Chapter 2

Shape Memory Wires in R^3

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Additional information is available at the end of the chapter

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Abstract

We propose a new model describing the dynamics of wire made of shape memory alloys, by combining an elastic curve theory and the Ginzburg-Landau theory. The wire is assumed to be a closed curve and is not to be stretched with deformation. The derived system of nonlinear partial differential equations consists of a thermoelastic system and a geometric evolution equation under the inextensible condition. We also show that the system has dual variation structure as well as a straight material case. The structure implies stability of infinitesimally stable stationary state in the Lyapunov sense.

Keywords: shape memory alloys, elastic curve, thermoelastic system, nonlinear partial differential equations, Ginzburg-Landau theory, phase transition, stability, dual variation principle

1. Introduction

Shape memory effect arises from the phase transition of lattice structure. Although there are many models for shape memory alloys, one of the classical model is proposed by Falk, which we call the **Falk model**. Falk applied the Ginzburg-Landau theory for phase transition to shape memory alloys by regarding shear strain ϵ as an order parameter (see e.g., [1]). That is, the Helmholz free energy density proposed by Falk is given by

$$\begin{aligned}\tilde{f}(\epsilon,\theta,\partial_x^2 u) &:= \frac{1}{2}|\partial_x^2 u|^2 + f(\epsilon,\theta) + f_0(\theta)\\ &= \frac{1}{2}|\partial_x^2 u|^2 + \left(\frac{1}{6}|\epsilon|^6 - \frac{1}{4}|\epsilon|^4 + \frac{1}{2}(\theta-\theta_c)|\epsilon|^2\right) + f_0(\theta)\end{aligned} \tag{1}$$

where u and θ are displacement and absolute temperature, respectively, and a positive constant θ_c denotes the critical temperature of the phase transition. We call the first term $\frac{|\partial_x^2 u|^2}{2}$ **curvature energy density**, the second term $f(\epsilon, \theta)$ **nonlinear elastic energy density** and the third term $f_0(\theta)$ **thermal energy density**. In other words, Falk represented the phase transition by using the form of the nonlinear elastic energy density. We also remark that for simplicity, all physical constants without the critical temperature are normalized by unity. The Falk model was proposed for straight materials. In the model, the material is built up by a stack of layers parallel to the so-called **habit plane** (see [2]) and assumed that the displacement u in that direction to depend only on a coordinate x perpendicular to the habit plane, that is, the variable x runs in the stacking direction. Then, the material conserves its volume.

From the point of the small deformation theory, we may use a linearized approximation relation $\epsilon = \partial_x u$. Moreover, we take f_0 as the following typical form:

$$f_0(\theta) := \theta - \theta\log\theta.$$

Then following a standard procedure of derivation of thermoelastic system (see e.g., [3]), we can derive the system of nonlinear partial differential equations called the **Falk model**:

$$\begin{cases}\partial_t^2 u + \partial_x^4 u = \partial_x\{(\partial_x u)^5 - (\partial_x u)^3 + (\theta-\theta_c)\partial_x u\},\\ \partial_t\theta - \partial_x^2\theta = \theta\partial_x u\,\partial_t\partial_x u.\end{cases} \tag{2}$$

Here, unknowns are displacement u and absolute temperature θ, and ∂_x and ∂_t represent partial differential operator with respect to x and t, respectively. The model is well known as one of classical models describing shape memory alloys. For the other models, we refer Fremond [4], Fremond-Miyazaki [5] and reference therein. The Falk model (Eq. (2)) has been studied actively in the mathematical literature. In the isothermal case, well-posedness, stability of solitary-wave solution, existence of steady state, travelling wave solution and invariant measure have been investigated by Fang-Grillakis [6], Falk-Laedke-Spatschek [7], Friedman-Sprekels [8], Garcke [9] and Tsutsumi-Yoshikawa [10], respectively. For the full system (Eq. (1)), the well-posedness results are found in, for example, [11–13] and so on, and numerical results are found in Hoffmann-Zou [14], Niezgodka-Sprekels [15] by finite element method and in Matus-Melnik-Wang-Rybak [16] and Yoshikawa [17, 18] by the finite difference method. In particular, in Ref. [19], the stability of steady state in the Lyapunov sense was shown. More precisely, the stationary state of Eq. (2) is expressed as a nonlocal nonlinear elliptic problem. If there exists a linearized stable critical point for the functional corresponding to the elliptic problem, then for each neighbourhood U of the equilibrium, we can find a neighbourhood W of the equilibrium such that the solution of Eq. (2) with the initial data in W stays in U for any time. The proof can be shown by the **dual variation principle** which appears in most of the models in non-equilibrium statistical thermodynamics (see [20]).

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The existence of several non-trivial steady states for low-temperature phase and low-energy case is proved in Ref. [21]. The numerical simulation given in Ref. [17] exactly indicates the properties mentioned above. The dual variation structure appears also in a multi-dimensional case [22]; however, well-posedness of the multi-dimensional model corresponding to Eq. (2) is still open in large initial data case due to the propagation of singularity. That is one of our motivations of this problem.

We mention mathematical studies on the motion of curves governed by geometric evolution equations. One of the typical objects is curve-shortening flow derived as an L^2 gradient flow for the length functional of curve γ:

$$L(\gamma) := \int_\gamma ds$$

where s denotes the arc length parameter of γ. By Gage [23], Gage and Hamilton [24] and Grayson [25], it is well known that the curve-shortening flow shrinks simple closed curves to a point in a finite time. Since the curve-shortening flow can be regarded as a one-dimensional case of mean curvature flow for surfaces, the flow is applicable to various mathematical analysis. For example, the curve-shortening flow plays an important role in studies on phase transition. We also mention the curve-straightening flow which has been attracted a great interest and studied actively in mathematical literature. The flow is derived as an L^2 gradient flow for the elastic energy

$$K(\gamma) := \frac{1}{2}\int_\gamma \kappa^2 \, ds$$

where κ denotes the scalar curvature of γ. It is well known that the flow is applicable to studies on elastic curve inspired by Bernoulli and Euler. Indeed, the curve-straightening flow under the length constraint $L(\gamma) \equiv C$ converges to a classical elastic curve so-called elastica. There is also an interest in the study on motion of curves governed by the L^2-gradient flow for E under the inextensible condition. Under the condition, the length constraint $L(\gamma) \equiv C$ is also satisfied for the condition means that the curve does not stretch. As we will state in Section 2.2, the constraint is imposed on each point of curves. Thus, a standard Lagrange multiplier theory does not work. Therefore, we have to make use of geometric properties of curves governed by the flow.

