex numbers essentially helped to avoid these math troubles, and about half of a century ago, quaternion algebra became a common tool serving for engineering goals of navigation and orientation. Indeed, the similarity transformation UqU^{-1} of a quaternion q performed with the help of auxiliary quaternion $U \equiv a + b\mathbf{q}$ geometrically leads to conical rotation of the vector part of q about an axis whose direction is determined by the unit Q-vector \mathbf{q} (e.g., [2]); the value of the instant rotation angle is computed as $2 \arctan(b/a)$. Below, we suggest a detailed analysis of this type of description of rotations.

5.2. Reorientation by a single rotation of the quaternion frame: Technique (ii)

Consider a 2×2 matrix (with complex-number components) $U \equiv \begin{pmatrix} x & z \\ w & y \end{pmatrix}$, belonging to a special linear group $U \in SL(2, C)$, $\det U = xy - wz = 1$. The multiplication law (6) is obviously form invariant under the similarity-type transformation

$$\mathbf{q}_{n'} = U\mathbf{q}_n U^{-1}. \quad (41)$$

One readily demonstrates that the matrix U is a biquaternion with the definable norm; indeed,

$$U = \begin{pmatrix} x & z \\ w & y \end{pmatrix} = \frac{x+y}{2} + \sqrt{1 - \left(\frac{x+y}{2}\right)^2} \mathbf{q} \equiv a + b\mathbf{q} \quad (42)$$

where

$$a = \frac{x+y}{2}, b = \sqrt{1 - \left(\frac{x+y}{2}\right)^2}, \quad (43)$$

and \mathbf{q} is a Q-vector unit

$$\mathbf{q} = \frac{1}{\sqrt{1-a^2}} \begin{pmatrix} \frac{x-y}{2} & z \\ w & -\frac{x-y}{2} \end{pmatrix}, \mathbf{q}^2 = -1. \quad (44)$$

The unit vector (44) represented through the constant basis (7) has the form; $\mathbf{q} = l_k \mathbf{q}_k = (b_k/b) \mathbf{q}_k$ where $l_k = b_k/b$ are components of a unit vector pointing in 3D space a vector with components b_k , then the condition $\det U = xy - wz = 1$ takes the form $a^2 + b^2 = 1$, $b^2 = b_k b_k$. This general biquaternion case will be used in subsequent studies when combined rotation-plus-translational motion is regarded (see Section 6). In this section, we consider only quaternion case: $a, b_k \in \mathbb{R}$, so the matrix U is unimodular if

$$a \equiv \cos \alpha, b = \sqrt{b_k b_k} \equiv \sin \alpha \quad (45)$$

therefore,

$$U = \cos \alpha + (\sin \alpha) l_n \mathbf{q}_n, U^{-1} = \cos \alpha - (\sin \alpha) l_n \mathbf{q}_n \quad (46)$$

with l_k representing cosines of angles between Q-vectors \mathbf{q}_k and the direction determined by \mathbf{q} . With the help of Eqs. (46) and (2), we reproduce the transformation (42) in the developed form

$$\begin{aligned} \mathbf{q}_{k'} &\equiv U \mathbf{q}_k U^{-1} = (\cos \alpha + \sin \alpha l_n \mathbf{q}_n) \mathbf{q}_k (\cos \alpha - \sin \alpha l_m \mathbf{q}_m) = \\ &= 2 \sin^2 \alpha l_k l_n \mathbf{q}_n + \cos 2\alpha \mathbf{q}_k + \sin 2\alpha l_n \varepsilon_{nkm} \mathbf{q}_m = \\ &= [l_k l_n + \cos 2\alpha (\delta_{kn} - l_k l_n) + \sin 2\alpha l_m \varepsilon_{mkn}] \mathbf{q}_n. \end{aligned} \quad (47)$$

Eq. (47) in fact interlinks the $SO(3, \mathbb{R})$ rotation matrix components and the parameters of $SU(2)$ transformations of a Q-frame [compare with (39)]. As well, Eq. (47) helps to make the following geometric analysis.

