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Stochastic Resonance and Related Topics

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Abstract

The stochastic resonance (SR) is the phenomenon which can emerge in nonlinear dynamic systems. In general, it is related with a bistable nonlinear system of Duffing type under additive excitation combining deterministic periodic force and Gaussian white noise. It manifests as a stable quasiperiodic interwell hopping between both stable states with a small random perturbation. Classical definition and basic features of SR are regarded. The most important methods of investigation outlined are: analytical, semi-analytical, and numerical procedures of governing physical systems or relevant Fokker-Planck equation. Stochastic simulation is mentioned and experimental way of results verification is recommended. Some areas in Engineering Dynamics related with SR are presented together with a particular demonstration observed in the aeroelastic stability. Interaction of stationary and quasiperiodic parts of the response is discussed. Some nonconventional definitions are outlined concerning alternative operators and driving processes are highlighted. The chapter shows a large potential of specific basic, applied and industrial research in SR. This strategy enables to formulate new ideas for both development of nonconventional measures for vibration damping and employment of SR in branches, where it represents an operating mode of the system itself. Weaknesses and empty areas where the research effort of SR should be oriented are indicated.

Keywords: stochastic resonance, post-critical processes, dynamic stability, Fokker-Planck equation, Galerkin approach

1. Introduction

The stochastic resonance (SR) is a phenomenon, which can be observed at certain nonlinear dynamic systems under combined excitation including mostly deterministic periodic force and random noise. The phenomenon of this type has been first observed and reported by Kramers,

see [1], investigating the interwell hopping in the Brownian motion. Some allusions can also be found in older resources devoted to stochastic processes and theory of stability (Lyapunov, Kolmogorov, Planck, and others).

The genuine phenomenon of SR has been discovered in early 1980s. The initiation point was probably two papers by Nicolis [2, 3] dealing with problems of climatic evolution. Other scientific and application areas followed that inspiration in due time, since it came to light that SR is a generic phenomenon. The idea of SR initiated remarkable cross-disciplinary interest bringing together nonlinear dynamics, statistical physics, information and communication theories, data analysis, life and medical sciences. Individual areas came to the use of SR phenomenon rather independently, and therefore, they introduced slightly different definitions and particular strategies in the first period. This transition time passed and many cross disciplines overlapping in their activities have been built at the unifying background developed by mathematics and theoretical physics. Despite this evolution, the historical aspects are still visible, due to fact that every branch still focuses on different needs, working in different scale and parameter intervals.

The term stochastic resonance was introduced probably in 1981 in informatics to describe the annoying noise in contemporary communication equipment that prevented to detect the weak useful signal. However, researchers recognized soon that under certain conditions, the noise can be helpful to enhance the device sensitivity.

The opportunity to employ SR in mechanics emerged only recently. SR appeared to be promising for modeling of certain post-critical effects in nonlinear dynamics, active vibration damping, feedback systems, biomechanics, etc. Therefore, it is worthy of presenting a certain overview to the community of rational and applied dynamics concerning strengths, weaknesses, and application possibilities of SR occurred in theoretical and applied disciplines.

The phenomenon itself manifests in the simplest case by a stable periodic hopping between two nearly constant limits perturbed by random noises. The occurrence of this phenomenon depends on certain combinations of input parameters, which can be determined theoretically and verified experimentally. The classical mathematical definition of SR follows from properties of the Duffing equation with the negative linear part of the stiffness (bistable system) under excitation by a Gaussian white noise together with a deterministic harmonic force with a fixed frequency. It should be highlighted that also more general definitions of SR exist and will be also briefly reported in this chapter. In particular, it considers various types of the random noise, shapes of the deterministic excitation component, types of oscillator nonlinearity (potential of internal forces), and finally also number of stable positions, which can exceed two or drop to one.

In terms of classical Engineering Dynamics, SR can be assumed as a dangerous effect accompanying a post-critical system response. Therefore, it should be eliminated by appropriate selection of parameters and operating conditions (plasma physics, aeroelasticity, rotating machines, etc.) in order to ensure the reliability of the system. On the other hand, SR can characterize the mode of a natural system we are observing, and therefore, it serves as a tool of its investigation (e.g., Brownian motion mentioned above). It can also represent an intentional operating mode of the

artificial system, and therefore, it should be considered as a useful state (special excitation or vibration damping devices, energy harvesting, etc.).

Nevertheless, many disciplines predominantly consider SR as a mechanism by which a system embedded in a noisy environment acquires an enhanced sensitivity toward small external signal, when the noise intensity reaches certain finite level. This phenomenon of boosting undetectable signals by resonating with added noise extends to many other systems, whether electromagnetic, physical, or biological, and is an area of intense research. This interpretation of SR shows that noise can play a positive role in systems either designed artificially or observed as a natural systems. Furthermore, SR and its variants can serve to understand many processes in various scales and temperature domains to understand various effects in solid state physics, biophysics, and electronics with possible application to design SR-inspired devices.

The study tries to mimic some excellent review studies published mainly in the areas of physics, informatics, and physiology with emphasis on Engineering Dynamics. See, for instance, papers [4–10], etc. Although their style is quite different, adequately with the branch they represent, they are full of valuable information and worthy to be studied. For reading are recommended problem-oriented monographs, e.g., [11, 12] or books including SR-devoted chapters, e.g., [13–15]. Additional information can be found also at numerous web sites, like popular Wikipedia, Scholarpedia, American Physical Society Sites, Encyclopedia of Maths, or Mathworks, see [16]. Doubtlessly, the largest source of primary information are leading journals edited by world societies of physics, electronics, informatics, and neurosciences. Moreover, lot of conference proceedings are available as well organized, e.g., by IEEE, APS, AIP, SIAM, or OSA.

Apart from this introductory remarks, the chapter consists of six sections (2–6). They have general or specific character oriented to particular disciplines. Section 2 introduces some overview of classical SR definitions, solution methods, and ways of its quantification. The following Section 3 estimates a possible future SR position in mechanics accompanied by a digest of a particular study performed in area of aeroelastic stability. Section 4 is devoted to SR-assisted energy harvesting as a discipline being very close to mechanics and having many joint features with that. Section 5 is unavoidably included for historical reasons dealing with climatology, where the modern SR appeared in the contemporary meaning of the term in early 1980. It gave an inspiration for all other branches, which are commonly discussed. Section 6 pays attention to nonconventional SR definitions dealing with alternative differential operators providing for instance, a possibility to abandon the bistable interwell hopping and to build SR on a monostable system. The use of nonGaussian driving noise is mentioned as well.

Concluding part No. 7 attempts to evaluate position of SR strategy and its strengths and weaknesses. With respect to the area of potential readers, it concentrates to a possible SR involvement in Engineering Dynamics. It means to eliminate dangerous SR-based phenomena occurring in industrial aerodynamics, dynamics of vehicles, and in whatever system endangered by dynamic stability loss and subsequent post-critical emergency regime. In the same time, SR can become the basis for the development of active equipment for vibration damping, earthquake resistance improvement, vehicle stabilization, etc. Let us take a note that SR phenomenon

appears in many additional disciplines of theoretical and applied physics, data mining, chemistry, neurophysiology, pattern recognition, etc., where many inherent extensions beyond the classical definition of SR have been developed and used. For more information, see review papers, e.g., [4, 17], where extensions into quantum stochastic resonance with specific applications are outlined.

2. Classical definition of stochastic resonance

In early 1980s, SR has been discovered as a generic phenomenon and the first classical definition has been introduced. Some modifications appeared in due time, but the basic version is still alive serving as the basis of SR mathematical modeling. There is a lot of resources reporting about SR from the viewpoint of the definition in a rigorous or loose interpretation, see for example well-known overview article [4] by Gammaitoni et al. and also review paper [17] by authors of this chapter. Note that although vast majority of cases use the classical definition, a number of problems need the special definition of the SR phenomenon regarding its basic philosophy or individual components. Such settings extending the classical definition will be briefly outlined in Section 6.

2.1. Phenomenon of stochastic resonance

In classical meaning, SR occurs in bistable systems with single degree of freedom (SDOF), when a small periodic force is applied together with a large broad band random noise, see **Figure 1**. The system response is driven by two excitation components resulting in a “system switch” between two stable states. Their positions are given by two wells of the system potential $V(u)$. Wells are separated by a barrier. Its height decisive for the switching is considered as a difference between maximum and minimum of the potential, see **Figure 1**.

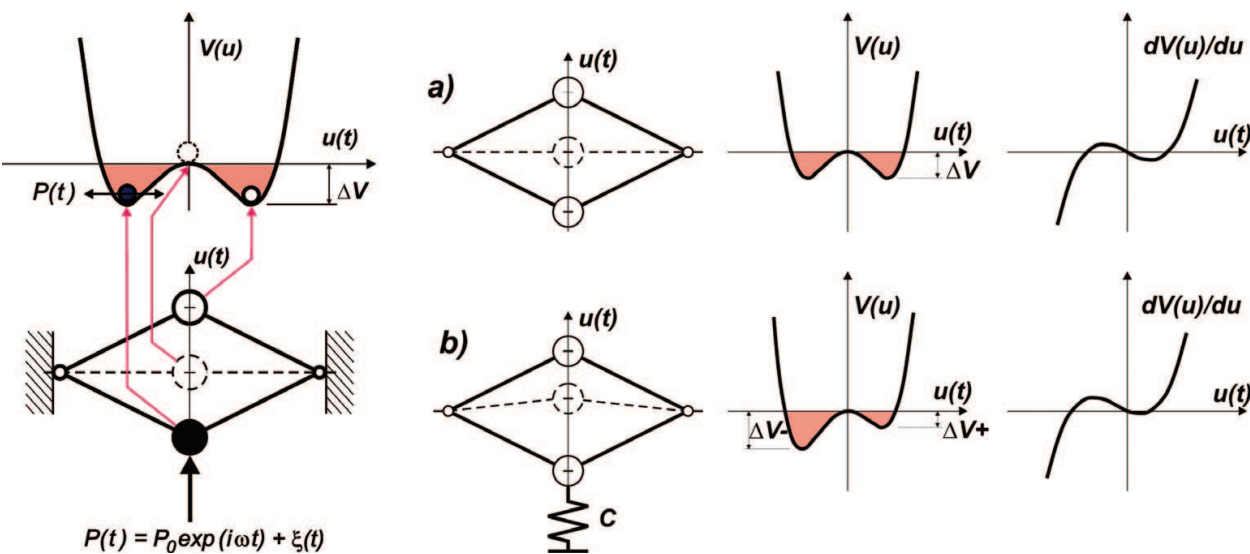


Figure 1. Bistable nonlinear system: (a) Symmetric potential; (b) Nonsymmetric potential.

In the absence of periodic forcing, the approximate frequency of escape from one well into the second is given by the following estimate published in the comprehensive study [1]:

$$\omega_e = \sqrt{2} \cdot \exp(-\Delta V/\sigma^2) \quad (1)$$

where σ^2 is the variance of the noise, and ΔV means the barrier separating potential minima (symmetric potential), see **Figure 1a**. For nonsymmetric potential, the symbols $\Delta V_-(u), \Delta V_+(u)$ in **Figure 1b** denote the left and right minima, respectively. In classical setting of SR, the Gaussian white noise is taken into account (for a couple of other variants, see Section 6).