The purpose of this chapter is to derive a mathematical model describing thermoelastic deformation of shape memory wire in $\boldsymbol{R}^3$. In particular, we regard the wire as a closed space curve satisfying the inextensible condition. From the physical point of view, it may be unnatural that the wire does not stretch. However, the contribution of this chapter is to adopt a geometric analysis into a classical thermoelastic theory with phase transition inspired by Falk.

2. Setting and derivation of equations

We denote the closed curve representing shape of wire by $\Gamma = \{\gamma(\xi)\colon \xi \in \Xi\}$, where the variable ξ is an arbitrary parameter not necessarily the arc length parameter. Let us define a displacement vector from a point ξ in an original shape $\Gamma^0 = \{\gamma^0(\xi)\colon \xi \in \Xi\}$ by $u(\xi)$ (see **Figure 1**); namely, it holds that

$$\gamma(\xi) = \gamma^0(\xi) + u(\xi)$$

for

- $\gamma(\xi) := \left(\gamma_1(\xi),\ \gamma_2(\xi),\ \gamma_3(\xi)\right)$: vector representing the shape,
- $\gamma^0(\xi) := \left(\gamma_1^0(\xi), \gamma_2^0(\xi), \gamma_3^0(\xi)\right)$: vector representing the original shape,
- $u(\xi) := \left(u_1(\xi), u_2(\xi), u_3(\xi)\right)$: displacement vector.

Throughout this chapter, we denote by L the length of Γ^0, and hence, the length of Γ is also L from the non-stretching assumption. To apply the idea by Falk, we need to determine the form of strain and free energy (Eq. (1)) suitable for this setting.

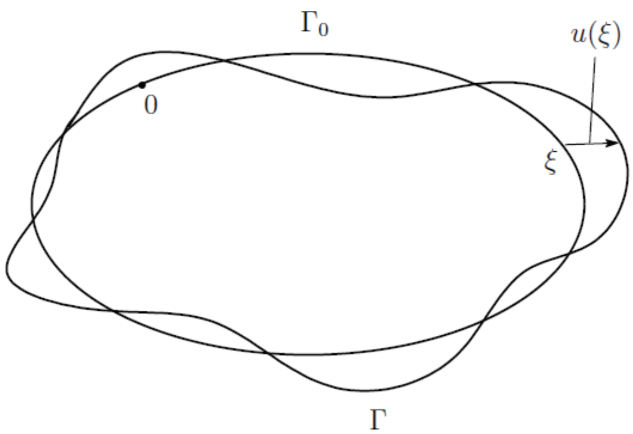

Figure 1. Original shape Γ_0 and deformed curve Γ.

2.1. Definition of strain

We first consider the strain. Let $\gamma^0(\xi)$ be a space closed curve, where ξ is a parameter (not necessary to be the arc length parameter). For $\gamma^0\ (\xi)$, we define the displacement vector by $u(\xi)$, and we denote $\gamma(\xi) = \gamma^0(\xi) + u(\xi)$. Since the relation "strain ≈ line element" holds, let us first pursue line element between $\gamma^0\ (\xi)$ and $\gamma\ (\xi)$. From the direct calculation, we have

$$\begin{aligned} |\gamma'(\xi)|^2 - |\gamma^{0'}(\xi)|^2 &= \{\gamma^{0'}(\xi) + u'(\xi)\} \cdot \{\gamma^{0'}(\xi) + u'(\xi)\} - \gamma^{0\prime}(\xi) \cdot \gamma^{0'}(\xi) \\ &= 2\gamma^{0'}(\xi) \cdot u'(\xi) + |u'(\xi)|^2. \end{aligned}$$

Here, if we assume a smallness of deformation, then we may assume

$$|\gamma^{0'}(\xi) \cdot u'(\xi)| \gg |u'(\xi)|^2.$$

From now on, we regard the strain as

$$\gamma^{0'}(\xi) \cdot u'(\xi).$$

2.2. Definition of energy functional

Let $\gamma^0(\xi)$ be an initial closed curve and $\gamma(\xi, t)$ denote a family of closed curves starting from $\gamma^0(\xi)$. Recalling that the arc length parameter $s(\xi, t)$of $\gamma(\xi, t)$ is given by

$$ds = \left|\frac{\partial\gamma}{\partial\xi}\right| d\xi,$$

we can write kinetic energy $M(\gamma)$ of γ as

$$M(\gamma) := \oint \left|\frac{\partial\gamma}{\partial t}\right|^2 \left|\frac{\partial\gamma}{\partial\xi}\right| d\xi.$$

In a similar manner, thermal energy is defined by

$$F_0(\gamma) := \oint f_0(\xi) \left|\frac{\partial\gamma}{\partial\xi}\right| d\xi,$$

and the curvature energy is expressed as

$$K(\gamma) := \oint \kappa^2 \left|\frac{\partial\gamma}{\partial\xi}\right| d\xi.$$

Observe that the scalar curvature κ is written as

$$\kappa = \left\{ \left|\frac{\partial\gamma}{\partial\xi}\right| \frac{\partial^2\gamma}{\partial\xi^2} - \left(\frac{\partial\gamma}{\partial\xi} \cdot \frac{\partial^2\gamma}{\partial\xi^2}\right) \frac{\partial\gamma}{\partial\xi} \right\} \left|\frac{\partial\gamma}{\partial\xi}\right|^{-3}.$$

Lastly, the nonlinear elastic energy density is given by

$$f(\partial_\xi \gamma,\ \theta; \partial_\xi \gamma^0) := \frac{1}{6}\left(\frac{\partial\gamma^0}{\partial\xi} \cdot \frac{\partial u}{\partial\xi}\right)^6 - \frac{1}{4}\left(\frac{\partial\gamma^0}{\partial\xi} \cdot \frac{\partial u}{\partial\xi}\right)^4 + \frac{1}{2}(\theta - \theta_c)\left(\frac{\partial\gamma^0}{\partial\xi} \cdot \frac{\partial u}{\partial\xi}\right)^2$$

and then the nonlinear elastic energy is written as

$$F(\partial_\xi\gamma, \theta; \partial_\xi\gamma^0) := \oint f(\partial_\xi\gamma,\ \theta; \partial_\xi \gamma^0) \left|\frac{\partial\gamma}{\partial\xi}\right| d\xi$$

where $u = \gamma - \gamma^0$. Thus, we obtain the Helmholtz energy for our setting as

$$H(\gamma, \theta, \gamma^0) := M(\gamma) + K(\gamma) + F(\partial_\xi\gamma, \theta; \partial_\xi\gamma^0) + F_0(\theta).$$