Multiplied by l_k (with summation in index k), Eq. (47) yields the equality $l_k \mathbf{q}_{k'} = l_n \mathbf{q}_n$, meaning that vectors of the transformed frame $\mathbf{q}_{k'}$ have the same projections onto vector l_k as the initial frame \mathbf{q}_k ; i.e., the transformation may be represented as a conical rotation about l_k , $\Phi \equiv 2\alpha$, which is angle of the rotation in the orthogonal plane with the metric $p_{kn} = \delta_{kn} - l_k l_n$ [see the

second term in Eq. (47)]. Let two unit vectors e_k, n_k form this plane $p_{kn} = e_k e_n + n_k n_n$, then $l_m \varepsilon_{mkn} = e_k n_n - e_n n_k$, and the $SO(3, R)$ -matrix comprised in Eq. (47) acquires the form

$$R_{kn} = l_k l_n + \cos \Phi (e_k e_n + n_k n_n) + \sin \Phi (e_k n_n - e_n n_k). \quad (48)$$

Introducing now two artificial unit vectors with complex number components $s_k \equiv (e_k + i n_k)/\sqrt{2}$ and $s_k^* \equiv (e_k - i n_k)/\sqrt{2}$, we get the final (canonical) expression

$$R_{kn} = l_k l_n + e^{i\Phi} s_k s_n^* + e^{-i\Phi} s_k^* s_n. \quad (49)$$

Eq. (49) is just an explicit formulation of the spectral theorem applied on a 3D orthogonal matrix. Since its determinant differs from zero, this matrix is nonsingular, all its eigenvalues $\lambda_{(i)}$ are different, so it is simple; therefore, it can be expanded into a series of projectors $C_{(i)}$ with $\lambda_{(i)}$ as coefficients

$$R = \sum_{i=1}^3 \lambda_{(i)} C_{(i)}. \quad (50)$$

Here, $\lambda_{(1)} = 1$, $\lambda_{(2)} = e^{i\Phi}$, $\lambda_{(3)} = e^{-i\Phi}$, $C_{(1)kn} = l_k l_n$, $C_{(2)kn} = s_k s_n^*$, $C_{(3)kn} = s_k^* s_n$; the projectors are idempotents $C_{(i)}^N = C_{(i)}$, N being a natural number, $\text{Tr} C_{(i)} = 1$, $\det C_{(i)} = 0$. It is important to note that the decomposition of a matrix R into the series (49, 50) necessarily leads to appearance of the complex-numbered 2D basis $s_k s_k^*$; we will indicate similar features in the fractal technique (iii) below.

The value of the single rotation angle follows from computation of the trace of the matrix (49)

$$\Phi = 2\alpha = \arccos\left(\frac{O_{kk} - 1}{2}\right); \quad (51)$$

antisymmetric part of the matrix yields the components of unit vector directing the rotation axis

$$l_j = i s_k s_m^* \varepsilon_{kmj} = \frac{O_{km} \varepsilon_{kmj}}{\sqrt{(3 - O_{nn})(1 + O_{nn})}}. \quad (52)$$

Eqs. (51) and (52) represent parameters of the single rotations, the angle Φ , and components l_j of the vector pointing the rotation axis, as functions of an arbitrary $SO(3, C)$ rotation angles, e.g., yaw, roll, and pitch $\{\psi, \varphi, \theta\}$, and parameters of an equivalent single rotation, the value of the angle Φ and components (in the initial frame) l_j of the vector pointing the rotation axis.

5.3. Reorientation as transformation of a fractal surface, technique (iii)

In Section 4, we demonstrated that each vector of any Q-triad \mathbf{q}_k is a linear combination of vector-covector direct products of its proper biorthogonal basis (ψ^\pm, φ^\pm) belonging to a

domain of complex-number valued 2D fractal space [see Eqs. (37)]. Then, rotation (reorientation) of the frame \mathbf{q}_k by the technique (ii) on the base of the transformation (42) induces specific type of the “interior” rotation on the fractal surface level [see Eq. (38)]

$$\psi'^{\pm} = U\psi^{\pm}, \varphi'^{\pm} = \varphi^{\pm}U^{-1}. \quad (53)$$

Further on, we use for the dyad the eigenvectors ψ^{\pm} [and eigenvectors as Hermitian conjugation of the vectors $\varphi^{\pm} = (\psi^{\pm})^T$] of \mathbf{q}_3 of any Q-triad, where respective eigenvalues being $\pm i$. In the simplest case of \mathbf{q}_3 from Eq. (7), the constant dyad is

$$\psi^+ = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \varphi^+ = \begin{pmatrix} 0 & 1 \end{pmatrix}, \psi^- = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \varphi^- = \begin{pmatrix} 1 & 0 \end{pmatrix}. \quad (54)$$

