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Epistemology and the Transformation of Knowledge in the Global Age: God and the Epistemology of Mathematics

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Abstract

Mathematics, as a scientific discipline, developed from the rather humble beginnings of practical counting and measurements. The Pythagoreans shifted this discipline to the ideal, intelligible world—the “Pythagorean paradise”—where it remains to this day. However, there have been doubts as to whether some of the more peculiar mathematical concepts (irrational numbers, zero, negative numbers, infinity...) also belong to this “Paradise”. Within Theo-Platonism of the fourth century, the Christian God legitimised the concept of infinity. God then acted as guarantor for the existence of infinity even in the nineteenth and twentieth centuries. Later, however, God was played down with explicit references to Him having been eliminated. He remained hidden, as it were, in the “supernatural axioms” of set theory. Attempts to “excommunicate” Him consistently from the foundation of mathematics had only a negligible impact on the mathematics itself. Was it due to the fact that those formal foundations of mathematics (the set theory) are not the true foundations, with the actual basis being in mathematical practice?

Keywords: God, epistemology of mathematics, infinity, set theories, alternative set theories

1. The origins of mathematics

1.1. Pythagoreanism and mathematical Platonism

Even prehistoric hunters numbered the game that they caught, as well as their cattle, wives, and children. Later, people measured their fields, they measured and counted the pieces of wood and stones when they built their houses and temples and manufactured things of

daily use. When currency was invented, calculations were necessary for how much, for what remains and for how much is owed.

This primitive pre-mathematics—the useful, practical tool—is the basis for “scientific” mathematics, as we understand it today. According to Proclus, it was Pythagoras who stripped mathematics from the servant position in crafts and trade and elevated it to the “liberal arts”. Mathematics shifted to the ideal, intelligible world, the world visible only to our inner sight, the world where everything is absolutely accurate. Mathematics has remained in this “Pythagorean paradise” to this day. More precisely, it stands between our daily, temporal, physical world (which supplies it with practical problems) and the higher, eternal, meta-physical world—the “Pythagorean paradise”. In the original Pythagorean view, this “higher” world also has a “supernatural nature”. Numbers not only expressed quantities but also bore deeper, metaphysical meaning. There is, however, a close mutual interaction between these two realms: the “higher” metaphysical world helps mathematics solve practical problems, and the physical world serves as a window through which flesh sparks from the “higher” world. This higher world, however, also carries problems of its own, to be discussed in a later section.

It was Plato who expanded and absolutised this Pythagorean paradise and created the realm of forms-ideas. This realm concerns not only mathematics but also spans above the entire mundane world. This conception of mathematics is called mathematical Platonism, though it should preferably be called Pythagoreanism—but history isn’t usually fair. (Especially astounding in this connection is the fact that, in Plato’s system, mathematical forms had a slightly lower degree of reality than other forms. See Plato’s Republic.)

It bears mentioning that many candidates for entry to that mathematical paradise were and still are of suspicious nature and their adoption is regarded with hesitation. Allow them inside, or rather not? During the time of the Pythagoreans, the most provocative question concerned irrational numbers. Such queer numbers were called *ALOGION*—“illogical”—or “*ALOGION KAI ANEIDON*”—“inexpressible and unthinkable”. More precisely, the Pythagoreans refused to even consider such peculiarities as numbers; they refused to attribute existence to them and didn’t allow them into their “mathematical paradise”. An even bigger problem presented itself with *APEIRON*, that is, infinity. The concept of infinity holds a place in ancient history but, as with all other old concepts, that of infinity was still very vague. The Pythagoreans then tried to clarify the meaning of *APEIRON* and integrate it into their mathematics. Their attempt, however, was unsuccessful. Infinity was also “*ALOGION KAI ANEIDON*”, that is, in contradiction to reason and non-restrictive.

Since rationality verbally and consequently also conceptually coincided with the numerical ratio—with the existence of rational numbers (in mathematics it still coincides, at least on a verbal level), the Pythagoreans concluded that rational numbers can express all quantities. A rational number can truly approximate any value with arbitrarily high accuracy. A physicist or an engineer would manage them easily. However, this “reasonable assumption” led Pythagoreans to the dispute, which was not only perceived as the first mathematics crisis but mainly as a crisis of reason-*LOGOS*.

Admission of such suspect concepts like infinity or irrational numbers would subvert their mathematical paradise. Since everything is a number and a defining limit, our world would

have collapsed along with the fall of the mathematical world—at least according to the Pythagoreans—however, this thesis is repeated even by some mathematicians today [1].

The issues of irrational numbers and infinity have the same denominator, as irrational numbers contain an “inner infinity”. Their writing as a fraction (LOGOS) would have to consist of an infinite numerator as well as denominator—if that were possible. Even their decimal notation would be infinite—if that were possible... (The Pythagoreans were obviously unaware of that at the time.)

Humans can't discover a new ocean until they have the courage to lose sight of the shore.

