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Intersecting Straight Lines: Titrimetric Applications

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Additional information is available at the end of the chapter

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Abstract

Plotting two straight line graphs from the experimental data and determining the point of their intersection solve a number of problems in analytical chemistry (i.e., potentiometric and conductometric titrations, the composition of metal-chelate complexes and binding interactions as ligand-protein). The relation between conductometric titration and the volume of titrant added leads to segmented linear titration curves, the endpoint being defined by the intersection of the two straight line segments. The estimation of the statistical uncertainty of the end point of intersecting straight lines is a topic scarcely treated in detail in a textbook or specialized analytical monographs. For this reason, a detailed treatment with that purpose in mind is addressed in this book chapter. The theoretical basis of a variety of methods such as first-order propagation of variance (random error propagation law), Fieller's theorem and two approaches based on intersecting confidence bands are explained in detail. Several experimental systems described in the literature are the subject of study, with the aim of gaining knowledge and experience in the application of the possible methods of uncertainty estimation. Finally, the developed theory has been applied to the conductivity measurements in triplicate in the titration of a mixture of hydrochloric acid and acetic acid with potassium hydroxide.

Keywords: titrimetric, straight lines, breakpoint

1. Introduction

Titrimetry is one of the oldest analytical methods [1], and it is still found [2–4] in a developing way. It plays an important role in various fields as well as routine studies [5–9], being used widely in the analytical laboratory given their simplicity, speed, accuracy, good reproducibility, and low cost. It is, together with gravimetry, one of the most used methods to determine chemical composition on the basis of chemical reactions (primary method).

Independent values of chemical quantities expressed in SI units are obtained through gravimetry and titrimetry (classical analysis).

In titrimetry, the quantity of tested components of a sample is assessed by the use of a solution of known concentration added to the sample, which reacts in a definite proportion. To identify the stoichiometric point, where equal amounts of titrant react with equal amounts of analyte, indicators are used in many cases to point out the end of the chemical reaction by a color change.

Information on reaction parameters is usually obtained from an analysis of the shape of the titration curve, whose shapes depend on some factors such as the reaction of titration, the monitored specie (indicator, titrant, analyte or formed product) as well as the chosen [10, 11] instrumental technique (**Table 1**) i.e., spectrophotometry, conductimetry or potentiometry, for instance. The importance of titrimetric analysis has increased with the advance of the instrumental method of end point detection.

Linear response functions are generally preferred, and when the response function is nonlinear, a linearization procedure has been commonly used with a suitable change of variables. Plotting two straight lines graphs from the experimental data and determining the point of their intersection solve a number of problems in analytical chemistry [10, 11] (**Table 1**). In segmented linear titration curves, the end point is defined by the intersection of the two straight segments. In some common examples in analytical chemistry (conductometric, spectrophotometric and amperometric titrations), this intersection lies beyond the linear ranges, and deviations from linearity are often observed directly at the end point. All curvature points should be excluded from the computation. The accuracy and precision of the results of a titrimetric determination are influenced not only by the nature of the titration reaction but also by the technique [10, 11] of the end-point location.

The problem of finding the breakpoint of two straight lines joined at some unknown point has a long statistical history [12–14] and has received considerable attention in the statistical literature. The problem in question is known by a variety of names (**Table 2**) [12–27]. Computer analysis [28–31], elimination of outliers [32, 33], and confidence limits for the abscissa [22, 34, 35] have been subject to study.

At the point of intersection (x_I), the two lines have the same ordinate. The estimation of statistical uncertainty of end points obtained from linear segmented titration is the subject

Technique	Measured property
Conductimetric titrations	Electrical conductivity
Potentiometric titrations	Potential of an indicator electrode
Spectrophotometric titrations	Absorbance
Amperometric titrations	Diffusion current at a polarizable indicator (dropping mercury or rotating platinum) electrode

Table 1. Instrumental end point detection techniques more widely applied.

Name	Authors
Breakpoint	Jones and Molitoris [12]; Shanubhogue et al. [15]
Changepoint	Csörgo and Horváth [16]; Krishanaiah and Miao [17]
Common intersection point	Rukhin [18]
Hockey stick regression	Yanagimoto and Yamamoto [19]
Intersystem crossing	Kita et al. [20]
Piecewise linear regression	Vieth [21]
Segmented regression	Piegorsch [22]
Transition	Bacon and Watts [23]
Two phase linear regression	Christensen [24]; Lee et al. [25];, Seber [26, 27]; Shaban [14]; Sprent [13]

Table 2. Names received in the literature for the intersecting point of two straight lines.

of this chapter. The topic is scarcely treated in [36, 37] analytical monographs. The method of least squares is the most common and appropriate choice and when the relative statistical uncertainties of the x data are negligible compared to the y data. Single linear regression or weighted linear regression may be applied depending on whether the variance of y is constant or varies from point to point with the magnitude of the response y , respectively.

The theoretical basis of a variety of method such as first-order propagation of variance for the abscissa or intersection, the application of Fieller’s method [38–43], and other methods based on intersecting hyperbolic confidence bands as weighted averages [57, 58] of the abscissas of the confidence hyperbolas at the ordinate of intersection will be dealt in detail in this book chapter. In addition, several experimental systems will be the subject of study, with the aims of gaining knowledge and experience in the application of these methods to uncertainty estimation.

2. Theory

V-shaped linear titration curves (**Table 1**) are well known in current analytical techniques such as conductimetry, radiometry, refractometry, spectrophotometric and amperometric titrations as well as in Gran’s plot. In this kind of titrations, the end point is usually located at the intersection of two lines when a certain property (conductance, absorbance, diffusion current) is plotted against the volume x of titrant added to the unknown sample containing the analyte.