If both component are acting, then the degree of switching is related with the noise intensity σ^2 , see a sample response in **Figure 2**. When the periodic force is small enough being unable to make the system response switch, the presence of a nonnegligible random component is required for it to happen. When the noise is small (small variance σ^2) very few switches occur, mainly at random with no significant periodicity in the system response—**Figure 2(a)**. When the noise is too strong, a large number of switches occur for each period of the periodic component, and the system response does not show remarkable periodicity—**Figure 2(c)**. Between these two conditions, there exists an optimal value of the noise intensity σ_0^2 that cooperatively concurs with the periodic forcing in order to make almost exactly one switch per period (a maximum in the signal-to-noise ratio)—**Figure 2(b)**. Amplitude of the response alternating component as a function of the noise level is outlined in **Figure 3**. Peakness of the maximum is given by the damping factor. If the damping is too high, the peak can completely disappear and SR vanishes.

The optimum of the noise level σ_0^2 is quantitatively determined by matching of two time scales:

- i. the period of the sinusoid (the deterministic time scale); and
- ii. the Kramers rate, Eq. (1)—average switch rate induced by the sole noise, which is the inverse of the stochastic time scale. It implicates the denomination “stochastic resonance”.

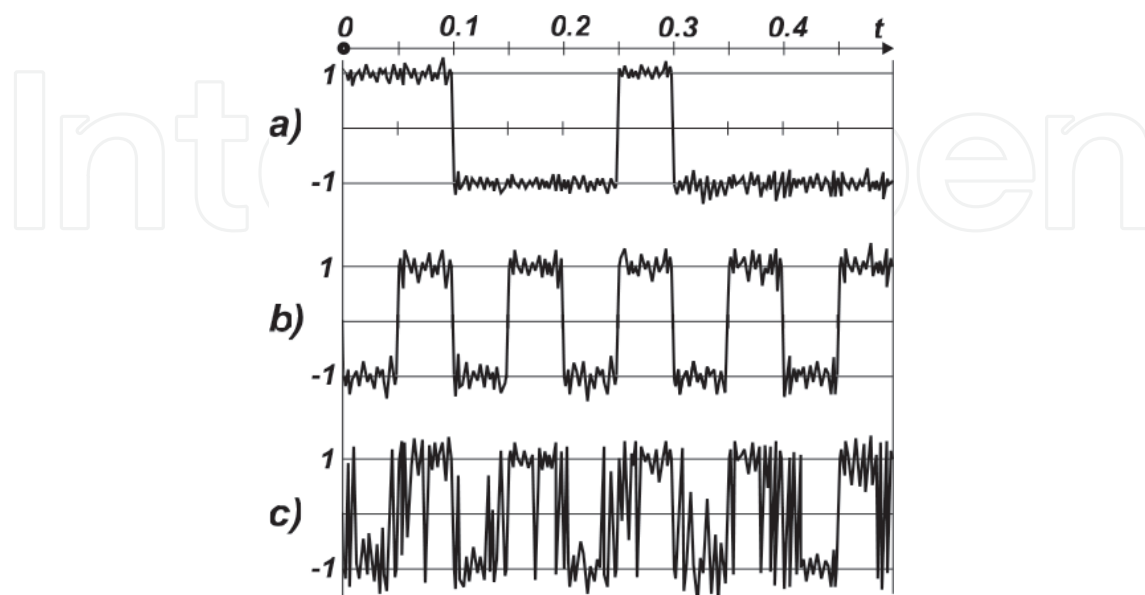


Figure 2. Time history of the system response for various noise variance: (a) low level; (b) optimal level σ_0^2 ; (c) high level.

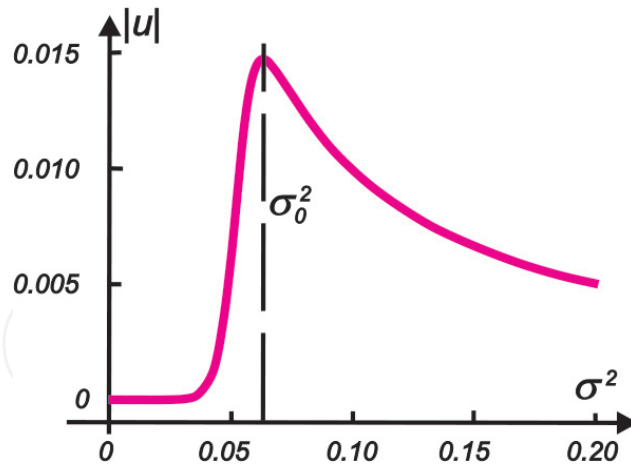


Figure 3. Amplitude of the system response alternating component due to simultaneous excitation by a weak periodic force and a random noise.

The Kramers formula, Eq. (1), is a result of theoretical and empirical investigation motivated by problems of nonlinear optics. Note that, in original resources, the absolute temperature T instead of the variance σ^2 is considered. The formula Eq. (1) is widely used and works very well. During the last decades, a number of areas of optics, quantum mechanics, chemistry, neurophysiology, etc., investigated this formula attempting to use the phenomenon of SR for the description of various effects arising in their branches using both experimental and theoretical ways of investigation, see, e.g., [18, 19].

The mathematical basis of the classical SR definition is related to the Duffing equation with negative linear part of the stiffness (in terms of mechanics). It is the most simple variant and it corresponds together with Gaussian white noise and deterministic harmonic force with a fixed frequency to the classical setting of SR. This configuration will be treated mostly throughout this chapter. Nevertheless, some generalizations and extensions beyond the classical formulation will be introduced in section 6 and furthermore at other remarked places.

Let us assume the nonlinear mass-unity SDOF oscillator written in a normal form:

$$\dot{u} = v; \quad \dot{v} = -2\omega_b \cdot v - V'(u) + P(t) + \xi(t). \quad (2)$$

$V(u)$ —potential commonly introduced in a form providing the Duffing equation:

$$V(u) = -\frac{\omega_0^2}{2}u^2 + \frac{\gamma^4}{4}u^4 \quad \Rightarrow \quad V'(u) = dV(u)/du = -\omega_0^2 \cdot u + \gamma^4 \cdot u^3 \quad (3)$$

$\xi(t)$ —Gaussian white noise of intensity $2\sigma^2$ respecting conditions:

$$\mathcal{E}\{\xi(t)\} = 0, \quad \mathcal{E}\{\xi(t)\xi(t')\} = 2\sigma^2 \cdot \delta(t - t'), \quad (4)$$

$\mathcal{E}\{\bullet\}$, $\delta(t)$ —operator of the mathematical mean value in Gaussian meaning and Dirac function, respectively,

$P(t) = P_o \exp(i\Omega t)$ —external harmonic force with frequency Ω . Amplitude P_o should be understood per unit mass.

Symbols ω_0 and ω_b have a usual meaning of the circular eigenfrequency and circular damping frequency of the associated linear system. The linear part of the $V'(u)$ is negatively making the system metastable in the origin, while the cubic part acts as stabilizing factor beyond a certain interval of displacement u . The system is drafted in **Figure 1** in two versions: (a) system with symmetric potential typical by an equivalent energy needed for hopping from the left into the right potential well and backwards; (b) system with asymmetric potential due to the supplementary linear string, which could be able (when rising its stiffness) to bring the oscillator to monostable state, see Section 6.1, where we will see that also the monostable system under certain circumstances is able to exhibit SR phenomenon.

2.2. Methods of stochastic resonance investigation

Theoretical approaches, either analytical or numerical, are mostly based on an assumption that random processes ruling inside the investigated system are of the Markov type. The primary requirement, namely the dependence of the process on its value only in one previous moment, is usually accomplished. In such a case, a large variety of methods are applicable for the investigation of SR phenomena.

Basically three type of solution procedures can be regarded:

(i) *Fokker-Planck (FP) equation*. It is the equation for cross probability density function (PDF) of the system response. Solution of this equation serves subsequently for the evaluation of various stochastic parameters like mean value, stochastic moments of adequate order, auto and cross correlation functions, probability flow, signal to noise ratio, mutual information etc. Concerning SR itself, the main indicators and parameters of this phenomenon can be evaluated and discussed in relation with the physical character of the problem, see subsection 2.3. So that, PDF is a certain “source function” to obtain all information needed.

Taking into account that random noise in the governing physical differential system, Eq. (2), has an additive character, no Wong-Zakai correction terms emerge, see, e.g., [20–22]. Then, the relevant FP equation, e.g., [23], can be easily written out:

$$\frac{\partial p(u, v, t)}{\partial t} = -\kappa_u \frac{\partial p(u, v, t)}{\partial u} + \frac{\partial}{\partial v} (\kappa_v p(u, v, t)) + \frac{1}{2} \kappa_{vv} \frac{\partial^2 p(u, v, t)}{\partial v^2}, \quad (5)$$

$$\begin{aligned} \kappa_u, \kappa_v &- \text{are drift coefficients: } \kappa_u = v; \quad \kappa_v = \kappa_v(t) = -2\omega_b \cdot v - V'(u) + P(t), \\ \kappa_{vv} &- \text{is a diffusion coefficient: } \kappa_{vv} = 2\sigma^2, \end{aligned} \quad (6)$$

together with boundary and initial conditions:

$$\lim_{u, v \rightarrow \pm\infty} p(u, v, t) = 0(a), \quad p(u, v, 0) = \delta(u, v)(b). \quad (7)$$

Solution of the above FP equation can be conducted using one of the following procedures:

(i-a) *Variational solution of Galerkin type.* In principle, it is a procedure of decomposition into stochastic moments (or cumulants) with Gaussian closure, e.g., [24]. The demonstration of this procedure is presented in subsection 3.2, where an application to stability analysis of the TDOF aeroelastic system is roughly outlined.

In general, for details of the Galerkin method on the basis of functional analysis rules, see, e.g., [25]. For details of particular solution, see [26–28], and other papers and monographs. The method is suitable namely for stationary solutions, but quasiperiodic solutions can be investigated as well, see, e.g., [29], where detailed procedure outlined above is presented.

(i-b) *Generalized Fourier method.* Decomposition into a series following eigen functions and values of FP operator.

$$p(u, v, t) = p_o(u, v) \cdot \varphi(t) \quad \Rightarrow \quad p(u, v, t) = \sum_{j=0}^N p_j(u, v) \cdot \varphi_j(t) \quad (8)$$

The series Eq. (8) can be substituted into the FPE Eq. (5). Due to the independency of $p_j(u, v)$ or $\varphi_j(t)$ on time or space variables, respectively, the part dependent on time only can be separated on the left side and that dependent on space variables on the right side. They can be equivalent only if both of them equal the same constant λ_j for each part of the series. It can be shown that λ_j are eigen values of the FP operator part, which is on the right side of Eq. (5). Subsequently, $p_j(u, v)$ are relevant eigen functions of this operator and finally $\varphi_j(t)$ are the simple exponential functions with the negative real part. Take a note that the $\lambda_0 = 0$, as the first part of the series Eq. (8) for $j = 0$ represents the stationary part of the FPE solution, provided the stationary part exists. In general, the occurrence of one or more positive real parts of λ_j can reveal positive, which would indicate an instable solution of FPE. However, it is not the case when investigating FPE used for modeling the SR phenomenon.

This approach is applicable rather in special cases with easy searching of eigen functions, when transition process is looked for. For example, see [30]. In general, searching for eigen functions of FP operator is a complex task, and it can prevent application of this method when more than SDOF system is analyzed.

(i-c) *Floquet theory.* Application of the Floquet theorem:

$$p(u, v, t) = p(u, v, t + T) \quad (9)$$

Suitable for equations with periodically variable coefficients, when transition nonperiodic process is investigated. See [30].

(i-d) *Finite element method (FEM) and other numerical procedures.* The FEM can be considered as a general numerical solution method of partial differential equation. It can be proved that FEM is well applicable for this purpose under certain circumstance, which are fulfilled regarding FPE. When constructing adequate elements, a care should be taken due to special properties of the FP operator. Significant problem originates from the fact of multi-dimensionality of space we are working with and a delicate character of initial conditions. Moreover, the non-self-adjointness

of the FP operator, special configuration of boundary conditions, etc., should be taken into account. These factors shift application of FEM in this case into a special area where a number of nonconventional problems should be solved.

