Chapter 7

Introduction to Parametric and Autoparametric Resonance

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Additional information is available at the end of the chapter

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Abstract

This chapter will give an introduction to linear and nonlinear oscillators and will propose literature to this topic. Most importantly, hands on examples with numerical simulations are illustrating oscillations and resonance phenomena and where useful, also analytical methods to treat nonlinear behavior are given.

Keywords: parametric resonance, autoparametric resonance, nonlinear vibration, Mathieu equation, Hopf bifurcation, Strutt diagram, nonlinear natural frequency, instability domain, basepoint excited primary and secondary system

1. Introduction

When a mechanical system has at least two vibrating components, the vibration of one of the components may influence the other component. This influence effect which might stabilize or destabilize the system is called autoparametric resonance. This chapter will introduce autoparametric resonance by examining hands on examples for such systems. In particular, basepoint excited systems are analyzed. Beside purely mechanical systems, also examples of an electrical system with two coupled resonators are investigated.

There are three main types of oscillation: (1) free oscillation, (2) forced excited oscillation and (3) self-excited oscillation.

Free oscillation is defined as temporal fluctuations of the state variables of a system. Such temporal fluctuations can be defined as deviations from a mean value. Vibrations are present in many mechanical systems and occur always in feedback systems. The concept of free oscillation is misleading since nearly all physical systems are subject to attenuation. However, it depends on the size (and thus the time). Exceptions could be, for example, orbit oscillations of planets (macroscopic) or oscillations of electrons (microscopic). The two systems mentioned are also subjected to a type of damping, since both systems cannot remain stable indefinitely, but for an extremely long time.

A forced excited spring mass system might be a mechanically forced oscillator. Such systems of translational motions are discussed in Sections 2 and 3. Beside translatory oscillations, rotatory oscillations and resonance is of vital interest to design engineers of aircraft turbines, etc. Unbalanced rotating machine parts are sources of unwanted vibrations and might resonate when excited accordingly.

Self-excited oscillation, also called as self-oscillation, self-induced, maintained or autonomous oscillation is known in electronics as parasitic oscillation and in mechanical engineering literature as hunting. Such systems are discussed in Section 3.

Table 1 depicts relevant parameters for characterization motion in translational and rotational structures. The parameters for displacement, velocity and acceleration have been written as absolute values – knowing that depending on the application, they might be vectors, depending on the chosen frame of reference. In the most general case, they form a four vector. The force is written as mass times acceleration (Newton's second law) and therefore force is also a vector. That brings us to Newton's first law, which states that an object that is at rest will stay at rest unless a force acts upon it or inversely an object will not change its velocity unless a force acts upon it. For completeness, also Newton's third law shall be given: Actio et Reactio – all forces between two objects exist in equal magnitude and opposite direction. A treaty to Newton's laws of dynamics can be found, for example in chapter 9 of volume I [1].

D'Alembert's principle is a statement of the fundamental classical laws of motion. It is the dynamic analogue to the principle of virtual work for applied forces in a static system and in fact is more general than Hamilton's principle, avoiding restriction to holonomic systems[1].

Translational			**Rotational**		
Symbol	Description	SI Unit	Symbol	Description	SI Unit
s	Displacement	m	φ	Angle	rad
$v = \frac{ds}{dt}$	Velocity	$\frac{\mathrm{m}}{\mathrm{s}}$	$\omega = \varphi \frac{d}{dt}$	Angular velocity	$\frac{\mathrm{rad}}{\mathrm{s}}$
$a = \frac{dv}{dt}$		$\frac{\mathrm{m}}{\mathrm{s}^2}$	$\alpha = \omega \frac{d}{dt}$	Angular acceleration	$\frac{\mathrm{rad}}{\mathrm{s}^2}$
m	Mass	kg	J	Inertia	kg m^2
$F = m\,a$	Force	N	$T = J\,\alpha$	Torque	Nm
$I = m\,v$	Momentum	Ns	$L = J\,\omega$	Angular momentum	Nms
$T = \frac{1}{2} m\,v^2$	Kinetic energy	Nm	$T = \frac{1}{2} J\,\omega^2$	Kinetic Energy	Nm
$U = \frac{1}{2} k\,y^2$	Potential energy	Nm	$U = \frac{1}{2} c\,\varphi^2$	Potential energy	Nm
$W = \int Fds$	Work	J	$W = \int Td\varphi$	Work	J
$P = F\,v$	Power	W	$P = J\,\omega$	Angular power	W

Table 1. Comparison of translational and rotational motion parameter characteristics.

[1]A holonomic constraint depends only on the coordinates and time and does not depend on velocities.

If the negative terms in accelerations are recognized as inertial forces, the statement of d'Alembert's principle becomes "the total virtual work of the impressed forces plus the inertial forces vanishes for reversible displacements". The principle does not apply for irreversible displacements, such as sliding friction, and more general specification of the irreversibility is required. A derivation of the Lagrangian equation of motion is well explained in Chapter 1 of [2] or [3]. In (1), the non-conservative energy term defined as the Lagrangian (L) is, composed of the kinetic energy T and the potential energy U.

$$L = T - U \tag{1}$$

In (2) the Lagrangian equation is given with generalized coordinates q_i of a dynamic system and dissipative generalized forces Q_i.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q_i}}\right) - \frac{\partial L}{\partial q_i} = Q_i \tag{2}$$

The sum of all kinetic energies T in the system, whether translational or rotational character (see also **Table 1**) needs to be included. The sum of all potential energies U in the system, whether it stems from the gravitation or energy from linear or nonlinear springs or whatever scalar field. Elastic potential energy from any linear or nonlinear spring can be obtained calculating its potential energy. The Lagrangian formalism can also be used for mechanical systems with mass explicitly dependent on position, see for example [4]. In the following chapters, all analyzed dynamical systems are derived using this elegant and powerful method.

Looking at translational (classic) mechanical springs, the displacement dependent force $F(q_t)$ can be written as shown in (3) using spring stiffness k_1 $\left(\frac{N}{m}\right)$, the generalized translational coordinates q_t and having also introduced a nonlinear spring term k_n and an exponent n for setting nonlinearity of spring. For linear springs, where the force is proportional to the displacement (Hooke's law), this nonlinear spring term will be zero.

$$F(q_t) = k_1\, q_t + k_n\, {q_t}^n \tag{3}$$

The elastic energy $E_{elastic}$ of the spring is obtained by integrating the exerted force over its covered path s.

$$E_{elastic}(s) = \int_0^s F(q_t)\mathrm{d}q_t = \int_0^s k_1\, q_t + k_n\, {q_t}^n = \frac{1}{2}k_1 + \frac{k_n}{1+n}\, {q_t}^{1+n}\Bigg|\begin{matrix} s \\ 0 \end{matrix}$$

$$E_{elastic}(s) = \frac{1}{2}k_1\, s^2 + \frac{k_n}{1+n}s^{1+n} \;\; with \;\; n > -1 \tag{4}$$

The elastic energy $E_{elastic}$ (a potential U) for a linear translational spring is given also in **Table 1**. For (rotational) torsion springs, the procedure is the same and a given spring torque can be expressed as shown in (5) with torsion coefficient D_1 $\left(\frac{Nm}{rad}\right)$, the generalized translational coordinates q_r and having also introduced a nonlinear torsion spring term D_n.

$$T(q_r) = D_1\, q_r + D_n\, q_r{}^n \tag{5}$$

The elastic rotational energy can be expressed as follows (6):

$$E_{elastic}(\varphi) = \int_0^{\varphi} T(q_r)\mathrm{d}q_r = \int_0^{\varphi} D_1\, q_r + D_n\, q_r{}^n \;\; \mathrm{d}q_r = \frac{1}{2}D_1\, q_r{}^2 + \frac{k_n}{1+n}D_n\, q_r{}^{1+n}\Big|_0^{\varphi}$$

$$E_{elastic}(\varphi) = \frac{1}{2}D_1\,\varphi^2 + \frac{1}{1+n}D_n\varphi^{1+n} \;\; with \;\; n > -1 \tag{6}$$

In **Figure 1**, sketches on the top show translational springs (from left to right linear, nonlinear and nonlinear unsymmetrical) and sketches on the bottom depict rotational springs (from left to right linear, nonlinear and nonlinear unsymmetrical). The origin is depicted with an O and the spring displacement is depicted as y and φ, respectively. For the translational magnetic spring systems, the lower and upper magnets are fixed to the reference frame with origin O. The nonlinear symmetric magnetic spring uses three identical block or disk magnets. Such magnetic springs have a nonlinear term $k_n > 0$ – forming so called hardening springs if $k_n < 0$ in literature referred to softening springs, see for example [5]). The nonlinear unsymmetrical translational magnetic spring is also shown in the neutral position and the displacement around the origin is unsymmetrical.

The rotational magnetic spring systems have a ferromagnetic stator fixed to the reference frame and a rotating hollow shaft carrying two permanent magnets, in this scenario also made of ferromagnetic material. The spring is drawn in the unstable equilibrium position. Exemplarily spring characteristics for such translational and rotational spring systems are depicted in **Figure 2**.