From now on, let $s \in \boldsymbol{R}/L\boldsymbol{Z} =: S_L^1$ be the arc length parameter of the initial closed curve $\gamma^0 = \gamma^0(s)$. It follows from the property of arc length parameter that $|\gamma^{0'}(s)| \equiv 1$. In a similar fashion to the above equation, $\gamma(s, t)$ means the closed curve deformed along evolution from $\gamma^0(s)$. Moreover, in what follows, we assume that $\gamma(s, t)$ satisfies

$$|\partial_s \gamma(s,t)| \equiv 1 \tag{3}$$

which means "s is arc length parameter of γ not only the initial time but also every time t". From the assumption, we can rewrite M, F_0 and F shortly as

$$M(\gamma) = \int_0^L |\ \partial_t \gamma\ |^2\ ds,$$

$$F_0(\theta) = \int_0^L f_0(\theta) ds,$$

$$F(\partial_s\ \gamma, \theta; \partial_s\ \gamma^0) = \int_0^L f(\partial_s \gamma, \theta; \partial_s \gamma^0)\ ds.$$

Moreover, since $\partial_s \gamma \cdot \partial_s^2 \gamma = 0$, from Eq. (3), it holds that

$$K(\gamma) = \int_0^L |\ \partial_s^2 \gamma|^2 ds.$$

Therefore, the Helmholtz free energy density is denoted by

$$H(t) := \frac{1}{2} ||\partial_t \gamma(\cdot, t)||^2_{L^2(S^1_L)} + \frac{1}{2} ||\partial_s^2 \gamma(\cdot, t)||^2_{L^2(S^1_L)} + F\Big(\partial_s \gamma(\cdot, t),\ \theta(\cdot, t); \partial_s \gamma^0(\cdot)\Big) + F_0\Big(\theta(\cdot, t)\Big).$$

We will explain that for the free energy under some assumptions, the following system of nonlinear partial differential equations is derived:

$$\begin{cases} \partial_t^2 \gamma + \partial_s^4 \gamma + \partial_s f_{,\partial_s \gamma}(\partial_s \gamma,\ \theta; \partial_s \gamma^0) - \partial_s\ \{(v - 2\ |\partial_s^2 \gamma|^2)\ \partial_s \gamma\} = 0, \\ -\ \partial_s^2 v\ +\ |\partial_s^2 \gamma|^2 v = 2\ |\partial_s^2 \gamma|^4 - |\partial_s^3 \gamma|^2 + |\partial_s \partial_t \gamma|^2 + \partial_s^2 f_{,\partial_s \gamma}(\partial_s \gamma,\ \theta; \partial_s \gamma^0) \cdot \partial_s \gamma, \\ \partial_t \theta\ - \partial_s^2 \theta = \theta\Big(\partial_s \gamma^0 \cdot \partial_s(\gamma - \gamma^0)\Big)(\partial_t \partial_s \gamma \cdot \partial_s \gamma^0) \end{cases} \tag{4}$$

where

$$\begin{aligned} f_{,\partial_s \gamma}(\partial_s \gamma,\ \theta; \partial_s\ \gamma^0) &= \left(\frac{\partial f}{\partial_s \gamma_1}, \frac{\partial f}{\partial_s \gamma_2}, \frac{\partial f}{\partial_s \gamma_3}\right) \\ &= \{(\ \partial_s \gamma^0 \cdot \partial_s u\ )^5 - (\ \partial_s \gamma^0 \cdot \partial_s u\ )^3 + (\theta - \theta_c) \partial_s \gamma^0 \cdot \partial_s u\ \} \partial_s \gamma^0. \end{aligned}$$

2.3. Equation of motion

By using the Hamilton principle, we derive an equation of the motion of γ. Namely, we will derive the Euler-Lagrange equation for the functional:

$$\tilde{H}(\gamma,\ \theta; \gamma^0) = \int_{t_2}^{t_1} \left\{ \frac{1}{2} ||\partial_t \gamma(\cdot, t)||^2_{L^2(S^1_L)} + \frac{1}{2} ||\partial_s^2 \gamma(\cdot, t)||^2_{L^2(S^1_L)} - F\Big(\partial_s \gamma(\cdot, t),\ \theta(\cdot, t); \partial_s \gamma^0(\cdot)\Big) - F_0\Big(\theta(\cdot, t)\Big) \right\} dt. \tag{5}$$

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Let us denote the variation of γ by

$$\gamma(s,t;\theta) := \gamma(s,t) + \varepsilon\,\varphi(s,t)$$

where ε is a sufficiently small positive parameter and φ is sufficiently smooth and satisfies $\varphi(s, t_1) = \varphi(s, t_2) = 0$. Moreover, from the assumption Eq. (3), it is also necessary to hold that

$$\frac{d}{d\varepsilon}|\partial_s\gamma(s,t;\varepsilon)|\bigg|_{\varepsilon=0} = 0.$$

Since

$$\frac{d}{d\varepsilon}|\partial_s\gamma(s,t;\theta)|\bigg|_{\varepsilon=0} = \partial_s\gamma(s,t)\cdot\partial_s\varphi(s,t)$$

φ has to satisfy

$$\partial_s\gamma(s,t)\cdot\partial_s\varphi(s,t) = 0$$

for any $s \in S_L^1$ and $t > 0$. Calculating the first variation of the energy functional, we have

$$\frac{d}{d\varepsilon}\tilde{H}(\gamma,\ \theta;\gamma^0)\bigg|_{\varepsilon=0} = \int_{t_1}^{t_2}\{\langle\partial_t\gamma,\partial_t\varphi\rangle - \langle\partial_s^2\gamma,\partial_s^2\varphi\rangle - \langle f_{,\partial_s\gamma}(\partial_s\gamma,\theta;\partial_s\gamma^0),\partial_s\varphi\rangle\}dt.$$

From the integral by parts, the right-hand side is rewritten as follows:

$$-\int_{t_1}^{t_2}\langle\partial_t^2\gamma + \partial_s^4\gamma - \partial_s f_{,\partial_s\gamma}(\partial_s\gamma,\theta;\partial_s\gamma^0),\varphi\rangle dt. \tag{6}$$

Then the integral Eq. (6) is equal to 0 for any φ satisfying $\varphi(s,t_1) = \varphi(s,t_2) = 0$ and $\partial_s\gamma\cdot\partial_s\varphi \equiv 0$. For the purpose, we define

$$V := \{\varphi \mid \partial_s\gamma\cdot\partial_s\varphi \equiv 0\}.$$

The orthogonal complement $V^\perp$ of the space V with respect to $L^2(S_L^1)$ inner product is given by

$$V^\perp = \{\partial_s(w\partial_s\gamma)w = w(s,t) \text{ is a scalar function}\}. \tag{7}$$

Here, we remark that in the case where γ is a curve embedded in three-dimensional space (not a planar curve), $\partial_s^2\gamma \neq \mathbf{0}$ has to be satisfied for every $(s,t) \in S_L^1 \times \mathbf{R}_+$. In the end of this section, we will show the reason why $V^\perp$ is given as above. Consequently, if for the direction $\partial_t^2\gamma + \partial_s^4\gamma - \partial_s f_{,\partial_s\gamma}(\partial_s\gamma,\theta;\partial_s\gamma^0)$, there exists a scalar function $w = w(s,\ t)$ such that

$$\partial_t^2\gamma + \partial_s^4\gamma - \partial_s f_{,\partial_s\gamma}(\partial_s\gamma, \theta; \partial_s\gamma^0) = \partial_s(w\partial_s\gamma) \tag{8}$$

then Eq. (6) is equal to 0.