Let N_1 observations on the first line

$$\hat{y}_1 = a_1 + b_1x \tag{1}$$

and N_2 observations on the second

$$\hat{y}_2 = a_2 + b_2x \tag{2}$$

where a_1, b_1, a_2, b_2 are the usual least squares estimates of the k th line ($k = 1, 2$), respectively. As it is stated in the introduction section when the relative statistical uncertainties of the x data are negligible compared to the y data, the use of the least squares method is the most common alternative. The ordinate variance can be considered on a priori grounds to vary systematically as a function of the position along the curve, so that weighted least squares analysis is appropriate. Formulae for calculating the intercept a , the slope b and their standard errors by weighted linear regression [59] are given in **Table 3**, where the analogy with simple linear regression (i.e., $w_i = 1$), is evident.

Note that in summation (1) and (2) by dividing by N_1 and N_2 , respectively, we get

$$\bar{y}_1 = a_1 + b_1\bar{x}_1 \tag{3}$$

$$\bar{y}_2 = a_2 + b_2\bar{x}_2 \tag{4}$$

At the point of intersection, the lines (1) and (2) have the same ordinate $\hat{y}_1 = \hat{y}_2$ and the abscissa of intersection (denotes by \hat{x}_I) is given by

$$a_1 + b_1\hat{x}_I = a_2 + b_2\hat{x}_I \tag{5}$$

$$\hat{x}_I = \frac{a_2 - a_1}{b_1 - b_2} = -\frac{\Delta a}{\Delta b} \tag{6}$$

Random error in the points produces uncertainty in the slopes and intercepts of the lines, and therefore in the point of intersection. The probability that a confidence interval contains the true value is equal to the confidence level (e.g., 95%).

<ul style="list-style-type: none"> Equation $\hat{y}_i = a + bx_i$ Weights $w_i = 1/s_i^2$ Explained sum of squares $SS_{\text{Reg}} = \sum w_i(\hat{y}_i - \bar{y})^2$ Residual sum of squares $SSE = \sum w_i(y_i - \hat{y}_i)^2$ Mean $\bar{x} = \sum w_i x_i / \sum w_i$ $\bar{y} = \sum w_i y_i / \sum w_i$ Sum of squares about the mean $S_{XX} = \sum w_i (x_i - \bar{x})^2$ $S_{YY} = \sum w_i (y_i - \bar{y})^2$ $S_{XY} = \sum w_i (x_i - \bar{x})(y_i - \bar{y})$ 	<ul style="list-style-type: none"> Slope $b = S_{XY}/S_{XX}$ Intercept $a = \bar{y} - b\bar{x}$ Weighted residuals $w_i^{1/2}(y_i - \hat{y}_i)$ Correlation coefficient $r = S_{XY}/\sqrt{S_{XX}S_{YY}}$ Standard errors $s_{y/x}^2 = \frac{SSE}{n-2} = \frac{S_{YY} - b^2 S_{XX}}{n-2}$ $s_a^2 = s_{y/x}^2 (\sum w_i x_i^2) / (S_{XX} \sum w_i)$ $s_b^2 = s_{y/x}^2 / S_{XX}$ $Cov(a, b) = \bar{x} s_{y/x}^2 / S_{XX}$
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Table 3. Formulae for calculating statistics for weighted linear regression.

3. First-order propagation of variance for $V[\hat{x}_I]$

The precision of the point of intersection and the corresponding statistical confidence interval can be found in the simplest way by considering the random error propagation law [60]. Some authors [61, 62] evaluate the uncertainty in \hat{x}_I on this way. First-order propagation of variance retains only first derivatives in the Taylor expansions and this procedure leads to

$$\begin{aligned} V[\hat{x}_I] &= \left(\frac{\partial \hat{x}_I}{\partial \Delta a}\right)^2 V[\Delta a] + \left(\frac{\partial \hat{x}_I}{\partial \Delta b}\right)^2 V[\Delta b] + 2\left(\frac{\partial \hat{x}_I}{\partial \Delta a}\right)\left(\frac{\partial \hat{x}_I}{\partial \Delta b}\right) \text{Cov}(\Delta a, \Delta b) \\ &= \left(\frac{\partial\left(\frac{-\Delta a}{\Delta b}\right)}{\partial \Delta a}\right)^2 V[\Delta a] + \left(\frac{\partial\left(\frac{-\Delta a}{\Delta b}\right)}{\partial \Delta b}\right)^2 V[\Delta b] + 2\left(\frac{\partial\left(\frac{-\Delta a}{\Delta b}\right)}{\partial \Delta a}\right)\left(\frac{\partial\left(\frac{-\Delta a}{\Delta b}\right)}{\partial \Delta b}\right) \text{Cov}(\Delta a, \Delta b) \quad (7) \\ &= \frac{V[\Delta a]}{\Delta b^2} + \frac{\Delta a^2 V[\Delta b]}{\Delta b^4} - 2\frac{\Delta a \text{Cov}(\Delta a, \Delta b)}{\Delta b^3} \end{aligned}$$

valid in those cases, in which the standard deviations of the ordinate data are a small fraction of their magnitude. Taking into account Eq. (6), Eq. (7) may be rewritten as follows

$$\begin{aligned} V[\hat{x}_I] &= \frac{1}{\Delta b^2} \left(V[\Delta a] + \frac{\Delta a^2}{\Delta b^2} V[\Delta b] - 2\frac{\Delta a}{\Delta b} \text{Cov}(\Delta a, \Delta b) \right) \\ &= \frac{V[\Delta a] + \hat{x}_I^2 V[\Delta b] + 2\hat{x}_I \text{Cov}(\Delta a, \Delta b)}{\Delta b^2} \quad (8) \end{aligned}$$

Then, the standard error estimate of \hat{x}_I is as follows

$$s(\hat{x}_I) = \sqrt{V[\hat{x}_I]} \quad (9)$$

The end point \hat{x}_I depends on four least squares parameters a_1, a_2, b_1, b_2 that are random variables. Segment one parameters depend only on measurements made along segment one and these are statistically independent of the measurements along segment two. However, Δa , and Δb are correlated random variables because each involves b_1 and b_2 . Note that, a_1 and a_2 are related to b_1 and b_2 by means of Eqs. (1) and (2).

