Chapter 7

Adaptive CUSUM for Steady State Normal Data

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Additional information is available at the end of the chapter

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Abstract

This chapter deals with monitoring plans that exploit temporal predictable trends by adjusting the cumulative sum (CUSUM) plan to be efficient for their early detection. The adjustment involves changing the amount of memory the chart retains to detect persistent changes in location early. The focus is on steady-state situations when either the shift size is known in advance or when it is unknown. Several options are explored using simulation studies, and an example of application is considered.

Keywords: average run length, early detection, monitoring, persistent trends, statistical process control

1. Introduction

The adaptive CUSUM of Sparks [12] exploits temporal predictable trends by adjusting its design to be efficient for the early detection of such trends. The adjustment involves changing the amount of memory the chart retains to detect persistent changes in location earlier.

In the zero-state case, Moustakides [6] proved that a step change of δ is best detected by using Page's [7] conventional CUSUM with the reference value $k = \delta/2$. Gan [3] demonstrated that the conventional CUSUM with $k = 0.5$ is optimal in the zero-state in their **Table 1** for a shift of one and standard normal distributed data. The adaptive CUSUM has been shown to be better at times at detecting small shifts in location than the conventional CUSUM with the optimal k value for that shift. For example, in the standard normal distribution case, the shift of 1 for the adaptive CUSUM is detected with an average run length (ARL); in **Table 3** of Sparks [12] is 9.29 or 9.34, while the zero state optimal conventional CUSUM with $k = 0.5$ has an ARL of 9.34 (see also [5] **Table 2**). Hence, the adaptive CUSUM can have smaller out-of-control ARLs than the best CUSUM in the zero-state situation. The reason for this is that, for smaller shifts, the adaptive CUSUM can exploit the steady-state situation by making use of the local knowledge about the size of its shift. For unknown large shifts this is more difficult because one often flags the change before it can be accurately predicted. In other words the adaptive CUSUM

Table 1. Optimal offset values (*k*) for detecting certain size shifts when the in-control ARL = 200.

(a)

k	0.2	0.2125	0.225	0.2375	0.25	0.2675	0.275	0.2875	0.3
δ									
0.5	16.724	**16.699**	16.739	16.827	16.847				
0.525	15.720	**15.716**	15.733	15.734	15.754				
0.55	14.786	14.801	14.793	**14.750**	14.774	14.888	14.904		
0.575			13.936	13.935	13.917	**13.907**	13.996	14.050	
0.6					13.224	13.261	**13.211**	13.226	13.219

(b)

k	0.2675	0.275	0.2875	0.3	0.3125	0.325	0.3375	0.35	0.3675
δ									
0.625	12.589	**12.509**	12.556	12.574	12.697				
0.65	11.880	**11.833**	11.874	11.919	12.017	12.051			
0.675	11.320	**11.253**	11.310	11.357	11.355	11.373	11.395		
0.7	10.755	10.750	**10.694**	10.768	10.814	10.818	10.822	10.827	
0.725	10.301	10.269	**10.236**	10.266	10.276	10.299	10.308	10.332	10.384

(c)

k	0.3	0.3125	0.325	0.3375	0.35	0.375	0.3875	0.4	0.4125
δ									
0.75	**9.777**	9.787	9.779	9.792	9.889	9.928			
0.775	9.351	**9.341**	9.372	9.352	9.379	9.441	9.482		
0.8	8.969	8.976	8.967	**8.934**	9.006	9.026	9.028	9.075	
0.825	8.617	8.617	8.605	**8.583**	8.586	8.650	8.651	8.692	8.736

(d)

k	0.35	0.3675	0.375	0.4	0.4125	0.425	0.4375	0.45	0.475
δ									
0.85	**8.249**	8.287	8.291	8.308	8.337	8.366			
0.875	7.964	7.969	7.973	**7.962**	7.970	8.046	8.077		
0.9	7.649	**7.648**	7.651	7.677	7.691	7.711	7.740	9.075	
0.95	7.101	**7.091**	7.110	7.114	7.110	7.144	7.146	8.692	8.736

(e)

k	0.375	0.4	0.425	0.45	0.475	0.5	0.525	0.55	0.575

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δ									
1.00	6.6253	**6.619**	6.620	6.641	6.655	6.713			
1.05	6.233	6.195	**6.187**	6.188	6.203	6.229	6.280		
1.10	5.860	5.796	5.812	**5.795**	5.797	5.818	5.823	5.899	
1.15	5.520	5.464	5.476	5.462	**5.458**	5.470	5.460	5.516	5.499

(f)

k	0.45	0.475	0.5	0.525	0.55	0.6	0.625	0.65	0.675
δ									
1.20	5.138	5.153	**5.137**	5.142	5.161	5.191			
1.25	4.864	4.854	4.863	**4.853**	4.870	4.883	4.909		
1.30	4.625	5.686	4.600	4.596	**4.592**	4.603	4.629	4.631	
1.35	4.403	4.401	4.362	4.367	4.372	**4.357**	4.365	4.385	4.396

(g)

k	0.5	0.5375	0.6	0.625	0.65	0.70	0.725	0.75	0.775
δ									
1.40	4.166	4.140	**4.130**	4.147	4.145	4.196			
1.45	3.968	3.964	3.938	**3.962**	3.945	3.969	3.989		
1.50	3.787	3.792	3.759	3.764	**3.760**	3.779	3.779	3.783	
1.55	3.647	3.626	3.587	3.598	3.591	**3.607**	3.609	3.609	3.609