The FPE is analyzed in an original evolutionary form which enables an analysis of transition effects starting the (nearly) Dirac type initial conditions. The FEM efficiency when solving FPE, which follows from the Duffing stochastic differential equation without external harmonic forces was already studied by the authors in [31]. With the periodic force taken into account, certain difficulties arise due to the time inhomogeneity of the corresponding stochastic process. Many results regarding FEM application on FP equation analysis can be found in [32] or [33]. For the most recent results concerning FEM application to SR problem, see [31], and additional details together with demonstrating examples, see [34].

The method is based on the approximation solution of Eq. (5) in the Galerkin-Petrov meaning on the piecewise smoothly bounded domain $\Psi \in u \times v$, in \mathbb{R}^d , $d = 2$. The initial conditions at $t = 0$ s for PDF are considered in a form of the Gauss distribution function with an initial system position at the point $u_0 = 0$, $v_0 = 0$. For a small value of standard deviation, it approaches the Dirac function as it is primarily requested.

After a spatial discretization of Ψ onto the rectangular finite elements using the bilinear approximation functions and implying boundary condition $p(\partial\Psi, t) = 0$, the system of ordinary differential equations emerges with global matrices \mathbf{M} , $\mathbf{S}(t)$ and vector of probability density values $\mathbf{P}(t)$ in nodes of the mesh.

Final differential system has the form as follows:

$$\mathbf{M} \cdot \dot{\mathbf{P}}(t) = \mathbf{S}(t) \cdot \mathbf{P}(t) \quad (10)$$

The matrix $\mathbf{S}(t)$ is time-dependent due to the periodic perturbation entering the drift term of FPE, and in the result, the solution oscillates periodically between the potential wells. In the regime of SR, the switchings are in phase with the external periodic signal $\mathbf{P}(t)$ and the mean residence time is closest to half the signal period $2\pi/\Omega$. Comparing the results obtained by means of FME with those following from the analytical investigation outlined above shows a good compatibility.

The efficiency of FEM is obvious as usual. It enables to investigate details, which are inaccessible using other methods. It applies especially to transition processes starting the excitation and response processes nearby the stability loss, when the Lyapunov exponent is floating around zero and boundary between local and global stability are ambiguous.

(ii) *Stochastic simulation—digital and analog.* Stochastic simulation is one of the most important methods of SR investigation. The basic idea is straightforward, the governing system Eq. (2) is subdued to numerical integration and subsequently probabilistic parameters including PDF are evaluated. However, extreme caution should be taken, as the differential system is stochastic. Because the system Eq. (2) includes only an additive noise, no Wong-Zakai correction terms are necessary, see [20–22]. However, the strategy of integration should be carefully controlled [35, 36], due to fact that we manipulate with the Ito system. In principle, the time

increment can be neither too long in order to prevent information loss, nor too short to keep the stochastic character of the output. Hence, the care should be taken during manipulations in the corrector phase of one step.

Results obtained in this manner are very important. They serve as a verification of semi-analytical results obtained using one of the procedures mentioned in the previous paragraph (i), and furthermore, the simulation is able to enter into small details, which remain hidden to methods mentioned in (i). It applies particularly to transition process if there is a need of their investigation. On the other hand, like every fully numerical method or simulation, it provides result for one set of parameters only. Like in experiments, it is difficult and laborious to obtain a broader overview.

Analog simulations have been very popular in the past wherever nonlinear differential equations were to be solved. However, they are still very attractive for researchers as they lie at the frontier between digital simulation and experiment. Their advantage is that the parameters can be easily and quickly tuned over a wide range of values and the response can be followed straightforwardly. Many review and technical papers have been published as for instance [37, 38], where the comparison of analog simulation of stochastic resonance with adiabatic theory has been performed. It should be appreciated now that a genuine analog simulation can be effectively emulated at digital computers using commercial software packages, see for instance McSimAPN package, visit <<http://www.edn.com>>. Moreover, actually whatever hybrid analysis enabling digital support of the analog simulation is possible.

(iii) *Experimental measurements.* SR has been observed in a wide variety of experiments involving electronic circuits, chemical reactions, semiconductor devices, nonlinear optical systems, magnetic systems, and superconducting quantum interference devices (SQUID). The general instruction for experimental procedures can be hardly recommended. They are always developed individually respecting specific character of every research activity. Anyway, be aware that many experiments do not serve for validation of theoretical results. Indeed, the strategy is often opposite. The purpose of the experiment is an initial recognition of the basic principle while the theoretical approach should verify subsequently its validity. It is very frequently observed particularly in neurophysiological experiments related with SR, see monograph [12] and papers [39–43] and others. Three popular examples of this type performed should be named: the mechanoreceptor cells of crayfish, the sensory hair cells of cricket, human visual perception. Another “inverse” experiments (preceding any theoretical modeling) can be seen in a wind tunnel. Here, the divergence instability of the prismatic bar in a cross flow has been observed in the view of SR without any previous theoretical background. A number of primary experimental studies are available also in plasma physics, optics, and in other branches, e.g., [44–46].

2.3. Quantification of stochastic resonance

Occurrence of SR is obviously indicated by periodic transition across the potential barrier which is synchronized in the mean with periodicity of the deterministic excitation component. The frequency should be close to that given by Kramers formula, Eq. (1). The phenomenon emerges markedly, when introducing the optimal noise amount under adequate damping

level, as it corresponds to **Figures 2** and **3**, otherwise the response is very small. This rather empirical identification is validated by theoretical means outlined above.

Internal character of the signal provided by SR can be inspected in particular cases using some useful parameters and functions:

(i) *Signal to noise ratio*. Very frequently used indicator. It is based on the power spectral density (PSD) attributes of the signal $u(t)$. A couple of variants can be found in literature, see, e.g., [21, 23], etc. Usually the ratio of PSD concerning the periodic signal being proportional to the integral of the Dirac function taken in a small neighborhood $\Delta\omega$ of its frequency $\pm \Omega$ and the total (PSD) integral at the same interval is considered. Symbolically expressed:

$$SNR(\omega) = PSD(\omega)/S_N, \quad S_N - \text{output background noise} \quad (11)$$

Strengths and shortcomings of the above expression are obvious. Spectral density $PSD(\omega)$ should be continuous and simple, otherwise Eq. (11) does not provide reliable results applicable in a practical analysis. Nevertheless, other variants of this procedure are evident. They can be based on a certain integral evaluation along the frequency axis, but they should be composed for particular cases.

(ii) *Residence time distribution and the first excursion probability*. Observing **Figure 2**, the output signal $u(t)$ is a random process. The time of residence in one basin and the jump to the other one can be regarded as a problem of time of the first excursion probability, see, e.g., [21] and many independent authors like [47], etc. Evaluation of individual periods of residence in one basin can serve as indication of SR stability and quality. This parameter gets an important information because the signal $u(t)$ suffers very often from nonintentional jumps within SR periods. Results provided are more reliable as a rule in comparison with (i), but the procedure in a particular case is much more laborious.

(iii) *Information entropy based indication*. Widely used in communication theories. This indicator is based on Boltzmann's entropy of information, see monograph [48]. Boltzmann's entropy is defined by the expression:

$$I(\varphi) = \int_{\mathbf{x}} p(\mathbf{x}, t) \cdot \lg p(\mathbf{x}, t) \, d\mathbf{x} \quad (12)$$

where $I(\varphi)$ denotes Boltzmann's entropy of probability and $p(x, t)$ is the cross-probability density of the system response. The procedures working with this tool are usually based on maximization of this entropy with auxiliary constraints, which is the governing dynamic system itself. In particular, PDF is written in a form of the multi-dimensional exponential (mostly a polynomial in a homogeneous form) with free coefficients. These coefficients are subsequently determined by means of the extreme searching using a suitable procedure (Fletcher-Powell, artificial neuronal network, etc.).

This procedure is very effective when impuls character of useful signal is considered, see the SR-focused paper by Neiman [49] or generally oriented [26], etc. As a large source of information

can doubtlessly serve relevant chapters in monographs [11] or [50]. A significant step forward to characterize the conventional SR by means of information theory tools has been put by [51], where SR in a nonlinear system driven by an aperiodic force has been studied. See also a number of other papers being more or less on the boundary between classical and nonconventional SR definitions, as for instance [52] dealing with SR capacity enhancement in an asymmetric binary channel.

(iv) *Statistics of local random processes in individual basins.* Random processes surrounding the mean value when residing in a basin is evaluated. Then random mean square root is evaluated and compared with the amplitude of the jumping process mean value. Rather special method which appears rarely in SR as a separate tool. If applied, it is more or less smoothly integrated with the analytical process. Its application can be observed more in areas working with more general SR definitions concerning the operator structure and driving noise type, see section 6.

(v) *Mutual information.* Let us denote $p_{\varphi\psi}(\varphi, \psi)$ the joint PDF of input and output processes $\varphi(t)$, $\psi(t)$. Being based on Shannon's theorem, see [53], mutual information between processes $\varphi(t)$, $\psi(t)$ is defined as the relative entropy between the joint PDF and the product of partial PDFs, see [48] or [54]:

$$I(\varphi, \psi) = \int_{\varphi, \psi} p_{\varphi\psi}(\varphi, \psi) \cdot \lg \left(\frac{p_{\varphi\psi}(\varphi, \psi)}{p_{\varphi}(\varphi)p_{\psi}(\psi)} \right) d\varphi d\psi \quad (13)$$

It seems that the mutual information is the most effective quantification parameter for assessment in suprathreshold stochastic resonance, see [11] and many more. Take a note that Eq. (13) basically represents a significant generalization of the Boltzmann's entropy procedure Eq. (12) with respect to conditional probability referring some intermediate state analogously with Bayesian updating.

3. Engineering Dynamics and stochastic resonance

It seems that Engineering Mechanics is now gradually discovering SR and is looking for areas of SR applicability. Nevertheless, some areas can already show off tangible results. Research activities are mostly the joint projects with physics, fluid mechanics, electronics, and medical disciples. Similarly like in other branches also in Engineering Mechanics, the direct and inverse tasks are investigated. Due to some delay, it can draw upon experience of other disciplines.

Let us outline some relevant areas of Engineering Mechanics where SR provides (or could provide) significant contribution in various points of the research and application. Then, we present briefly a sample problem of aeroelastic stability related with SR.

3.1. Areas in dynamics related with stochastic resonance

Engineering Dynamics of discrete and continuous systems in classical meaning of the term can come into contact with SR roughly in three areas:

(i) *nonlinear SDOF, multi degree of freedom (MDOF) or possibly continuous dynamic systems subdued to combination of periodic and random excitation.* A number of problems arising in flow structure interaction can be tackled using various models of SR type, e.g., slender structures in a cross flow, soft large roofs, high speed channels with streaming fluids, and propagating solitary waves, etc. Some more examples can be found among other systems with significant Duffing type nonlinearity with meta-stable point of origin, even those more complicated nonlinear system (Van der Pol, Rayleigh, etc.) can exhibit SR effects. They emerge usually after entering into a post-critical regime stabilized by certain nonlinear forces. A sample problem of aeroelastic stability will be shortly looked through.

(ii) *Experimental measurements of weak signals below threshold limit.* Subthreshold signal sensing, recording, and filtering is rather a cross discipline widespread nearly everywhere.

Signal sensing and subsequent data processing is a wide area pervading all scientific and engineering disciplines. Hence, relevant problems attracted many researchers all the time. The aim has always been to speed up digitizing frequency, increase resolution and reliability, and to diminish as much as possible differences between input and output processes.