- Andre Gide

According to the legend, the Pythagoreans tried to keep the existence of incommensurability (irrational numbers) a secret. It contradicted their basic idea that the world is rationally tangible. It is also said that the life of Hippasus, the (alleged) discoverer of incommensurability, ended in the depths of the sea. Apparently, he was drowned by the gods because he revealed the Pythagorean secret. Perhaps he was drowned by the Pythagoreans themselves. The story is even alive today as it is witnessed by another variant of the legend, authored by the contemporary mathematician, Rudy Rucker [2]. He suggests that Pythagoras himself was behind the death of Hippasus. Rucker, however, neglected to mention that Hippasus lived a hundred years after Pythagoras. Be that as it may, the story is more important than the facts inasmuch as “poetry is more philosophical than history”, as stated by Aristotle [3]. The aforementioned legends can be understood symbolically: poor Hippasus drowned in the very APEIRON that is hidden in the concept of irrational numbers. Just as the body can drown in the boundless vastness of the ocean, the soul can be drowned in the immense APEIRON. APEIRON is a metaphor for the ocean, and the ocean is a metaphor for APEIRON-infinity.¹

1.2. Actual and potential infinity

According to Aristotle's view, we should consider two types of infinity: the actual one which is present here and now, and the potential one which is present only in a possibility. In his *Physics* [4], he says that infinity could be understood similarly as a finite number, that is, the quantity of some units or some size. This would correspond to actual infinity. Although in nature, we don't encounter such limitless quantities and we apparently don't need it in mathematics either. For example, we can imagine dividing a segment into an infinite number of parts, but no one can carry out such a task. Therefore, it is necessary to reject the existence of actual infinity. In essence, only the potential infinity (DYNAMIKOI) remains. (For details see Refs. [5, 6].)

The concepts of actuality and potentiality (ENTELECHEIA and DYNAMIS) were introduced by Aristotle already in his *Metaphysics* [5, 7]. He illustrated them in the example of marble or bronze. These materials are a statue in their potentiality–possibility. All that is necessary is to cut away the superfluous stone, cast the metal in the mould. The possibility will be fulfilled, realised. However, in the case of infinity, Aristotle's analogy of the statue is clearly lacking. Unlike the “actual statue” in which the “potential statue” (i.e. a piece of stone) can be

¹In 1991, a minor planet was named after Hippasus (1991 XG1, cat. no. 17492). So now Hippasus floats in the infinite ocean of cosmic space.

remodelled, potential infinity can't be "remodelled"—as actual infinity does not exist! Aristotle himself was aware of this issue:

We must not take "potentially" here in the same way as that in which, if it is possible for this to be a statue, it actually will be a statue, and suppose that there is an infinite which will be in actual operation.

Aristotle, Physics III. 6. 20 [4]

Despite these problems, Aristotle accepted the existence of potential infinity as did all other ancient mathematicians. (Actual infinity, as we will see, was not afforded the same acceptance for quite a long time.) For example, Aristotle considered the number of points on a line as potentially infinite. In Aristotle's view, a line doesn't "consist" of an infinite number of points nor can it be divided into an infinite number of points. (This view was already pointed out by Zeno of Elea in his APORIA on a bisection of a line.) We can, however, create points on the line without limitation—by cutting it in half over and over, for example. We will always reach only a finite number of points, in this case. Aristotle's opinion actually defines what the concept of existence means in the case of points on a line (as well as other geometric shapes, of course). In this manner, Aristotle introduced the category of potentiality into mathematics.

However, using the concept of "potentiality" regarding infinity is problematic. Potential infinity isn't something that can be achieved or something that we can approach. "Approaching infinity" is just receding from finite limits—from certain finite numbers and values. Potential infinity can't be defined by actual infinity as it is only the negation of the finite. It is not defined positively nor is it "something" I can "have in front of me" in some sense; it is not an object. Potential infinity is a mere designation of a process which is altogether impossible to be finished in principle. Its reality is, in essence, only "borrowed".

In fact, potential infinity has only a borrowed reality as it still refers to actual infinity, thanks to which it is only possible.

Georg Cantor [8]

So-called potential infinity is but an ongoing process of enlargement,² a movement in a general sense. Although movement isn't actual nor potential, according to Aristotle! Wouldn't it be, therefore, more accurate to speak of processual infinity?

1.3. God approaches the world of mathematics

As we have seen, the Pythagorean God (perhaps there was only one God in the Pythagorean religion) protected people against paradoxes and confusions. The role of the Christian God was different, however. Although not a mathematician, it was St. Augustine who first summoned God to the realm of mathematics. Augustine attempted to combine Platonism and Christian theology on the philosophical plane. In his "Theo-Platonism", he placed Plato's realm of forms (ideas) in the mind of God. The existence of infinity Augustine justified on the grounds of God's omniscience. Since God is all-knowing, He must know all numbers; hence, He must also be cognizant of infinity [9]. In this manner of thinking, actual infinity exists—at

²In the case of infinite small (infinitesimal), it is infinite reduction.

the very least in the mind of God. Variations of this theological “proof of infinity” have survived to the present day, though this “proof” is open to question even in terms of theology:

1. It was Thomas Aquinas who, in his attempts to rationalize theology, demonstrated that not even God can know concepts that are intrinsically contradictory, that is, nonsense. So if actual infinity were a dubious concept, not even God could help it [10].
2. The second objection: If (actual) infinity exists in front of God’s eyes, does it mean it should exist also for us mortals? How does this relate to the story of the forbidden fruit of knowledge?
3. And perhaps the most serious objection: is it appropriate to suppose mathematics could be dependent on theological arguments?