(h)

k	0.65	0.70	0.725	0.75	0.775	0.80	0.825	0.85	0.875
δ									
1.60	**3.434**	3.443	3.441	3.445	3.440	3.441			
1.65	3.288	3.290	**3.279**	3.286	3.290	3.290	3.317		
1.70	3.161	3.160	**3.148**	3.154	3.152	3.155	3.164	3.164	
1.75	3.047	3.040	3.034	3.024	**3.021**	3.024	3.035	3.038	3.040

(i)

k	0.775	0.80	0.825	0.85	0.875	0.90	0.925	0.95	0.975
δ									
1.80	**2.891**	2.906	2.905	2.901	2.915	2.928			
1.85	2.796	2.795	**2.794**	2.795	2.799	2.806	2.804		
1.90	2.695	2.690	2.694	**2.686**	2.688	2.694	2.692	2.697	
1.95	2.606	2.600	2.598	2.595	2.597	**2.593**	2.598	2.598	2.600

Table 2. Optimal offset values (*k*) for detecting certain size shifts when the in-control ARL = 800.

				(a)					
k	0.2	0.2125	0.225	0.2375	0.25	0.2675	0.275	0.2875	0.3
δ									
0.5	26.563	26.568	26.496	**26.449**	26.543				
0.525	24.847	24.746	24.613	**24.586**	24.626				
0.55	23.307	23.038	23.010	22.951	**22.900**	22.953	23.010		
0.575	21.930	21.757	21.642	21.473	**21.447**	21.511	21.506	21.505	
0.6					20.141	21.144	20.119	**20.061**	20.236
				(b)					
k	0.2675	0.275	0.2875	0.3	0.3125	0.325	0.3375	0.35	0.3675
δ									
0.625	18.929	**18.905**	18.940	18.982	19.091				
0.65	17.877	17.833	**17.798**	17.881	17.854	18.034			
0.675	16.912	16.883	16.872	**16.829**	16.921	16.899	16.964		
0.7	16.076	16.026	16.054	15.979	16.022	**15.928**	16.033	16.021	
0.725	15.296	15.206	15.203	15.142	15.178	**15.098**	15.172	15.188	15.200
				(c)					
k	0.3	0.3125	0.325	0.3375	0.35	0.375	0.3875	0.4	0.4125
δ									
0.75	14.435	14.365	14.415	**14.326**	14.329	14.405			
0.775	13.707	13.703	13.749	**13.679**	13.679	13.697	13.740		
0.8	13.170	13.113	13.074	13.041	**13.039**	13.085	13.046	13.097	
0.825	12.586	12.540	12.521	12.450	**12.414**	12.428	12.467	12.468	12.477
				(d)					
k	0.35	0.3675	0.375	0.4	0.4125	0.425	0.4375	0.45	0.475
δ									
0.85	11.892	11.873	**11.855**	11.893	11.895	11.925			
0.875	11.390	11.377	11.364	**11.364**	11.373	11.374	11.434		
0.9	10.923	10.900	10.900	10.891	**10.889**	10.886	10.875	10.935	
0.95	10.128	10.079	10.014	10.013	**9.999**	10.017	10.004	10.069	10.092
				(e)					
k	0.375	0.4	0.425	0.45	0.475	0.5	0.525	0.55	0.575

δ									
1.00	9.379	9.309	9.272	**9.263**	9.289	9.329			
1.05	8.757	8.670	8.622	8.628	**8.595**	8.635	8.632		
1.10	8.206	8.112	8.050	8.028	8.025	**8.019**	8.029	8.053	
1.15	7.742	7.631	7.551	7.514	7.499	7.478	**7.477**	7.503	7.513

(f)

k	0.45	0.475	0.5	0.525	0.55	0.6	0.625	0.65	0.675
δ									
1.20	7.093	7.045	7.017	7.020	**7.000**	7.039			
1.25	6.676	6.642	6.609	**6.567**	6.578	6.575	6.602		
1.30	6.334	6.295	6.260	6.196	**6.194**	6.195	6.203	6.220	
1.35	6.013	5.971	5.942	5.869	5.857	**5.848**	5.854	5.849	5.865

(g)

k	0.5	0.5375	0.6	0.625	0.65	0.70	0.725	0.75	0.775
δ									
1.40	5.640	5.577	5.533	**5.509**	5.519	5.538			
1.45	5.375	5.308	5.254	5.233	**5.223**	5.228	5.252		
1.50	5.143	5.076	4.983	4.979	4.980	**4.969**	4.981	4.992	
1.55	4.913	4.831	4.756	4.747	4.726	4.725	**4.717**	4.725	4.738

(h)

k	0.65	0.70	0.725	0.75	0.775	0.80	0.825	0.85	0.875
δ									
1.60	4.512	4.499	4.504	**4.498**	4.499	4.509			
1.65	4.316	4.284	4.287	4.280	4.294	**4.274**	4.302		
1.70	4.159	4.114	4.097	4.099	4.099	4.096	**4.083**	4.098	
1.75	3.986	3.945	3.931	3.932	3.916	3.911	3.913	**3.910**	3.928

(when it can) exploits the steady-state situation to improve the zero-state performance. This, however, becomes more difficult for larger shifts. It only works for the smaller shifts where the steady state is reached while we are trying to accumulate enough memory to detect the change. For large shifts there is generally insufficient information on these shifts after its occurrence to exploit it before it is detected.