It has been recognized in the past that a weak signal being below the threshold limit of a sensor, can be boosted by adding white noise to the useful signal, see **Figure 4**. For details, see [11]. The sum of both signals can overcome the threshold limit and hence to be detectable by the sensor. Then, random component is filtered out to effectively detect original, previously undetectable signal. Many general studies and special-oriented variants have been performed to detect subthreshold signals using a driving random signal, let us name a few of them [24, 55–58] following various attributes of SR employment in weak signal recognition and reliable recording.

The qualitative jump forward in this strategy brought the suprathreshold stochastic resonance (SSR). The phenomenon of SSR has been discovered by N. Stocks. The first paper informing about SSR is the review paper [8] published in 1999. As the primary source can serve [59], which appeared 1 year later and authored solely by Stocks. Since then, many articles have been published about SSR. Probably, the most comprehensive explanation can be found in the monograph by McDonnell et al. [11].

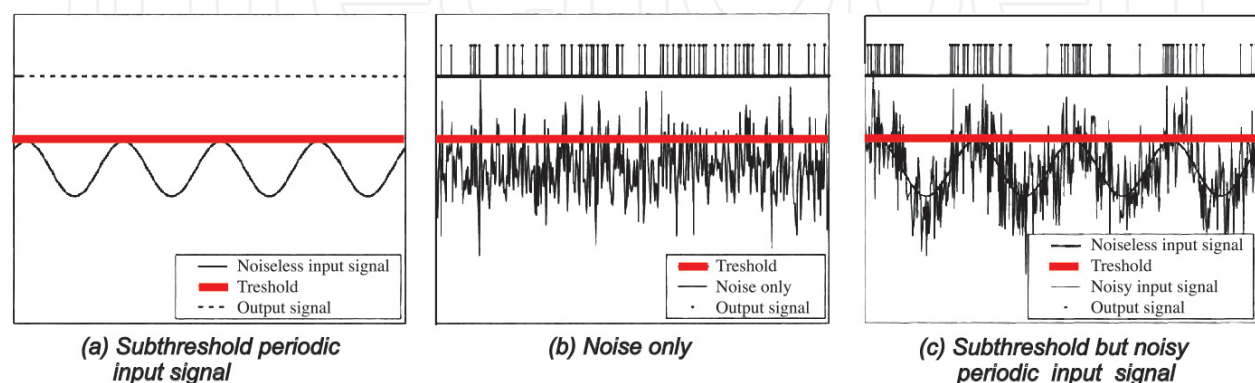


Figure 4. Experimental measurements of weak signals below threshold limit, see [11].

(iii) *Biomechanics*. Very wide domain gathering experts of many areas constituted interdisciplinary teams worldwide. Domains like heart dynamics, blood streaming, muscle system functionality, and vocal folds are followed. However, predominantly problems of the human skeleton are tackled, see for instance [42, 60]. Here, substantial attention is paid to SR in which noise enhances the response of a nonlinear system to weak signals in various biological sensory systems. In the same time, it has been recognized that adding low magnitude periodic vibration greatly enhances the bone formation in response to loading, which is definitely an excellent contribution of SR for the osteogenic processes. An outcome of these activities are among others the therapies of the whole-body through vibration training on a chair rising in elderly individuals [61, 62]. Very sophisticated stochastic analysis of discrete data sets provided by measuring records has been performed in order to bring an exact evidence of the meaningful healing procedure.

Let us take a note beyond limit of this study. Biomechanics is not far from various medical branches, where a wide range of modern special implants based on the SR principle is successfully used. In particular cochlear stimulators, oftalmological adaptors, pacemakers and others, see for instance [4] or [17] where also many additional references can be found.

3.2. Sample problem of aeroelastic stability

With reference to wind tunnel observations in a wind channel, it seems that SR is promising as a theoretical model inherent for several aeroelastic post-critical effects arising at a prismatic beam in a cross flow. Dealing with relevant projects, these post-critical effects should be carefully investigated in order to eliminate any danger of the bridge deck collapse due to aeroelastic effects. In particular, the divergence or buffeting of a bridge deck can be modeled as a post-critical process of the SR type at an SDOF or two degree of freedom (TDOF) system, see **Figure 5**. For details, see [63]. In **Figure 5(a)**, we can see outline of the TDOF system investigated. **Figure 5(b)** exhibits the stability diagram itself. White or dark fields indicate stable or instable zones, respectively. The stability limits are plotted in the plane of heaving and pitching eigen frequencies ω_u^2 and ω_φ^2 . **Figure 5(c)** shows value of the flutter frequency Ω^2 with respect to position on the parabola with axes x_1, x_2 in **Figure 5(b)**.

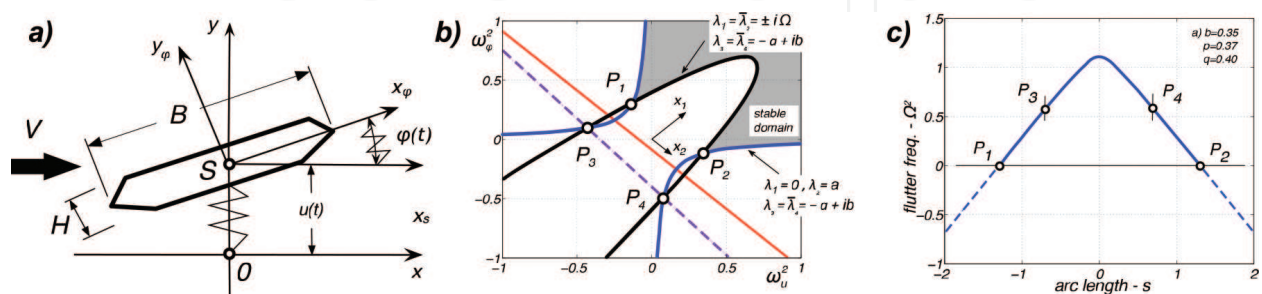


Figure 5. Stability diagram of the TDOF aeroelastic system: (a) TDOF aeroelastic system, (b) stability diagram in ω_u^2 and ω_φ^2 coordinates, (c) flutter frequency Ω^2 as function of a position on the parabolic part of stability limits.

Paper [64] is referred for details and further references. Anyway, let us revisit Eqs. (2–4) for basic mathematical model. Three theoretical solution ways have been followed. FP equation together with boundary conditions is written out in Eqs. (6, 7).

(i) *Semi-analytical solution of FP equation.* Galerkin type procedure has been applied in order to respect non-self-adjointness of the FP operator, see [25]. With respect to the linearity of the FP equation, the basic periodicity of the PDF should be equivalent with the frequency of the deterministic excitation component Ω and its integer multiples. See formulation Eqs. (2–4) together with Eqs. (5–7).

Therefore, the series can be written in the following form:

$$p(u, v, t) = p_o(u, v) \sum_{j=0}^J q_j(u, v) \cdot \exp(ij\Omega t) \quad (14)$$

where Ω is the harmonic excitation frequency. The series Eq. (14) represents an approach of a weak solution of FP equation, which repeats in the period $T = 2\pi/\Omega$. It gives a true picture of solution within one period, but cannot express any influence of initial conditions.

In Eq. (14), $p_o(u, v)$ means the solution of FPE Eq. (5) for $P_0 = 0$, it means that the deterministic part of excitation vanishes and the external excitation is limited to random component only. The solution is time independent (solution of the Boltzmann type). For details, see, e.g., [26–28], and other papers and monographs, see also **Figure 6** for symmetric and nonsymmetric potentials V :

$$p_o(u, v) = D \cdot \exp\left(-\frac{2\omega_b}{\sigma^2} H(u, v)\right). \quad (15)$$

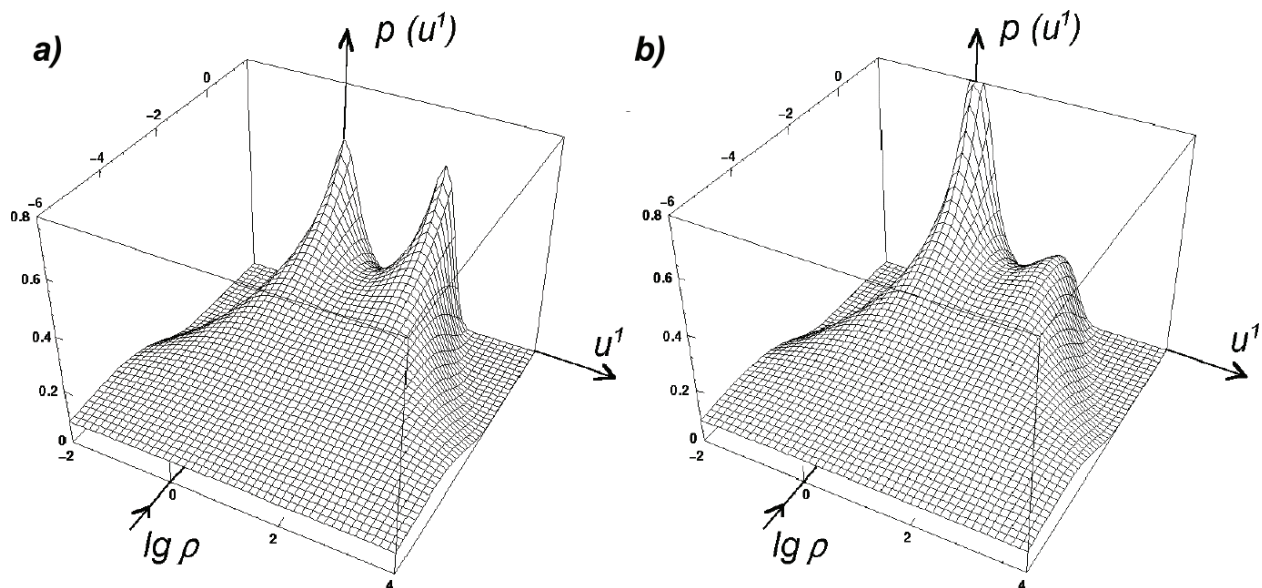


Figure 6. Response PDF of the system excited by white noise only: (a) Symmetric potential; (b) Nonsymmetric potential.

In the above expression, D is the normalizing constant, and $H(u, v)$ represents the Hamiltonian function of the basic system. In particular:

$$H(u, v) = \frac{1}{2}v^2 + V(u) = \frac{1}{2}v^2 - \frac{1}{2}\omega_0^2 u^2 + \frac{1}{4}\gamma^4 u^4 \quad (16)$$

The unknown functions $q_j(u, v)$ in Eq. (14) can be searched for using the generalized method of stochastic moments as it can be found, in [23]. For additional details, see [29]. Using the Galerkin approach, the expression (14) is substituted into Eq. (5) and the whole equation is subsequently multiplied by the testing functions $\alpha(u, v)$.