1.4. Arriaga, God and actual infinity

The concept of infinity was addressed by many philosophers and mathematicians. Most of them surrendered before it due to its paradoxical and contradictory nature. (Descartes, Galileo...)

But what is meant by “paradoxical nature”? This is simply a contradiction to our presentiment of how things should be. It is a contradiction to our subconscious philosophy of mathematics and in conflict with our prejudices. But what about “real, that is, inner contradiction”? This means mutual conflict of accepted assumptions. However, in the case that we have two contradictory assumptions, does it mean therefore that just the assumption of the existence of infinity should be avoided?

One of those who were not afraid of accepting infinity was Spanish Jesuit Rodrigo de Arriaga (1592–1667).³ He worked in the first half of the seventeenth century as a professor in Clementinum College in Prague.⁴ (As a curiosity, it bears mentioning that Arriaga illustrated the existence of actual infinity by way of the number of angels.) Arriaga accepted the existence of actual infinity and divided it into five types: four secular (infinity in number, extension, intensity and perfection-quality) and one divine which is related to God only [11, 12]. He formulated many interesting and farsighted approaches to infinity; however, he did not create a consistent mathematical theory of infinity.

2. The birth of set theory

2.1. Bernard Bolzano, the grandfather of set theory

One of the most interesting thinkers to undertake the problem of infinity was Bohemian scholar Bernard Bolzano (1781–1848).

³Arriaga tried to reconcile the atomist and Aristotelian approaches [11, 12].

⁴Clementinum College later joined with Charles University, the oldest University west of the Alps.

Bolzano studied at Clementinum College, where the spirit of Rodrigo de Arriaga lived on. Upon completion of his studies of philosophy and mathematics, he was still concerned with matters of theology. He was a lengthy hesitation. He was influenced by the ideas of the Enlightenment and approached all of the supernatural with scepticism. Wasn't it all only myth and delusion? Wasn't it in contradiction with history? Eventually, Bolzano acquiesced to the ethos of the moral and practical implications of religion—that faith in God brings good to people and he aimed to take part in the dissemination of good help. On 7th April, 1805, he was ordained a priest, and ten days later, he graduated as a doctor of philosophy. Bolzano preached at the Church of the Holy Saviour (a part of Clementinum College) and taught at the newly established department of religious studies. His position was rather complicated, however, as there was a strong anticlerical sentiment among the students, and Bolzano was booed. Eventually, Bolzano gained control of the situation and became a popular teacher. Nevertheless, he soon faced resistance from the opposite side. Bolzano's idiosyncratic views and his approach to religion were met with contempt at the Vienna suzerain. A professor of religious science paid by the Austrian State was thoroughly expected to teach only what he had been instructed. He should not independently invent, even if it were to be new evidence of God's existence. This trend of inventing new theory originated solely, according to the commission, in philosophers' vain delusions of grandeur.⁵ The affair led to a ban on Bolzano's teaching and publishing. Subpoenas were delivered to Bolzano on Christmas Day, 1819 [13–15].

Bolzano wrote a number of treatises in the field of philosophy, sociology, mathematics and logic. In his *Wissenschaftslehre* (Theory of Science), Bolzano defined the concept of the set (*die Menge*), a concept that would become central to future mathematics [16]. Bolzano's final work was *Paradoxes of Infinity* (published 1851, three years after the author's death) [17]. On the basis of theological considerations, he acknowledged actual infinity and formulated some original approaches to it.

Bolzano justified the existence of infinity roughly as follows: It is certain that there is at least one true sentence, one truth. Therefore, it is also true that there is one truth. Furthermore, it is true that there is truth concerning the existence of truth etc. An omniscient God must know all these truths; therefore, He must also know infinity. (The question remains: why doesn't Bolzano say that God knows all the numbers? It could be proposed that He knows them all, but not necessarily. He must, however, know all truths, etc.) The argument is perplexing, especially when we consider how sceptical Bolzano had been regarding the supernatural. Perhaps he concluded that belief in God also brings about "mathematical good"?

In Bolzano's approach, there existed many sizes of infinity: the infinity of points on a longer line is greater than that of the shorter line; the infinite number of points in a square is bigger by order than the number of points on a line, etc. He failed, however, to give complete mathematical form to his approach (the possibility of which is still uncertain). And he also did not realise the crucial role of infinity in mathematics.

Bolzano had the misfortune of working in isolation, without contact with mathematical society. Subsequently, his provocative ideas nearly fell into oblivion. It took more than a hundred

⁵Many of Bolzano's attitudes are reminiscent of the opinions of Hans Küng.

years for new approaches to set theory to emerge in the middle of the twentieth century, whereby some of Bolzano's notions found new life (or were, perhaps, reinvented).

2.2. Cantor and his set theory

There is no doubt that the genuine father of the set theory was the German mathematician Georg Cantor (1842–1918). His theory of sets constituted a revolution in mathematics. To this day, most mathematicians regard his set theory as the undisputed “gospel” truth, the image of objective reality.

Cantor's criterion of equivalence of sets is one-to-one correspondence (bijection). This criterion represents the basic difference from Bolzano's approach. (According to Bolzano, a one-to-one correspondence exists even between infinite sets of various sizes. This peculiarity represented one of Bolzano's paradoxes of infinity.)