The variances of Δa and Δb are given by

$$V[\Delta a] = V[a_1 - a_2] = V[a_1] + V[a_2] \quad (10)$$

$$V[\Delta b] = V[b_1 - b_2] = V[b_1] + V[b_2] \quad (11)$$

and for the covariance between Δa and Δb , we get [63]

$$\begin{aligned} \text{Cov}(\Delta a, \Delta b) &= \text{Cov}(a_1 - a_2, b_1 - b_2) \\ &= \text{Cov}(\bar{y}_1 - b_1 \bar{x}_1 - (\bar{y}_2 - b_2 \bar{x}_2), b_1 - b_2) = \text{Cov}(\bar{y}_1 - \bar{y}_2 - b_1 \bar{x}_1 + b_2 \bar{x}_2, b_1 - b_2) \\ &= \text{Cov}(\bar{y}_1 - \bar{y}_2, b_1 - b_2) - \text{Cov}(b_1 \bar{x}_1 - b_2 \bar{x}_2, b_1 - b_2) \\ &= -\text{Cov}(b_1 \bar{x}_1 - b_2 \bar{x}_2, b_1 - b_2) \\ &= -\bar{x}_1 V[b_1] - \bar{x}_2 V[b_2] \quad (12) \end{aligned}$$

It should be noted that in the calculations, the variance regression estimates from both line segments are pooled into a single s_p^2 , by using the following formula which weights each contribution according to the corresponding [64–67] degrees of freedom

$$s_p^2 = \frac{\sum_{i=1}^{N_1} (y_{1i} - \bar{y}_1)^2 + \sum_{i=1}^{N_2} (y_{2i} - \bar{y}_2)^2}{(N_1 - 2) + (N_2 - 2)} = \frac{(N_1 - 2)s_1^2 + (N_2 - 2)s_2^2}{N_1 + N_2 - 4} \tag{13}$$

The standard deviation in Eq. (13) is calculated on the assumption that the $s_{(y/x)}$ values for the two lines are sufficiently similar to be pooled.

From expression in **Table 3** for the variance of the intercept ($s_a^2 = V[a]$), we may derive

$$\begin{aligned} V[a] &= s^2 \left[\frac{\sum w_i x_i^2}{S_{XX} (\sum w_i)} \right] = s^2 \left[\frac{S_{XX} + \frac{(\sum w_i x_i)^2}{\sum w_i}}{S_{XX} (\sum w_i)} \right] = s^2 \left[\frac{S_{XX} + (\sum w_i) \left(\frac{\sum w_i x_i}{\sum w_i} \right)^2}{S_{XX} (\sum w_i)} \right] \\ &= s^2 \left[\frac{S_{XX} + (\sum w_i) \bar{x}^2}{S_{XX} (\sum w_i)} \right] = s^2 \left[\frac{1}{\sum w_i} + \frac{\bar{x}^2}{S_{XX}} \right] \end{aligned} \tag{14}$$

in which s^2 is $s_{(y/x)}$ in **Table 3**; $\sum w_i$ is the sum of weights, which simply reduces to N , the number of points if the non-weighted least squares analysis is used. Taking into account Eq. (14), Eqs. (10) and (11) lead to

$$V[\Delta a] = V[a_1] + V[a_2] = \left[\frac{1}{(\sum w)_1} + \frac{\bar{x}_1^2}{(S_{XX})_1} \right] s_1^2 + \left[\frac{1}{(\sum w)_2} + \frac{\bar{x}_1^2}{(S_{XX})_2} \right] s_2^2 \tag{15}$$

$$V[\Delta a] = \left[\frac{1}{(\sum w)_1} + \frac{1}{(\sum w)_2} + \frac{\bar{x}_1^2}{(S_{XX})_1} + \frac{\bar{x}_1^2}{(S_{XX})_2} \right] s_p^2 \tag{16}$$

$$V[\Delta b] = V[b_1] + V[b_2] = \frac{s_1^2}{(S_{XX})_1} + \frac{s_1^2}{(S_{XX})_2} = \left[\frac{1}{(S_{XX})_1} + \frac{1}{(S_{XX})_2} \right] s_p^2 \tag{17}$$

$$Cov(\Delta a, \Delta b) = -\bar{x}_1 V[b_1] - \bar{x}_2 V[b_2] = - \left[\frac{\bar{x}_1}{(S_{XX})_1} + \frac{\bar{x}_2}{(S_{XX})_2} \right] s_p^2 \tag{18}$$

Once the values of $V[\Delta a]$, $V[\Delta b]$ and $Cov[\Delta a, \Delta b]$ are known from Eqs. (16), (17) and (18), respectively, the estimate of the variance of the intersection abscissa of the two straight lines, $V[\hat{x}_I]$, is calculated by applying Eq. (8).

4. Confidence interval on the abscissa of the point of intersection of two fitted linear regressions

The use of confidence intervals is another alternative to express the statistical uncertainty of \hat{x}_I . This method depends on the distribution function of the random variable \hat{x}_I . If the ordinates y_i are assumed to have Gaussian (normal) distribution, the least squares parameters as well as Δa , and Δb are also normally distributed [68]. However, \hat{x}_I , which even is regarded as the ratio of two normally distributed variables, is not normally distributed and, indeed, becomes more and more skewed [69] as the variance levels increase. For sufficiently small variance though, \hat{x}_I is approximately normally distributed. Under these circumstances, confidence intervals may be calculated from the standard deviation of \hat{x}_I , which is also accurate only when variances are small.