Automation and sensor devices that measure very frequently means that data stream in these days in real-time, and therefore steady-state situations have now become more common than when the CUSUM was first advocated by Page [7]. Most applications in environmental sciences are steady state since the process cannot be stopped. The majority of service processes, although can be stopped, are hardly ever stopped and restarted. Thus, they may be referred to as steady-state processes.

For this reason zero-state processes are less common, thus, revealing a scientific area that needs to be further researched.

2. Literature review

Sparks's [12] adaptive CUSUM improved the CUSUM early detection performance by appropriately adjusting the reference value k to improve its early detection performance. This paper will introduce and elaborate on a different approach to optimise equilibrium conditions and draw on observed outcomes. Jiang et al. [5] followed Sparks [12] in using the zero-state optimal reference value of the shift value divided by 2, but introduced a weighting function for the departures of the control variable from the zero-state optimal reference value. In particular, open-ended work should focus in optimising the CUSUM in steady-state situations (even for known shifts).

This paper starts by introducing the conventional CUSUM and the adaptive CUSUM statistics. It derives the thresholds for the CUSUM plans in steady-state situations for high-sided signals only. Low-sided charts can be established by symmetry and two-sided charts can be applied by simultaneously applying two one-sided charts and halving the in-control ARL of the high-sided chart. The high-sided charts for steady-state situations are designed to deliver a specific in-control ARL of either 100, 200, 300, …, 1000 (see Appendix A). Monitoring plans are defined in Sparks [14]. If the location is known in advance then this paper establishes the reference value closest to the best plan for the steady-state situation. A simulation study is carried out to find the CUSUM p best for the early detection of a known location shift.

Methods that compete with the adaptive CUSUM in terms of performance involve the simultaneous application of multiple CUSUMs with differing levels of memory [4, 12], combining Shewhart and CUSUM charts [8, 11], the adaptive EWMA [1] and multiple moving averages [13]. Ryu et al. [9] assumes the shift is known and optimises the CUSUM plan without mentioning whether it is based on zero or steady state, and therefore it must be viewed as competing methodology. However, this paper's contribution is on improving the out-of-control performance of the adaptive CUSUM plan in the steady-state situation and provides formulae to estimate the thresholds for the high-sided conventional CUSUM in steady-state situations.

3. CUSUM and adaptive CUSUM plans

Let y_t the process variable measured at time t which has mean μ and variance σ^2. Define the standardised score as $z_t = (y_t - \mu)/\sigma$. Then Page's CUSUM plan for high-sided location changes is given by

$$C_t^U(k) = \max\left(0, C_{t-1}^U + z_t - k\right) \tag{1}$$

where k is referred to as the reference value that determines the level of past memory held by the CUSUM statistic. The resetting to zero of the CUSUM statistic is the process that controls the memory in the plan. Large values of k will make the CUSUM statistic operate like the memoryless Shewhart chart by frequently resetting to zero. Smaller values of k retain more historical information in the plan by resetting to zero less often. Therefore, practitioners would like to have large values of k when the shift is large and small values of k when shifts are small. Small values of k allow the CUSUM to accumulate more information thus having sufficient power to detect small shifts. The conventional CUSUM statistic signals an unusual location shift on the high side whenever

$$C_t^U(k) > h(k) \tag{2}$$

where $h(k)$ is the positive valued threshold that delivers a specified in-control ARL in the steady-state situation. Appendix A provides models for accurately predicting the thresholds for the conventional CUSUM in the steady-state situation.

The adaptive CUSUM allows the reference value to change over time t and is given by using the adaptive CUSUM statistic:

$$AC_t^U(k) = \max\left(0, AC_{t-1}^U + (z_t - k_t)/h(k_t)\right) \tag{3}$$

and flags an out-breaks whenever this exceeds a threshold of approximately 1. The challenge in practice is how to change k_t over time t to improve the early detection performance of the plan. An alternative approach that is explored in this paper is how to select $(z_t - k_t)/h(k_t)$ to improve the early detection performance of the plan.

Sparks's [12] plan was based on the hypothesis that the zero-state optimal setting was going to be optimal in the steady-state situation. This is however, not the case. The examples that illustrate this are reported in **Figure 1(a)–(c)**.

Figure 1 plots the conventional CUSUM divided by its threshold (i.e., $C_t^U/h(k)$ for $k=0.25$ and 0.2125 both designed to have an in-control ARL of 200) for 100 observations from a normal distribution. The first 80 observations are in-control standard normal data and the last 20 normally observations that are shifted on the high side by 0.5. Note that k = 0.25 is the value which is the zero-state optimal value established by Moustakides [6] for this shift, while k = 0.2125 is a better alternative in the steady-state situation. Note that prior to the change point at time = 81 the $C_t^U(k)/h(k)$ with $k=0.25$ is almost identical to $C_t^U(k)/h(k)$ with $k=0.2125$ but after a near missed signal at time 71 the $C_t^U(k)/h(k)$ with $k=0.2125$ is higher than the $C_t^U(k)/h(k)$ values with $k=0.25$. This increase is enough to flag this change in the last 20 observations while the conventional CUSUM fails to signal.

Figure 1(b) illustrates that fact that the CUSUM plan with $k=0.2125$ is less likely to reset to zero than the CUSUM plan with $k=0.25$ and therefore is likely to flag the change in the last 20 observations sooner than the CUSUM with $k=0.25$.