Next, we derive the equation for w. From the assumption Eq. (3), we see that

$$0 = \partial_t^2|\partial_s\gamma|^2 = 2\partial_s\partial_t^2\gamma \cdot \partial_s\gamma + 2|\partial_s\partial_t\gamma|^2$$

It follows from Eq. (8) that

$$\{-\partial_s^5\gamma + \partial_s^2 f_{,\partial_s\gamma}(\partial_s\gamma, \theta; \partial_s\gamma^0) + \partial_s^2(w\partial_s\gamma)\} \cdot \partial_s\gamma = -|\partial_s\partial_t\gamma|^2. \tag{9}$$

Differentiating Eq. (3), we obtain

$$\begin{aligned}
&\partial_s\gamma \cdot \partial_s^2\gamma = 0,\\
&\partial_s\gamma \cdot \partial_s^3\gamma = -|\partial_s^2\gamma|^2,\\
&\partial_s\gamma \cdot \partial_s^4\gamma = -\frac{3}{2}\partial_s(|\partial_s^2\gamma|^2),\\
&\partial_s\gamma \cdot \partial_s^5\gamma = -\ 2\partial_s^2(|\partial_s^2\gamma|^2) +\ |\partial_s^3\gamma|^2.
\end{aligned}$$

By the relations, we will rewrite Eq. (9). It follows from the direct calculation that

$$\partial_s^2\ (w\partial_s\gamma) \cdot \partial_s\gamma = \partial_s^2 w\ -\ w|\partial_s^2\gamma|^2.$$

Since from the definition

$$\begin{aligned}
&f_{,\partial_s\gamma}(\partial_s\gamma,\ \theta; \partial_s\gamma^0)\\
&= \left\{ \left(\ \partial_s\gamma^0 \cdot \partial_s(\gamma - \gamma^0)\ \right)^5 - \left(\ \partial_s\gamma^0 \cdot \partial_s(\gamma - \gamma^0)\ \right)^3 + (\theta - \theta_c)\left(\partial_s\gamma^0 \cdot \partial_s(\gamma - \gamma^0)\right) \right\}\partial_s\gamma^0
\end{aligned}$$

we also obtain

$$\begin{aligned}
&\partial_s^2 f_{,\partial_s\gamma}(\partial_s\gamma,\ \theta; \partial_s\gamma^0) \cdot \partial_s\gamma\\
&= \partial_s^2\left[\left\{ \left(\partial_s\gamma^0 \cdot \partial_s(\gamma - \gamma^0)\right)^5 - \left(\partial_s\gamma^0 \cdot \partial_s(\gamma - \gamma^0)\right)^3 + (\theta - \theta_c)\left(\partial_s\gamma^0 \cdot \partial_s(\gamma - \gamma^0)\right) \right\}\partial_s\gamma^0 \right] \cdot \partial_s\gamma.
\end{aligned}$$

Therefore, substituting these into Eq. (9), we find

$$\begin{aligned}
0 = 2\partial_s^2(|\partial_s^2\gamma|^2) - |\partial_s^3\gamma|^2 + \partial_s^2 w - w|\partial_s^2\gamma|^2 + |\partial_s\partial_t\gamma|^2 + \partial_s^2\Bigg[\Bigg\{ \left(\partial_s\gamma^0 \cdot \partial_s(\gamma - \gamma^0)\right)^5\\
-\left(\ \partial_s\gamma^0 \cdot \partial_s\ (\gamma - \gamma^0)\right)^3 + (\theta - \theta_c)\left(\partial_s\ \gamma^0 \cdot \partial_s\ (\gamma - \gamma^0)\right) \Bigg\}\partial_s\gamma^0 \Bigg] \cdot \partial_s\gamma.
\end{aligned}$$

Here setting the new unknown v by $v := w + 2|\partial_s^2\gamma|^2$, we can rewrite the equation as follows:

$$-\partial_s^2 v + |\partial_s^2\gamma|^2 v = 2\,|\partial_s^2\gamma|^4 - |\partial_s^3\gamma|^2 + |\partial_s\partial_t\gamma|^2$$
$$+\partial_s^2\left[\left\{\left(\partial_s\gamma^0\cdot\partial_s(\gamma-\gamma^0)\right)^5 - \left(\partial_s\gamma^0\cdot\partial_s(\gamma-\gamma^0)\right)^3 + (\theta-\theta_c)\left(\partial_s\gamma^0\cdot\partial_s(\gamma-\gamma^0)\right)\right\}\partial_s\gamma^0\right]\cdot\partial_s\gamma.$$

Consequently, under given temperature θ, the equation of motion is given by

$$\begin{cases} \partial_t^2\gamma + \partial_s^4\gamma - \partial_s f_{,\partial_s\gamma}(\partial_s\gamma,\theta;\partial_s\gamma^0) - \partial_s\{(v-2|\partial_s^2\gamma|^2)\partial_s\gamma\} = 0, \\ -\partial_s^2 v + |\partial_s^2\gamma|^2 v = 2|\partial_s^2\gamma|^4 - |\partial_s^3\gamma|^2 + |\partial_s\partial_t\gamma|^2 - \partial_s^2 f_{,\partial_s\gamma}(\partial_s\gamma,\ \theta;\partial_s\gamma^0)\cdot\partial_s\gamma. \end{cases}$$

At the rest of this section, we prove that the orthogonal complement of V is given by Eq. (7).