However, the construction of the confidence interval (limits) for the equivalence point by using the Student's t -test

$$\hat{x}_I \pm t_{\alpha/2} s(\hat{x}_I) \quad (19)$$

where $t_{\alpha/2}$ is the Student's t statistics at the $1 - \alpha$ confidence level (i.e., leaving an area of $\alpha/2$ to the right) and for the number of degrees of freedom ($N_1 + N_2 - 4$) inherent in the standard deviation of \hat{x}_I , could be misleading. Note that because \hat{x}_I involves the ratio of random variables, first-order propagation of variance is not exact [69]. Evidently, \hat{x}_I is a random variable not normally distributed unless $s(\hat{x}_I)$ is small enough. When the variances of the responses are not necessarily small, a solution to this problem is to apply the called Fieller's theorem [38–43]. Another point of view is focused on the problem in the calculation of the limits of the confidence intervals by using the confidence bands for the two segmented branches.

5. The Fieller's theorem

This theorem [38–43] is supported by two capital premises:

- i. Any linear combination z of normally distributed random variables is itself normally distributed.
- ii. If the standardized variable $\frac{z}{\sqrt{V[z]}}$ is distributed as $N(0, 1)$, then z is distributed as t .

Consider now any pair of individual line segments written as a difference z as follows

$$z = [a_1 + b_1 x_I] - [a_2 + b_2 x_I] = \Delta a + x_I \Delta b \quad (20)$$

Note that for any such pair of lines, the difference z is not, in general, zero, because the "best" end point cannot be the one for each pair of lines of the collection. However, the mean $\langle z \rangle$ of all these z values is zero and z are normally distributed because it is formed as a linear combination of normally distributed variables. Taking into account that a_1 , a_2 , b_1 and b_2 are normally

distributed, then z will be normally distributed. Then, in the vicinity of the intersection point, z has zero mean and its variance is

$$V[z] = V[\Delta a + x_I \Delta b] = V[\Delta a] + x_I^2 V[\Delta b] + 2x_I Cov(\Delta a, \Delta b) \tag{21}$$

and therefore, $\frac{z}{\sqrt{\hat{V}[z]}}$ is distributed as $N(0, 1)$ and according to (ii)

$$\frac{z}{\sqrt{\hat{V}[z]}} = t \tag{22}$$

This is called Fieller’s Theorem [34, 38]. The development of Eq. (11) leads to the equation

$$\frac{(\Delta a + x_I \Delta b)^2}{V[\Delta a] + x_I^2 V[\Delta b] + 2x_I Cov(\Delta a, \Delta b)} = t^2 \tag{23}$$

which on rearrangement leads to

$$(\Delta a)^2 + 2x_I \Delta a \Delta b + x_I^2 (\Delta b)^2 = t^2 V[\Delta a] + t^2 x_I^2 V[\Delta b] + 2t^2 x_I Cov(\Delta a, \Delta b) \tag{24}$$

which may be factored as

$$\left((\Delta a)^2 - t^2 V[\Delta a] \right) + 2x_I \left(\Delta a \Delta b - t^2 Cov(\Delta a, \Delta b) \right) + x_I^2 \left((\Delta b)^2 - t^2 V[\Delta b] \right) = 0 \tag{25}$$

The solution of Eq. (25) gives the confidence limits for x_I estimated, where $t_{\alpha/2}$ is the appropriate value of the Student distribution at a α significance level (confidence level $1 - \alpha$) for $N_1 + N_2 - 4$ degrees of freedom. Note that in Eqs. (21), (23), (24) and (25), the corresponding values of $V[\Delta a]$, $V[\Delta b]$ and $Cov[\Delta a, \Delta b]$ are given by Eqs. (16), (17) and (18), respectively, as in the first-order propagation of variance for $V[\hat{x}_I]$.

The first and last groups of symbols enclosed in braces in Eq. (25) has the form of hypothesis tests, that is, two-tailed tests, for significant difference of intercepts and significant difference of slopes, respectively. When the hypothesis test for different slope fails, the coefficient of x_I^2

Topic	Reference
Arrhenius plot	Cook and Charnock [44]; Han [45]; Puterman et al. [46]
Calibration curves	Baxter [47]; Bonate [48]; Mandel y Linning [49]; Schwartz [50–52]
Estimation of safe doses	Yanagimoto and Yanamoto [19]
Estimation of uncertainty in binding constants	Almansa López et al. [53]
Models for biologic half-life data	Lee et al. [25]
Position and confidence limits of an extremum	Asuero and Recamales [54]; Heilbronner [55]
Standard addition method	Franke et al. [56]

Table 4. Some applications of Fieller theorem in analytical chemistry.

becomes negative finding two complex roots [22], so Fieller confidence interval embraces the entire x -axis (the lower and upper limits should strictly be set to $-\infty$ and ∞ , respectively) at the chosen level of confidence.

This method has been extensively described in some other contexts in analytical and chemical literature (Table 4).

6. Use of hyperbolic confidence bands for the two linear branches

Several procedures dealing with hyperbolic confidence bands approximate them by straight lines and give symmetric confidence intervals for estimated x_I [58, 61, 70–72]. Evidently, the best confidence interval would be obtained by the projection on the abscissa of the surface between the four hyperbolic arcs [73].