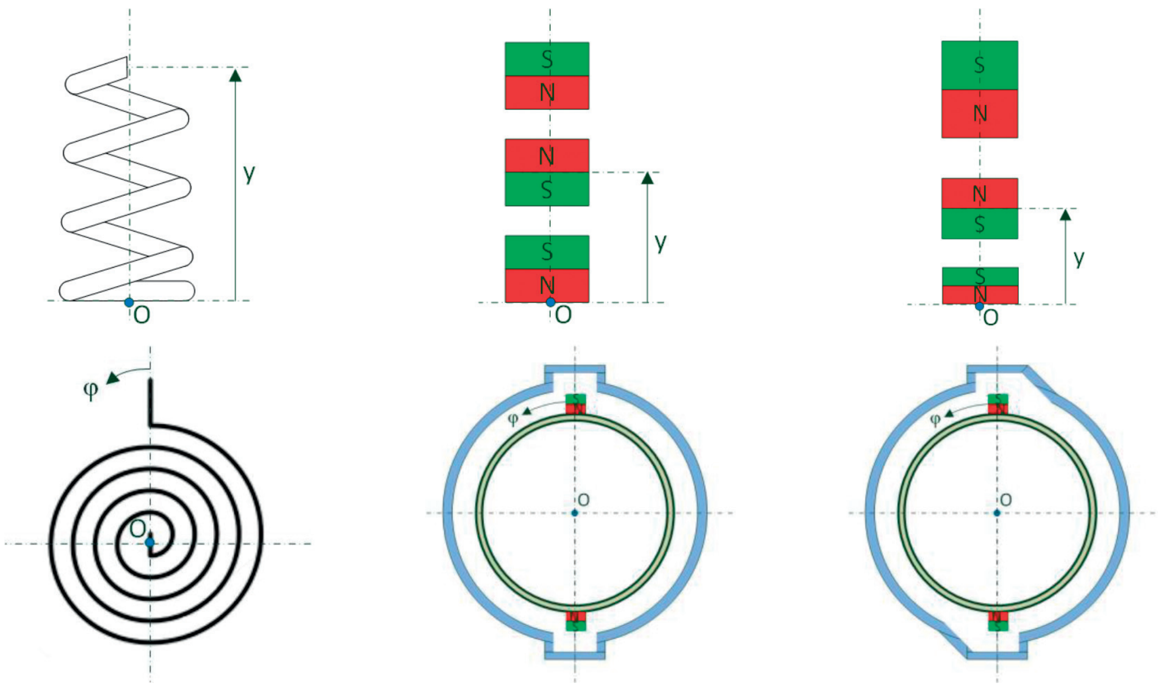


Figure 1. Sketches of translational and rotational spring systems.

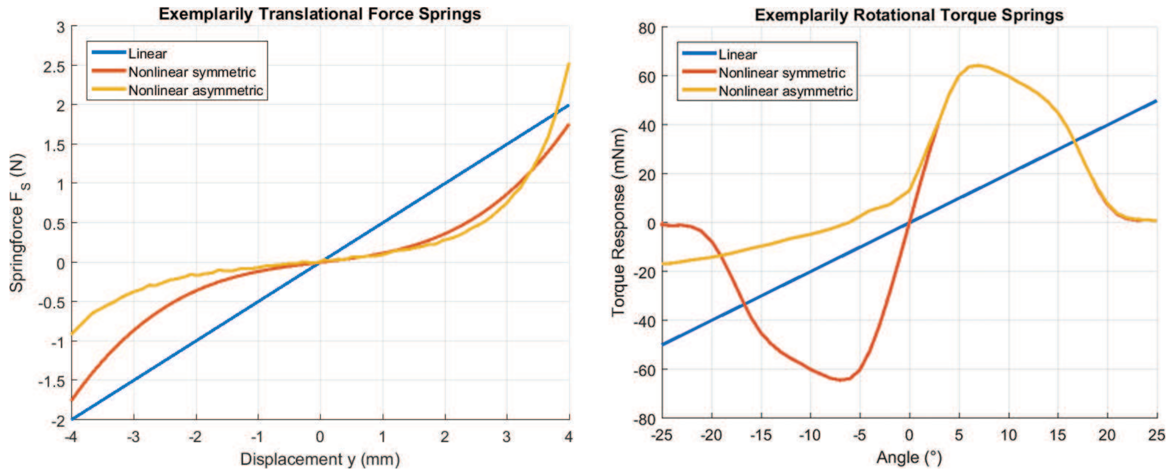


Figure 2. Exemplarily displacement-force signals (l) and angular displacement-torque signals (r) of the shown spring systems of **Figure 1**.

Note that integrating the force or torque response will add up to zero for rotational and translational spring systems. The given exemplarily spring characteristics of drawn spring systems in **Figure 1** are shown in **Figure 2**. On the left-hand side, exemplarily translational spring characteristics are shown and on the right-hand side rotational spring characteristics are depicted. The linear case where the force and torque are proportional to the displacement is shown in blue. In red, symmetric nonlinear – here for translational and rotational systems a permanent magnet system is shown, but also mechanical spring systems could be envisaged. The curves of the nonlinear asymmetric cases are depicted in orange. In the appendix A.1, more simulations have been depicted for translational symmetric spring systems using ring magnets.

Equivalence of electrical and mechanical systems are shown in **Table 2**. On the left-hand side, a mechanical system with only one degree of freedom in y direction is shown and its equivalent electrical structure with charge q and current i on the right-hand side. Kinetic energies are denoted with T_T (translational kinetic energy) and T_L (inductive kinetic energy), potentials are written as U_S (spring potential) and U_C (capacitive potential) and the gravitational potential denoted as U_G and the DC battery voltage is U_B. Non-conservative components are in the mechanical system the viscous damping force and in the electrical system the electrical resistor.

The Lagrange energy function is shown for the mechanical system in (7) and its equivalent electrical system in (8).

$$L_{mech} = T_T - (U_S + U_G) = \frac{1}{2}\, m\, y'^2 + \frac{1}{2}\, k\, y^2 - m\, g\, y \tag{7}$$

$$L_{el} = T_L - (U_C + U_B) = \frac{1}{2}\, L\, q'^2 + \frac{1}{2}\frac{1}{C}\, q^2 - U_0\, q \tag{8}$$

Applying the Lagrangian formalism (1) and (2) to these SDoF systems, will lead to the resulting DE's as shown in (9) and (10). Note that the sign of the (viscous) damping must be introduced always with a negative sign using the generalized coordinates – as its velocity is

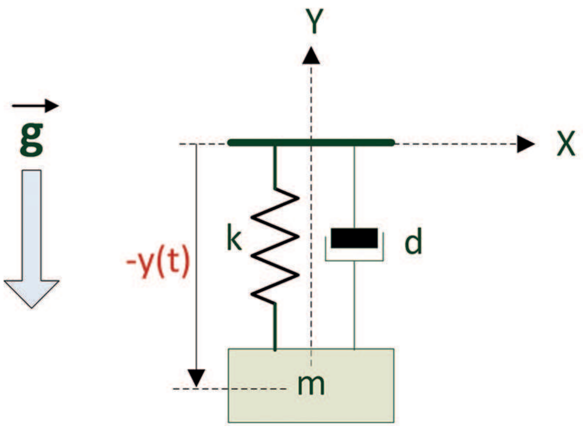
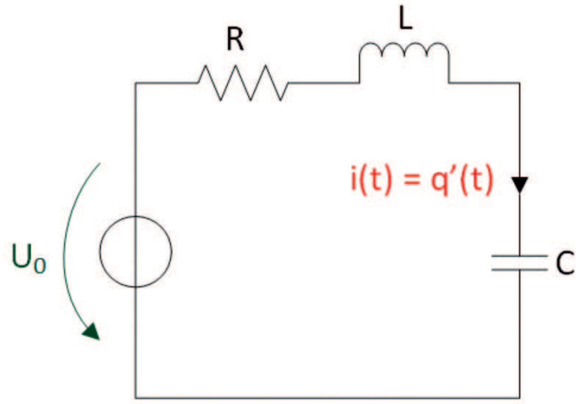
Mechanical System	Electrical System
	
Mechanical kinetic energy:	Electrical kinetic energy:
$T_T = \frac{1}{2} m\, y'^2$	$T_L = \frac{1}{2} L\, q'^2$
Mechanical potential energies:	Electrical potential energies:
$U_S = \frac{1}{2} k\, y^2$	$U_C = \frac{1}{2}\frac{1}{C}\, q^2$
$U_G = m\, g\, y$	$U_B = U_0\, q$
Mechanical damping force:	Electrical damping voltage:
$F_d = d\, y'$	$V_d = R\, q'$

Table 2. Equivalence of electrical and mechanical systems.

always opposing the system velocity. In the electrical circuit, having the flowing charge velocity q' for example, current i defined in clockwise direction, the battery voltage, as it is a source, must act in the opposite direction and therefore this potential energy must be introduced with a negative sign.

$$m\, y'' + d\, y' + k\, y = g\, m \tag{9}$$

$$L\, q'' + R\, q' + \frac{1}{C}\, q = U_0 \tag{10}$$

Both systems (9) a force DE, (10) a voltage DE, belong to the same class of ordinary linear second-order DE. Resonance frequency for the mechanical system (9) is $\omega_{mech}{}^2 = \frac{k}{m}$ and for the electrical system (10) $\omega_{el}{}^2 = \frac{1}{L\,C}$.