The testing functions $\alpha(u, v)$ and unknown functions $q_j(u, v)$ are assumed to have a following advantageous form:

$$\alpha(u, v) = \alpha_{r,s}(u, v) = u^r \cdot H_s(\beta v) ; \quad r = 0, \dots, R ; \quad s = 0, \dots, S \quad (17)$$

$$q_j(u, v) = \sum_{k,l=0}^{R,S} q_{j,kl} u^k \cdot H_l(\beta v) \quad (18)$$

where $H_s(\beta v)$ are l'Hermite polynomials and $\beta = \sqrt{\omega_b/\sigma^2}$. After applying the mathematical mean operator with respect to probability density function $p_o(u, v)$, see Eq. (15), and employing orthogonality of l'Hermite polynomials, the linear algebraic system for unknown coefficients $q_{j;kl}$ arises ($q_o(u, v) = 1$, $q_{-1}(u, v) \equiv 0$):

$$2\beta(ij\Omega + 2\omega_b s)\mathbf{A}\mathbf{q}_{j,s} - 2(s+1)\mathbf{C}\mathbf{q}_{j,s+1} + \mathbf{B}\mathbf{q}_{j,s-1} = 2\beta^2 P_o \mathbf{A}\mathbf{q}_{j-1,s-1} \quad (19)$$

where $\mathbf{q}_{j,s} = [q_{j,0s}, q_{j,1s}, \dots, q_{j,Rs}]^T$ —column vector ($R+1$ components) and $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{R}^{(R+1) \times (R+1)}$ —square arrays containing moments:

$$A_{r,k} = \int_{-\infty}^{\infty} u^{r+k} \Phi(u) du ; \quad B_{r,k} = \int_{-\infty}^{\infty} k u^{r+k-1} \Phi(u) du ; \quad C_{r,k} = \int_{-\infty}^{\infty} r u^{r+k-1} \Phi(u) du \quad (20)$$

$$\Phi(u) = \exp\left(\beta\omega_0^2 u^2 - \frac{1}{2}\gamma^4 u^4\right)$$

Function $\Phi(u)$ is symmetric with respect to zero and therefore $A_{r,k} = 0$ for odd $r+k$, while $B_{r,k}$, $C_{r,k}$ vanish for even $r+k$.

For each j , the three-term recurrence formula Eq. (19) forms an algebraic system of size $(S+1)(R+1) \times (S+1)(R+1)$ for all unknown coefficients $q_{j,rs}$. The block diagonal of the system matrix consists from scaled regular matrices \mathbf{A} , see Eq. (20), and thus it is invertible.

The resulting probability density function varies in time with periodicity, which corresponds to the frequency of external loading Ω . The individual peaks alternate but the lower peak never vanishes completely, see **Figure 7**. The computed joint probability density is shown in the **Figure 7(a)**, the corresponding curve for the displacement variable u (section for $v = 0$) is in the **Figure 7(b)**. The solid line shows the computed time-dependent probability density for $t = 30$, the dashed line corresponds to the stationary solution of the Boltzmann type $p_o(u, v)$, see Eq. (15).

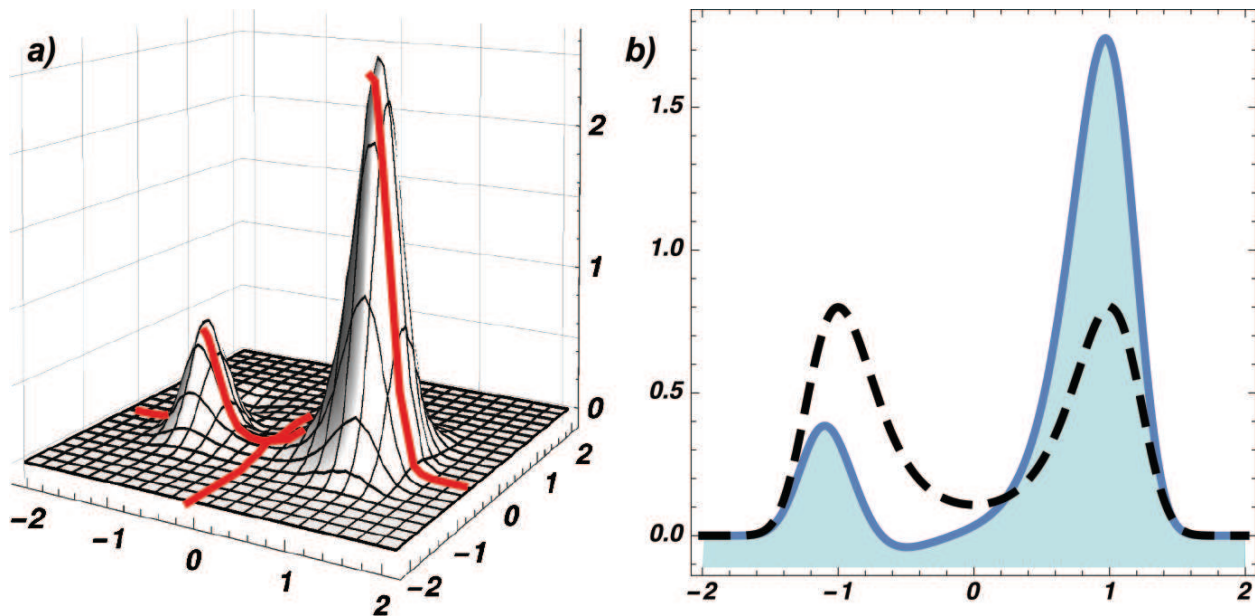


Figure 7. (a) PDF according to relation Eq. (14) for $t = 30$; (b) the corresponding cross section for $v = 0$ during the transition period starting initial condition of the Dirac type (solid line with filling) and stationary solution of the Boltzmann type (dashed).

(ii) *Solution of FP equation using FEM.* Solution procedure is based on the approximate solution of Eq. (5) in the Galerin-Petrov meaning on the piecewise smoothly bounded domain $\Psi \in u \times v$, in \mathbb{R}^d , $d = 2$. The initial conditions at $t = 0$ s for PDF are considered in a form of very pointed Gaussian distribution function with an initial system position at the point $u_0 = 0$, $v_0 = 0$. For a small values of standard deviation, it approaches to the Dirac function as it is primarily requested. The system of ordinary differential equations emerge with global matrices \mathbf{M} , $\mathbf{K}(t)$ and vector of PD values $\mathbf{p}(t)$ in nodes of the mesh:

$$\mathbf{M}\dot{\mathbf{p}}(t) = \mathbf{K}(t)\mathbf{p}(t). \quad (21)$$

The matrix $\mathbf{K}(t)$ is time-dependent due to the periodic perturbation entering the drift term of FP equation, and in the result, the solution oscillates periodically between the potential wells. In the regime of SR, the switchings are in phase with the external periodic signal $P(t)$, and the mean residence time is closest to half of the signal period $2\pi/\Omega$.

Some results of numerical analysis are depicted in **Figure 8**. Comparison of those with semi-analytic results plotted in **Figure 7** shows a perfect coincidence.

(iii) *Stochastic simulation.* Differential system Eq. (2) has been repeatedly solved numerically respecting its stochastic character, see [35], with the same parameter setting as used before. The white noise was simulated as a finite sum of harmonic functions with uniformly distributed random frequencies $\omega_i \in (0, \omega_{max})$ ($\omega_{max} = 10 \text{ rad. s}^{-1}$) and phases $\varphi_i \in (0, 2\pi)$:

$$\xi(t) = \sqrt{2}\sigma \sum_{i=1}^N \cos(\omega_i t + \varphi_i) \quad (22)$$

The results of the SR analysis are illustrated in **Figure 9**, which presents the signal to noise ratio — **Figure 9(a)** as the function of the noise intensity expressed by $2\sigma^2 = \kappa_{vv}$ and the results (Fourier spectra) of the stochastic simulations using the basic system Eq. (2) and **Figure 9(b)**. In the individual spectral lines, it can be seen in the influence of rising the white noise intensity, which acts together with a harmonic force onto the system. For a very low level of the noise, the harmonic component is hardly able to overcome the interwell barrier, and therefore, only

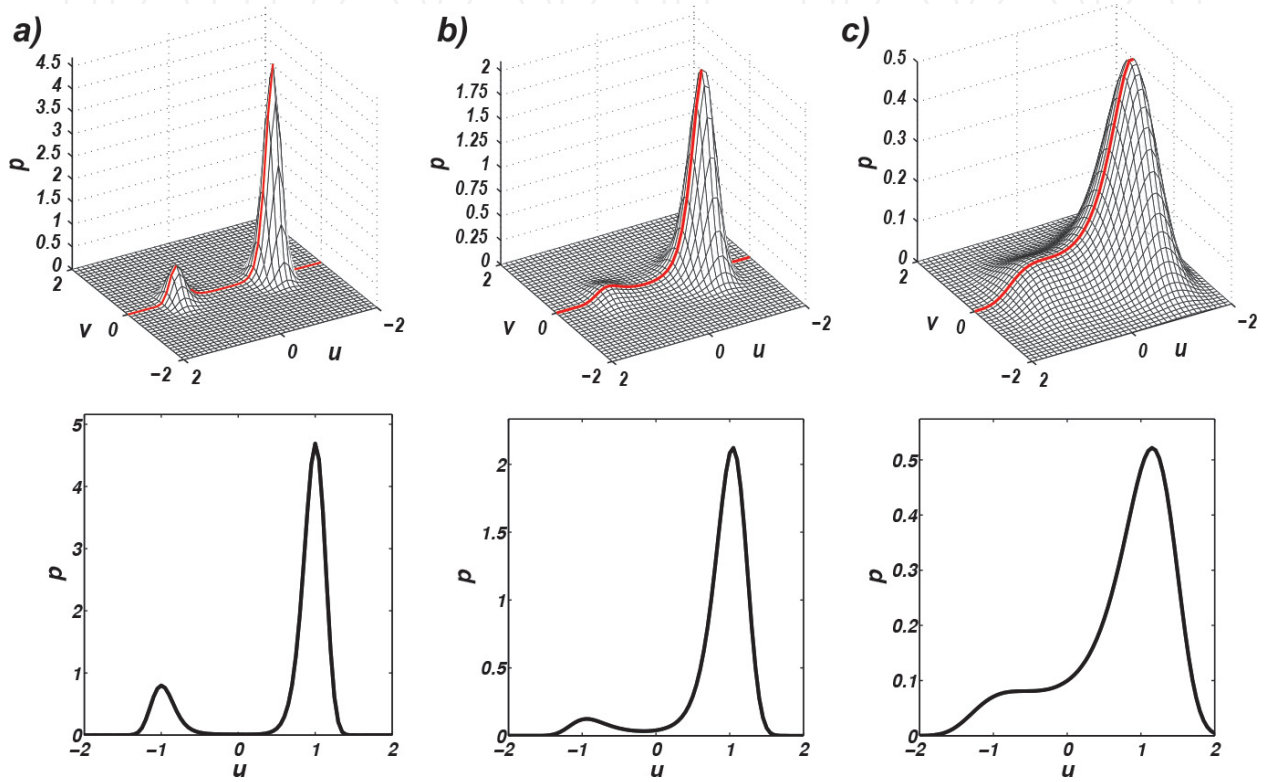


Figure 8. Axonometric and sectional display of the PDF at the highest value of probability of residing in selected potential well: (a) $\kappa_{vv} = 0.10$; (b) $\kappa_{vv} = 0.25$ —stochastic resonance; (c) $\kappa_{vv} = 1.0$; the lower pictures are vertical cross-sections of surfaces in the upper row for $v = 0$, see highlighted curves in red.

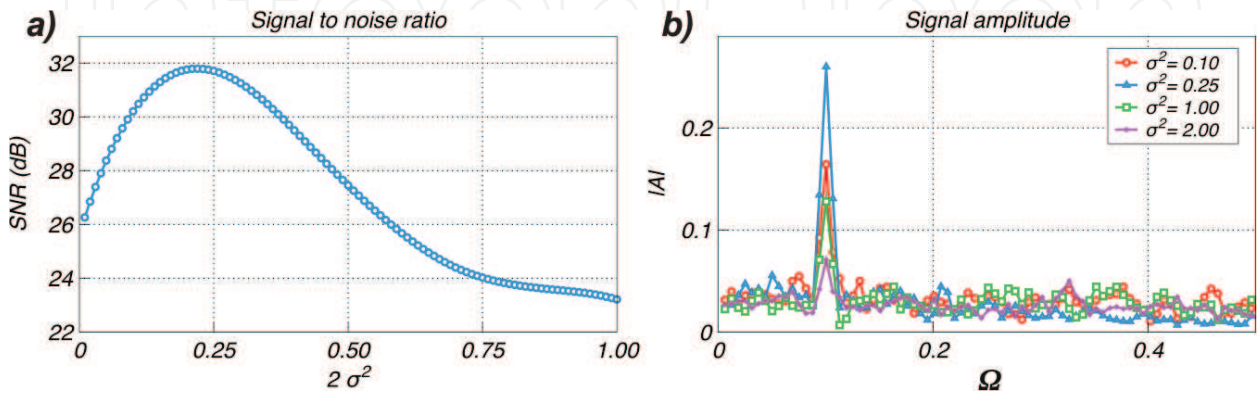


Figure 9. Results of stochastic simulation: (a) the signal to noise ratio as the function of various noise intensity (σ^2) due to SR; (b) Fourier spectra of the response obtained by numerical solution.

seldom irregular jumps between stable points occur, as it has been already demonstrated in **Figure 2**.