Because a confidence band, bounded by two hyperbolic arcs, is associated with each regression line, it is obvious that the point of intersection, x_I , is only a mean value, with which a certain confidence interval is associated. If the signal values both before and after the point of intersection are normally distributed around the line with a constant standard deviation, the point of intersection and its statistical confidence interval will be estimated by the projection of the intersection onto the abscissa. The confidence interval (x_l, x_u) for the true value of the equivalence point is given by the projection on the abscissa of the common surface delimited by the four hyperbolic arcs.

For the first line, we get:

$$y_{01} \pm t_1 s_{y_{01}} \quad (26)$$

and for the second line:

$$y_{02} \pm t_2 s_{y_{02}} \quad (27)$$

t_1 and t_2 are the corresponding t Student values for $\alpha/2 = 0,05$ and $N_1 - 2$ and $N_2 - 2$ degrees of freedom, respectively. Hence, the lower value x_l of the confidence interval is obtained by solving the following equation:

$$y_{01} - t_1 s_{y_{01}} = y_{02} + t_2 s_{y_{02}} \quad (28)$$

The higher value x_u is obtained from the equation:

$$y_{01} + t_1 s_{y_{01}} = y_{02} - t_2 s_{y_{02}} \quad (29)$$

From Eqs. (1) and (3) we get

$$\hat{y}_1 = \bar{y}_1 + b_1(x - \bar{x}_1) \quad (30)$$

and then the variance of the fitted y_1 value will be given by

$$\begin{aligned} V[\hat{y}_1] &= V[\bar{y}_1] + (x - \bar{x}_1)^2 V[b_1] = \frac{V[y_1]}{\left(\sum w\right)_1} + (x - \bar{x}_1)^2 \frac{V[y_1]}{(S_{XX})_1} \\ &= \left(\frac{1}{\left(\sum w\right)_1} + \frac{(x - \bar{x}_1)^2}{(S_{XX})_1} \right) V[y_1] \end{aligned} \tag{31}$$

Note that the variance of the (weighted) mean of the values

$$\begin{aligned} V[\bar{y}_1] &= V \left[\frac{\left(\sum w\right)_1 y_1}{\left(\sum w\right)_1} \right] = V \left[\frac{\left(\sum \left((\sqrt{w})_1 (\sqrt{w})_1\right) y_1\right)}{\left(\sum w\right)_1} \right] \\ &= \frac{\left(\sum w\right)_1 V[(\sqrt{w})_1 y_1]}{\left(\left(\sum w\right)_1\right)^2} = \frac{V[y_1]}{\left(\sum w\right)_1} \end{aligned} \tag{32}$$

and that the mean y_1 value and the slope b_1 are uncorrelated random variables (property, which was also applied in Eq. (12) without further demonstration) as shown as follows. Taking into account that

$$\begin{aligned} \bar{y}_1 &= \sum b_1 (\sqrt{w})_1 y_1 \text{ and } b_1 = \sum c_1 (\sqrt{w})_1 y_1 \text{ where} \\ b_1 &= \frac{(\sqrt{w})_1}{\left(\sum w\right)_1} \quad c_1 = \frac{(\sqrt{w})_1 (x - \bar{x}_1)}{(S_{XX})_1} \end{aligned} \tag{33}$$

and then

$$Cov(b_1, c_1) = \left(\sum a_1 c_1\right) V[(\sqrt{w})_1 y_1] = \left(\sum \frac{(\sqrt{w})_1 (x - \bar{x}_1)}{(S_{XX})_1 \left(\sum w\right)_1}\right) V[y] = 0 \tag{34}$$

From Eq. (31), we get for the standard error of the fitted value

$$s_{y01} = V[\hat{y}_{1i}] = \left(\sqrt{\frac{1}{\left(\sum w\right)_1} + \frac{(x - \bar{x}_1)^2}{(S_{XX})_1}} \right) s_1 \left(y_{01} = \hat{y}_{1i}; \quad s_1 = \sqrt{V[y_1]} \right) \tag{35}$$

Thus, the lower value, x_l , for the confidence interval is obtained by solving the equation

$$a_1 + b_1 x_l + t_1 s_1 \sqrt{\frac{1}{\left(\sum w\right)_1} + \frac{(x_l - \bar{x}_1)^2}{(S_{XX})_1}} = a_2 + b_2 x_l + t_2 s_2 \sqrt{\frac{1}{\left(\sum w\right)_2} + \frac{(x_l - \bar{x}_2)^2}{(S_{XX})_2}} \tag{36}$$

by, for example, successive approximations with an Excel spreadsheet. The higher value x_u is obtained in the same way from the equation that follows also by successive approximations

$$a_1 + b_1x_u + t_1s_1\sqrt{\frac{1}{\left(\sum w\right)_1} + \frac{(x_u - \bar{x}_1)^2}{(S_{XX})_1}} = a_2 + b_2x_u + t_2s_2\sqrt{\frac{1}{\left(\sum w\right)_2} + \frac{(x_u - \bar{x}_2)^2}{(S_{XX})_2}} \quad (37)$$

The point of view of Liteanu et al. [57, 58] is very interesting: the authors consider the point of intersection x_I as belonging to the linear regression before the equivalence point. Then, a certain interval is associated with it. If it is regarded as belonging to the linear regression after the equivalence point, however, another interval is associated with it. As the equivalence point belongs concurrently to both linear regressions, the confidence interval of the two segments can be got by taking the weighted averages of the branches of the two separate sets of confidence intervals. So, we obtain the ultimate confidence interval (x_I^l, x_I^u) where

$$\frac{(N_1 - 2)(x_I)_1^l + (N_2 - 2)(x_I)_2^l}{N_1 + N_2 - 4} = x_I^l \quad (38)$$

$$\frac{(N_1 - 2)(x_I)_1^u + (N_2 - 2)(x_I)_2^u}{N_1 + N_2 - 4} = x_I^u \quad (39)$$

The two values of the limits of the confidence interval will be given by the two solutions of the equations

$$y_I = a_1 + b_1(x_I)_1 \pm t_1s_1\sqrt{\frac{1}{\left(\sum w\right)_1} + \frac{\left((x_I)_1 - \bar{x}_1\right)^2}{(S_{XX})_1}} \quad (40)$$

$$y_I = a_2 + b_2(x_I)_2 \pm t_2s_2\sqrt{\frac{1}{\left(\sum w\right)_2} + \frac{\left((x_I)_2 - \bar{x}_2\right)^2}{(S_{XX})_2}} \quad (41)$$

As the estimation method used assumes the worst case in combining random error of the two lines, the derived confidence limits are on the pessimistic (i.e., realistic) side.