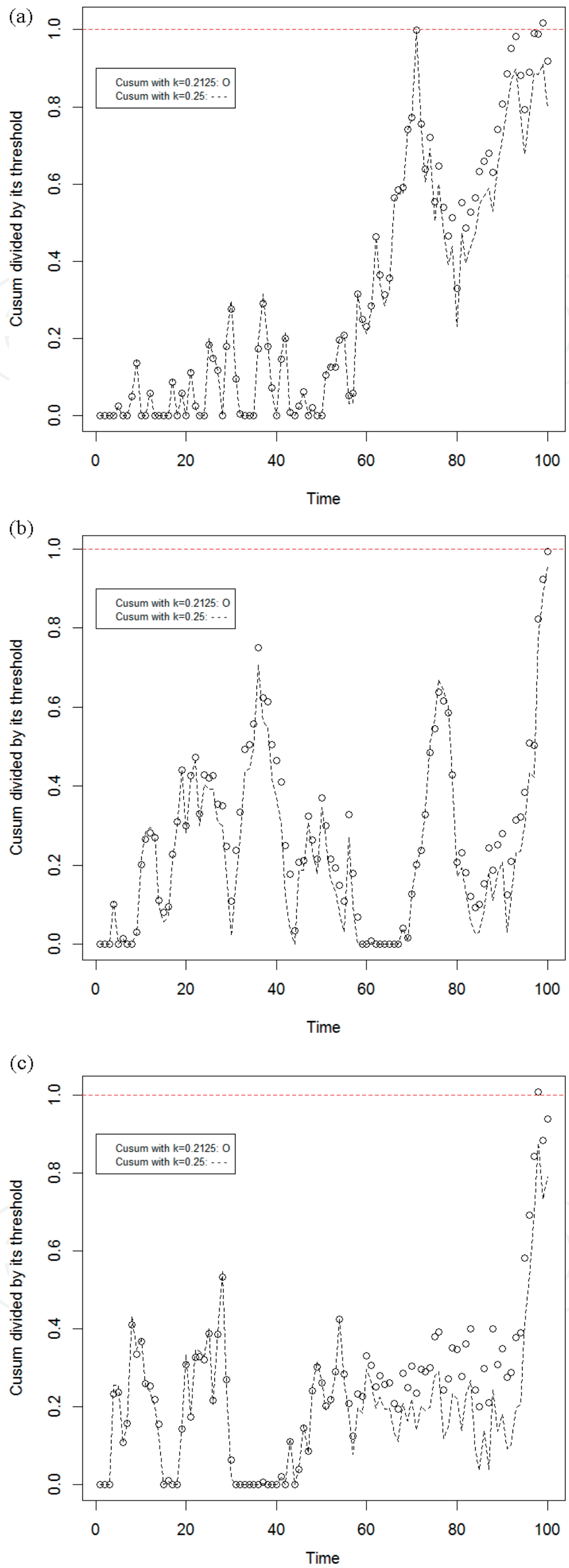

Figure 1. (a) Example 1 of when zero state optimal $k = 0.25$ does not do better than $k = 0.2125$. (b) Example 2 of when $k = 0.2125$ resets the CUSUM to zero than $k = 0.25$ and there has a better chance of detecting small shifts early. (c) Example 2 of when zero state optimal $k = 0.25$ does not do better than $k = 0.2125$.

Figure 1(c) exemplifies that the CUSUM plans with $k = 0.2125$ and $k = 0.25$ are almost identical for the first 60 in-control observations, but once the change occurs CUSUM with $k = 0.2125$ accelerates to the threshold quicker than the CUSUM plan with $k = 0.25$, and thus flagging this shift earlier. Extensive simulated examples not reported in this paper revealed that these plans, on most occasions, are almost identical. However, in a few examples as illustrated in **Figure 1(a)–(c)** the plan with $k = 0.2125$ exploits the situation better by being less likely to rest to zero and thus, more likely to flag an out-break in a steady-state situation earlier.

This begs the question of what reference values k in the steady-state situations are better at detecting location changes from the in-control mean than k equal to shift divided by 2 that is optimal for the zero-state.

4. Near optimal steady-state plans when the shift is known

A simulation study was carried out that started with running through 25 in-control observations before generating the out-of-control situations. This was designed to simulate a steady-state situation prior to the change point. The thresholds for this process are given in Appendix A for the standard normal distribution. There is no loss of generality by assuming mean of zero and variance of one, however the results only apply to normally distributed data. The smallest out-of-control ARLs for various scenarios are presented in **Table 1** for in-control ARL = 200, and for in-control ARL = 800 in **Table 2**.

The reference value with the smallest out-of-control ARL is highlighted in bold text, e.g., for in-control ARL = 200 and a location shift of $\delta = 0.5$ the near optimal steady state k is 0.2125 with an out-of-control ARL = 16.699 while the zero state optimal in the steady-state situation $k = 0.25$ delivers an out-of-control ARL = 16.847 (see **Table 1(a)**). In most cases the last entry in the rows of **Tables 1** and **2** is the zero-state optimal value of k equal to the location change divided by 2. Notice that $k = \delta/2$ is never the plan with the smallest out-of-control ARL—the k with the smallest out-of-control ARL is always smaller than $\delta/2$; in other words the better plan which resets the CUSUM statistic to zero a little less often.