2. Linear resonance systems

2.1. Linear single degree of freedom systems

In this section, a linear basepoint excited single degree of freedom systems is discussed. The lumped parameter model for the examined system (**Figure 3**) consists of a linear oscillator with

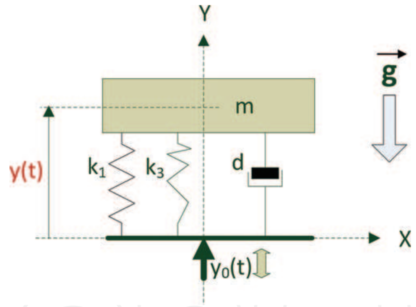


Figure 3. Linear single degree of freedom (SDoF) spring mass damper model of a resonant harmonic basepoint excited oscillator.

mass m damping factor d, a linear spring with a spring rate k_1 and an external basepoint excited harmonic force with amplitude $y_0(t)$. System coordinate origin is placed at the basepoint excitation.

The kinetic energy (11) of this system and the potential energy (12) form the non-dissipative energy of this system. The dissipative force F_d of this system – we consider only viscous friction – is shown in (13) and the driving force F_0 (14) assuming a harmonic basepoint excitation with amplitude A and driving frequency ω.

$$T = \frac{1}{2}\, m\, y'^2 \tag{11}$$

$$U = m\, g\, y + \frac{1}{2}\, k_1\, y^2 \tag{12}$$

$$F_d = \mathrm{d}\;\; y' \tag{13}$$

$$F_0 = m\frac{\mathrm{d}^2}{\mathrm{d}t^2}(A\;\cos\,\omega t) = -mA\omega^2 \cos\,\omega t \tag{14}$$

Applying the Lagrangian formalism (1) and (2), we deal with SDoF system, will lead to the resulting DE shown in (15). Note that the sign of the viscous damping must be introduced with a negative sign using the generalized coordinates. The driving force F_0, as it is a harmonic signal, can be introduced with a positive or a negative sign, resulting in a phase shift of 180°.

$$m\, y'' - (-mg - k_1 y) = -F_d - F_0 \tag{15}$$

$$m\, y'' + d\, y' + k_1 y + mg = mA\omega^2 \cos\,\omega t \tag{16}$$

Introducing dimensionless notation, by using a dimensionless time τ, a dimensionless system resonance frequency Ω, the damping factor ξ_1 and the gravity offset term ϱ (17) and setting the dimensionless displacement u (18).

$$\tau = \mathrm{t}\,\omega_1;\, \Omega = \frac{\omega}{\omega_1};\, {\omega_1}^2 = \frac{\mathrm{k}_1}{\mathrm{m}};\, \xi_1 = \frac{d}{2\, m\, \omega_1};\, \varrho = \frac{g}{A\, {\omega_1}^2} \tag{17}$$

$$\text{path } u(\tau) = \frac{y(t)}{A} \tag{18}$$

By replacing parameters of (16) with (17, 18), we obtain (19). We can drop the gravity offset term ϱ, as it will only add a non-time dependent offset to the solution u (20).

$$u'' + 2\xi_1 u' + u + \varrho = \Omega^2 \cos(\Omega\tau) \tag{19}$$

$$u'' + 2\xi_1 u' + u = \Omega^2 \cos(\Omega\tau) \tag{20}$$

Frequency domain behavior is obtained by applying the Laplace Transformation (21–24) and by replacing $s = j\Omega$ we obtain the frequency response (25).

$$U(s) = \mathcal{L}\{u(\tau)\} = \int_0^\infty u(\tau)e^{-s\tau}\mathrm{d}\tau \tag{21}$$

$$\mathcal{L}\left\{u'' + 2\xi_1 u' + u\right\} = -\frac{\mathrm{d}^2}{\mathrm{d}t^2}\cos(\Omega\tau) \tag{22}$$

$$U(s)s^2 + 2\xi_1 U(s)s + U(s) = s^2 Y_0(s) \tag{23}$$

$$G(\mathrm{s}) = \frac{U(s)}{Y_0(s)} = \frac{\mathrm{s}^2}{1 + 2\xi_1\mathrm{s} + \mathrm{s}^2} \tag{24}$$

$$G(\xi_1, \Omega) = \frac{U(j\Omega)}{Y_0(j\Omega)} = \left|\frac{\Omega^2}{1 + \Omega^2 + j2\xi_1\Omega}\right| \tag{25}$$

G represents the relative motion of the oscillation. As long as the excitation frequency can be represented by a Fourier series of harmonic functions, this obtained solution is valid and a very powerful result. (24) and (25) are represented in **Figure 4**. The advantage of the Bode Plot

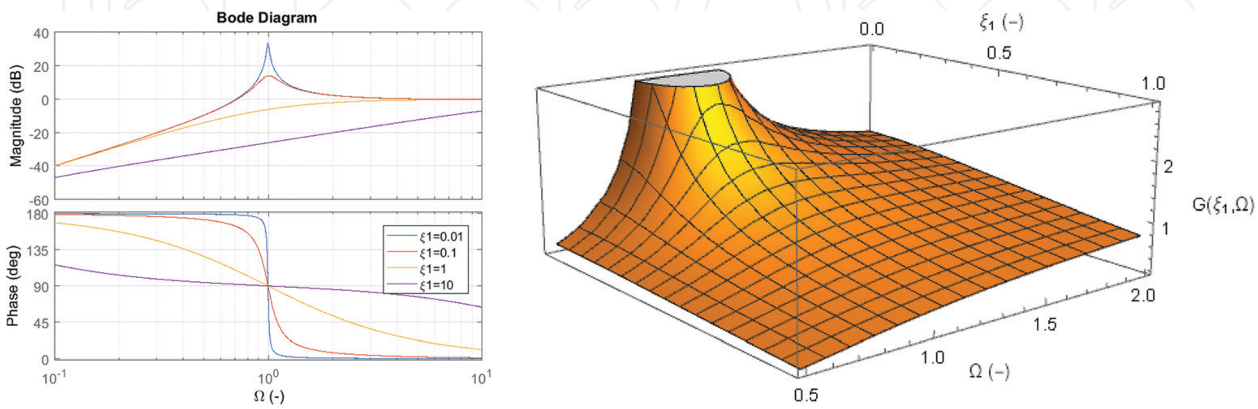


Figure 4. Representation of frequency response of a linear SDoF system using (24) Bode diagram (left) and absolute value representation of (25) (right).

representation is to have also the phase shown. As smaller the dimensionless damping ξ_1 become, as larger becomes the scaled resonance at the dimensionless frequency ration Ω.

2.2. Linear two degree of freedom systems (2DoF systems)

In this section, a linear basepoint excited two degree of freedom systems is discussed. The lumped parameter model for the examined system (**Figure 5**) consists of two linear oscillators with mass m_1 and m_2, damping factors d_1 and d_2, linear springs with spring rates k_1 and k_2 and an external basepoint excited harmonic force with amplitude $y_0(t)$. System coordinate origin is placed at the basepoint excitation.

$$T = \frac{1}{2}\, m_1\, {y_1}'^2 + \frac{1}{2}\, m_2 {y_2}'^2 \tag{26}$$

$$U = m_1\, g\, y_1 + m_2\, g\, y_2 + \frac{1}{2}\, k_1\, {y_1}^2 + \frac{1}{2}\, k_2\, (y_2 - y_1)^2 \tag{27}$$

$$F_d = d_1\, {y_1}' + d_2\, {y_2}' \tag{28}$$

$$F_0 = (m_1 + m_2)\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\, A\cos \omega t) = -(m_1 + m_2)A\omega^2 \cos \omega t \tag{29}$$

Applying the Lagrangian formalism (1) and (2) to this 2DoF problem, we obtain the coupled DE system shown in (30) and (31).

$$m_1\, {y_1}'' + d_1 y_1' + k_1 y_1 - k_2 (y_2 - y_1) + m_1 g = (m_1 + m_2) A\omega^2 \cos \omega t \tag{30}$$

$$m_2\, {y_2}'' + d_2 y_2' + k_2 (y_2 - y_1) + m_2 g = 0 \tag{31}$$

DE system shown in (30), (31) is represented dimensionless in equation DE system (32), (33) using the dimensionless parameters of (34), (35).