Normalization and orthogonality conditions are identically satisfied. The matrix U , as a quaternion (46), is expressible in terms of the fractal basis

$$U = \cos \alpha + l_n \mathbf{q}_n \sin \alpha = \cos \alpha + [-l_1 i(\psi^+ \varphi^- + \psi^- \varphi^+) + l_2(\psi^+ \varphi^- - \psi^- \varphi^+) + l_3 i(\psi^+ \varphi^+ - \psi^- \varphi^-)] \sin \alpha, \quad (55a)$$

$$U^{-1} = \cos \alpha - [-l_1 i(\psi^+ \varphi^- + \psi^- \varphi^+) + l_2(\psi^+ \varphi^- - \psi^- \varphi^+) + l_3 i(\psi^+ \varphi^+ - \psi^- \varphi^-)] \sin \alpha. \quad (55b)$$

Therefore, Eq. (53) takes the form

$$\psi'^+ = (\cos \alpha + il_3 \sin \alpha)\psi^+ - \sin \alpha(il_1 + l_2)\psi^-, \quad (56a)$$

$$\psi'^- = \sin \alpha(-il_1 + l_2)\psi^+ + (\cos \alpha - il_3 \sin \alpha)\psi^-, \quad (56b)$$

$$\varphi'^+ = (\cos \alpha + il_3 \sin \alpha)\varphi^+ - \sin \alpha(il_1 + l_2)\varphi^-, \quad (56c)$$

$$\varphi'^- = \sin \alpha(il_1 + l_2)\varphi^+ + (\cos \alpha + il_3 \sin \alpha)\varphi^-. \quad (56d)$$

Eq. (56) shows that the nonlinear problem formulated within the technique (ii), on the fractal surface level, is reduced to a linear task of the 2D basis rotation.

To get technological formulas convenient for fast numerical computation, we denote the final values of the new 2D basis as

$$A \equiv \cos \alpha + il_3 \sin \alpha, \quad B \equiv \sin \alpha(il_1 + l_2). \quad (57)$$

Then, we notice that only one new dyad vector is to be computed,

$$\psi'^+ \equiv A\psi^+ - B\psi^-. \quad (58a)$$

The second vector ψ'^- and the co-vectors are simply expressed through the factors (57) and their complex conjugation

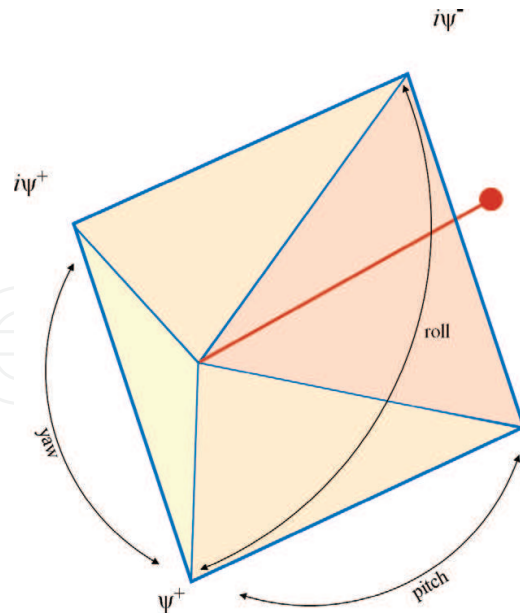


Figure 1. Fractal “joystick tool”.

$$\psi'^{-} = B^* \psi^{+} + A^* \psi^{-}, \varphi'^{+} \equiv A^* \varphi^{+} - B^* \varphi^{-}, \varphi'^{-} = B \varphi^{+} + A \varphi^{-}. \quad (58b)$$

This helps to represent the 3D reorientation processes “subgeometrically”, on the 2D fractal level, as a displacement of a “joystick” tool (see [12] and **Figure 1**).

2D complex-numbered space can be imaged as a pyramid (with no base) consisting of one real, one imaginary, and two mixed real-imaginary joined surfaces. The joystick has one of its end matched with the pyramid’s top by a hinge; a certain shift of the stick gives components of a new dyad vectors and co-vectors. From these fractal elements, a new Q-frame providing the assigned reorientation of the spacecraft is straightforwardly built.

All reorientation parameters providing operations in the fractal space are in fact the components of the matrix $U \in SU(2)$; therefore, the unit vector directing the axis of instant rotation is given by Eq. (52); the fractal rotation angle is

$$\alpha = \arccos \left(\frac{\sqrt{1 + O_{k'k}}}{2} \right). \quad (59)$$

Eqs. (59), (52), (56), and (37) suggest a very simple algorithm for computation of all parameters of a single rotation and resulting matrices of a reoriented Q-triad describing new orientation of a spacecraft.

The technological scheme of the reorientation procedure can be briefly outlined as the following steps:

- A spacecraft reorientation is assigned by a series of simple rotations [Eq. (40)].
- Components of the rotation axis vector are computed [Eq. (52)].