On rearrangement Eq. (40) and squaring, we have

$$\left(y_I - a_1 - b_1(x_I)_1\right)^2 = t_1^2s_1^2\left(\frac{1}{\left(\sum w\right)_1} + \frac{\left((x_I)_1 - \bar{x}_1\right)^2}{(S_{XX})_1}\right) \quad (42)$$

which by simple algebra it may be ordered in powers of x_1 as

$$\begin{aligned} &\left(b_1^2 - \frac{t_1^2s_1^2}{(S_{XX})_1}\right)(x_I)_1^2 - 2\left(b_1(y_I - a_1) - t^2\frac{\bar{x}_1s_1^2}{(S_{XX})_1}\right)(x_I)_1 \\ &+ (y_I - a_1^2) - t_1^2s_1^2\left(\frac{1}{\left(\sum w\right)_1} + \frac{\bar{x}^2}{(S_{XX})_1}\right) = 0 \end{aligned} \quad (43)$$

and taking into account the values of $V[b_1]$, $Cov[a_1, b_1]$ and $V[a_1]$ (see **Table 3**), we get finally

$$(b_1^2 - t_1^2 V[b_1])(x_I)_1^2 - 2(b_1(y_I - a_1) + t_1^2 Cov(a_1, b_1))(x_I)_1 + (y_I - a_1)^2 - t_1^2 V[a_1] = 0 \quad (44)$$

whose roots give the two values of $(x_I)_1$. Since the point of intersection x_I belongs to one of the response functions, then a certain confidence interval is associated with it.

Similarly, if it is regarded as belonging to the other response function, there is another confidence interval associated with it

$$(b_2^2 - t_2^2 V[b_2])(x_I)_2^2 - 2(b_2(y_I - a_2) + t_2^2 Cov(a_2, b_2))(x_I)_2 + (y_I - a_2)^2 - t_2^2 V[a_2] = 0 \quad (45)$$

Because the intersection point belongs concomitantly to the two response functions, the two segments which together compose the confidence interval, will be obtained by averaging the segments of the two separate confidence intervals, Eqs. (40) and (41). The two values of the limits of the confidence interval will be the two solutions of the second degree Eqs. (44) and (45).

The bands mentioned in this section are [63] for the ordinate of the true line at only a single point. If we desire the confidence bands for the entire line, the critical constant $\sqrt{2F_{2,n-2}^\alpha}$ should be substituted for $t_{\alpha/2}$, originating wider bands.

7. Statistical uncertainty of endpoint differences

When we are dealing with the titration of a mixture of a strong and a weak acid that is, hydrochloric and acetic acids, then if x_I is the volume at which the straight lines one and two intersect and x_{II} the volume at which the two and three lines intersect, the difference $x_{II} - x_I$ denoted as Δx , is given by

$$\Delta x = \hat{x}_{II} - \hat{x}_I = \frac{-\Delta a_2}{\Delta b_2} + \frac{\Delta a_1}{\Delta b_1} \quad (46)$$

By multiplying Δx by the molarity of titrant, we have the amount in millimoles of the second acid, that is, acetic acid, in the reaction mixture.

First-order propagation of variance applied to Δx leads to [65] the following expression

$$V[\Delta x] = V[\hat{x}_I] + V[\hat{x}_{II}] + Cov(\Delta a_1, \Delta a_2) + Cov(\Delta a_1, \Delta b_2) + Cov(\Delta a_2, \Delta b_1) + Cov(\Delta b_1, \Delta b_2) \quad (47)$$

where

$$Cov(\Delta a_1, \Delta a_2) = 2 \left(\frac{V[\bar{y}_2] + \bar{x}_2^2 V[b_2]}{\Delta b_1 \Delta b_2} \right) \quad (48)$$

$$Cov(\Delta a_1, \Delta b_2) = -2 \left(\frac{\bar{x}_2 \hat{x}_{II} V[b_2]}{\Delta b_1 \Delta b_2} \right) \quad (49)$$

$$Cov(\Delta a_2, \Delta b_1) = -2 \left(\frac{\bar{x}_2 \hat{x}_I V[b_2]}{\Delta b_1 \Delta b_2} \right) \quad (50)$$

$$Cov(\Delta b_1, \Delta b_2) = 2 \left(\frac{\hat{x}_I \hat{x}_{II} V[b_2]}{\Delta b_1 \Delta b_2} \right) \quad (51)$$

The standard error estimate is given by

$$s_{\Delta x} = \sqrt{V[\hat{x}_I] + V[\hat{x}_{II}] + \sum_4 Cov} \quad (52)$$

where \sum_4 is the sum of Eqs. (48)–(51).