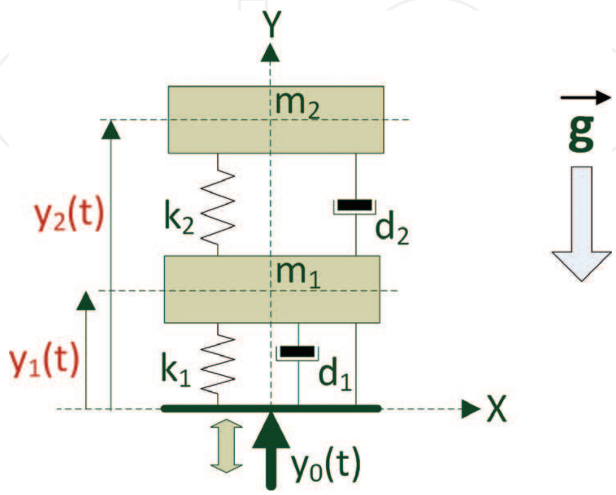


Figure 5. Linear 2DoF spring mass damper model of a resonant harmonic basepoint excited oscillator.

$$u''(\tau) + 2\xi_1 u'(\tau) + u(\tau) + \lambda_m {\Omega_0}^2 u(\tau) - \lambda_m {\Omega_0}^2 v(\tau) + \varrho = \Omega^2 \cos(\tau\Omega) \tag{32}$$

$$v''(\tau) + 2\xi_2 \Omega_0 v'(\tau) + {\Omega_0}^2 v(\tau) - {\Omega_0}^2 u(\tau) + \varrho = 0 \tag{33}$$

Similar to dimensionless parameters of (17), (18), the dimensionless time τ, a dimensionless system resonance frequency Ω, damping factors ξ_1 and ξ_2, a gravity offset term ϱ, system resonance frequencies of each oscillator ω_1 and ω_2 plus a mass ratio λ_m and an oscillator frequency ratio Ω_0 plus dimensionless displacements u and v.

$$\tau = t\,\omega_1; \Omega = \frac{\omega}{\omega_1}; {\omega_1}^2 = \frac{k_1}{m_1}; {\omega_2}^2 = \frac{k_2}{m_2}; \xi_1 = \frac{d_1}{2\,m_1\,\omega_1}; \xi_2 = \frac{d_2}{2\,m_2\,\omega_2}; \Omega_0 = \frac{\omega_2}{\omega_1}; \lambda_m = \frac{m_2}{m_1}; \varrho = \frac{g}{A\,{\omega_1}^2} \tag{34}$$

$$\text{path } u(\tau) = \frac{y_1(t)}{A} \text{ and path } v(\tau) = \frac{y_2(t)}{A} \tag{35}$$

The frequency response of this coupled oscillator system can again be obtained using the Laplace transformation introduced in (21). The system in the frequency domain is shown in (36) and (37) using the same steps as shown in the SDoF system (22)–(25).

$$U(s)s^2 + 2\xi_1 U(s)s + U(s)\left(1 + \lambda_m {\Omega_0}^2\right) - \lambda_m {\Omega_0}^2 \mathrm{V}(s) + \varrho = s^2 Y_0(s) \tag{36}$$

$$V(s)s^2 + 2\xi_2 \Omega_0\, V(s)s + V(s){\Omega_0}^2 - U(s){\Omega_0}^2 + \varrho = 0 \tag{37}$$

As (36) and (37) represent two algebraic equations, $U(s)$ and $V(s)$ can be separated, resulting in (38) and (39). Note that the gravity term ϱ in the numerator will introduce an additional damping of the transfer function.

$$U(s) = -\frac{-\rho\lambda_m {\Omega_0}^2 + \left(s^2 + 2s\xi_2\Omega_0 + {\Omega_0}^2\right)\left(-\rho + s^2 Y_0(s)\right)}{\lambda_m {\Omega_0}^4 - \left(s^2 + 2s\xi_2\Omega_0 + {\Omega_0}^2\right)\left(1 + s^2 + 2s\xi_1 + \lambda_m {\Omega_0}^2\right)} \tag{38}$$

$$V(s) = \frac{-\rho\left(1 + s^2 + 2s\xi_1 + (1+\lambda_m){\Omega_0}^2\right) + s^2 {\Omega_0}^2 Y_0(s)}{s^4 + {\Omega_0}^2 + 2s^3(\xi_1 + \xi_2\Omega_0) + 2s\Omega_0\left(\xi_2 + \xi_1\Omega_0 + \lambda_m \xi_2 {\Omega_0}^2\right) + s^2(1 + \Omega_0(4\xi_1\xi_2 + \Omega_0 + \lambda_m\Omega_0))} \tag{39}$$

Figure 6 depicts the relative oscillation response in the frequency (left) and time domain (right) of the derived 2DoF system. The frequency response is given as a dimensionless ratio Ω, see also (34). The dimensionless simulation parameters have been set exemplarily to $\xi_1 = \xi_2 = 0.021$, $\lambda_m = 0.42$ and $\Omega_0 = 1$. As we have two resonators with same system frequencies $\omega_1 = \omega_2 = 169\,\frac{rad}{s}$, two resonances will occur.This system reaches resonances at 0.71 Ω (19Hz) and at 1.41 Ω (38Hz) for $\Omega_0 = 1$ and 0.82 Ω and at 6.1 Ω for $\Omega_0 = 5$ (dashed lines).

The time-domain response from this coupled DE system with lumped parameter model **Figure 5** and (32) and (33) is shown in **Figure 7**. On the left-hand side, the ^dimensionless basepoint acceleration signal is given and its dimensionless response signals of first (blue) and second (red) DoF, simulating 50 periods and starting with settled initial conditions (amplitude

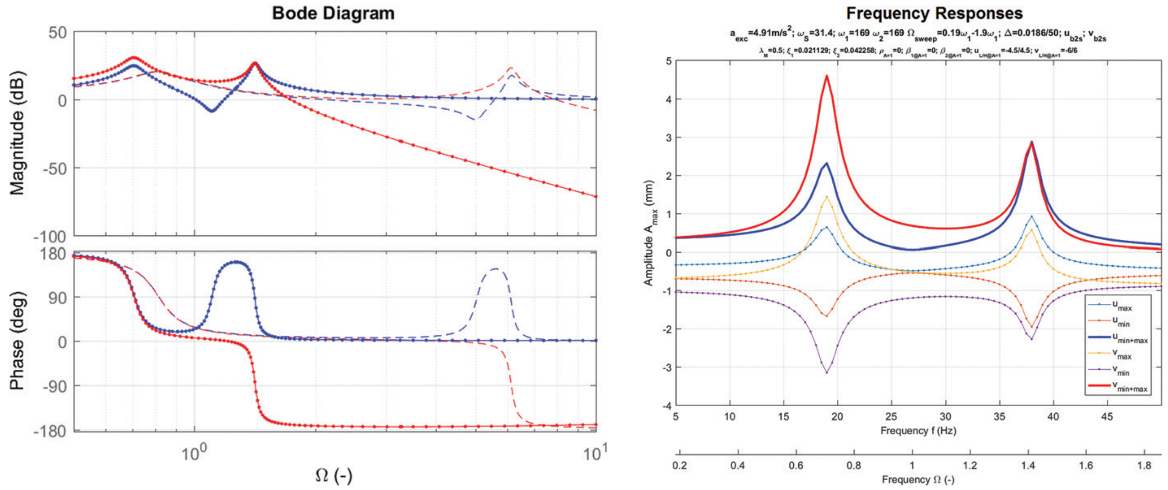


Figure 6. Bode diagram of a linear 2DoF system represented by (38) and (39) (left); first oscillator with mass m_1 in blue and second with m_2 in red with $\Omega_0 = 1$ and $\Omega_0 = 5$ (dashed lines) and constructed frequency response using time domain signals (right).

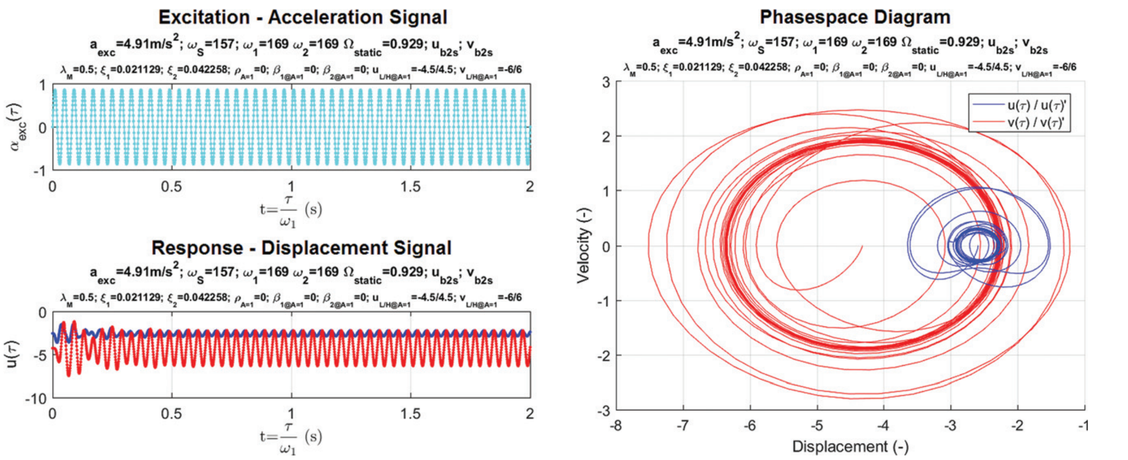


Figure 7. Numerical simulation results of the linear basepoint excited 2DoF system shown in (32) and (33) using a constant acceleration of 0.5g and a basepoint excitation of $\omega = 25\ Hz$.

of $\hat{y}_{1start} < 1\mu m$ and $\hat{y}_{2start} < 1\mu m$). The main simulation parameters are shown in the heading, a variant of this setup using nonlinear springs is given in the appendix A.3.