Cantor also assumed that there are different sizes of infinite sets but he defined their size differently than Bolzano. (Different sizes of infinity in Bolzano's concept are equally large in Cantor's theory.) He marked the size of infinities with the so-called cardinal numbers, cardinalities or briefly, cardinals. The cardinal number of the “smallest infinity” was marked as aleph-zero. This infinity denotes the quantity of natural numbers. According to Cantor, it also corresponds to the quantity of rational numbers. (Cantor found a one-to-one correspondence between the sets of natural and rational numbers.) On the other hand, the size of the set of real numbers is greater, the cardinality of real numbers being aleph one. Cantor proved that the one-to-one correspondence between real and natural numbers didn't exist. Soon, Cantor discovered how to construct larger and larger infinite sets. In Cantor's view, there were an infinite number of cardinalities. The infinities grow ever skyward to an infinite infinity—the Absolute. This greatest infinity is impossible to capture by human means; by mathematics, this infinity belongs only to God [18, 19].

The fundamentals of set theory were created in their entirety in one mind—the mind of Georg Cantor. Cantor's original intention was, however, not so ambitious, i.e. the study of the so-called fundamental series.⁶ He was surprised by the world he had opened (invented or discovered?) to mathematicians—the world which David Hilbert called “Cantor's mathematical paradise”.

To understand the important role religion played in the development of set theory, let's stop for a moment to examine Cantor's life course. Born in St. Petersburg, he with his parents moved in 1856 to Germany. His mother was a Catholic and his father a Lutheran. Deep religious faith was reflected in all of Cantor's life as well as in his work. Georg was gifted not only in mathematics but also in music and art and went to study mathematics in Zurich. After his father's death, he moved to Berlin and then to Göttingen to study. In 1867, he defended his doctorate, and at 34 years of age, he became a full professor of mathematics at the University of Halle.

However, Cantor's life also had a downside. Shortly after his 39th birthday, his mental illness developed. He never freed himself of the manic and depressive attacks and was repeatedly hospitalized at the neurology clinic. The trigger for the first seizure was a conflict with his former teacher, Leopold Kronecker, who considered Cantor's infinities insane and a source

⁶These series are also called the Cauchy or Bolzano-Cauchy series.

of many contradictions. Cantor couldn't find understanding even from Henri Poincaré who, at first, enthusiastically accepted his theory. He turned away from Cantor's work, however, when contradictions appeared. Regarding Cantor's theory, Poincaré said that it represented a "lethal infection for mathematics". Even Cantor's close friend Mittag-Leffler discouraged him from publishing the results, stating that they wouldn't be understood as they were too ahead of their time. Cantor surmised that his focus on mathematics was too narrow and didn't lead to a proper valuation. He began looking for new interests that would help him overcome his deep depression. In 1899, he was again in crisis. Within a short period, his mother, brother and his talented, thirteen-year-old only son all died. Cantor reassessed his life and regretted that he hadn't dedicated himself to music instead of mathematics. He wished to leave teaching and requested the job of librarian. He attached a curious remark to his application: he stated that he had some important findings regarding the early English kings, knowledge that would certainly shock the British Government. If his request stood without response, he would join the service of Czar Nicholas II as a native Russian. After further hospitalization in 1905, he announced that he received the inspiration from above to re-study the Bible without prejudices and with open eyes. "Enlightened from above", he wrote down the interesting moments from the history of Christianity. His final admittance to the clinic began in May 1917. He didn't live to be released from the clinic or to see the end of the war. His earthly journey ended on the feast of the Epiphany, 6th January, 1918.

Cantor had many opponents among mathematicians. Conversely, he found kinship among Catholic theologians. God's mind is infinite, and therefore, it must also contain infinite sets. Actually, St. Augustine argued similarly, and, in the nineteenth century, Bolzano, Dedekind, and neo-Thomist, Constantin Gutberlet also formulated similar "proofs of infinity".⁷ Cantor himself believed that the existence of infinities is warranted by God—after all, it wouldn't be worthy of the Almighty to create only finite sets. This calls to mind the "heretical argument" of Giordano Bruno, who used nearly the same words to justify the infinity of the universe! In the case of Cantor, however, theologians also started to worry: Will Cantor's theory perhaps lead to the identification of God with the infinite? What would be the consequences? Fortunately, it was no longer the sixteenth century and heretics didn't end in flames. Cantor's eloquence dispelled these doubts. Prominent theologian Johannes Baptiste Cardinal Franzelin commented that the concept of transfinite infinity—as he understood it—didn't hold any danger for religious truths. Ironically, set theory was then recognized sooner by theologians than by mathematicians.

The wisdom that we teach comes from God, full of secrets. Years ago, God predestined it for our glory.

St. Paul, First Corinthians 2.7

Paradoxically, Cantor himself couldn't explain his success rationally. He stated that the theory of transfinite numbers was told to him by a "powerful energy"—who other than God? He claimed that he was merely a messenger chosen to proclaim that truth to humanity. How these events mirror those in the life of St. Augustine! He also couldn't explain the reason for his sudden life turn—leaving a debauched life for an inclination to Christianity—and so also attributed it to a divine intervention! Cantor gradually sought God's intervention in his entire

⁷For the climate of that time, a return of the church to the teachings of Thomas Aquinas was characteristic. Neo-Thomism tried to reconcile theology with an exact science and philosophy.

life. Eventually, he began to understand his set theory as a “theory of everything”, as it concerned not only mathematics but also the world of the Divine and physics.