Attempts to derive confidence limits for Δx as we get in the previous Fieller's theorem section fails because the quantity analogous to z of Eq. (20) involves products of random variables. Therefore, this quantity is not normally distributed and so exact confidence limits cannot be found in terms of Student's t distribution. Because in this case the exact confidence limits cannot be calculated, we use the small variance confidence interval

$$C.I. = 2t_{\alpha/2} s_{\Delta x} \quad (53)$$

8. Application to experimental system

A bibliographic search allows us to demonstrate the importance of conductivity measurements despite their antiquity. The general fundamentals of this technique are collected in Gelhaus and Lacourse (2005) [74] and Gzybkoski (2002) [75]. Its importance in the educational literature has been highlighted [76, 77] and many examples have been recently published in the Journal of Chemical Education i.e., studies on sulfate determination [78]; the identification and quantification of an unknown acid [79], electrolyte polymers [80, 81], acid and basic constants determinations [82], its use in general chemistry [83], microcomputer interface [84] and conductometric-potentiometric titrations [85]. An accurate method of determining conductivity in acid-base reactions [86], the acid-base properties of weak electrolytes [87], and those of polybasic organic acids [88] have also been recently subject of study.

The relation between conductometric and the volume of titrant added leads to segmented linear titration curves, the endpoint being defined by the intersection of the two straight lines segments. What follows is the application of the possible methods of uncertainty estimation of the endpoint of data described in the literature as well as experimental measurements carried out in the laboratory.

8.1. Conductometric titration of 100 mL of a mixture of acids with potassium hydroxide 0.100 M

Table 5 shows the data [conductance ($1/R$), volume (x)] published by Carter et al. [69]; Schwartz and Gelb [65]) and corresponding to the conductometric titration of a mixture of acids, perchloric acid and acetic acid with potassium hydroxide 0.100 M as titrant agent. The points recorded belong to the three branches of the titration curve; the first (branch A) corresponds to the neutralization of perchloric acid, the second (branch B) to the neutralization of acetic acid, and the third (branch C) to the excess of potassium hydroxide.

Let us focus first on the perchloric acid titration. The plot of conductance data ($1/R$) versus volume (x), in general, is not linear due to the dilution effect of the titrant. So that, as it is carried out in the usual way, it is plotted the product $(1/R)(100 + x)$ versus x (see Figure 1).

Firstly, Schwartz and Gelb [65] select 13 points, six (volume 4–14 mL) for branch A and seven (volume 20–32 mL) for branch B. The points near to the endpoint of perchloric acid are deviating from linearity and discarded in the first instance. It is also considered that the data have a different variance $V \cdot (100 + x_i)^2$, being the weighting factor $(100 + x_i)^{-2}$ (see Table 5).

In the case of acetic titration, six points (volume 35–44 mL) are selected for branch C, at first. The points of branch B near to the acetic acid endpoint are discarded. Figures 2 and 3 show the straight line segments with the corresponding selected points.

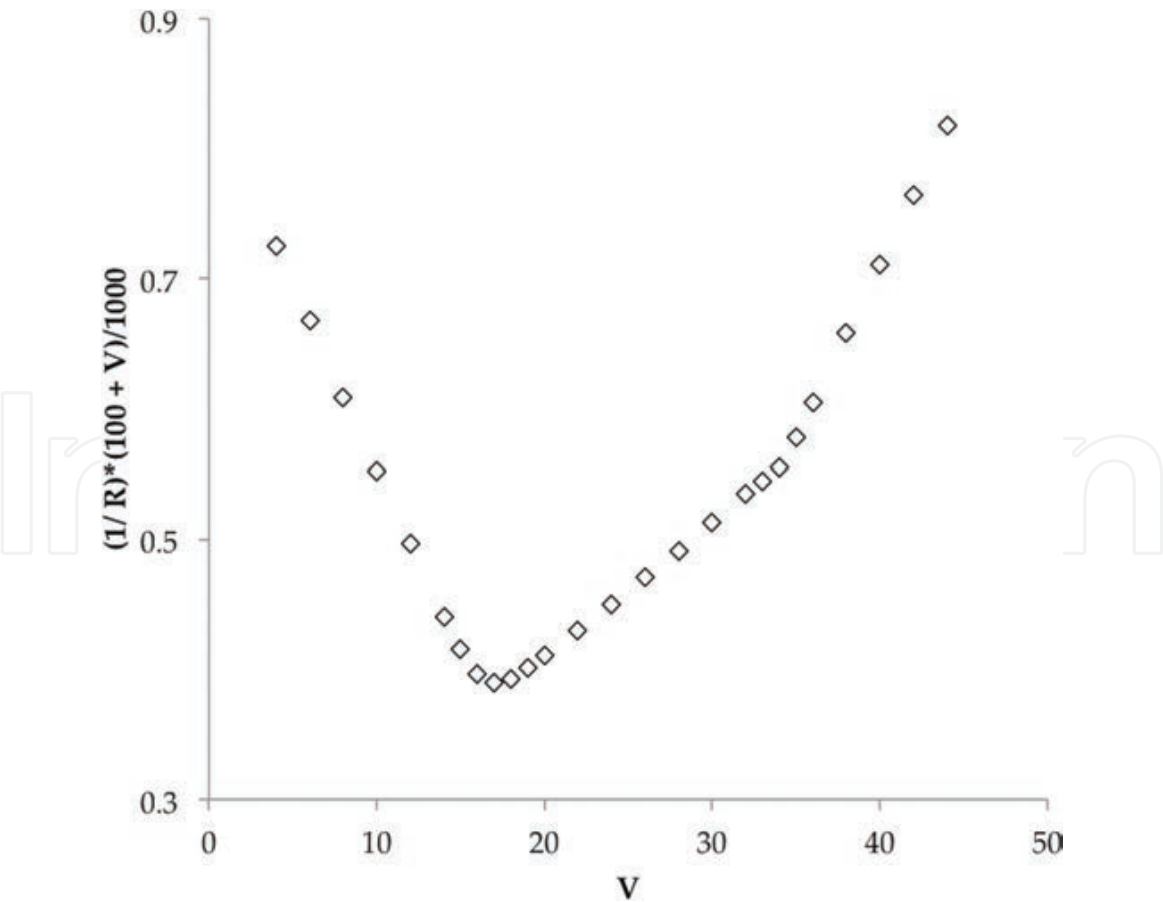


Figure 1. Conductometric titration of a mixture of perchloric and acetic acids with potassium hydroxide (data shown in Table 5).