The optimal reference value is reported in bold text in **Table 2**, for example, for in-control ARL = 800 and a location shift of $\delta = 0.5$ the near optimal steady state k is 0.2375 with an out-of-control ARL = 26.449 while the zero state optimal in the steady-state situation $k = 0.25$ delivers an out-of-control ARL = 26.543 (see **Table 1(a)**). Notice that relative to **Table 1**, $k = \delta/2$ is closer to the plan with the smallest out-of-control ARL than in **Table 1**, that is, the k with the smallest out-of-control ARL is always smaller than $\delta/2$ but now the difference between the k with the smallest out-of-control ARL and $\delta/2$ is less than was found in **Table 1**. For this reason we expect less relative gain by optimising the adaptive CUSUM for the steady-state situation with larger in-control ARL.

5. Improving adaptive CUSUM performance for the steady-state situation

The EWMA statistic in Sparks [12] and Jiang et al. [5] is used to forecast the change δ. However, this forecast always under-estimates the change in location. This bias in prediction is more severe for large shifts where only a few observations can be used to optimise the CUSUM before the change is signalled. For this reason the EWMA statistic is thresholded to not fall below a certain minimum values, e.g.,

$$SP_t = \max\left(\delta_{min}, \alpha y_t + (1-\alpha)SP_{t-1}\right) \tag{4}$$

where $0<\alpha<1$, δ_{min} is the smallest positive location change one wishes to detect early and $SP_0=\delta_{min}$. This paper takes $\delta_{min}=0.5$ and $\alpha=0.2$. Since this forecast is biased and the change in location is unknown in advance it is difficult to know what value to use for the reference value k_t given the knowledge of SP_t. Sparks [12] used $k_t=SP_{t-1}/2$ based on the assumption that this was the optimal zero-state situation. For additional information of adaptive plans see [10, 16, 18].

Given SP_t under predicts the change and the optimal k_t for in steady-state situation is generally lower than $SP_{t-1}/2$ (the EWMA one time ahead forecast divided by 2) or $SP_t/2$ (the local smoothed value) this may be a good compromise strategy. When a change occurs then generally $k=SP_t/2$ is less biased for this change than $k=SP_{t-1}/2$.

In other words the local smoothed value SP_t is used to establish k rather than the step-ahead forecast SP_{t-1}. This section explores whether this is a better alternative than the forecast. The comparisons of columns 2 and 3 in **Tables 3–8** indicate that using the reference value equal to $SP_t/2$ becomes less attractive as in-control ARL increases, for example, for in-control ARL equal to 100 it has the smaller out-of-control ARL in most cases, but when the in-control ARL = 800 it

Table 3. Comparison of adaptive CUSUM plans for in-control ARL = 100.

Delta	$\alpha=0.2$	$\alpha=0.2$	$\alpha=0.6$	$\alpha=0.7$
	$h_{adj}=1.2271$	$h_{adj}=0.9215$	$h_{opt}=1.005$	$h_{opt}=1.172$
	$k=SP_t/2$	$k=SP_{t-1}/2$		
0.00	100.03	100.09	100.00	100.00
0.50	12.70	13.59	**12.56**	12.59
0.75	7.72	7.82	**7.71**	7.72
1.00	5.42	**5.41**	5.50	5.50
1.25	4.14	**4.08**	4.30	4.24
1.50	3.35	**3.31**	3.52	3.45
1.75	2.82	**2.79**	2.99	2.90
2.00	**2.43**	2.44	2.62	2.51
2.25	**2.15**	2.19	2.32	2.19
2.50	**1.93**	2.00	2.10	1.99
2.75	**1.75**	1.86	1.92	1.80
3.00	**1.61**	1.75	1.77	1.65

Table 4. Comparison of adaptive CUSUM plans for in-control ARL = 200.

Delta	$\alpha = 0.2$	$\alpha = 0.2$	$\alpha = 0.6$	$\alpha = 0.7$
	$h_{adj} = 1.2877$	$h_{adj} = 0.9215$	$h_{opt} = 1.132637$	$h_{opt} = 1.312637$
	$k = SP_t/2$	$k = SP_{t-1}/2$		
0.00	199.979	200.897	200.328	200.006
0.50	17.009	18.291	**15.380**	15.641
0.75	9.996	9.971	**9.122**	9.255
1.00	6.944	6.662	**6.453**	6.488
1.25	5.240	**4.948**	4.996	**4.948**
1.50	4.179	**3.949**	3.990	3.984
1.75	3.472	**3.303**	3.367	3.325
2.00	2.968	2.864	2.928	**2.855**
2.25	2.594	2.541	2.580	**2.510**
2.50	2.306	2.303	2.317	**2.231**
2.75	2.085	2.126	2.116	**2.024**
3.00	1.904	1.984	1.952	**1.852**

Table 5. Comparison of adaptive CUSUM plans for in-control ARL = 300.

Delta	$\alpha = 0.2$	$\alpha = 0.2$	$\alpha = 0.6$	$\alpha = 0.7$
	$h_{adj} = 1.2877$	$h_{adj} = 0.9215$	$h_{opt} = 1.168285$	$h_{opt} = 1.351279$
	$k = SP_t/2$	$k = SP_{t-1}/2$		
0.00	300.089	300.581	300.682	300.858
0.50	19.689	21.272	**19.662**	19.931
0.75	11.453	**11.324**	11.549	11.758
1.00	7.851	**7.414**	8.064	8.162
1.25	5.904	**5.473**	6.119	6.170
1.50	4.698	**4.341**	4.922	4.921
1.75	3.861	**3.615**	4.103	4.059
2.00	3.290	**3.101**	3.518	3.448
2.25	2.876	**2.757**	3.090	2.991
2.50	2.543	**2.491**	2.749	2.653
2.75	2.286	**2.282**	2.488	2.380
3.00	**2.084**	2.123	2.281	2.168

Table 6. Comparison of adaptive CUSUM plans for in-control ARL = 400.