Cantor constantly sought a career at the prestigious University of Göttingen or Berlin. Backroom scheming by his enemies, however, obstructed his plans. Cantor fell under the impression that he was being persecuted, that someone wanted to silence him because he had revealed uncomfortable secrets. In the end, he came to the conclusion that God was the reason his career ambitions weren’t fulfilled:

Now, however, I thank the Almighty and most good God that He still denied the fulfilment of my desire to obtain a better place at the University of Berlin or Göttingen, because this way I was forced with deeper insight into theology to serve Him and His Catholic Church better than I would have been able to with my exclusive passion for mathematics.

Georg Cantor, letter from 1894 [20]

Cantor died in January 1918. His arch-rival Leopold Kronecker (+1891) was also dead and so were other of Cantor’s opponents. The dispute of finitism and transfinitism was slowly fading. The onset generation of mathematicians accepted and further developed Cantor’s mathematical results.⁸ Together with these results, they also subconsciously accepted God hidden in Cantor’s presumptions.

Contradictions soon emerged in Cantor’s original (so-called naïve) theory, which obliged mathematicians to underpin it by systems of axioms. Problematical sets were excluded. However, it soon became apparent that the complete axiomatisation of the theory was not possible (as was demonstrated by Kurt Gödel). Even the “corrected” axiomatic theories gave rise to a number of paradoxes, that is, contradictions with an intuitive opinion, with our “unconscious philosophy”. They may be viewed positively as remarkable discoveries from the higher, “divine world of mathematics”, or negatively as a sign of the theory’s detachment from reality.

However, after some initial hesitation, the majority of mathematicians accepted Cantor’s theory. “Cantor’s paradise” represented a fragment of a much spacious paradise which had been opened by ancient Pythagoreans. To this day, most mathematicians do not hesitate to recognize the verity of Cantor’s approach, and their vision is to base the whole of mathematics on the set theory. That endeavour culminated in the works published under the name of Nicolas Bourbaki, which surfaced during the period between the thirties and the eighties of the twentieth century.

3. Mathematics returns to the earthly world

3.1. First attempts

Cantor’s mathematical Platonism admits the existence of many mathematical objects for which there is no example or equivalent in the real world. Despite this, many significant mathematicians spent the most prolific years of their lives in Cantor’s paradise. The myth was

⁸Cantor’s historiographical and theological speculations remained only a contemporary curiosity, same as Cantor’s reasoning on the border between physics and metaphysics.

so seductive that mathematicians liked to believe that the question of infinity was definitively settled. If not indeed settled, then at least it had been caught with the right end.

However, physicists were aware that Cantor's paradise has very little or even nothing to do with the world around us, nothing in common with physics [21]. Additionally, also some mathematicians gradually started to realise the need to abandon that beautiful myth in order to create a set theory (and thus a theoretical basis for the whole of mathematics) in a simpler, more worldly, more secular fashion.

Most of Cantor's "supernatural" concepts, however, are unusable not only in the physical world but are also problematic in terms of mathematics itself. As an illustration, we can mention the concept of the set of real numbers. According to Cantor, real numbers form a set of cardinality aleph one, and therefore, there are many more of them than natural (and rational) numbers. However, the vast majority of these real numbers are "random"—numbers that can't be expressed by any (finite) decimal notation, or by any other mathematical means. They are the so-called uncalculable numbers.⁹ By title alone, "an uncalculable number" appears to be an oxymoron or even nonsense. Isn't it just a "maths joke", as declared in 1927 by Émile Borel? [22] Or is it referring to the unlimited mathematical God? Borel wasn't alone in questioning the meaningful existence of these "uncalculable numbers" and was joined by Charles Peirce and Kurt Gödel. At first, Cantor himself also hesitated. Finally, he decided to accept them and reached the uncountable, innumerable infinity. In a relatively simple way, Cantor later reached even greater infinities. All he needed to do was to create powers set, that is, sets of all subsets of the given set. This power set has a greater cardinality than the original set. In the creation of power sets, it is possible to continue without end. This is guaranteed by the so-called Cantor theorem, which is de facto an axiom-definition¹⁰ assigning existence to these sets.¹¹ The existence ensured by this "mathematical aid", however, is quite doubtful. It is just a proof of existence, unconstructive definition because we have to our disposal no real construction. "It's possible to do it, but no one can do it" (perhaps God).¹² This is a typical situation with many mathematical theorems, axioms, and proofs. However, is it legitimate to refer to God? What then is the reality of those infinities which create the "Tower of Babel"?¹³

Another flaw in Cantor's theory was that, on its basis, it wasn't possible to build a theory of infinitesimals.¹⁴ These infinitely small quantities already introduced in the seventeenth century by Newton and Leibniz were based solely on intuition. Mathematicians, however, failed to establish this concept exactly. So, they resorted to the definition using limits, that is, the proven potential-processual infinity. However, physicists, as well as some mathematicians,

⁹It was also Bolzano, who considered the existence of such numbers. According to Vopěnka: "Bolzano completed the real numbers". [20], p. 255

¹⁰As pointed out by David Hilbert, axioms are from another view definitions.