Lemma 1. *Let $\gamma(s,t)$ be a smooth curve in $\boldsymbol{R}^3$ and $s\in S_L^1$ be an arc length parameter of γ for any t. Suppose that $\partial_s^2\gamma\neq\boldsymbol{0}$ holds for all $(s,t)\in S_L^1\times\boldsymbol{R}_+$ then the orthogonal complement $V^\perp$ of the space $V=\{\varphi|\partial_s\gamma\cdot\partial_s\varphi\equiv 0\}$ with respect to $L^2(S_L^1)$ inner product is represented by*

$$V^\perp = \{\partial_s(w\partial_s\gamma)w = w(s,t) \text{ is a scalar function}\}.$$

Proof. Observe that $\partial_s\gamma$, $\partial_s^2\gamma$ and the outer product $\partial_s\gamma\times\partial_s^2\gamma$ are orthogonal each other. Under the assumption $\partial_s^2\gamma\neq\boldsymbol{0}$, a coordinate system defined on γ consists of the vectors. Then arbitrary vector $\eta=\eta(s,t)$ can be represented as

$$\eta(s,t) = \eta_1(s,t)\partial_s\gamma(s,t) + \eta_2(s,t)\,\partial_s^2\gamma(s,t) + \eta_3(s,t)\,\partial_s\gamma(s,t)\times\partial_s^2\gamma(s,t).$$

If we assume additionally $\eta\in V$, then we obtain

$$\begin{aligned} 0 &= \partial_s\eta\cdot\partial_s\gamma \\ &= \{\partial_s\eta_1\partial_s\gamma + \eta_1\partial_s^2\gamma + \partial_s\eta_2\partial_s^2\gamma + \eta_2\partial_s^3\gamma + \partial_s\eta_3\partial_s\gamma\times\partial_s^2\gamma + \eta_3\,(\partial_s^2\gamma\times\partial_s^2\gamma + \partial_s\gamma\times\partial_s^3\gamma)\}\cdot\partial_s\gamma \\ &= \partial_s\eta_1 + \eta_2\partial_s\gamma\cdot\partial_s^3\gamma \\ &= \partial_s\eta_1 - \eta_2|\partial_s^2\gamma|^2 \end{aligned}$$

that is

$$\partial_s\eta_1 = |\partial_s^2\gamma|^2\eta_2. \tag{10}$$

In other words, an element of V consists of η_1, η_2 satisfying Eq. (10) and arbitrary η_3. However, we remark that we cannot take η_2 freely. Indeed, in order to verify L periodicity of η_1, η_2, we have to show the following condition:

$$\int_0^L |\partial_s^2\gamma|^2\eta_2 ds = 0. \tag{11}$$

If $\zeta(s,t) = \zeta_1(s,t)\partial_s\gamma(s,t) + \zeta_2(s,t)\partial_s^2\gamma(s,t) + \zeta_3(s,t)\partial_s\gamma(s,t) \times \partial_s^2\gamma(s,t)$satisfies $\langle\zeta, \ \eta\rangle = 0$, then we see that η_1 and η_2 satisfy Eqs. (10) and (11) and any η_3 satisfies

$$\int_0^L \{\zeta_1\eta_1 + \zeta_2\eta_2|\partial_s^2\gamma|^2 + \zeta_3\eta_3(\partial_s\gamma \times \partial_s^2\gamma)^2\}ds = 0. \tag{12}$$

In particular, if we assume $\eta_2 \equiv 0$ and $\eta_3 \equiv 0$, then we infer from Eq. (10) that $\eta_1 \equiv C$ holds true. Then, we deduce from Eq. (12) that

$$\int_0^L \zeta_1 ds = 0.$$

Now we define

$$\varphi(s,t) = \zeta_1(0,t) + \int_0^s \zeta_1(s,t)ds.$$

Then, φ has the period L and satisfies $\partial_s\varphi = \zeta_1$. Substituting it into Eq. (12) and using Eq. (10), we have

$$\begin{aligned} 0 &= \int_0^L \{\partial_s\varphi\ \eta_1 + \zeta_2\eta_2|\partial_s^2\gamma|^2 + \zeta_3\eta_3(\partial_s\gamma \times \partial_s^2\gamma)^2\}ds \\ &= \int_0^L \{-\varphi\partial_s\eta_1 + \zeta_2\eta_2|\partial_s^2\gamma|^2 + \zeta_3\eta_3(\partial_s\gamma \times \partial_s^2\gamma)^2\}ds \\ &= \int_0^L \{-\varphi\ |\partial_s^2\gamma|^2\eta_2 + \zeta_2\eta_2|\partial_s^2\gamma|^2 + \zeta_3\eta_3(\partial_s\gamma \times \partial_s^2\gamma)^2\}ds \\ &= \int_0^L \{(-\varphi + \zeta_2)\eta_2|\partial_s^2\gamma|^2 + \zeta_3\eta_3(\partial_s\gamma \times \partial_s^2\gamma)^2\}ds. \end{aligned}$$

Recalling Eq. (11), we see that the vector-valued function (η_2, η_3) is orthogonal with $(|\partial_s^2\gamma|^2, 0)$ in the sense of L^2 inner product. Therefore, there exists some function $\mu = \mu(t)$ depending only on t such that

$$\left(\ \{-\varphi + \zeta_2\}\ |\partial_s^2\gamma|^2,\ (\partial_s\gamma \times \partial_s^2\gamma)^2\zeta_3\right) = \mu(|\partial_s^2\gamma|^2, 0)$$

that is,

$$-\varphi + \zeta_2 = \mu,\ \ \zeta_3 \equiv 0. \tag{13}$$

Setting

$$\mu + \varphi(s,t) = w(s,t),$$

the function $w(s, t)$ is the L periodic function and satisfies $\partial_s w = \zeta_1$. It follows from Eq. (13) that

$$\zeta_2(s,t) = w(s,t).$$

Then $\zeta(s)$ is orthogonal with elements of V with respect to L^2 inner product. Thus, we get

$$\zeta(s,t) = \partial_s w(s,t)\partial_s\gamma(s,t) + w(s,t)\partial_s^2\gamma(s,t) = \partial_s\Big(w(s,t)\partial_s\gamma(s)\Big)$$

which completes the proof.

Q.E.D.

Remark 1. The assumption $\partial_s^2\gamma \neq \mathbf{0}$ in Lemma 1 means that the curvature is always non-zero. If $\partial_s^2\gamma = \mathbf{0}$ at some point, we cannot determine the tangential vector $\partial_s\gamma$ at the point uniquely. Therefore, we need the assumption in order to give a coordinate system at every point of Γ. On the other hand, in the case of a planar curve, we do not need the assumption. Indeed, by rotating the tangential vector, we can construct a coordinate system.

Remark 2. We mention the elastic flow with the inextensible condition (Eq. (3)), more precisely, L^2 gradient flow for $K(\gamma)$ under the constraint (Eq. (3)). To the best of our knowledge, the problem was first considered by N. Koiso [26] for planar closed curves. With the aid of smoothing effect of the elastic energy E, the Cauchy problem on the elastic flow has a unique classical solution and the solution converges to an equilibrium state as $t \to \infty$ in the C^∞-topology. The result can be extended to the following case: (i) L^2 gradient flow for E under the area-preserving condition and (C) [27] and (ii) L^2gradient flow for Tadjbakhsh-Odeh energy functional under the constraint (C) [28]. Moreover, the result [26] was also extended to the case of space curves [29].