Delta	$\alpha = 0.2$	$\alpha = 0.2$	$\alpha = 0.6$	$\alpha = 0.7$
	$h_{adj} = 1.386$	$h_{adj} = 0.9215$	$h_{opt} = 1.267766$	$h_{opt} = 1.484168$
	$k = SP_t/2$	$k = SP_{t-1}/2$		
0.00	398.492	399.897	400.127	400.369
0.50	21.831	23.357	**21.662**	22.120
0.75	12.875	**12.201**	12.670	12.999
1.00	8.896	**7.991**	8.776	8.992
1.25	6.663	**5.902**	6.644	6.745
1.50	5.269	**4.648**	5.306	5.345
1.75	4.333	**3.872**	4.413	4.401
2.00	3.674	**3.326**	3.776	3.726
2.25	3.187	**2.941**	3.302	3.233
2.50	2.815	**2.649**	2.938	2.845
2.75	2.512	**2.421**	2.648	2.558
3.00	2.288	**2.249**	2.416	2.316

Table 7. Comparison of adaptive CUSUM plans for in-control ARL = 600.

Delta	$\alpha = 0.2$	$\alpha = 0.2$	$\alpha = 0.6$	$\alpha = 0.7$
	$h_{adj} = 1.4311$	$h_{adj} = 0.907$	$h_{opt} = 1.266595$	$h_{opt} = 1.47165$
	$k = SP_t/2$	$k = SP_{t-1}/2$		
0.00	599.579	599.797	600.319	600.402
0.50	25.226	26.808	**24.801**	25.365
0.75	14.740	**13.650**	14.375	14.879
1.00	10.039	**8.825**	9.876	10.193
1.25	7.500	**6.430**	7.421	7.603
1.50	5.900	**5.045**	5.919	6.008
1.75	4.837	**4.168**	4.888	4.926
2.00	4.077	**3.149**	4.162	4.142
2.25	3.526	**3.187**	3.622	3.577
2.50	3.107	**2.833**	3.205	3.143
2.75	2.778	**2.590**	2.881	2.801
3.00	2.513	**2.392**	2.621	2.538

Table 8. Comparison of adaptive CUSUM plans for in-control ARL = 800.

Delta	$\alpha = 0.2$	$\alpha = 0.2$	$\alpha = 0.6$	$\alpha = 0.7$
	$h_{adj} = 1.4311$	$h_{adj} = 0.909$	$h_{opt} = 1.3042$	$h_{opt} = 1.5196$
	$k = SP_t/2$	$k = SP_{t-1}/2$		
0.00	799.979	800.213	800.279	800.312
0.50	27.772	29.368	**27.129**	27.928
0.75	16.047	**14.722**	15.608	16.195
1.00	10.956	**9.404**	10.694	11.093
1.25	8.158	**6.823**	7.998	8.265
1.50	6.395	**5.343**	6.332	6.493
1.75	5.216	**4.139**	5.212	5.289
2.00	4.387	**3.759**	4.435	4.455
2.25	3.783	**3.305**	3.840	3.821
2.50	3.316	**2.963**	3.391	3.325
2.75	2.961	**2.708**	3.043	2.981
3.00	2.675	**2.501**	2.767	2.693

is only preferred when delta = 0.5. As such, selecting $k = SP_t/2$ is preferred if the in-control ARL = 100. However, its preference soon drops off as the in-control ARL increases from 200.

5.1. Attempts to improve on the adaptive plan of Sparks [12] in steady-state situations

Recall the adaptive CUSUM

$$AC_t^U(k) = \max\left(0, AC_{t-1}^U + (z_t - k_t)/h(k_t)\right) \tag{5}$$

Now the Signal-to-Noise Ratio, SNR, $(z_t - k_t)/h(k_t)$ will be selected that will improve the detection performance of the plan. The EWMA smoothed trend in the z_t is given by

$$E_t = \alpha z_t + (1 - \alpha)E_{t-1} \tag{6}$$

Next, k_t is chosen such that the Signal-to-Noise Ratio $(z_t - k_t)/h(k_t)$ is a maximum, denote

$$SNR_t = max_k\left(\frac{z_t - k}{h(k)}\right) \tag{7}$$

for positive k values. The k is restricted to be greater than 0.22 in this paper which means we are less interested in location shifts less than 0.5 standard deviations. Note that $SNR_t < 0$ whenever $z_t < 0.22$. The new adaptive CUSUM statistic is now defined by

$$NC_t^U(k) = \max\left(0, NC_{t-1}^U + SNR_t\right) \tag{8}$$

The threshold for this CUSUM is expected to be larger than 1. Therefore an increase in location is flagged when

$$NC_t^U(k) > h_{opt} \tag{9}$$

where h_{opt} is selected to deliver a specified in-control ARL. The results in **Tables 3–8** outline the performance of this plan relative to the traditional adaptive CUSUM plan of Sparks [12] in the case where the in-control ARL = 100, 200, 300, 400, 600 and 800 (in the 3rd column).

Table 3 indicates that the user should select the EWMA weights to be 0.7 to improve on the traditional adaptive CUSUM plan when $0 < \delta \le 0.75$ and $\delta \ge 2.25$ for in-control ARL = 200, but for all in-control ARL tried (in-control ARL$\neq$200) there is no advantage in using this plan in all cases except when $\delta = 0.5$.