¹¹Later, mathematicians introduced axioms-definitions which were even "wilder" and allowed to "construct" (but rather prove the existence of) even larger infinities.

¹²There are many more of these non-constructive axioms in the theory, namely the axiom of choice.

¹³A similar situation occurs with mathematical functions, most of which also can't be described mathematically or otherwise. How is it then with their reality? [21].

¹⁴Cantor himself denied this possibility: "infinitely small did not exist"! However, Abraham Robinson later managed to construct the concept of infinitely small within the framework of Cantor's theory. His construction was, however, complicated and without any practical impact.

continued to use the non-exact, intuitive definition, and they successfully worked with it [23]. What does this mean for the importance of the foundations of mathematics?

Despite these reservations which remained only in the subconscious of mathematicians, the higher infinities were eventually adopted. Some mathematicians, however, didn't accept them, and their world didn't collapse. In a certain way, their world simplified.¹⁵ These mathematicians and physicists felt the need to get rid of a mythical Divine Implementer who (more or less manifestly) guaranteed the veracity of axioms and so also the existence of (actual) infinities. This idea was advanced by the Dutch mathematician Luzien Brouwer (1881–1966), the founder of the doctrine of mathematical intuitionism (close to constructivism) [24]. He rejected the concept of actual infinity as non-evident and non-intuitive and, by human means, non-constructible.¹⁶ With the refusal of Cantor's approach, Brouwer also eliminated many paradoxes. For example, for Brouwer, the mathematical continuum (line) isn't anything completed; it's "a media of free emergence".¹⁷ He also considered the dividing sphere as described by the paradox of Banach and Tarski as nonsense.¹⁸

With Brouwer, mathematics receives the highest possible intuitive clarity... With pain, a mathematician watches how the majority of theories climbing to a height fades in fog.

Herman Weyl

However, as it happens in a revolution, a new totalitarianism often takes hold in the name of freedom. Brouwer replaced the obsolete "bent" facts with new facts, intuitionistic facts which he understood equally dogmatically. He didn't avoid such oddities as the previously discussed concept of objectively existing random, that is, non-quantifiable numbers.¹⁹ The old myth is replaced by a new one and age-old prejudices with those new and unused. Brouwer condemned Platonism so he could quietly go back to it through the back door. However, before the return, he had swept out a lot of old junk from "Plato's world". Being revolutionary on the one hand and inconsistent on the other became fatal for Brouwer. The intuitionists' approach to mathematics was so idiosyncratic that most mathematicians rejected intuitionism as a "Bolshevist menace", as British mathematician Frank Ramsey (1903–1930) claimed. Mathematicians returned to their proven "rigid mathematical truths", to their traditionally sanctified myths and prejudices, to their mathematical God and to the "supernatural" axioms. They also returned to their "tower of infinities".

Brouwer also met with incomprehension in David Hilbert. In 1928, Hilbert withdrew him from the editorial board of the prestigious *Mathematische Annalen*. Brouwer then loudly

¹⁵The words "in a certain way" are alibism, of course. Something was simplified and conversely, something became more complicated.

¹⁶However, most present-day constructive mathematicians accept the reality of countable infinite sets. Nevertheless, there are exceptions, see Alexander Esenin-Volpin for a counter-example.

¹⁷At first glance it would seem that similar concepts were also held by Aristotle. Brouwer, however, understood "freedom" as independence from any rules. This inadvertently led him into a similar situation as Cantor with his "incalculable numbers".

¹⁸The paradoxical claim that a sphere can be divided into five parts and shifting them and turning them can create two spheres of identical size as the original. Although this can be proved no one can do it. So, what does such a purely existential proof show?

¹⁹It seems that it was the inconsistency that led to Brouwer's undoing.

doubted Hilbert's mental health. Albert Einstein was also in the dispute. From the perspective of philosopher and physicist, he labelled the dispute a farce and trifling.

3.2. Vopěnka's excommunication of God

We have demonstrated the crucial role God played in the epistemology of higher mathematics. Mathematics was perhaps the last scientific discipline dependent on divine inspiration. In the 1970s, a prominent role in systematic attempts to strip away theological motivations was played by Czech mathematician Petr Vopěnka [25, 26, 27]. He created quite a different set theory based on different assumptions. In his approach, Alternative Set Theory, Vopěnka demonstrated that set theory could be founded in quite a different way without Cantor's paradise, without the "Tower of Babel of infinities", even without an omniscient God. Vopěnka's new approach was not a manifestation of atheism, however, but only a demonstration of the fact that God (whether He exists or not) should play no role in human mathematics.²⁰ God's knowledge is for us inaccessible and mathematics isn't part of theology. Cantor's "higher" infinities are but a projection of our unabridged fantasy and divine inspiration.

Vopěnka did not ascribe reality to infinities of higher cardinals and formulated the concept of so-called general collapse, according to which it is possible to find a one-to-one correspondence between all infinities. (So, all infinite sets must be equivalent.) This step is in accordance with Bolzano's view, but in strong contradiction with Cantor's approach. However, the contradiction disappears when we reject ascribing reality to these incalculable, innumerable "real" numbers—to these "mathematical jokes".