- The angle of fractal rotation is computed [Eq. (59)].
- The dyad and resulting Q-triad are computed [Eqs. (56), (37), much simpler than in Eq. (49)].
- If the computed and assigned frames match, then the rotation parameters are sent to the operational systems realizing the reorientation.

The study suggested in Section 5 gives detailed analysis of math mechanisms linking two different approaches to solution of an object's reorientation task, a consequent 3D rotations described by matrices and a single rotation about an instant axis described by matrices. We like to emphasize importance (and original form) of Eqs. (48) and (49) explicitly demonstrating the projector-eigenvalue decomposition of any $SO(3, R)$ matrix, so immediately giving technological values of the single rotation. Another novel math feature of the problem is its connection with subgeometric properties of a fractal complex number surface.

However, thorough analysis of the Q-math reveals its additional, and important, option quite helpful in operational tasks. Namely, extension of the groups $SO(3, R)$ and $SU(2)$ to the rotations with complex parameters, $SO(3, C)$ and $SL(2, C)$, respectively, with the vector-quaternion version of relativity theory taken into account, may open a possibility not only reorient but as well simultaneously endow a spacecraft with velocity assigned in value and direction. Apparently, this math tool matching rotations and accelerations, if possible in 3D space, should exist as fractal mechanism. Designing of such original (and exotic) operational instrument is a challenging task; it is in detail analyzed in the next section.

6. Hyperbolic rotations and a fractal pyramid

In this section, we essentially extend the methods briefly described above. The crucial point of the extension is introduction of an imaginary parameter of rotation, thus involving hyperbolic functions. We assume that this action will result in possibility to control not only orientation, but as well dynamics of the spacecraft. We will prove the assumption within extended formulation of the technique (iii).

But at first, to make the picture more clear, we show it in framework of 3D serial rotations [technique (i)], and for simplicity, we implement just one supplement plane hyperbolic rotation about one axis

$$O_3^{i\eta} = \begin{pmatrix} \cos(i\eta) & -\sin(i\eta) & 0 \\ \sin(i\eta) & \cos(i\eta) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh \eta & -i \sinh \eta & 0 \\ i \sinh \eta & \cosh \eta & 0 \\ 0 & 0 & 1 \end{pmatrix} \equiv H_3^\eta, \quad (60)$$

so that hyperbolic functions are introduced. Then, complete rotational operator is

$$O = H_3^\eta R \quad (61)$$

We rewrite the operator (61) in the spinor-type form where the tilde denotes some initial basis

$$U = \left(\cosh \frac{\eta}{2} - i \sinh \frac{\eta}{2} \mathbf{q}_3 \right) \left(\cos \frac{\Phi}{2} + \sin \frac{\Phi}{2} l_{\tilde{k}} \mathbf{q}_{\tilde{k}} \right), \quad (62)$$

and the components of the instant rotation axis vector given by Eq. (52). It is important to note that in the computation procedure, we have to deal with vectors belonging to the same frame. Therefore, we express $\mathbf{q}_{3'} = R_{3n} \mathbf{q}_n$ and make multiplication in Eq. (62) to obtain

$$U = \cosh \frac{\eta}{2} \cos \frac{\Phi}{2} - i \sinh \frac{\eta}{2} R_{3\tilde{n}} l_{\tilde{n}} + \left[\cosh \frac{\eta}{2} \sin \frac{\Phi}{2} l_{\tilde{n}} - i \sinh \frac{\eta}{2} \left(\cos \frac{\Phi}{2} R_{3\tilde{n}} + \sin \frac{\Phi}{2} R_{3\tilde{j}} l_{\tilde{m}} \varepsilon_{jmn} \right) \right] \mathbf{q}_n. \quad (63)$$

This expression is again a quaternion and we denote it as

$$U = \cos \Theta + (\sin \Theta) \mathbf{q}, \quad U^{-1} = \cos \Theta - (\sin \Theta) \mathbf{q} \quad (64)$$

where

$$\cos \Theta \equiv \cosh \frac{\eta}{2} \cos \frac{\Phi}{2} - i \sinh \eta \sin \frac{\Phi}{2} R_{3\tilde{n}} l_{\tilde{n}}, \quad (65)$$

$$\sin \Theta \mathbf{q} \equiv \left[\cosh \frac{\eta}{2} \sin \frac{\Phi}{2} l_{\tilde{n}} - i \sinh \frac{\eta}{2} \left(\cos \frac{\Phi}{2} R_{3\tilde{n}} + \sin \frac{\Phi}{2} R_{3\tilde{j}} l_{\tilde{m}} \varepsilon_{jmn} \right) \right] \mathbf{q}_n, \quad (66)$$