2.4. Derivation of heat equation

In this subsection, we study the energy law. We confirm thermal energy conservation law (the first law of thermodynamics) and the increasing law of entropy (the second law of thermodynamics). To begin with, we consider the first law of thermodynamics. According to Ref. [30], thermal energy conservation law for thermoelastic system is given by

$$\theta\partial_t S + \nabla\cdot q = h \tag{14}$$

where S, q and h are entropy, thermal velocity and external heat, respectively. In our setting, thermal transfer occurs only on wire, and the wire does not expand. Then, we may regard $\nabla\cdot q$ as $\partial_s q$ as the same as one-dimensional case, where s is necessary to be arc length parameter. By the same reason, the Fourier law $q = \nabla\theta$ is replaced by

$$q = -\partial_s\theta. \tag{15}$$

The Helmholtz free energy density

$$\begin{aligned}\tilde{f} &= \tilde{f}(\partial_s^2\gamma,\ \partial_s\gamma,\ \theta\,;\gamma^0)\\ &= \frac{1}{2}|\partial_s^2\gamma\,|^2 + f(\partial_s\gamma,\ \theta\,;\gamma^0) + f_0(\theta)\end{aligned}$$

and the entropy S are connected with the relation

$$S = -\frac{\partial\tilde{f}}{\partial\theta}.$$

Then, the conservation law, Eq. (14), is rewritten as follows:

$$-\,\theta\, f_0''(\theta)\,\partial_t\theta\; - \partial_s^2\theta = \theta\Big(\partial_s\gamma^0\cdot\partial_s(\gamma-\gamma^0)\Big)(\partial_t\partial_s\gamma\cdot\partial_s\gamma^0) + h. \tag{16}$$

We note that the Clausius-Duhem inequality holds automatically:

$$\partial_t S + \partial_s\left(\frac{q}{\theta}\right) \geq \frac{h}{\theta}\ .$$

Indeed, we observe from Eqs. (14) and (15) that

$$\begin{aligned}\partial_t S + \partial_s\left(\frac{q}{\theta}\right) &= \frac{h-\partial_s q}{\theta} + \partial_s\left(\frac{q}{\theta}\right)\\ &= \frac{h}{\theta} - \frac{q\partial_s\theta}{\theta^2}\\ &= \frac{h}{\theta} + |\frac{\partial_s\theta}{\theta}|^2 \geq \frac{h}{\theta}\ .\end{aligned}$$

The Clausius-Duhem inequality corresponds to the second law of thermodynamics. For more precise information of the inequality, we refer to, for example, 1.11 of chapter 4 in Ref. [2].

Here, we assume external heat source $h = 0$ and adopt the well-known form:

$$f_0(\theta) = \theta - \theta\log\theta.$$

Then since $f_0{}''(\theta) = -1/\theta,$ Eq. (16) is reduced to

$$\partial_t\theta - \partial_s^2\theta = \theta\Big(\partial_s\gamma^0\cdot\partial_s(\gamma-\gamma^0)\Big)(\partial_t\partial_s\gamma\cdot\partial_s\gamma^0).$$

We thus obtain the system of equation as Eq. (4).

3. Dual variation structure

Let us rewrite Eq. (4):

$$\partial_t^2\gamma + \partial_s^4\gamma\; + \partial_s f_{,\partial_s\gamma}(\partial_s\gamma,\ \theta;\partial_s\gamma^0) - \partial_s\{(v - 2\,|\partial_s^2\gamma|^2)\partial_s\gamma\} = 0, \tag{17}$$

$$-\,\partial_s^2 v + |\partial_s^2\gamma|^2 v = 2|\partial_s^2\gamma|^4 - |\partial_s^3\gamma|^2 + |\partial_s\partial_t\gamma|^2 + \partial_s^2 f_{,\partial_s\gamma}(\partial_s\gamma,\ \theta;\partial_s\gamma^0)\cdot\partial_s\gamma, \tag{18}$$

$$\partial_t\theta - \partial_s^2\theta = \theta\Big(\partial_s\gamma^0\cdot\partial_s(\gamma-\gamma^0)\Big)(\partial_t\partial_s\gamma\cdot\partial_s\gamma^0),\ (t,s)\in(0,T)\times S_L^1, \tag{19}$$

$$\gamma(0,s)=\gamma_0(s),\ \partial_t\gamma(0,s)=\gamma_1,\ \theta(0,s)=\theta_0,\ s\in S_L^1. \tag{20}$$

In this section, we show that the problem in Eqs. (17)–(20) has also dual variation structure as well as the problem (Eq. (2)). The structure plays an important role to prove the dynamical stability of infinitesimally stable stationary state.

We assume that the system has sufficiently smooth solution (γ, θ) satisfying $\theta > 0$. Then initial data also has to satisfy

$$|\partial_s\gamma_0| = 1,\ \partial_s\gamma_0\cdot\partial_s\gamma_1 = 0.$$

Setting

$$\begin{aligned} f_1(\partial_s\gamma)\ &:= \frac{1}{2}\left(\frac{\partial\gamma^0}{\partial s}\cdot\frac{\partial(\gamma-\gamma^0)}{\partial s}\right)^2,\\ f_2(\partial_s\gamma)\ &:= \frac{1}{6}\left(\frac{\partial\gamma^0}{\partial s}\cdot\frac{\partial(\gamma-\gamma^0)}{\partial s}\right)^6 - \frac{1}{4}\left(\frac{\partial\gamma^0}{\partial s}\cdot\frac{\partial(\gamma-\gamma^0)}{\partial s}\right)^4 - \frac{\theta_c}{2}\left(\frac{\partial\gamma^0}{\partial s}\cdot\frac{\partial(\gamma-\gamma^0)}{\partial s}\right)^2, \end{aligned}$$

we have the relation $f = \theta f_1 + f_2$. Multiplying Eq. (17) by $\partial_t\gamma$ and integrating it with respect to s, we obtain

$$\frac{d}{dt}\left(\frac{1}{2}||\partial_t\gamma||_{L^2}^2 + \frac{1}{2}||\partial_s^2\gamma||_{L^2}^2\right) = -\langle f_{,\partial_s\gamma},\ \partial_s\partial_t\gamma\rangle = -\frac{d}{dt}\int_0^L f_2(\partial_s\gamma)ds - \int_0^L \theta\partial_t f_1(\partial_s\gamma)ds.$$