And in his Alternative set theory, Vopěnka demonstrated that parallel concepts of infinite ensembles can be constructed. And also, that exact construction of infinitesimals is possible. On these ideas, he started to formulate new foundations of mathematical analysis [28–36]. He also derived philosophical consequences of his approach.

Does Vopěnka's approach mean that the Platonic realm of mathematical forms does not exist? That this realm (Cantor's or even the Pythagorean paradise) is only our fantasy, a non-obligatory notion? Does it mean that we can build up different mathematical realms and also different foundations of maths? Does it mean that we might accept the Aristotelean viewpoint (i.e. nominalism), in which Plato's ideas-forms are but our common names? Or does it mean that we should accept "postmodern" approach according to which "everything goes"?

However, Vopěnka's answer to these questions was emphatic no! He did not accept these courageous views. The reason was perhaps psychological. While young, he flirted with phenomenological philosophy and empiricism. When mature, he abandoned such ideas and returned to the "certainties of Platonism" or, more precisely, to certainties of "his own mathematical Platonism". He relied on the existence of "a single eternal mathematical realm", albeit no longer with the enticing recesses of that "ornamental neo-Baroque superstructure"—Cantor's paradise of sets. Shortly before Vopěnka left this world, he even stopped considering his Alternative Theory (with a capital "A") to being an alternative (with a lower case "a") and

²⁰Attracts motivated by atheism really appeared at the beginning of twentieth century. They originated by renowned French mathematicians.

regarded it as the only possible one. He completely rejected classical Cantorian set theory and renamed his own the “New set theory” because “there exists no alternative”. He replaced the old “religion of mathematics” with a new one, which was less divine and less metaphysical, but again, the only true one.

Vopěnka’s intellectual development reminds us of Nicolas Copernicus. Copernicus moved the centre of the universe from the Earth to the Sun and by this step (perhaps unconsciously) raised the question as to whether there actually is any centre at all. However, Copernicus himself was afraid of this overly revolutionary idea and he did not leave the bounded, final universe closed to the sphere of fixed stars. Vopěnka later cancelled Cantor’s theory as the sole basis of mathematics. By this step, he evoked the idea of whether mathematics should have a single, metaphysical (metamathematical) basis or whether it can be established with various alternative (better said, parallel) ways. Even the name of his theory directly points to alternativity. However, like Copernicus, Vopěnka also backed and marked his own theory as the only acceptable and therefore renamed it “The New” because “there is no alternative”, as Vopěnka proclaimed [37].

What about other mathematicians? We must admit that the impact of Vopěnka’s ideas was very limited.²¹ Tradition is firmly rooted in our collective unconsciousness and tends to be stronger than reason. Thanks to tradition, the old mathematical “religion” still survives, and the new approaches have only a negligible impact. Time will tell for how long.

3.3. Religion, mathematics and pragmatism

Within the framework of mathematics, physics and what we call “reality”, the nature of infinity is non-apparent and metaphysical, much like the nature of God. It is left to us to admit its existence or not. And the one we choose, the one we create is the one we are stuck with. Our choice cannot be entirely arbitrary, of course. It depends on the overall concept of mathematics, and possibly also on that of physics. So, it depends on the philosophy of mathematics and physics, that is, metaphysics. The choice of a metaphysical system is dependent not only on our rational considerations but also on our subconscious beliefs, on our faith, on our “religion” (or religion without quotation marks). However, it should first and foremost take into consideration its utility in real mathematics and physics, that is, how it can be used in practice. Infinity is primarily our tool to understand the world which is finite. This does not imply short-sighted pragmatism, of course. The usefulness of our tools might only be apparent after some period of time, and there is no way to determine it in advance. Reliance on intuition is the only possibility.

Let us recall the hesitant approach of young Bolzano towards religion. Under the influence of the Enlightenment and rationalism, he doubted the supernatural origin and content of the Biblical message as likely a myth, a fallacy. It seemed that Bolzano doubted the very basis of faith. However, he overcame his doubts and eventually became a preacher of that faith. He realised that the supposed divine origin isn’t what was the most important. More fundamental is the good that religion brings to mankind. What seems to be the real foundation is only a formal basis, while the true foundation lies elsewhere. Therefore, “not true but useful” or even “truth = usefulness”. This exact sentiment is echoed in the philosophy of pragmatism.

²¹So poor was the impact of other “alternative set theories”. For the overview see [38].

Imagine: If it is proved that no miracles happened and that everything is just a myth, then the church wouldn't collapse.²² However, if religion has nothing to say to people, it cannot dispense the "good", the cathedrals will become dead monuments over time and the church itself will die. The strength and weakness of faith lie in the practice of human life. A practice which elevates people, which is beneficial to them, which leads them to good and successful ends, such practice is also the right foundation and the right argument for faith and religion.

...it is quite indifferent if a certain doctrine of the church was established later, or if its formation and expansion are also due to a fallacy.

In religion, especially in God's revelation, it is not at all about what a thing itself is but what conception of it elevates us the most.

Bernard Bolzano, Autobiography [13]

Objective truth, in which the feature satisfying human desires doesn't have any role whatsoever, doesn't exist...