parameter Θ being a complex number. One straightforwardly verifies fulfilling the identity

$$\begin{aligned} \cos^2 \Theta + (\sin^2 \Theta) \mathbf{q}^2 = & \left(\cosh \frac{\eta}{2} \cos \frac{\Phi}{2} - i \sinh \eta \sin \frac{\Phi}{2} R_{3\tilde{n}} l_{\tilde{n}} \right) \left(\cosh \frac{\eta}{2} \cos \frac{\Phi}{2} - i \sinh \eta \sin \frac{\Phi}{2} R_{3\tilde{p}} l_{\tilde{p}} \right) + \\ & + \left[\cosh \frac{\eta}{2} \sin \frac{\Phi}{2} l_{\tilde{n}} - i \sinh \frac{\eta}{2} \left(\cos \frac{\Phi}{2} R_{3\tilde{n}} + \sin \frac{\Phi}{2} R_{3\tilde{j}} l_{\tilde{m}} \varepsilon_{jmn} \right) \right] \times \\ & \times \left[\cosh \frac{\eta}{2} \sin \frac{\Phi}{2} l_{\tilde{n}} - i \sinh \frac{\eta}{2} \left(\cosh \frac{\Phi}{2} R_{3\tilde{n}} + \sin \frac{\Phi}{2} R_{3\tilde{l}} l_{\tilde{p}} \varepsilon_{lps} \right) \right] \mathbf{q}_{\tilde{n}} \mathbf{q}_{\tilde{s}} = 1. \end{aligned} \quad (67)$$

Expression for the vector-directing axis of the single rotation is found from Eqs. (65) and (66)

$$l_n = \frac{\cosh \frac{\eta}{2}, \sin \frac{\Phi}{2} l_{\tilde{n}}, -i, \sinh \frac{\eta}{2}, \left(\cos \frac{\Phi}{2} R_{3\tilde{n}} + \sin \frac{\Phi}{2} R_{3\tilde{j}} l_{\tilde{m}} \varepsilon_{jmn} \right)}{\sqrt{1 - \left(\cosh \frac{\eta}{2} \cos \frac{\Phi}{2} - i \sinh \eta \sin \frac{\Phi}{2} R_{3\tilde{n}} l_{\tilde{n}} \right) \left(\cosh \frac{\eta}{2} \cos \frac{\Phi}{2} - i \sinh \eta \sin \frac{\Phi}{2} R_{3\tilde{p}} l_{\tilde{p}} \right)}}. \quad (68)$$

Eq. (61) represents an operator performing the serial rotation, and Eqs. (65), (68) give parameters of a single rotation. Physical content of this rotation is easily revealed when the mapping

is made in the fractal surface format, and then returned into 3D space. Despite seeming complexity of the given expressions, the final calculation is shown to be very simple.

So, following the ideology of geometrization of the algebraic actions, we plunge into the fractal medium, and we consider the technique (iii). We rewrite fractal mapping with the operator (62) in the form

$$\psi^\pm = U\psi'^\pm = \left(\cosh \frac{\eta}{2} - i \sinh \frac{\eta}{2} \mathbf{q}_3 \right) \psi'^\pm \quad (69)$$

where the intermediate dyad is a result of the real rotation (similar with the covectors)

$$\psi'^\pm = \left(\cos \frac{\Phi}{2} + \sin \frac{\Phi}{2} l_k \mathbf{q}_k \right) \tilde{\psi}^\pm. \quad (70)$$

We also stress that all dyad elements used in the computations are always the eigenvectors (eigencovectors) of the quaternion unit \mathbf{q}_3

$$\mathbf{q}_3 \psi^+ = +i\psi^+, \quad \mathbf{q}_3 \psi^- = -i\psi^+, \quad \varphi^+ \mathbf{q}_3 = +i\varphi^+, \quad \varphi^- \mathbf{q}_3 = -i\varphi^-; \quad (71)$$

hence, Eq. (69) produces a new fractal basis simply multiplying the intermediate dyad by an exponent

$$\psi^+ = \left(\cosh \frac{\eta}{2} + \sinh \frac{\eta}{2} \right) \psi'^+ = e^{\eta/2} \psi'^+, \quad \psi^- = e^{-\eta/2} \psi'^-, \quad \varphi^+ = e^{-\eta/2} \varphi'^+, \quad \varphi^- = e^{\eta/2} \varphi'^-. \quad (72)$$

By other words, one dyad vector and one co-vector (here ψ^+ and φ^-) become longer, and the others (ψ^- and φ^+) become shorter, all of them though preserving unit length, i.e., rescaled.