Integrating Eq. (19), we find

$$\frac{d}{dt}\int_0^L\theta ds = \int_0^L\theta\partial_t f_1(\partial_s\gamma)ds.$$

Then for the quantity

$$E(\gamma,\partial_t\gamma,\theta) := \frac{1}{2}||\partial_t\gamma||_{L^2}^2 + \frac{1}{2}||\partial_s^2\gamma||_{L^2}^2 + \int_0^L\theta ds + \int_0^L f_2(\partial_s\gamma)ds,$$

it holds that

$$\frac{d}{dt}E(\gamma,\partial_t\gamma,\theta) = 0.$$

Moreover, for the quantity

$$W(\partial_s\gamma,\theta) := \int_0^L\{f_1(\partial_s\gamma) - \log\theta\}ds$$

we deduce from Eq. (19) that

$$\frac{d}{dt}W(\partial_s\gamma,\theta)=\int_0^L\frac{\partial_s^2\theta}{\theta}ds=-\int_0^L|\frac{\partial_s\theta}{\theta}|^2ds\le 0.$$

Finally, we confirm an important structure of the system (17)–(20). We denote the stationary state of θ by $\overline{\theta}>0$, and the corresponding equilibrium by γ satisfy the following constraint:

$$b\equiv E(\gamma_0,\ \gamma_1,\ \theta_0)=E(\gamma,\ 0,\ \overline{\theta}),$$
$$\partial_s^4\gamma=\partial_s\{\overline{\theta}f_{1,\partial_s\gamma}+f_{2,\partial_s\gamma}\}.$$

Eliminating $\overline{\theta}$ by the relation

$$b=L\overline{\theta}+\frac{1}{2}||\partial_s^2\gamma||_{L^2}^2+\int_0^L f_2(\partial_s\gamma)\ ds$$

the stationary state of this problem satisfies the following nonlinear nonlocal problem:

$$\partial_s^4\gamma=\partial_s\left\{\frac{1}{L}\left(b-\frac{1}{2}||\partial_s^2\gamma||_{L^2}^2-\int_0^L f_2(\partial_s\gamma)ds\right)f_{1,\partial_s\gamma}(\partial_s\gamma)+f_{2,\partial_s\gamma}(\partial_s\gamma)\right\}. \tag{21}$$

Eq. (21) is derived as the Euler-Lagrange equation of the functional

$$J_b(y):=\frac{1}{L}\int_0^L f_1(y)ds-\log\left(b-\frac{1}{2}||\partial_s y||_{L^2}^2-\int_0^L f_2(y)ds\right)+\log L$$

where $y=\ \partial_s\gamma\in H^2(S_L^1,S^2)$ and

$$S^2:=\{\omega\in\boldsymbol{R}^3||\omega|=1\}.$$

We remark that the following relation between J_b and W holds true:

$$W(\partial_s\gamma,\theta)\ge LJ_b(\partial_s\gamma).$$

The relation is called **semi-unfolding minimality**. Thus, if $(\gamma,\ \theta)$ is a non-stationary state, $b=E(\gamma_0,\gamma_1,\theta_0),\ \overline{y}=\partial_s\overline{\gamma}$ is a linearized stable critical point of $J_b=J_b(y),\ y\in\ H^1(S_L^1,\ S^2)$ and $\overline{\theta}>\ 0$ is a constant satisfying $E(\overline{\gamma},0,\overline{\theta})\ =\ b,$ then it holds that for $y=\partial_s\gamma$

$$J_b(y)-J_b(\overline{y})\le W(\partial_s\gamma_0,\ \theta_0)-W(\overline{y},\overline{\theta}).$$

By this structure, we can infer that any infinitesimally stable stationary state is dynamically stable, that is, stable in the Lyapunov sense. A critical point $\overline{y}=\partial_s\overline{\gamma}$ of J_b for $\overline{\gamma}\in H^2(S_L^1,S^2)$ is **infinitesimally stable** if there exists $\varepsilon_0>0$ such that any $\varepsilon_1\in\left(0,\ \frac{\varepsilon_0}{2}\right]$ admits $\delta_0>0$ such that if $||\partial_s(\gamma-\overline{\gamma})||_{H^1}<\varepsilon_0$ and $J_b(\partial_s\gamma)-\ J_b(\partial_s\overline{\gamma})<\delta_0$ then

$$||\partial_s(\gamma - \overline{\gamma})||_{H^1} < \varepsilon_1.$$

The definition of infinitesimally stable is obviously weaker than the one of well-known linearized stable which means that for a critical point $\overline{y} = \partial_s \overline{\gamma}$ of J_b, the quadratic form

$$Q_y(w, w) = \frac{d}{d\varepsilon^2} J_b(\overline{y} + \varepsilon w)|_{\varepsilon=0}$$

is a positive definite for any $w \in H^2(S_L^1, S^2)$.

Theorem 1. *Assume that $\overline{\theta} > 0$ is a constant and that $\overline{\gamma}$ is an infinitesimally stable critical point of J_b with constraint $\partial_s\gamma \in H^2(S_L^1, S^2)$. Then $(\overline{\gamma}, \overline{\theta})$ is a dynamically stable in the sense that for any $\varepsilon > 0$, there exists $\delta > 0$ such that if*

$$E(\gamma_0,\ \gamma_1,\ \theta_0) =\ b, ||\partial_s(\gamma_0 - \overline{\gamma})||_{H^1} < \delta,\ |\frac{1}{L}\int_0^L \log\theta_0(s)ds - \log\overline{\theta}| < \delta \tag{22}$$

then

$$\sup_{t\geq 0}||\partial_s\Big(\gamma(t) - \overline{\gamma}\Big)||_{H^1} < \varepsilon,\ |\frac{1}{L}\int_0^L \log\theta(t, s)ds - \log\overline{\theta}| < \varepsilon.$$

Proof. We first show the semi-unfolding minimality. From the energy conservation law, we see that

$$b\ = \frac{1}{2}||\partial_t\gamma||_{L^2}^2 + \frac{1}{2}||\partial_s^2\gamma||_{L^2}^2 + \int_0^L \theta ds + \int_0^L f_2(\partial_s\gamma)ds.$$

It follows from the Jensen inequality that

$$\frac{1}{L}\int_0^L \log\theta ds \leq \log\left(\frac{1}{L}\int_0^L \theta ds\right).$$

Then we have

$$\begin{aligned}
W(\partial_s\gamma,\ \theta) &\geq \int_0^L f_1(\partial_s\gamma)ds - \log\left(\frac{1}{L}\int_0^L \theta ds\right) \\
&\geq \int_0^L f_1(\partial_s\gamma)ds - L\log\frac{1}{L}\left(b - \frac{1}{2}||\partial_s y||_{L^2}^2 - \int_0^L f_2(y)ds\right) \\
&= LJ_b(\partial_s\gamma).
\end{aligned}$$