Independent truth is only the dead heart of an empty tree.

William James, Pragmatism [39]

Similarly, I believe that it is advisable to use the Bolzano pragmatic way to approach science, including mathematics. The formal "ontological under-building" of mathematics—the set theory—is a monumental work on which mathematicians practiced their art. Yet it is only a formal basis. What is important is what mathematics brings to mankind and what it could bring even if it were built on a different basis. If mathematicians discovered (or rather accepted) that the set theory is just a myth or fallacy and that there are no infinities, the mathematics wouldn't collapse. If, however, mathematics couldn't be used in the earthly world, if it weren't beneficial for practice (therefore not providing "Bolzano's good"), it would become a purposeless game, such as chess, checkers and go. The strength of mathematics lies in the practice of life and it is this practice—that is real, applied mathematics—that is the most real justification and foundation for the entire of mathematics.

The above-mentioned conclusions can be illustrated by two examples. First, we can examine the pragmatic approach of Abraham Robinson, the founder of non-standard analysis. In 1964, he concluded that actual infinity doesn't exist—neither in a real nor in an ideal sense. However, Robinson reassures us: the absence of actual infinity means nothing for practice and for mathematics! We are supposed to ignore this fact and pretend that nothing is happening! [40]

The second example brings us back to Petr Vopěnka. He approached the expected absence of actual infinity (represented by infinite sets) quite differently. He deemed it necessary to redo all the foundations of mathematics. And he undertook this ambitious task himself. The effect, however, wasn't as significant as he probably expected. Those new foundations did not provide much that couldn't be obtained from the disputed traditional foundations.

²²I mean mainly Catholicism. Newer denominations such as Protestantism and Islam approach the miracles in a more sceptical way. Finally, Mohammed performed no miracles and "rational" Thomas Aquinas thought it was a symptom of his inferiority!

What do these cases demonstrate regarding the importance of the foundations of mathematics? What does that say about the conception of infinity?

There may be very different mathematical foundations as well as different superficial details if the results needed for the real world can rely on them.

Richard W. Hamming [41]

We have found a strange footprint on the shores of the unknown. We have devised profound theories, one after another, to account for its origins. At last, we have succeeded in reconstructing the creature that made the footprint. And lo! It is our own.

Arthur Eddington [42]

4. Conclusion: science, theology and epistemology

Throughout history, we are aware of many examples wherein theology hampered the development of philosophy and science. This applies especially to the fundamentalistically understood religions which were doing a disservice to astronomy and to all natural science, in fact. As we have seen, the situation was different in mathematics. Monotheistic religion served with an inspiring vision of an all-knowing God who was transformed into “the god of mathematicians”.

Mathematicians relied on God; He was the guarantor of the metaphysical world of mathematics. Throughout the advent of modern times, mathematicians slowly began to realise that mathematics isn't a part of or an extension of theology. They gradually left the theological “proofs of infinity”; more specifically, they stopped talking about God. However, God survived in the depths of their collective unconsciousness. Most mathematicians never got rid of this hidden God or “supernatural concepts”—axioms guaranteeing the existence of large, highly “unnatural” infinities, the continuum hypotheses, the axiom of choice, etc. By declaring these “holy truths” through their mouths, they only provided God with a disguise.

In the mid-twentieth century, some mathematicians (or rather philosophers of mathematics) timidly began to realise the limitation of this Theo-Platonic approach. They emboldened themselves and tried to formulate the “God-free” foundations of mathematics. Nevertheless, on a deeper philosophical layer, Platonism still was hidden there (Brouwer, Vopěnka...). One “mathematical religion” was exchanged for another. One that was less theological and less metaphysical, perhaps “more scientific”, but again it was the only real and true. Despite this effort towards new foundations of mathematics, the practical effect was negligible. Mathematicians are conservative people; like most of us, they don't like to abandon their prejudices. However, there is a more rational argument: their mathematics “works”. However, this is not an argument against different foundations of mathematic, because “working mathematics” can also be based completely differently. The reason is that the factual foundation of mathematics is the real mathematics, that is, applied maths.

But, were the substantial effort of Georg Cantor and other mathematicians all for naught? Was it all leading to a dead end? I am not of this view. Even if Cantor's paradise vanished for good,

even if it became a mere myth or simply a beautiful dream, the metaphysical excursions of nineteenth and twentieth century mathematics would not have been in vain. Neither would the pilgrimage to “God’s mind”, and to “divine infinities”. This journey provided a great deal of experience and inspiring insights. Mathematics returns from these voyages changed, much richer, more experienced and much stronger.

Although I accept the practice as the ultimate criterion, I am aware that the first-pragmatism isn’t sufficient for the foundation of science. Neither mathematics nor science can do without a metaphysical overlap, without “myth” or “religion”. We as people can’t do without it either. Only metaphysics, the “irradiation of a spiritual light”, provides an understanding of things and incorporate them to science and to our life. Only metaphysics creates sense and provides a story.

Just as the light of the sun irradiates the organ of vision and things visible, enabling the former to see and the latter to be seen, so too the irradiation of a spiritual light brings the mind into relation with that which is intelligible.

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²³Commentary on Aristotle’s *Posterior Analytics*, i.17 as quoted by Stevenson [43].

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