This primitive mapping has clear physical sense concerning kinematic of a spacecraft. To reveal it, we, using Eq. (75), build an “imaginary constituent” of the 3D frame vector $\mathbf{q}_{1'}$ as in Eq. (37b).

$$\mathbf{q}_{1'} = -i(\psi'^+ \varphi'^- + \psi'^- \varphi'^+) = -i(e^\eta \psi^+ \varphi^- + e^{-\eta} \psi^- \varphi^+). \quad (73)$$

However from Eqs. (37b, c), we find

$$\psi^+ \varphi^- = \frac{1}{2}(i\mathbf{q}_1 + \mathbf{q}_2), \quad \psi^- \varphi^+ = \frac{1}{2}(i\mathbf{q}_1 - \mathbf{q}_2); \quad (74)$$

substitution of the Eq. (74) into Eq. (73) yields

$$i\mathbf{q}_{1'} = \cosh \eta (i\mathbf{q}_1 + \tanh \eta \mathbf{q}_2). \quad (75a)$$

Eq. (75a) rewritten in terms of the Pauli-type matrices [as in Eqs. (20), (22)] $\mathbf{p} \equiv i\mathbf{q}$ has the form

$$\mathbf{p}_{1'} = \cosh \eta (\mathbf{p}_1 + \tanh \eta \mathbf{q}_2). \quad (75b)$$

Using results of Section 3, we associate the hyperbolic functions with the time ratio

$$\cosh \eta = \frac{dt}{dt'} \quad (76)$$

(linking time dt of an immobile frame and proper time dt' of moving spacecraft) and with the relative velocity ratio (c is speed of light).

$$\tanh \eta = V/c. \quad (77)$$

Then, Eq. (75b) takes the form of “vector interval” of quaternion version of relativity theory (23)

$$dt' \mathbf{p}_{1'} = dt \left(\mathbf{p}_1 + \frac{V}{c} \mathbf{q}_2 \right); \quad (78)$$

when squared, it gives the spacecraft’s special relativistic space-time interval linked with the frame at rest by the Lorentz (hyperbolic) transformation

$$dt'^2 = dt^2 (1 - V^2/c^2) \quad (79)$$

describing kinematics of a frame moving along \mathbf{q}_2 with velocity V , while the vectors \mathbf{p}_1 (or $\mathbf{p}_{1'}$) play the role of direction of time in the immobile (or moving) spacecraft. It is always possible to choose the direction \mathbf{q}_2 as pointing the “yaw” of a spacecraft. In particular, the velocity can be small sufficiently to reduce the calculations into classical format

$$V/c = \tanh \eta \approx \eta \quad (80)$$

besides, the velocity modulus may be variable in time; hence, the spacecraft is accelerated.