3. Nonlinear resonance systems

In the introduction, Section 1, we distinguished three cases of vibration. The class forced excitation will be further investigated in this section. In **Figure 8** five systems are depicted that can potentially exhibit parametric resonance effects. The term parametric means that of cases where the external excitation appears as a time varying modification of a system parameter. A "normal" forced excitation system whether linear or nonlinear, will respond to the excitation

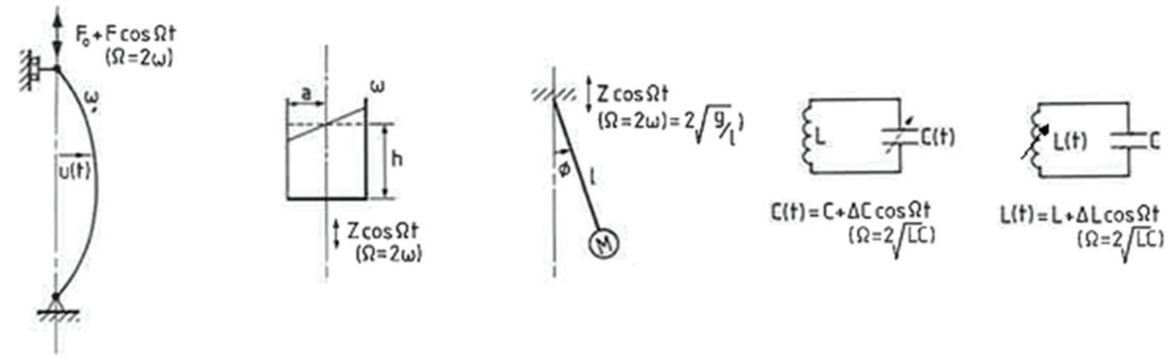


Figure 8. Examples of physical systems exhibiting potential parametric resonance effects, adapted after [2].

with or without resonance using the energy fed into it and no time varying modification of a system parameter might excite additionally the system.

The five depicted systems in **Figure 8** might show an exponential amplitude growth when excited externally in presence of a system damping factor. In the two electrical systems on the right-hand side, any of the three components R (here not drawn), L or C that is parametrically excited will respond with an exponential amplitude growth, if the mathematical physical system model has at least one degree of freedom of the Mathieu DE (40) or the Hill DE (41).

$$q'' + q(a + b\cos\,\Omega t) = 0 \tag{40}$$

The Hill differential equation is a generalized form of (40), in which the harmonic function is replaced with any periodic function, shown in (41).

$$q'' + q\left(a + f_p(t)\right) = 0 \tag{41}$$

It is most interestingly that any system parameter including also damping factors with time varying influence of a system parameter will result in an exponential growth of the response amplitude. To give a concrete example of this behavior, we consider here the example from Section 3.2 and inspect the resulting (dimensioned) DE system with (62) as primary system and (63) as secondary system of such a behavior.

The primary system has no such configuration, but the secondary system (63) is of Mathieu type. To simplify the treated system, we use instead of the basepoint excitation y_0 the primary system y, compare also the lumped parameter model in **Figure 12** (in an experiment we would simply make the stiffness k of the system very large, for example, replacing the spring with a fixed stiff rod). Now the new induced basepoint excitation y will excite the secondary system directly. As y is appearing in (63) as acceleration, we adjust this basepoint excitation simply in form of an acceleration (42).

$$y = A\cos\,(\omega t) \rightarrow y'' = A\,\omega^2\cos\,(\omega t) \tag{42}$$

Writing (63) as an acceleration DE (dividing by $m_2\,l$) and inserting it the acceleration of (42), it is read (43).

$$l\,\varphi'' + \frac{D}{m_2\,l}\,\varphi' + g\sin\,\varphi + A\,\omega^2\cos\,(\omega t)\,\sin\,\varphi = 0 \tag{43}$$

Rearranging the terms and setting $\sin\varphi \cong \varphi$, we get (44), which is a Mathieu type DE with parameters $a = \frac{g}{l}$ and $b = -\frac{A\ \omega^2}{l}$.

$$\varphi'' + \frac{D}{m_2}\,\varphi' + \sin\,\varphi\left(\frac{g}{l} + \frac{A\,\omega^2}{l}\cos\,(\omega t)\right) = 0 \tag{44}$$

For generating parametric resonances, the (natural) system frequency needs to be coupled with the excitation frequency ω. Using the same nomenclature as in (64),we define the pendulum system frequency $\omega_2{}^2 = \frac{g}{l}$. For generating parametric resonances for which the angle $\varphi(t)$ is growing exponentially, a frequency ratio $\omega : \omega_2 = 1:1$ is sufficient (as well as the ratio $\omega : \omega_2 = 2:1$), see also left-hand side of **Figure 9**. In case of letting the displacement term be

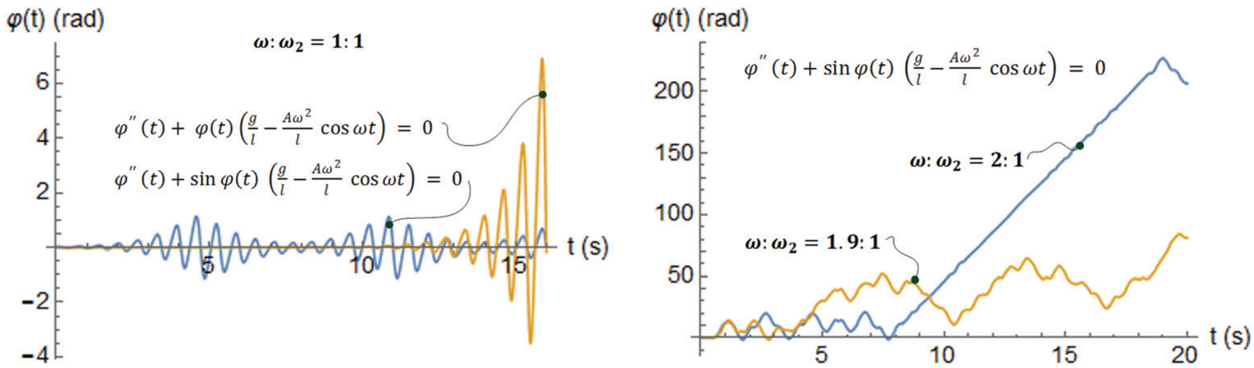


Figure 9. Nonlinear single degree of freedom (SDoF) spring mass damper model of a resonant harmonic basepoint excited oscillator.

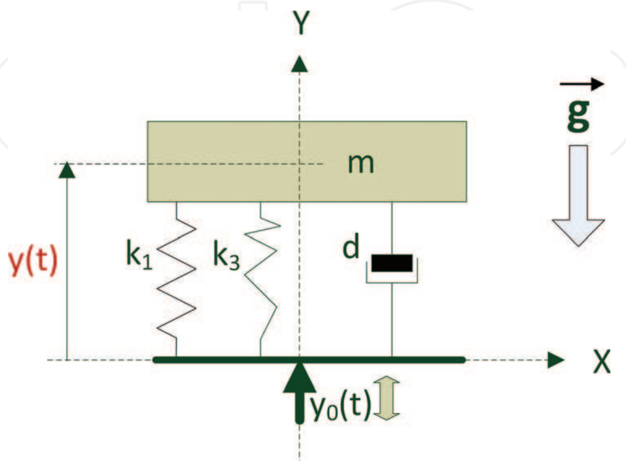


Figure 10. Response signals of a parametrically excited pendulum examining DE (47) with keeping damping term $D = 0$, $l = 108.1$ mm, $A = 100$ mm and $\omega_1 = 10$ *rad*.

harmonic (left-and right-hand side of **Figures 9**), the frequency ratio must be very close to a 2:1 ratio to have a large amplitude response. The ratio tolerance for having a large growth has a band width of ca. 1 rad to keep a large amplitude growth going.

Note that the response signal in orange on the left-hand side of **Figure 10** is using an approximated linear displacement function $\sin(\varphi(t)) \rightarrow \varphi(t)$ and is scaled down by factor 10^{-4}.The generated beat frequency signal is obtained using the exact harmonic displacement function.

3.1. Nonlinear single degree of freedom systems

Similar to the case in Section 2.1, also a SDoF system will be discussed, but this time a linear and a nonlinear spring will be present. The nonlinearity of this spring shall have the form shown in (3) having a nonlinear exponent $n = 2$ and $k_3 > 0$, a parameterization like that is generally used for a magnetic spring (see also top middle sketch in **Figure 1** and appendix A.1). The lumped parameter model for the examined system (**Figure 10**) consists beside this spring system with linear spring rate k_1 and nonlinear spring rate k_3 of an oscillator mass m, a viscous damping factor d and an external basepoint excited harmonic force with amplitude $y_0(t)$. System coordinate origin is placed at the basepoint excitation.