6. Example of application

The example of application is the nitrogen dioxide (NO_2) values at Liverpool (a suburb in the western part of Sydney, Australia). Nitrogen dioxide primary gets into the air from the burning of fuel. High exposure to this can cause respiratory problems such as asthma (see WHO [17]). Nitrogen dioxide reacts with other chemicals in the air to form both particulate matter and ozone (see [2]). Both of these are harmful to humans and possibly animals when inhaled.

The data was downloaded from New South Wales (Australia) Heritage Foundation website on air pollution. Data ranged from the beginning of 2010 to the end of March 2017 and were daily averages.

The data up to the end of August 2016 were used as training data to fit both the (in-control) mean and standard deviation of the normal distribution using gamlss library in R [15]. The model had explanatory variables as time in days, day-of-the-week and harmonics. Harmonics are included because there were strong seasonal influences on nitrogen dioxide values at Liverpool. The qq-normal plot of standardised residuals of this model indicated that the normal assumption for the residuals was appropriate. This fitted model was then used to predict the mean and standard deviation for the period on 1 September 2016–31 March 2017 (taken as the expected value and standard deviation for in-control data).

The actual daily average nitrogen dioxide measures were standardised by subtracting their fitted mean and dividing by the fitted standard deviation. The adaptive CUSUM was then applied to these standardised scores to see if these values had increased significantly from expect during the period 1 September 2016–31 March 2017. The plan was designed to deliver an in-control ARL of 200. Whenever the chart flagged a significant increase the adaptive CUSUM was set equal to zero to see if the nitrogen dioxide levels remained significantly higher than expected.

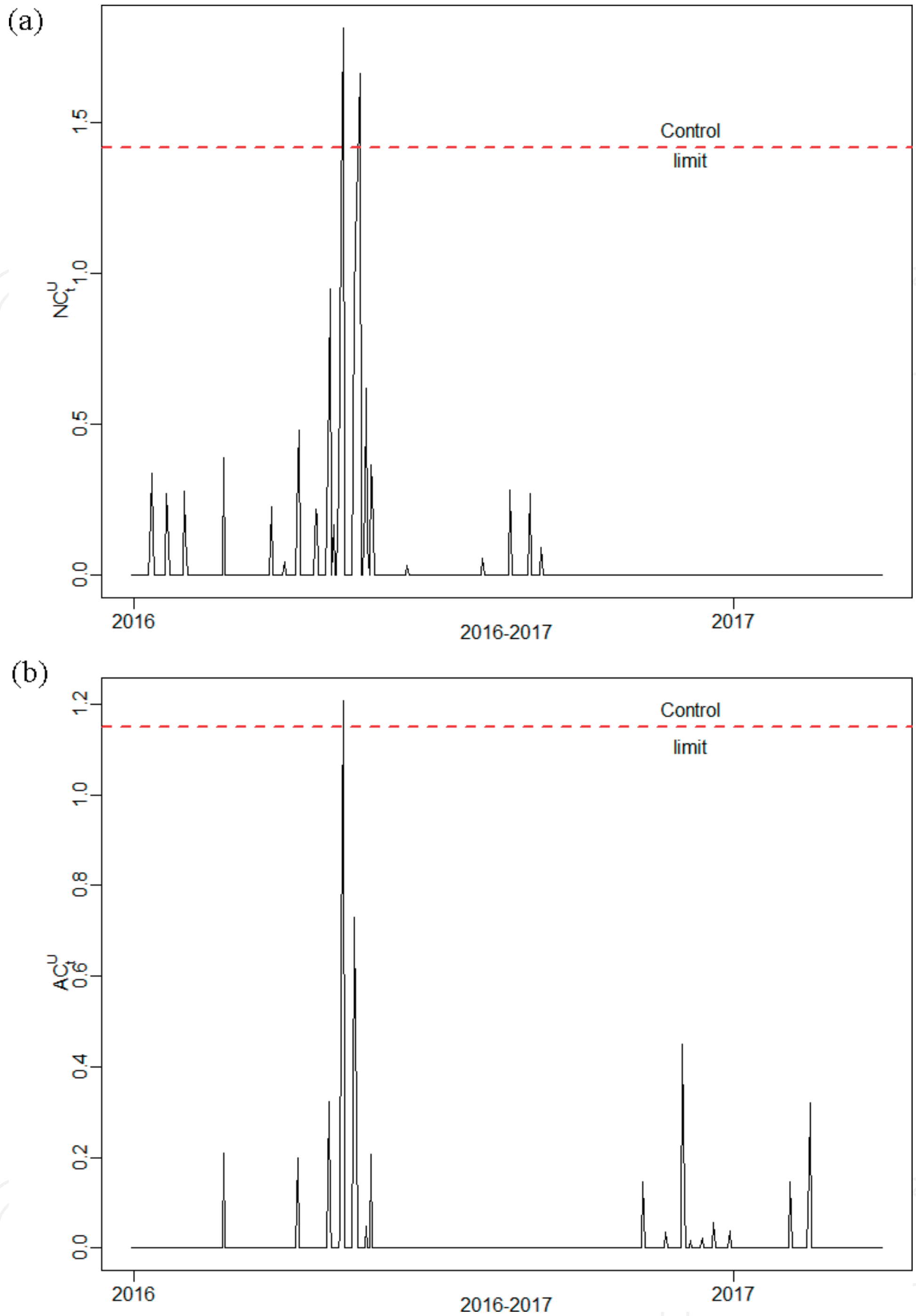

Figure 2. (a) The adaptive CUSUM $NC_t^U(k)$ advocated in this paper for in-control ARL = 200. (b) The adaptive CUSUM of Sparks [12].