In local regimes, the system response is relatively small and nearly linear. Optimal ratio of the noise intensity (σ_0^2), and the amplitude of the harmonic force results for its certain frequency in the system response containing visible spectral peaks (amplification) corresponding with the frequency of the external harmonic modulation. The single peak (in the case of colored noise more peaks may appear) and thus the “optimal” noise strength can be identified.

4. Energy harvesting

A number of sources of harvestable ambient energy exist, including waste heat, vibration, electromagnetic waves, wind, flowing water, solar energy, human motion, and others. They can serve for powering remote wireless sensors, controllers, stimulators in a number of technological and biological applications, without any battery or wiring complements. Therefore, energy harvesting (EH) has emerged as a discipline with the goal of fabricating devices that can generate electrical power by exploiting ambient waste energy, for instance see [65]—ambient vibration, [66]—thermo gradient [67]. Basically following ideas are used: piezoelectric layered parts, magnetic levitation, magneto-rheologic hydraulic elements, ball screw systems, impact systems, and other principles. A pioneering work highlighting theoretical aspects of EH and challenging other authors is [68]. The adequate model follows from SDOF bistable system:

$$\ddot{u} + 2\omega_b\dot{u} + \omega_0^2u \left(1 - \frac{l}{\sqrt{u^2 + d^2}}\right) = P_0 \cos \Omega t + h\xi(t) \quad (23)$$

where l, d are dimension characterizing von Mises truss—remembers the system in **Figure 1a**. A couple of modified equations are also used in order to facilitate the insight into the system. The most frequent is the relevant Duffing system with negative linear part of stiffness.

4.1. Small scale energy production and measuring system feeding

A few electro-mechanical principles are used for this purpose. Typically, a cantilevered beam with a piezoelectric strip is used to transform vibrational energy into electrical energy through damping, see **Figure 10**. This figure has been taken over from [69], where many details and systematic background can be found. For small displacements of the beam, peak power generation in the mechanism will occur when the natural frequency of the beam is tuned to the peak of the vibration noise spectrum. Briefly speaking, SR despite being counter-intuitive phenomenon proved to be effective to enhance vibrational EH by adding periodic forcing to a vibration excited energy harvesting. A review of EH suitable piezoelectric materials together with adequate shaping and comprehensive experience in practice can be found in [9]. The most frequent applications cover human stimulation feeding, measuring and transducer system feeding, traffic control feeding, and many other devices with consumption approximately less than 1.0W.

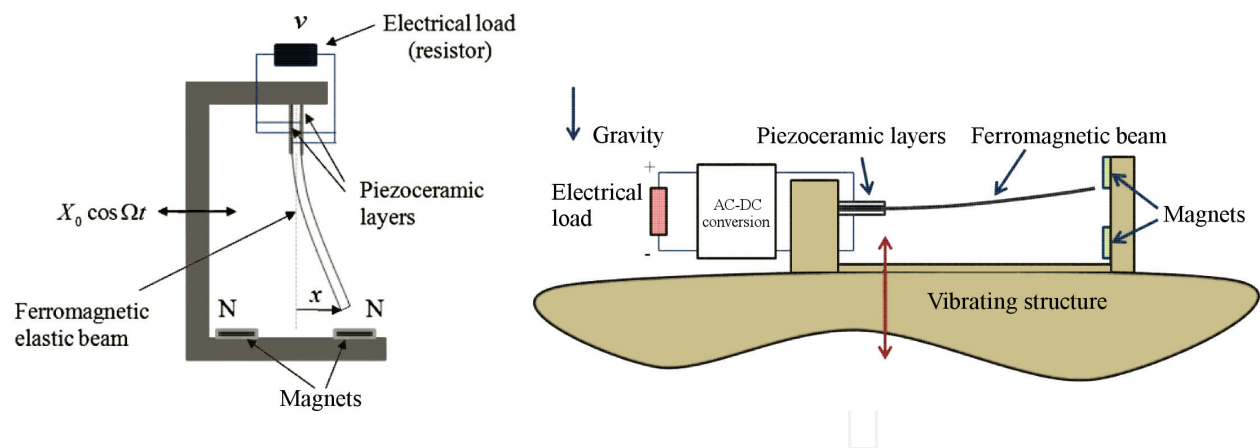


Figure 10. Small scale energy production for capture local feeding, see [69].

4.2. Large scale application and vibration damping

While the first generation of EH devices has been intended for the low power consumption, subsequently an idea of SR-assisted EH application in large scale systems appeared. These systems usually combine auxiliary power production and vibration suppressing in large scale engineering systems and suppose to work with energy approx 1 – 100kW. Energy is gained from vehicles and transport means operation, vibration of civil and mechanical engineering systems, and other resources.

Comprehensive review of the contemporary knowledge regarding EH in large scale facilities is presented in papers [70] and [71]. Relevant principles are based again on EH assisted by SR phenomena. Possibilities and practical aspects of vibration damping using SR support EH are widely discussed in engineering oriented journals, see Figure 11. A number of other facilities is based or supported using this principle. Let us name a few: floating floor, railroad track vertical deflection, vehicle suspension, ocean energy harvesting, and many others.

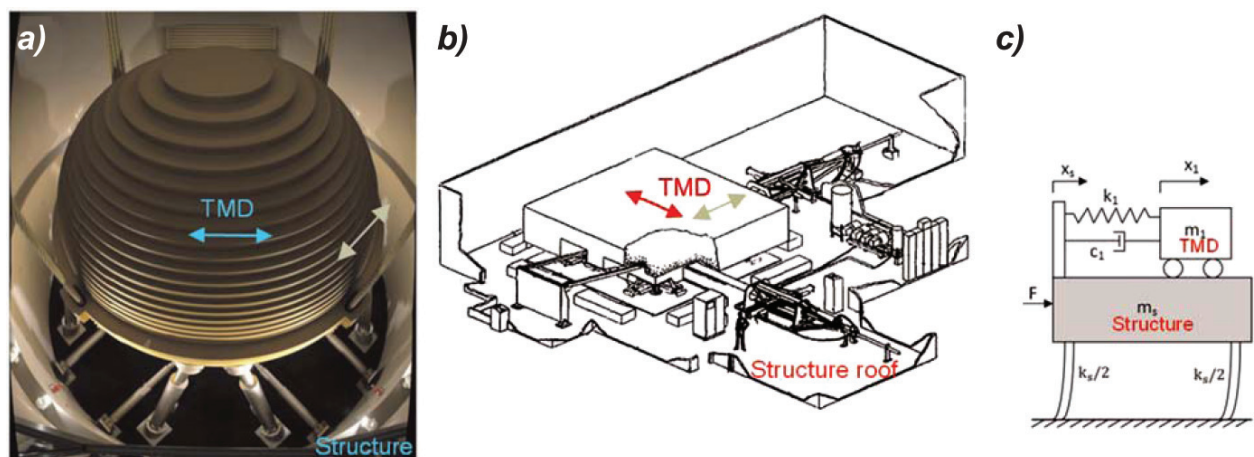


Figure 11. Large scale energy supply of the active TMD, see [70].

5. Climatology

The position of SR in climatology is specific in comparison with other disciplines. It is worthy to be highlighted in the separate section, although apparently it is a bit far from engineering mechanics. The reason is that researchers in climatology demonstrated the first systematic genuine SR in contemporary meaning. This concept was introduced in 1981–1982 by C. Nicolis, see [2, 3], dealing with the problem of climatic changes during the Quaternary. Approximately in the same time appeared papers by Benzi et al. dealing with similar topics [72, 73] preferring a bit more theoretical aspects of the SR phenomena. So that, this pioneering step came from the apparently exotic context of the Earth's climate evolution of the periodic recurrence of Earth's ice ages. For some summary of the starting period and physical motivation analyzing the physical essentials of climatological changes in view of SR, see Scholarpedia [74], other encyclopedia co-authored by C. and G. Nicolis and also a couple of review articles, e.g., [4, 5].

5.1. Physical motivation

It has been known that the climatic system possesses a very pronounced internal variability. A striking illustration is provided by the last glaciation which reached its peak some 18,000 years ago, leading to mean global temperatures of some degrees lower than the present ones and a total ice volume more than twice its present value.

Going further back in the past, it is realized that glaciation has covered, in an intermittent fashion, much of the Quaternary era. Statistical data analysis shows that the glacial/interglacial transitions that have marked the last hundred thousand years display an average periodicity of 10,000 years, indeed. To this process is superimposed a considerable, random looking variability of Sun flux. The conventional explanation was that variations in the eccentricity of Earth's orbital path occurred with a period of about 10^5 years. So that, the energy flux Q impacting the Earth can be characterized as follows:

$$Q = Q_0(1 + \varepsilon \cdot \sin \omega t), \quad (24)$$

where $\varepsilon \approx 0.001$, $\omega \approx 2\pi/10^5 \text{ years}^{-1}$. This process caused the year average temperature to shift dramatically and produces the ice volume changes on the Earth, which randomly oscillates between limits $30 - 60 \times 10^6 \text{ km}^3$, see **Figure 12a**.

However, it sounds strange, since the only known time scale in this range is that of the changes in time of the eccentricity of the Earth's orbit around the sun, as a result of the perturbing action of the other solar system bodies. This perturbation modifies the total amount of solar energy received by the Earth but the magnitude of this astronomical effect is exceedingly small, about 0.1%, see above. So that, the measured variation in the eccentricity had a relatively small amplitude compared to the dramatic temperature change. Therefore a question arose, whether one can identify in the Earth-atmosphere-cryosphere system any mechanism capable of enhancing its sensitivity to such small external time-dependent forcing.

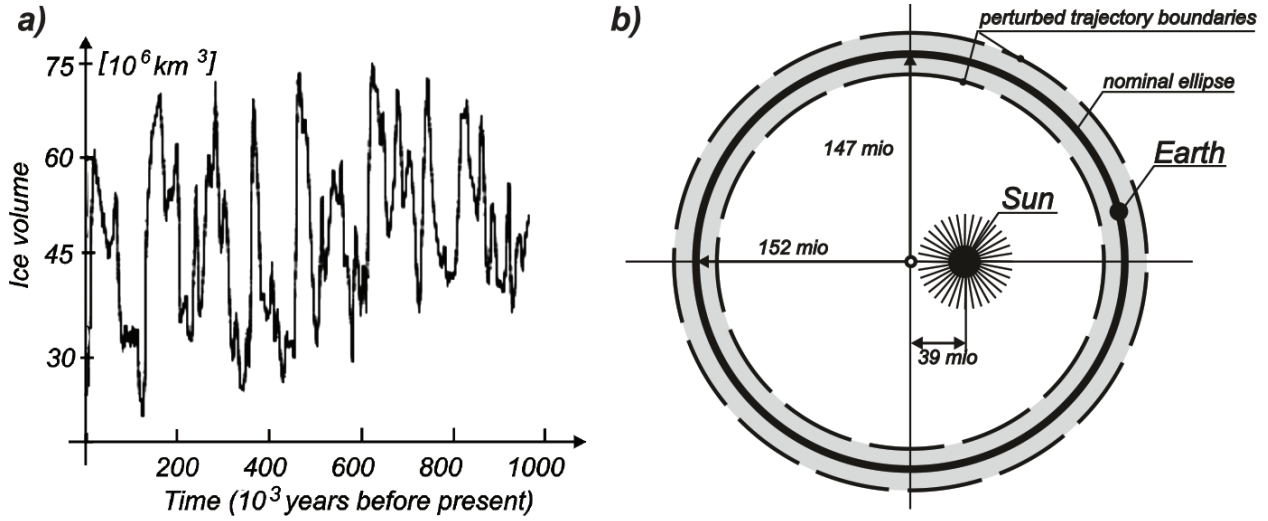


Figure 12. (a) Ice volume on the Earth surface in the past, see [5]. (b) Earth trajectory around the sun.