The elastic energy of this nonlinear spring system with $n = 2$ will lead to the following spring energy, see also derivation in (4).

$$E_{elastic}(y) = \frac{1}{2}k_1 y^2 + \frac{1}{4}k_3\, y^4 \tag{45}$$

Adding up all kinetic energies and all potential energies, disturbances in form of a viscous damping and a basepoint excited force is given in (46)–(49).

$$T = \frac{1}{2}\, m\, {y'}^2 \tag{46}$$

$$U = m\, g\, y + \frac{1}{2}\, k_1\, y^2 + \frac{1}{4}k_3\, y^4 \tag{47}$$

$$F_d = d\;\; y' \tag{48}$$

$$F_0 = m\frac{d^2}{dt^2}(A\;\cos\omega t) = -mA\omega^2 \cos\omega t \tag{49}$$

Applying the Lagrangian formalism (1) and (2), we deal again with a SDoF system, will lead to the resulting DE shown in (50), similar to the result derived in Section 2.1 – but here we have now introduced a nonlinear spring system.

$$m\, y'' + d\, y' + k_1 y + k_3 y^3 + mg = mA\omega^2 \cos\omega t \tag{50}$$

Introducing dimensionless notation, by using a dimensionless time τ , a dimensionless system resonance frequency Ω, the damping factor ξ_1 and the gravity offset term ϱ (51) and setting the dimensionless displacement u (52).

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$$\tau = t\,\omega_1; \Omega = \frac{\omega}{\omega_1}; {\omega_1}^2 = \frac{k_1}{m}; \beta = \frac{k_3}{k_1}\,A^2; \xi_1 = \frac{d}{2\,m\,\omega_1}; \varrho = \frac{g}{A\,{\omega_1}^2} \tag{51}$$

$$\text{path } u(\tau) = \frac{y(t)}{A} \tag{52}$$

By replacing parameters of (50) with (51), (52) we obtain (53), including the dimensionless gravity term ϱ as well.

$$u'' + 2\xi_1 u' + u + \beta u^3 + \varrho = \Omega^2 \cos\,(\Omega\tau) \tag{53}$$

DE (53) is nonlinear, as we have also a path dependent function to the power of 3 and a dimensionless factor β which is generally small. If this factor β is positive, we deal with a nonlinear spring hardening system, if β is negative, it is a spring softening system. Unfortunately, such a system cannot be examined using the Laplace or Laplace-like transformation, such as [6], as this transformation can deal only with linear functions, respectively, nonlinear quadratic functions. For solving this nonlinear so-called Duffing DE, there are several methods available, such as averaging method or the harmonic balancing method. The averaging method assumes that a solution of the DE can be obtained using harmonic functions In Ref. [7], chapter 9.3, a general solution for nonlinearity terms with a positive integer exponent βu^n is obtained using the averaging method. Another method to get analytic solutions is, as said, the harmonic balance method, which is well explained in the textbook [8], chapter 2.3.4; the DE case of (53) is discussed in the same book, chapter 4.1 and there are many research papers to discuss this nonlinear DE, see for example [9, 10]. Note that this case is of nonlinear nature, as it includes the nonlinear term βu^3, but cannot exert parametric resonance. However, there are also many research papers where such nonlinearities coupled with a Mathieu DE are discussed, see for example [11].

The depicted **Figure 11** shows simulation results of DE (53). The time domain behavior (top left) and its dimensionless phase space behavior (top right) is shown with simulating 50 periods with settled initial conditions (amplitude of $\hat{y} < 1\mu m$). The top row depicts one simulation point in the bottom row, where a frequency sweep has been done, sweeping the basepoint excitation from ω = 5…40Hz and keeping the acceleration signal constant at 0.5 g. To make sure that only non-transient amplitudes are selected to create the frequency response, only in the last 5 periods (out of 50) the maximal and minimal value is selected). The top row is using a constant angular excitation of 25 Hz and depicts only one simulation point of the generated frequency response. The main simulation parameters are shown in the heading, a variant of this simulation is given in the appendix A.3. Note the shown simulated data are taken from a validated electromagnetic SDoF vibration energy harvester system by the author.

3.2. Nonlinear two degree of freedom systems

Let us consider the lumped parameter model in **Figure 12**. A pendulum with a stiff rod of length l and mass m_2 suspended on a spring damper system with mass m_1 and stiffness k and

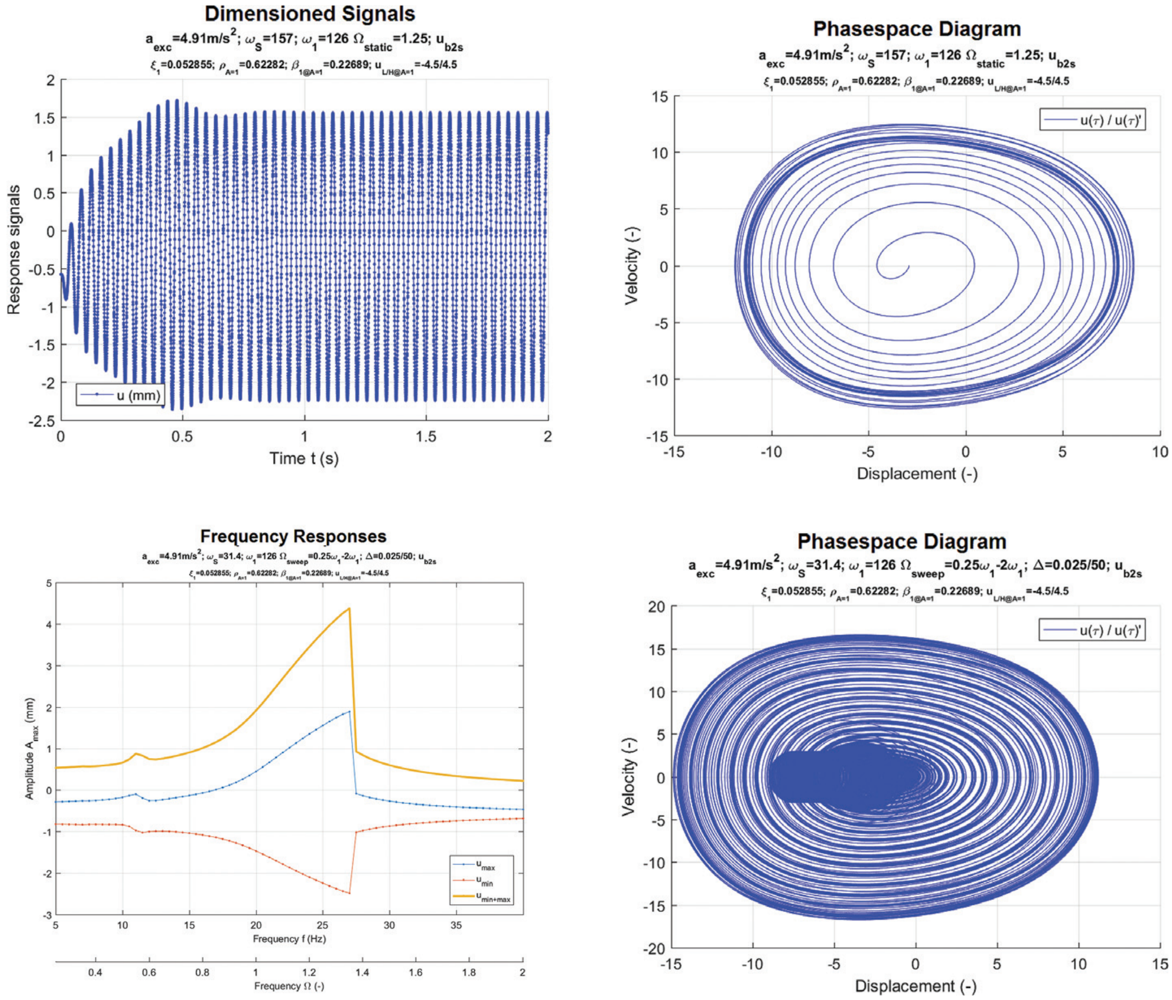

Figure 11. Numerical simulation results of the nonlinear basepoint excited SDoF system shown in (53) using a constant acceleration of 0.5 g and a basepoint excitation of ω = 25 **Hz** (top row) and its sweep behavior ω = 5…40 **Hz** (bottom row).

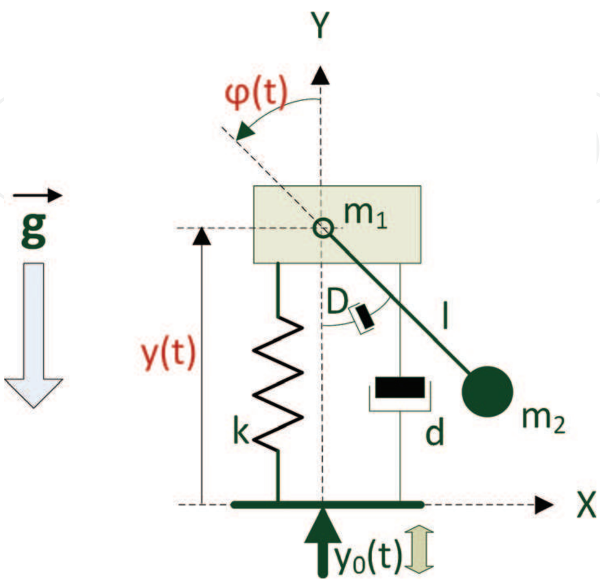

Figure 12. Nonlinear two degree of freedom (2DoF) spring mass damper model.

damping factor d. Mass m_1 can move only in the depicted y direction and pendulum only in the X-Y plane.