Figure 2a adaptive CUSUM values as advocated in this paper for in-control ARL = 200 is plotted against the date for the period.

Figure 2b is the adaptive CUSUM of Sparks [12]. Both signal an increase in nitrogen dioxides on 8 May 2016, but the adaptive CUSUM values $NC_t^U(k)$ signals again on the 18 May 2016 after starting the CUSUM again at zero. The traditional adaptive CUSUM of Sparks [12] failed to signal a second time (**Figure 2b**).

7. Conclusions and further work

Although the new adaptive CUSUM has promise, the SNR_t proved too volatile to be efficient. There may be merit in establishing a smoother version of SNR_t that is less noisy. If future location shifts are known, then this paper offers the mean of selecting an optimal adaptive CUSUM plan.

A. Appendix

In-control ARL	Fitted model for $h(k)$
100	$h(k) = 0.3794337 - 2.9630562\log(k) + 1.9600587k - 0.8024828k^2 + 0.9033659\log(k)k$
200	$h(k) = -2.828476 - 4.867645\log(k) + 4.704948k - 1.827205k \times \log(k)$
300	$h(k) = -3.574586 - 5.639812\log(k) + 5.650032k - 2.177893k \times \log(k)$
400	$h(k) = -4.39191859 - 6.32066081\log(k) + 6.67882498k - 0.06873969k^2 - 2.50144146k \times \log(k)$
500	$h(k) = -5.44288223 - 7.02105471\log(k) + 7.68319645k + 0.08688656k^2 - 3.22718165k \times \log(k)$
600	$h(k) = -6.602602 - 7.687670\log(k) + 8.825719k + 0.196071k^2 - 3.996312k \times \log(k)$
700	$h(k) = -8.0773942 - 8.4028196\log(k) + 9.9107572k + 0.6595734k^2 - 5.3895806k \times \log(k)$
800	$h(k) = -8.9383214 - 8.9021000\log(k) + 10.7584243k + 0.7361421k^2 - 5.9389200k \times \log(k)$
900	$h(k) = -9.0757848 - 9.1040296\log(k) + 10.9300859k + 0.7607276k^2 - 6.0276468\,k \times \log(k)$
1000	$h(k) = -8.84991553 - 9.31320632\log(k) + 11.10662579k + 0.40968650k^2 - 5.33730283k \times \log(k) + \text{as . factor}$ $(k < 0.675) \times (-0.11040543 + 0.09135357\,k + 0.11203402\,k^2)$

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References

[1] Capizzi G, Masarotto G. An adaptive exponentially weighted moving average control chart. Technometrics. 2003;**45**(3):199-207

[2] Gamon LF, White JM, Wille U. Oxidative damage of aromatic dipeptides by the environmental oxidants NO_2 and O_3. Organic & Biomolecular Chemistry. 2014;**12**(41):8280-8287

[3] Gan FF. An optimal design of CUSUM quality control charts. Journal of Quality Technology. 1991;**23**(4):279-286

[4] Han D, Tsung F, Hu X, Wang K. CUSUM and EWMA multi-charts for detecting a range of mean shifts. Statistica Sinica. 2007;**17**(3):1139-1164

[5] Jiang W, Shu L, Apley DW. Adaptive CUSUM procedures with EWMA-based shift estimators. IIE Transactions. 2008;**40**(10):992-1003

[6] Moustakides G. Optimal stopping times for detecting changes in distributions. The Annals of Statistics. 1986;**14**(4):1379-1387

[7] Page ES. Continuous inspection schemes. Biometrika. 1954;**41**(1/2):100-115

[8] Page ES. Cumulative sum schemes using gauging. Technometrics. 1962;**4**(1):97-109

[9] Ryu JH, Wan H, Kim S. Optimal design of a CUSUM chart for a mean shift of unknown size. Journal of Quality Technology. 2010;**42**(3):311

[10] Shu J, Jiang W, Wu Z. Adaptive CUSUM procedures with Markovian Mean Estimation. Computational Statistics and Data Analysis. 2008;**52**(9):4395-4409

[11] Souza GP, Samohyl RW (2008). Monitoring forecast errors with combined CUSUM and Shewhart control charts. In: International Symposium of Forecasting. 26

[12] Sparks RS. CUSUM charts for signalling varying location shifts. Journal of Quality Technology. 2000;**32**(2):157-171

[13] Sparks RS. A group of moving averages control plan for signaling varying location shifts. Quality Engineering. 2003;**15**(4):519-532

[14] Sparks RS. Shewhart dispersion charts made easy for mild to moderately autocorrelated normally distributed data. Quality Engineering. 2017 in press. http://www.tandfonline.com/doi/abs/10.1080/08982112.2017.1311415

[15] Stasinopoulos DM, Rigby RA. Generalized additive models for location scale and shape (GAMLSS) in R. Journal of Statistical Software. 2007;**23**(7):1-46

[16] Stoumbos ZG, Mittenthal J, Runger GC. Steady-state-optimal adaptive control charts based on variable sampling intervals. Stochastic Analysis and Applications. 2001;**19**(6):1025-1057

[17] WHO Regional Office for Europe. Health aspects of air Pollution. Chapter 7. In: Nitrogen dioxide. 2003

[18] Wu Z, Jiao J, Yang M, Liu Y, Wang Z. An enhanced adaptive CUSUM control chart. IIE Transactions. 2009;**41**(7):642-653