The search of a response to this question led to the concept of SR, which has been developed out of an effort to understand how the Earth's climate oscillates periodically between two relatively stable global temperature states, one "normal" and the other an "ice age" state. In other words, a theoretical explanation has been elaborated to show that the temperature change due to the weak eccentricity oscillation and added stochastic variation due to the unpredictable energy output of the sun (known as the solar constant) could cause the temperature to move in a nonlinear fashion between two stable dynamic states. Specifically, glaciation cycles are viewed as transitions between glacial and inter-glacial states that are somehow managing to capture the periodicity of the astronomical signal, even though they are actually made possible by the environmental noise rather than by the signal itself. Note that also dynamics of the Earth as a deformable body should be taken into account, see [75], as an indirect source of periodic processes involved.

5.2. Mathematical modeling

The orbit of Earth around the sun is not exactly elliptical, as it is commonly reported. The shape of its trajectory is complex following a form of a spiral. This trajectory is stable within a basin having a form of a closed strip. Its width is approximately 10^7 km, see **Figure 12b**, and exhibits a character of deterministic chaotic attractor. Earth trajectory takes place within the shadow area, see **Figure 12b**. The Lyapunov exponent mostly oscillates nearby 0.

The basic setting of SR in climatology started with SDOF nondynamic system subjected to a stochastic excitation and weak harmonic forcing. It corresponds formally to the Langevin equation of the first order in the form with suppressed inertia term due to high damping (adiabatic approach), compare with Eqs (26) or (27), Section 6.1:

$$\begin{aligned} \frac{d^2 u}{dt^2} + \frac{\partial V(u)}{\partial u} &= \eta(t) + Q_0(1 + \varepsilon \exp(i\omega t)) \\ \dot{\eta} + a \cdot \eta &= \xi(t), \quad \text{or} \quad \ddot{\eta} + a\dot{\eta} + b\eta = \xi(t) \end{aligned} \quad (25)$$

where it has been denoted: $V(u)$ —conventional symmetric quartic potential, $\eta(t)$ —exponentially correlated random process, $\xi(t)$ — α stable white noise.

Potential $V(u)$ is considered usually in conventional symmetric quartic form, but also various nonsymmetric variants are regarded in order to respect specific anomalous situations, see also sections 2 and 3. Compare Eq. (25) with FitzHugh-Nagumo equations, see [76, 77]. The contemporary research uses more sophisticated models respecting the space distribution. However, the basic mechanism concerning the time coordinate following Eq. (25) is kept.

Further research in 1990s has been focused to abrupt glacial climatic changes. It has been conducted in view to SR phenomenon related with these changes. Results appeared successively during last 2 decades, see, e.g., [5, 78], and later [79, 80] reflecting furthermore specific attributes of the chaotic dynamics. On the basis of SR, many more studies have been published dealing with general and specific themes. See, e.g., [81] discussing SR in the North Atlantic and a large series of articles by Ditlevsens (senior and junior), e.g., [82] dealing with the rapid climate shifts observed in the glacial climate.

Take a note that the statistical properties of relevant processes are adequately characterized by α -stable processes and so they are widely used in this discipline. For theoretical background see, e.g., monographs [83, 84] and some problem specific papers, see subsection 6.2.

6. Alternative operators and driving processes

The most common SR definition is based on the Duffing equation with the negative linear part of stiffness being excited by an appropriate combination of a harmonic and Gaussian white noise signals. However, it came to light that a few different definitions of SR are possible being based on an alternative differential system or using other driving noise than the Gaussian one. It revealed that many cases can be treated much more effectively than under classical definitions. Application of this background is very wide, and it can be concluded that starting investigation of a particular problem a suitable definition should be carefully selected. So that they can be actually found everywhere in physics, life, and social disciplines.

6.1. Alternative differential operators

Despite of classical definitions of SR, some nonconventional inherent settings appeared together with excellent applications in general theory, nano-scale systems, neurophysiology, etc. Using the linear response theory, some alternative types of SR turned out. For details, see the original papers by Dykman [19, 85, 86], Luchinsky [7, 8], and other authors. They identified SR existence in quite different systems from those commonly studied to date, which are typical by a static double-well potential and being excited by a force equal to the sum of periodic and driving stochastic components.

(i) *SR in a monostable system.* The SR can be observed in a monostable nonlinear Duffing oscillator being driven by additive Gaussian white noise $\xi(t)$ of intensity σ . Let us assume the nonlinear mass-unity SDOF oscillator:

$$\ddot{u} + 2\omega_b \dot{u} + \frac{\partial V(u)}{\partial u} = \xi(t) + P_0 \exp(i\Omega t), \quad V(u) = \frac{\omega_0^2}{2} u^2 + \frac{\gamma^4}{4} u^4 + Bu, \quad (26)$$

$$\omega_b \ll 1, \quad \mathcal{E}\{\xi(t)\} = 0, \quad \mathcal{E}\{\xi(t)\xi(t')\} = 4\omega_b \sigma^2 \cdot \delta(t - t').$$

Note that the potential $V(u)$ possesses the positive quadratic part and therefore the derivative $\partial V(u)/\partial u$ (providing the stiffness force in mechanical system) is a monotonous function. Therefore, the system is monostable unlike conventional systems exhibiting SR. Moreover, the system Eq. (26) is nonsymmetric due to linear term in the potential. It can be understood as a constant external force pre-stressing the system, see **Figure 13(a)**.

The first variant $|B| \leq 0.43$: The eigen frequency is rising monotonously with increasing energy (or the square of response amplitude). In absence of periodic force and under small noise intensity σ , the peak of the response variance, spanning around the eigen frequency $\omega_0(E)$ in an excitation level E , has the width which is approximately given by ω_b , see, e.g., [21] or [23] (in other word Lorentzian peak). That small periodic force inserted on the right side of Eq. (26) will be amplified significantly and therefore SR emerges. The most considerable increment corresponds to the frequency $\Omega = \omega_0(E)$.

The second variant $|B| > 0.43$: The eigen frequency is no more monotonous and exhibits a minimum for a certain $E > 0$. Without periodic force, the system response is given by a narrow spectral density with a maximum at the frequency ω_m lying in the point $d\omega_0(E)/dE = 0$. The ω_b is very small, and therefore, in this point the extremely sharp variance of width approximately $\omega_b^{1/2}$ arises and increases nearly exponentially with rising σ . So that for Ω close to ω_m , the SR phenomenon can be expected. It comes to light that the second variant leads to more significant SR phenomenon.

(ii) *SR in a bistable system with periodically modulated noise*. Potential of the system is similar to the classical version, in particular its quadratic part is negative and hence the system is bistable again. Linear part of the potential is retained. Damping is high (system is over-damped) and therefore the inertia term can be neglected. The system behavior is modeled as follows:

$$\dot{u} + \frac{\partial V(u)}{\partial u} = f(t) \equiv \xi(t) \left(\frac{1}{2} P_0 \exp(i\Omega t) + 1 \right), \quad V(u) = -\frac{\omega_0^2}{2} u^2 + \frac{\gamma^4}{4} u^4 + Bu, \quad (27)$$

Unlike Eq. (26), a harmonically modulated white noise is applied on the right side. Parameter B characterizes again the asymmetry of the potential. For $-2/(3\sqrt{3}) < B < 2/(3\sqrt{3})$, the potential possesses two minima. Simple manipulation gets the intensity of the driving force, see **Figure 13(b)**. As the amplitude P_0 is considered small, its square can be neglected. So the real part reads:

$$\mathcal{E}\{f(t)f(t')\} = 2\sigma^2 \delta(t - t')(1 + P_0 \cos(\Omega t)) \quad (28)$$

and we can see that the intensity of the driving force is periodic. Herewith the phenomenon of SR type emerges.

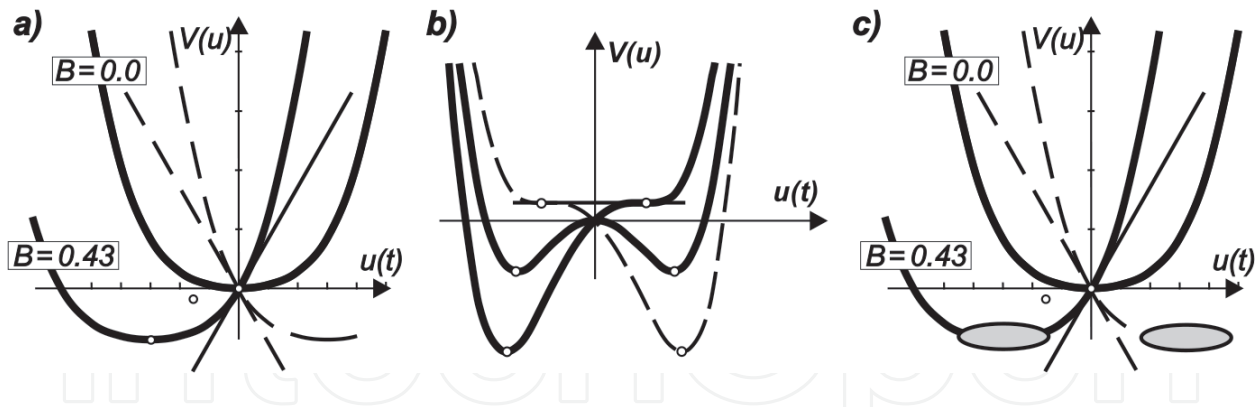


Figure 13. Alternative operators: (a) monostable system; (b) bistable system with periodically modulated noise; (c) system with coexisting periodic attractors.

(iii) *SR in a system with coexisting periodic attractors.* The third form of nonconventional SR is entirely different form of bistability. The SR concept can be based on coexisting stable states having the form of periodic or chaotic attractors, if there are any. The coexisting attractors are not static, but periodic. Theoretical analysis of these more involved situations draw on the existence (for relevant systems) of generalized potentials, not necessarily analytic in the state variables, possessing local minima on the corresponding attractors. For simplicity, the case where the period of vibration for each of the two attractors is the same can be considered and, consequently, it can be assumed that they correspond to two different stable states of forced vibration induced by an external periodic field driving the system, see **Figure 13(c)**. This interesting approach has been proposed in [87] where chaotic SR is studied to enhance attractors reconstruction using an appropriated random additional noise.