Governing equations are derived using again the Lagrange formalism. Considering the frame of reference at the origin shown in **Figure 12** and defining in Cartesian coordinates first the two degrees of freedom vector $\boldsymbol{r_y}$ and $\boldsymbol{r_\varphi}$ (54).

$$\boldsymbol{r_y} = \begin{pmatrix} 0 \\ y(t) \end{pmatrix} \text{ and } \boldsymbol{r_\varphi} = \begin{pmatrix} l \sin \varphi(t) \\ y(t) - l \cos \varphi(t) \end{pmatrix} \tag{54}$$

The kinetic energy for both degrees of freedom are shown in (55, 56).

$$T_y = \frac{1}{2} m_1 \left(\left(\frac{d}{dt} \boldsymbol{r}_{y.x} \right)^2 + \left(\boldsymbol{\frac{d}{dt}} \boldsymbol{r}_{y.y} \right)^2 \right) = \frac{1}{2} m_1 y\prime^2 \tag{55}$$

$$T_\varphi = \frac{1}{2} m_2 \left(\left(\frac{d}{dt} \boldsymbol{r}_{\varphi.x} \right)^2 + \left(\boldsymbol{\frac{d}{dt}} \boldsymbol{r}_{\varphi.y} \right)^2 \right) = \frac{1}{2} m_2 l^2 \cos(\varphi)^2 \varphi'^2 + \frac{1}{2} m_2 (y' + l \sin \varphi \, \varphi')^2 \tag{56}$$

The potential energies derived from the same vectors lead to (57) and (58).

$$U_y = m_1 g \, y + \frac{1}{2} k \, y^2 \tag{57}$$

$$U_\varphi = m_2 g \, (y - l \cos \varphi) \tag{58}$$

The Lagrange energy function L becomes:

$$L = T_y + T_\varphi - (U_y + U_\varphi) \tag{59}$$

The viscous friction for both degree of freedoms is given in (60) and the basepoint excited driving force is given in (61).

$$F_{dy} = d \, y' \text{ and } T_{d\varphi} = D \, \varphi' \tag{60}$$

$$F_0 = -(m_1 + m_2) \frac{d^2}{dt^2} (A \cos \omega t) = (m_1 + m_2) A \omega^2 \cos \omega t \tag{61}$$

Applying the Lagrange formalism (2) for both degrees $q_1 = y$ and $q_2 = \varphi$ lead to the coupled DE system of (62), (63), representing a force DE respectively a torque DE. On the right-hand side of those DE's, the defined viscous frictions of (60) and the basepoint excitation (61) is present.

$$(m_1 + m_2) y'' + g(m_1 + m_2) + ky + l m_2 \left(\varphi'^2 \cos \varphi + \varphi'' \sin \varphi \right) = -d \, y' + F_0 \tag{62}$$

$$m_2 l^2 \varphi'' + g l m_2 \sin \varphi + l m_2 y'' \sin \varphi = -D \, \varphi' \tag{63}$$

The parameters for non dimensionalization are given in (64, (65). Note that the reference system frequency ω_1 is set to the 1. DoF (also called primary, the mass spring system) and the second system frequency ω_2 to the 2. DoF (the pendulum – also called secondary system). The excitation frequency is associated to ω.

$$\tau = t\,\omega_1;\, \lambda_m = \frac{m_2}{m_1+m_2} = \frac{m_2}{m};\, \Omega = \frac{\omega}{\omega_1};\, {\omega_1}^2 = \frac{k}{m};\, {\omega_2}^2 = \frac{g}{l};\, {\Omega_0}^2 = \frac{{\omega_2}^2}{{\omega_1}^2};\, \lambda_l = \frac{l}{A};$$

$$\xi_1 = \frac{d}{2\,m_1\,\omega_1};\, \xi_2 = \frac{D}{2\,m_2\,l^2\,\omega_2};\, \varrho = \frac{g}{A\,{\omega_1}^2} \tag{64}$$

$$\text{path } u(\tau) = \frac{y(t)}{A} \text{ and angle } \theta(\tau) = \frac{\varphi(t)}{\varphi_0} \tag{65}$$

$$u^{''} + 2\xi_1 u' + u + \lambda_m \lambda_l \left(\theta'^2 \cos\,\theta + \theta^{''} \sin\,\theta\right) + \varrho = \Omega^2 \cos\,(\Omega\tau) \tag{66}$$

$$\theta^{''} + 2\xi_2 \Omega_0\,\theta' + \sin\,\theta\left({\Omega_0}^2 + {\lambda_l}^{-1}\,u^{''}\right) = 0 \tag{67}$$

Figure 13 depicts left the excitation (magenta) and the time response signals of y (blue) and φ (red) and its phase space behavior on the right-hand side.The parameters are chosen in such a way,that the pendulum starts rotate.The excitation primary system has a resonance ratio of $\omega : \omega_1 = 1 : 1$ and secondary primary frequency ratio is $\omega_2 : \omega_1 = 2 : 1$. The stability of such a pendulum is described in chapter 4.4 of [7], where so called semi trivial and nontrivial solutions for this system are discussed.

A treaty of such a system, a kinetic energy harvesting device, with additionally a nonlinear spring system on the primary and an electromagnetic harvester on the secondary system is given in Ref. [12]. Note that the derivation of the system equations there have been made without the Lagrangian formalism and the found system equations are equivalent.

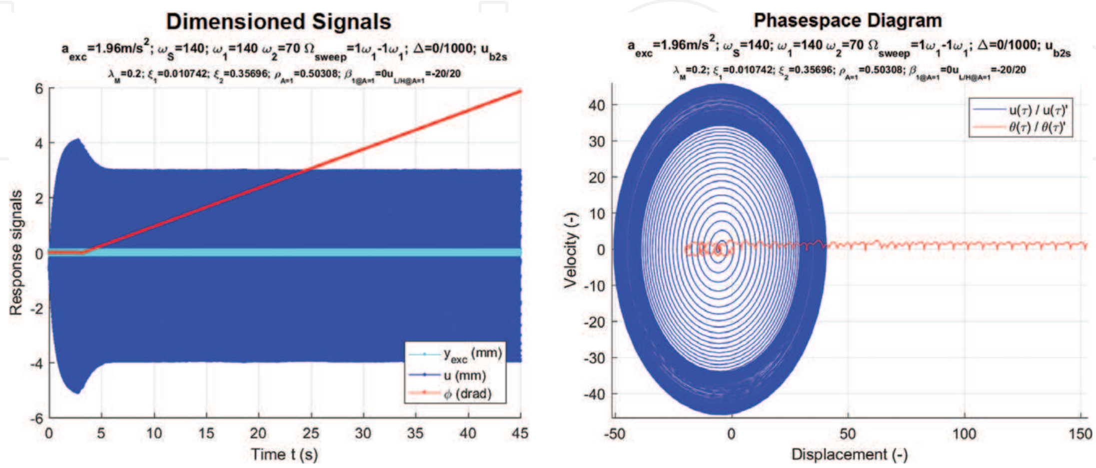

Figure 13. Numerical simulation results of the nonlinear basepoint excited 2DoF system shown in (66, 67) using a constant acceleration of 0.2 g and a constant basepoint excitation of f_{exc} = 22.22 Hz.

4. Conclusions

In the introduction, we showed the equivalence of rotary and translatory mechanical systems as well as the equivalence of mechanical and electrical resonance systems. Also, a brief introduction to the Lagrangian formalism is given. In preparation to nonlinear resonance systems, also rotational and translational springs are discussed. Three classes of spring systems have been identified: linear springs nonlinear symmetric springs and nonlinear asymmetric springs. Throughout the chapter further readings are proposed.

In Section 2, linear resonance systems with one and two degrees of freedom have been investigated using basepoint excited systems. Using the Laplace Transformation is most useful to analyze any linear resonance system with a periodic excitation.

Section 3 deals with nonlinear resonance systems. When in such a dynamical system one of the resulting DE's is of Mathieu or Hill type, the response amplitude of such a system might grow exponentially. This is exemplary demonstrated in Section 3.2 identifying the system differential equations of a basepoint excited two degree of freedom system. Some dynamic properties of such a system is demonstrated.

A. Appendix

A.1. Nonlinear symmetric spring systems

Using instead of disk magnets ring magnets, strong nonlinearities can be generated. The following series in **Figure A1** depicts a few simulation cases. Some of shown simulation cases have been validated and proven experimentally.