We have thus completed to show the semi-unfolding minimality. Recall that $\overline{\gamma} \in H^2(S_L^1, S^2)$ is an infinitesimally stable critical point of $J_b(\partial_s\gamma)$. Thus, we find $\varepsilon_0 > 0$such that any $\varepsilon_1 \in (0,\ \frac{\varepsilon_0}{2}]$ admits $\delta_0 > 0$ such that if $||\partial_s(\gamma - \overline{\gamma})||_{H^1} < \varepsilon_0$ and $J_b(\partial_s\gamma) -\ J_b(\partial_s\overline{\gamma}) < \delta_0$ then

$$||\partial_s(\gamma - \overline{\gamma})||_{H^1} < \varepsilon_1. \tag{23}$$

From the above properties, it holds that

$$J_b\left(\partial_s\gamma(t)\right) \leq \frac{1}{L}W\left(\partial_s\gamma(t),\ \theta(t)\right) \leq \frac{1}{L}W(\partial_s\gamma_0,\ \theta_0). \tag{24}$$

Moreover, for the constant $\overline{\theta} > 0,$ we obtain

$$W(\partial_s\overline{\gamma},\ \overline{\theta}) = L\left[\frac{1}{L}\int_0^L f_1(\partial_s\gamma)ds\ - \log\frac{1}{L}\left(b - \frac{1}{2}||\partial_s\overline{y}||_{L^2}^2 - \int_0^L f_2(\overline{y})ds\right)\right] = LJ_b(\partial_s\ \overline{\gamma}),$$

namely,

$$J_b(\partial_s\overline{\gamma}) = \frac{1}{L}W(\partial_s\overline{\gamma}, \overline{\theta}). \tag{25}$$

Given $\varepsilon > 0$, setting $\delta \in \left(0, \frac{\varepsilon_0}{2}\right]$ and satisfying Eq. (22), we have

$$\begin{aligned}\frac{1}{L}|W(\partial_s\gamma_0,\ \theta_0) - W(\partial_s\overline{\gamma}, \overline{\theta})| \leq \frac{1}{L}\left(\|f_{1,\ \partial_s\gamma}(\partial_s\gamma_0)\|_{L^\infty} + \|f_{1,\ \partial_s\gamma}(\partial_s\overline{\gamma})\|_{L^\infty}\right)\|\partial_s(\gamma_0 - \overline{\gamma})\|_{L^1} \\ +|\frac{1}{L}\int_0^L \log\theta_0 ds - \log\overline{\theta}| < \min(\delta_0, \varepsilon).\end{aligned} \tag{26}$$

Therefore, it follows from Eq. (24) to Eq. (26) that

$$J_b\left(\partial_s\gamma(t)\right) - J_b(\partial_s\overline{\gamma}) = \frac{1}{L}\left(W(\partial_s\gamma_0,\ \theta_0) - W(\partial_s\overline{\gamma}, \overline{\theta})\right) < \delta_0.$$

If $||\partial_s(\gamma(t) - \overline{\gamma})||_{H^1} = \delta(\leq \varepsilon_0/2\ < \varepsilon_0)$, then we apply Eq. (23) for $\varepsilon_1 = \delta$, and hence

$$||\partial_s\left(\gamma(t) - \overline{\gamma}\right)||_{H^1} < \delta,$$

which is a contradiction. Thus, we have

$$||\partial_s\left(\gamma(t) - \overline{\gamma}\right)||_{H^1} \neq \delta.$$

Here, from $\gamma \in C\left([0, \infty); H^2\right)$ and $\|\partial_s(\gamma_0 - \overline{\gamma})\|_{H^1} < \delta,$ it follows that

$$||\partial_s\left(\gamma(t) - \overline{\gamma}\right)||_{H^1} < \delta \tag{27}$$

for any $t \geq 0$.

From the semi-unfolding minimality (Eqs. (24) and (25)) and the linearized stability of J_b, we observe that

$$W(\partial_s\gamma,\theta) \geq LJ_b(\partial_s\gamma) \geq LJ_b(\partial_s\overline{\gamma}) = W(\partial_s\overline{\gamma},\overline{\theta}).$$

Then, combining Eq. (26) with Eq. (27), we have

$$\left|\frac{1}{L}\int_0^L \log\theta(t,s)ds - \log\overline{\theta}\right| \leq \frac{1}{L}\Big(W(\partial_s\gamma,\theta) - W(\partial_s\overline{\gamma},\overline{\theta})\Big) + \frac{1}{L}\left|\int_0^L \{f_1(\partial_s\gamma) - f_1(\partial_s\overline{\gamma})\}ds\right|$$
$$\leq \frac{1}{L}\Big(W(\partial_s\gamma_0,\theta_0) - W(\partial_s\overline{\gamma},\overline{\theta})\Big) + C||\partial_s(\gamma-\overline{\gamma})||_{H^1} \leq \varepsilon + C\delta \leq 2\varepsilon$$

where δ is small enough such that $\delta < \varepsilon / C$. This completes the proof.

Q.E.D.

Remark 3. Both the existence of solution for evolution equations (Eq. (4)) and non-trivial solutions for stationary problem (Eq. (21)) are open problems. In the straight material case (i.e. the problem (2)), smooth solution for Eq. (2) is assured in [11] (we also refer to chapter 5 in [2]). The existence results of non-trivial solution for stationary problem (Eq. (21)) in low-temperature and low-energy cases can be found in [21, 31].

4. Concluding remarks

In this chapter, we propose the new mathematical model describing the movement of wire made of shape memory alloys. The derived system of nonlinear partial differential equations is a thermoelastic system with phase transition and non-stretching constraint. The Falk model (Eq. (2)) represents the dynamics for crystal as a stack of layers, whose displacement is restricted to move only on one direction. On the other hand, our model describes the dynamics of wire. We emphasize that our model allows the displacement of each direction. Thus, our model may describe a more realistic motion of wire made of shape memory alloys. Moreover, it is also interesting to regard our model as a mathematical problem on elastic curve with heat conduction. To the best of our knowledge, there is no result considering such a mathematical problem.

We mention the mathematical contribution of the present chapter. We prove the dynamical stability of an infinitesimally stable stationary state by finding the dual variation structure in our model. This property shall be applicable, for example, to assure the strength of not only a wire in an original shape but also a deformed wire. Indeed, in the straight material case, namely in the Falk model (Eq. (2)), numerical simulation shows the stability in this sense (see [17]).

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