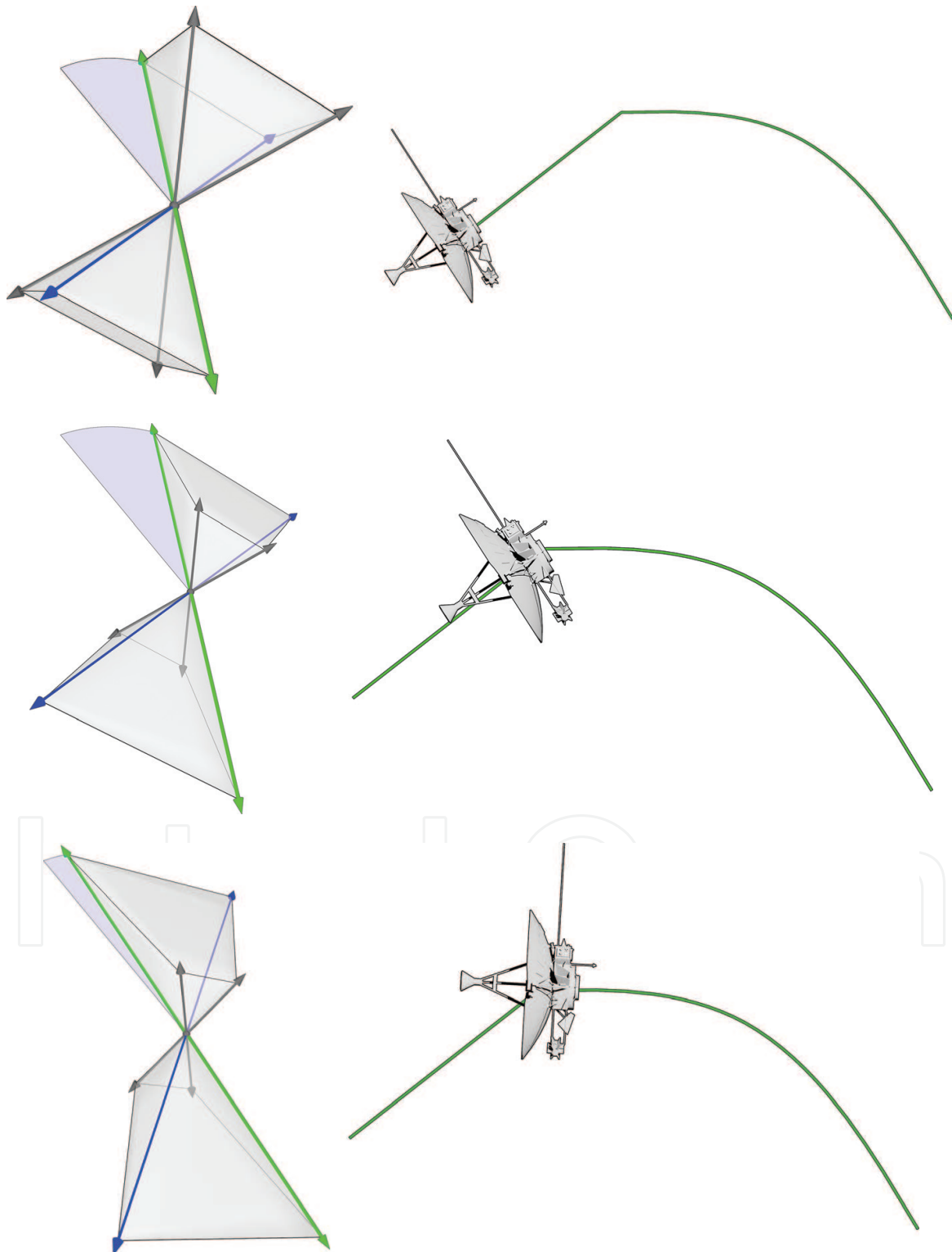
So, introducing imaginary rotation angles, we obtain a possibility to control an arbitrary space reorientation of a spacecraft with variation of its velocity in the direction that can be as well changing with time (In this sample, the vector \mathbf{q}_2 is in fact permanently rotating.)

This math tool has two important properties. First, a spacecraft endowed by the tool with a velocity is initially described as a relativistic system; one comes to the classical mechanics considering the hyperbolic parameter small. Second, the tool accelerates the spacecraft always in the direction of the frame vector appointed to indicate “yaw”; if this vector rotates, changing the yaw, the acceleration arrow changes with it; i.e., the spacecraft is accelerated along a curve line. These properties can be useful in real motion control.

On the 2D fractal level, the spacecraft’s more complex 3D motion comprising reorientation and acceleration is accompanied by respective rotation and deformation of the mentioned above fractal pyramid. Here, this subgeometric image of the math instrument necessarily enriches a

simpler model of the joystick, and moreover, to make the picture symmetric, we show positive and negative directions of the pyramid (see **Figure 2**).

Computations providing the spacecraft's reorientation and acceleration are performed on the fractal level by Eq. (58) with the functions A, B generalized as