The under-damped nonlinear oscillator to be considered provides a well-known simple, but nontrivial, example of a system that behaves in just this way; its bistability under periodic, nearly resonant driving has been investigated in the context of nonlinear optics and in experiments on a confined relativistic electron excited by cyclotron resonant radiation. The particular model we treat, the nearly-resonantly-driven, under-damped, single-well Duffing oscillator with additive noise, which serves as an archetype for the study of fluctuation phenomena associated with coexisting periodic attractors, is described by

$$\ddot{u} + 2\omega_b \dot{u} + \frac{\partial V(u)}{\partial u} = \xi(t) + P_0 \exp(i\Omega t), \quad V(u) = \frac{\omega_0^2}{2} u^2 + \frac{\gamma^4}{4} u^4 + Bu, \quad (29)$$

$$\omega_b \ll 1, \quad \mathcal{E}\{\xi(t)\} = 0, \quad \mathcal{E}\{\xi(t)\xi(t')\} = 4\omega_b \sigma^2 \cdot \delta(t - t').$$

The appearance of new types of SR in systems far from the conventional static double-well potential shows that SR is a very general phenomenon. In other words, there are many physical situations where noise can be used to increase the response of a system to periodic driving. The effect is not confined to systems with coexisting static stable states. Correspondingly, SR may be more widespread in nature, and potentially of wider relevance in science and technology, than has hitherto been appreciated.

(iv) *Logistic map*. Let us note that each of differential operators above can be formulated in term of its discretized variant. Then the whole stochastic differential system can be rewritten in form of a logistic map:

$$\mathbf{u}_{i+1} = (\mathbf{u}_i, \mathbf{u}_{i-1}, \dots, t_i, t_{i-1}, \dots) \quad (30)$$

where \mathbf{u}_i is the state vector of the system in i -th point. This scheme includes an explicit time point to indicate that additive excitation (deterministic, stochastic in time) is acting. This discretized version is widely used if the immediate stochastic simulation is foreseen. Anyway, a care should be taken and Ito system is to be formulated the first respecting principles of manipulation with stochastic processes, see, e.g., [20–22]. These operations related with SR are very close to optimal (suboptimal) filtering and other stochastic data treatment. They can provide valuable contribution to SR application especially in numerical processing. This concerns particularly one-pass filtering where evaluation processes with SR algorithms are very close. For details see, for instance, [23] and other monographs, where even more general models than Eq. (30) are formulated.

6.2. NonGaussian driving noise

Although Gaussian random noise is mostly used as driving component, there approved well also other than Gaussian processes. This finding results from the inherent nature of a number of processes originally characterized by different PDF.

(i) *α -stable processes*. A number of papers deal with α -stable processes in the role of SR generator. For comprehensive acquainting with α -stable and other useful nonGaussian processes monographs [83, 84] are recommended, see **Figure 14**. Indeed, α -stable processes are suitable for use in nondynamical application, see, e.g., [88]. Authors thoroughly analyzed specific attributes of this class of problems and show doubtless advantages of α -stable instead Gaussian processes in certain nondynamical cases. They obtained these conclusions by means of theoretical and experimental procedures with white and arbitrarily colored noise. Further contribution being neurophysiology motivated are papers [40, 41]. These large

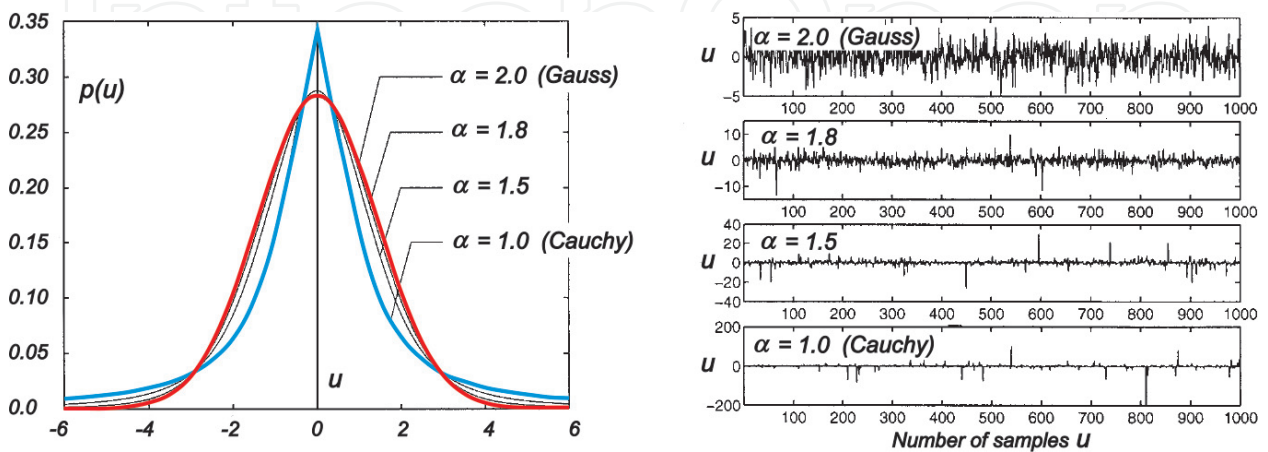


Figure 14. α -stable, Gaussian, and Cauchy processes.

studies treat problems of robust SR and adaptive SR in noisy neurons based on mutual information assessment.

(ii) *Impuls chains–Poisson driven processes*. Concerning nonGaussian driving noise, the impuls chains and various Poisson driven processes can be used in special cases [46]. A couple of authors investigated the basis of signal detection and adaptation in impulsive driving noise in framework of plasma physic, see [89].

(iii) *Colored noise*. Employment of colored noise is studied in review and particular problem focused articles. This noise being used intentionally has an effect, which is close to window filtering. It can be adjusted suitably to instantaneous needs. If it follows from the frequency limited “white noise” (finite correlation times), then influence of this low pass filter should be examined. The role of such physically realistic noise is studied for exponentially correlated Gaussian noise with constant intensity, see, e.g., [46, 90], etc. In principle, in over-damped dynamics (first order equation of SDOF system), the role of colored noise generally results in a reduction of SR efficiency. In contrast, finite inertia effects (second order equation of SDOF system), induced by moderate damping, tends to increase SR system response.

(iv) *High frequency deterministic signal*. Interesting idea is to use a high frequency deterministic signal in a meaning of a driving noise instead a random noise. For this reason, the final phenomenon is called vibrational resonance (VR), see [91]. This phenomenon analogous with SR occurs when the excitation frequency is well separated from the forcing frequency of the potential well. This setting approved very well when machine vibration is treated. Machine vibration is never truly stochastic, this provides a mechanism to link stochastic resonance to real mechanical devices, such as those used for vibrational energy harvesting, see section 3 or 4 referring among others about vibration damping.

6.3. Some other nonconventional settings

Let us briefly remark some specific SR settings. They are valuable not only for the area where they usually have been evolved, but serve as a possible inspiration for the whole SR community. Although for the full understanding, the adequate papers should be studied, have a look at some of them:

(i) *Useful signal has the impulsive or rectangular form*. Driving random signal is still Gaussian white noise. Amplification and distortion of a periodic rectangular driving signal by a noisy bistable system has been studied in [92]. Impulsive signals emerging in plasma physics are thoroughly reported in a series of publications by Nurujjaman et al. see [89]. Anyway, these papers attract attention also beyond plasma physics being interesting from general methodological points of view.

(ii) *SR in systems exhibiting chaos*. Dynamical systems in the regime of deterministic chaos evolve under certain conditions through a sequence of intermittent jumps between two preferred regions of phase space and without the intervention of a driving noise. Such systems, which give rise to multi-modal probability distributions, display an enhanced sensitivity to external periodic forcings through a stochastic resonance-like mechanism, see, e.g., [87]. For

further reading about chaotic response of deterministic systems, see monographs e.g., [93] or [94].

Let us include to this paragraph also reference to the adaptive SR, see, e.g., [41]. This approach seems to be promising as it makes possible to change parameters of the system dynamically during signal transmission in noisy neurons ambiance. The main goal of that concept consists in the fact that fuzzy and other adaptive systems can learn to induce SR based only on samples from the process. Application in other fields like control of electro-hydraulic testing equipment or smart control of vibration damping are obvious.

(iii) *Slowly varying parameters.* In many systems, the dynamics in the absence of both noise and forcing is controlled by a number of parameters λ_i describing the constraints acting from the external world. Ordinarily these parameters are assumed to remain constant, but there are situations where this strategy constitutes an oversimplification (gradual switching on/off a device, man-biosphere-climate interactions, etc.). In the absence of external periodic forcing, the simultaneous action of noise and of a slow variation of λ_i in the form of a ramp may lead to freezing of the system in a preferred state by practically quenching the transitions across the barrier. The interaction between SR and the action of the ramp provides an alternative method for the control of the transition rates by allowing the system to perform (transiently) a certain number of transitions (depending on the forcing frequency and the noise strength) prior to quenching.

7. Conclusion

The chapter tried to indicate the essence of SR. This is for the first view counter-intuitive phenomenon brings a large impact on physical, biological, and engineering systems. It is clear that SR is generic enough to be observable in a large variety of systems. The SR emerges in all scales, we can imagine. It governs the processes from nuclear fusion in the sun to the intra-atomic structures on the level of quantum mechanics. Amazing results of the basic research have been achieved and excellent industrial programs have been launched being based on many variants of SR. This concept of SR enabled to obtain an insight and exact description of many effects in macro and micro (nano) world and to fight successfully against various undesirable phenomena in engineering. It resulted in many actually nonreplaceable products of signal sensing and processing, medical instruments, and treatment procedures. Many SR-inspired neurophysiological implants represent cornerstones at the field.

The SR can be perceived as a natural phenomenon ruling inside of certain dynamic systems. In such a case, it can act either positively as for instance to help stabilize the dynamic system and therefore, to improve the system reliability or oppositely it can affect the system negatively, e.g., as a strong periodic exciting force, which is necessary to be avoided. The second view of SR understanding is considered in active synthesis and manipulation with the noise. Addition of appropriate dose of (mostly) random noise onto the useful signal provides a significant increase of sensitivity and reliability of the equipment and enlarge its ability of data sensing,

processing, and possibly their usage in a feedback. The same is valid concerning an increase of information transfer capacity and reliability.

The chapter outlines a short history of SR. An overview of SR utilization in various disciplines in physical, life, and social sciences is briefly looked through. Some possibilities of modeling in dynamics using SR strategy are indicated. Mathematical treatment and the most popular solution methods of investigation are pointed out including semi-analytic, numerical, simulation based and experimental approaches. Nevertheless, aspects related with Engineering Dynamics make intentionally a core of the chapter. Also the section dealing with energy harvesting has been highlighted as it shares many joint attributes with dynamics itself.

The phenomenon of SR in whatever variant is worthy to be employed in Engineering Dynamics having a large potential of specific basic research as well as of engineering applications. Industrial aerodynamics seems to be promising wide branch where several effects of stability loss could be explained as effects related with SR. This approach approved to describe the divergence stability loss in the nonlinear formulation of a slender beam post-critical behavior in a cross flow. Additional problems are waiting for similar type of theoretical description and subsequent experimental verification. The same probably emerge at area of panel flutter, various variants of buffeting, etc. This strategy could enable to formulate new ideas for development of nonconventional measures for vibration damping. Another area of SR application prove to be problems of vehicle dynamic stability and its post-critical behavior. Similarly like in aeroelasticity the results obtained can be used for development of new generation of vibration quenching devices of both passive and actively controlled types.

It should be highlighted that adequate experiments will be absolutely necessary. However, they should be newly proposed and performed properly, as they will differ in many ways from conventional experiments. On the other hand, a lot of inspiration at both theoretical as well as experimental fields can be taken from solid state physics and energy harvesting area.

Let us be aware that SR is a challenging discipline for Engineering Dynamics offering a large variety of possibilities of new developments at theoretical as well as experimental platform. It could significantly enhance the top areas of nonlinear and stochastic dynamics closely related with Computational Mechanics, which is very advanced and widely used in comparison with other fields of numerical analysis. It provides strong support to Engineering Dynamics, which stands on the threshold to enter the field of research and application of SR.

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