A.2. Variant of linear 2DoF system

Instead of using linear springs, magnetic nonlinear springs can be used (see also a selection of such spring characteristics in A.1). Using nonlinear springs and making the system nonlinear (instead of having only a linear spring term, we have for each spring also a term of the form $\beta_1\, u^3$ and $\beta_2\, v^3$). The relative response signals of such a system is depicted in **Figure A2**.

It is interesting, that the relative motion of the 2. DoF is responding with resonance between 19…25.5 Hz.The first degree of freedom has a nonlinear spring hardening behavior, reaching A_{pp} = 7.9 mm at 25.5 Hz.

A.3. Variant of nonlinear SDoF system

A created tool by the author in Matlab/Simulink has been used to simulate many basepoint excited SDoF or nDoF systems with rotational or translational or mixed structures.

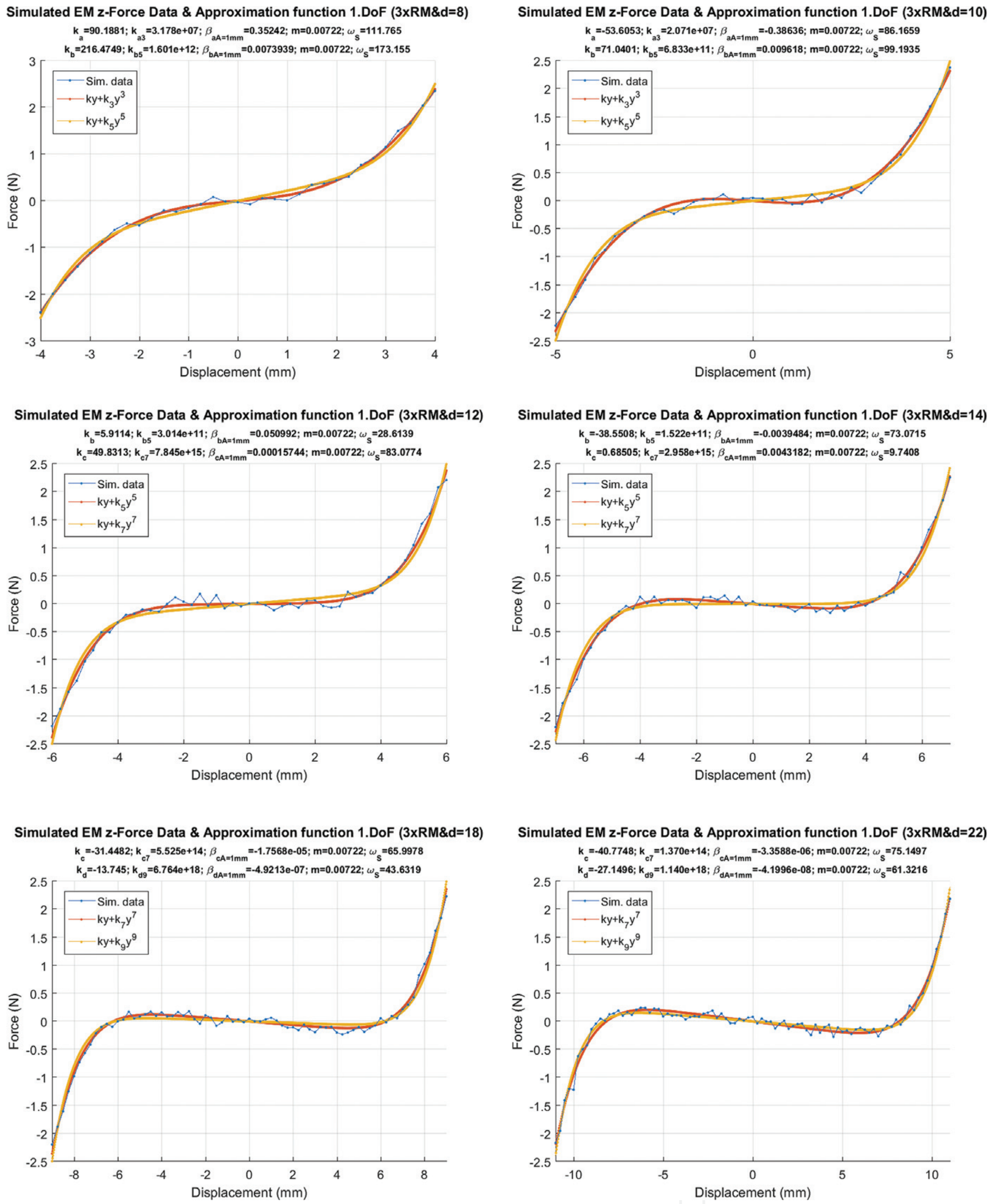

Figure A1. Spring force behavior of ring magnets using different distances of non-movable magnets.

It allows to simulate such systems with constant amplitude or constant acceleration, can handle hard or soft-impact of the oscillating proof mass(es). In addition, one sided spring characteristics can be simulated, see also **Figure A3** – a feature that is especially interesting in relation with magnetic springs. Main disadvantage of such one-sided bound springs is the fact, that they need to be installed upright. The behavior of such a one-sided magnetic spring is depicted in **Figure A3**. It has a frequency response similar to a softening spring. The maximal

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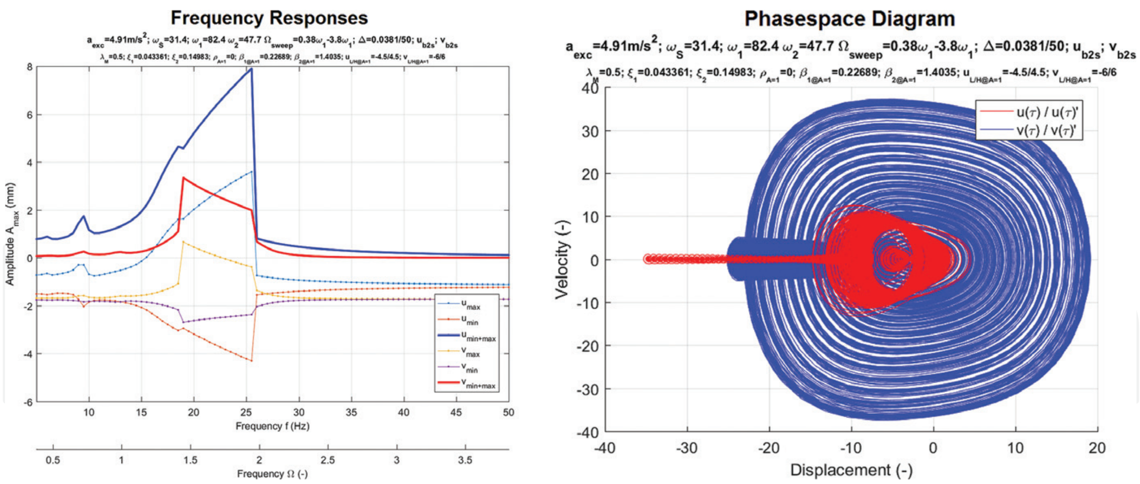

Figure A2. Response signals of a 2DoF system using lumped parameter model in **Figure 5**: Instead of having linear springs, also nonlinear springs are present.

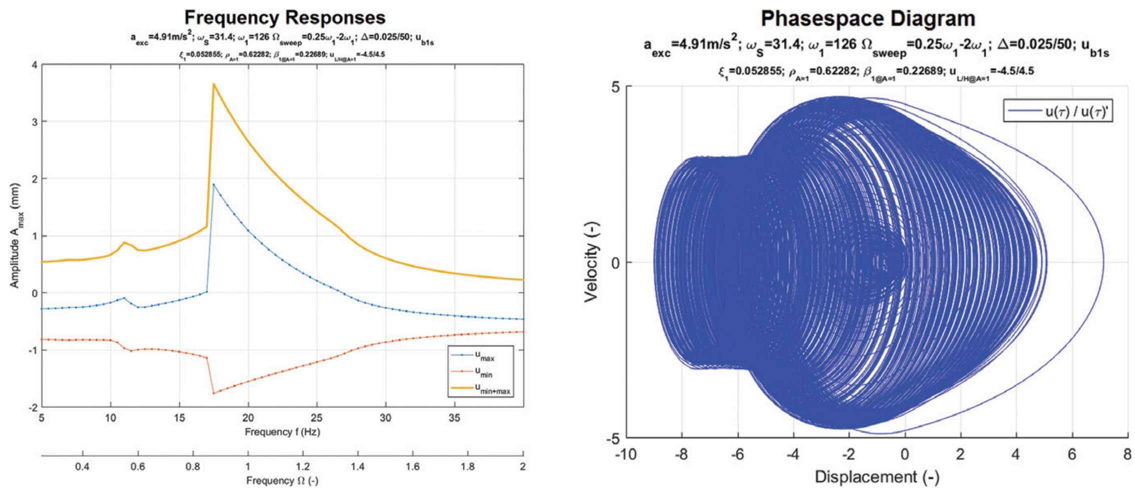

Figure A3. Response signals of a one-sided bound magnetic spring.

amplitude of 3.65 mm occurs at 17.5 Hz (the two-sided classical hardening magnetic spring reaches an amplitude maximum of 4.3 mm at 27 Hz). Such a spring system could also be analytically described, by introducing for example continuous piecewise functions.

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