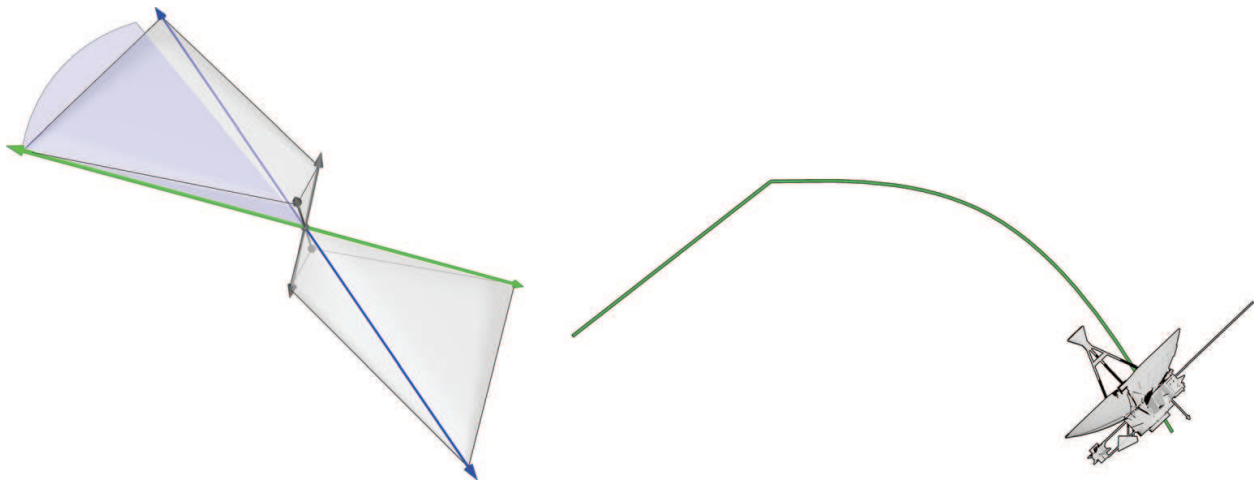


Figure 2. Case (a): The spacecraft performs a 3D rotation, the pyramid is tilted by respective halfangle. Rotations and displacements of a spacecraft (Pioneer-10) accompanied by respective 2D rotations and deformations of the fractal pyramid. Case (b): The reoriented spacecraft rectilinearly moves with some velocity, and the tilted pyramid is distorted: Two its edges become shorter, and the other two edges become longer. Case (c): The spacecraft (“free-framed”) is reoriented by another angle, and the distorted pyramid as tilted by respective halfangle. Case (d): The spacecraft moves along a curve trajectory with changing velocity (accelerated), and the pyramid is subject to permanent respective tilt and distortion.

$$A \equiv \left(\cos \frac{\Phi}{2} + il_3 \sin \frac{\Phi}{2} \right) e^{\eta/2}, \quad B \equiv \sin \frac{\Phi}{2} (i l_1 + l_2) e^{-\eta/2}, \quad (81a)$$

with hyperbolic conjugation ($(\oplus : e^{\pm\eta/2} \rightarrow e^{\mp\eta/2})$), similar to the complex conjugation, introduced, e.g.,

$$A^{\oplus} \equiv \left(\cos \frac{\Phi}{2} + il_3 \sin \frac{\Phi}{2} \right) e^{-\eta/2}, \quad B^{*\oplus} \equiv \sin \frac{\Phi}{2} (-i l_1 + l_2) e^{\eta/2}, \quad (81b)$$

where vector l_k directs axis of the single space rotation by angle Φ . Then (as in Section 5), only one equation is to be solved, e.g., that determining the dyad vector

$$\psi'^+ = A \psi^+ - B \psi^-, \quad (82a)$$

and rest of the dyad elements is found by primitive math actions

$$\psi'^- A^* \psi^+ + B^* \psi^-, \quad \varphi'^+ = A^{*\oplus} \varphi^+ - B^{*\oplus} \varphi^-, \quad \varphi'^- = A^{\oplus} \varphi^+ + B^{\oplus} \varphi^-. \quad (82b)$$

Eqs. (82), (37) immediately give expressions of all spacecraft's frame vectors, thus solving the reorientation and acceleration problem in explicit form.

One straightforwardly finds that use of the fractal technique (iii) essentially simplifies computation procedures. In paper [13], we compare math difficulty of the discussed three techniques in solution of the simple problem of the spacecraft's one-plane space rotation and acceleration. It is demonstrated there that the techniques (i) and (ii) demand solution of at least seven equations, among them are matrix equations, while the fractal technique (iii) suggests solution of only four relatively simple algebraic equations.

7. Technological scheme and concluding remarks

A sketch of technological scheme aimed to realize mixed rotation-acceleration maneuver of a spacecraft can be suggested as the following consequence of actions fit for any mentioned above approach.

- The initial and final parameters of reorientation and acceleration are assigned and memorized.
- Parameters as functions of time must be determined and input.
- Time intervals are divided into standard steps (quantized), the standard input.
- Process of computation of quantum steps starts resulting in obtaining of a series of related parameter values describing the orientation and velocity of the spacecraft's frame.
- The data of each step are transmitted to the systems changing the spacecraft orientation and velocity until the assigned values are achieved.

And we emphasize two most important results of this study.

First, we succeeded to show that an extrarotation by an imaginary angle entails endowing a spacecraft with a (relativistic) velocity, hence in addition to reorientation, to accelerate it. This math observation seems to be a novel one since no similar information is met in related literature.

Second, we show that the most mathematically economical way to compute operational parameters needed for realization of the maneuver is to utilize the "fractal pyramid" technique (definitely a new tool) comprising minimal number of math actions, where major of them are simple algorithms, other approaches having no such advantages.

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