1/R	x	y	W _i	1/R	x	y	W _i
6.975	4	0.7254	9.246E-05	3.633	24	0.4505	6.504E-05
6.305	6	0.6683	8.900E-05	3.742	26	0.4715	6.299E-05
5.638	8	0.6089	8.573E-05	3.840	28	0.4915	6.104E-05
5.020	10	0.5522	8.264E-05	3.946	30	0.5130	5.917E-05
4.432	12	0.4964	7.972E-05	4.052	32	0.5349	5.739E-05
3.865	14	0.4406	7.695E-05	4.097	33	0.5449	5.653E-05
3.610	15	0.4152	7.561E-05	4.145	34	0.5554	5.569E-05
3.415	16	0.3961	7.432E-05	4.280	35	0.5778	5.487E-05
3.328	17	0.3894	7.305E-05	4.445	36	0.6045	5.407E-05
3.330	18	0.3929	7.182E-05	4.772	38	0.6585	5.251E-05
3.370	19	0.4010	7.062E-05	5.080	40	0.7112	5.102E-05
3.420	20	0.4104	6.944E-05	5.380	42	0.7640	4.959E-05
3.522	22	0.4297	6.719E-05	5.680	44	0.8179	4.823E-05

Table 5. Data conductance (1/R) and volume (x) corresponding to the titration of a mixture of perchloric acid and acetic acid with potassium hydroxide.

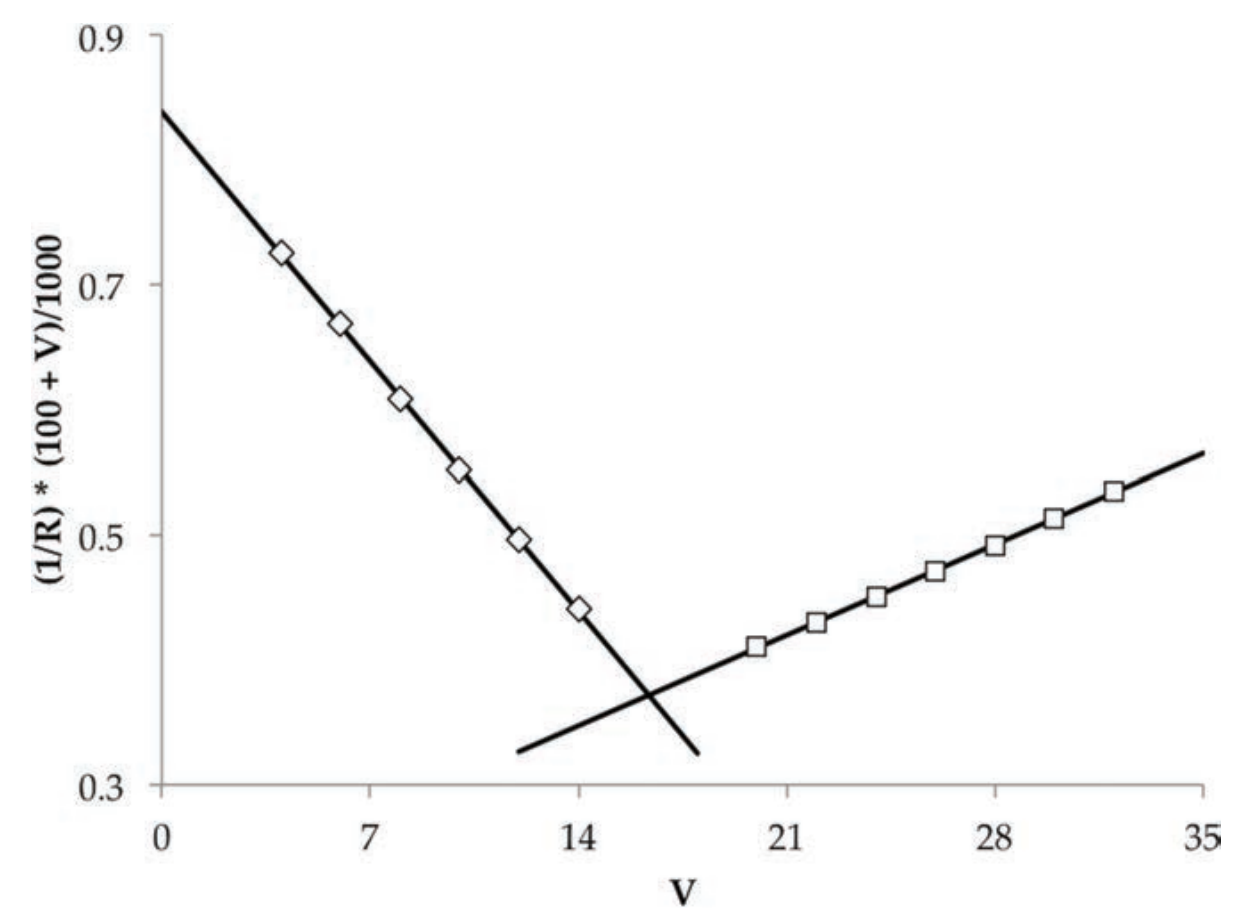


Figure 2. Conductometric titration of perchloric acid in the mixture (branches A and B).

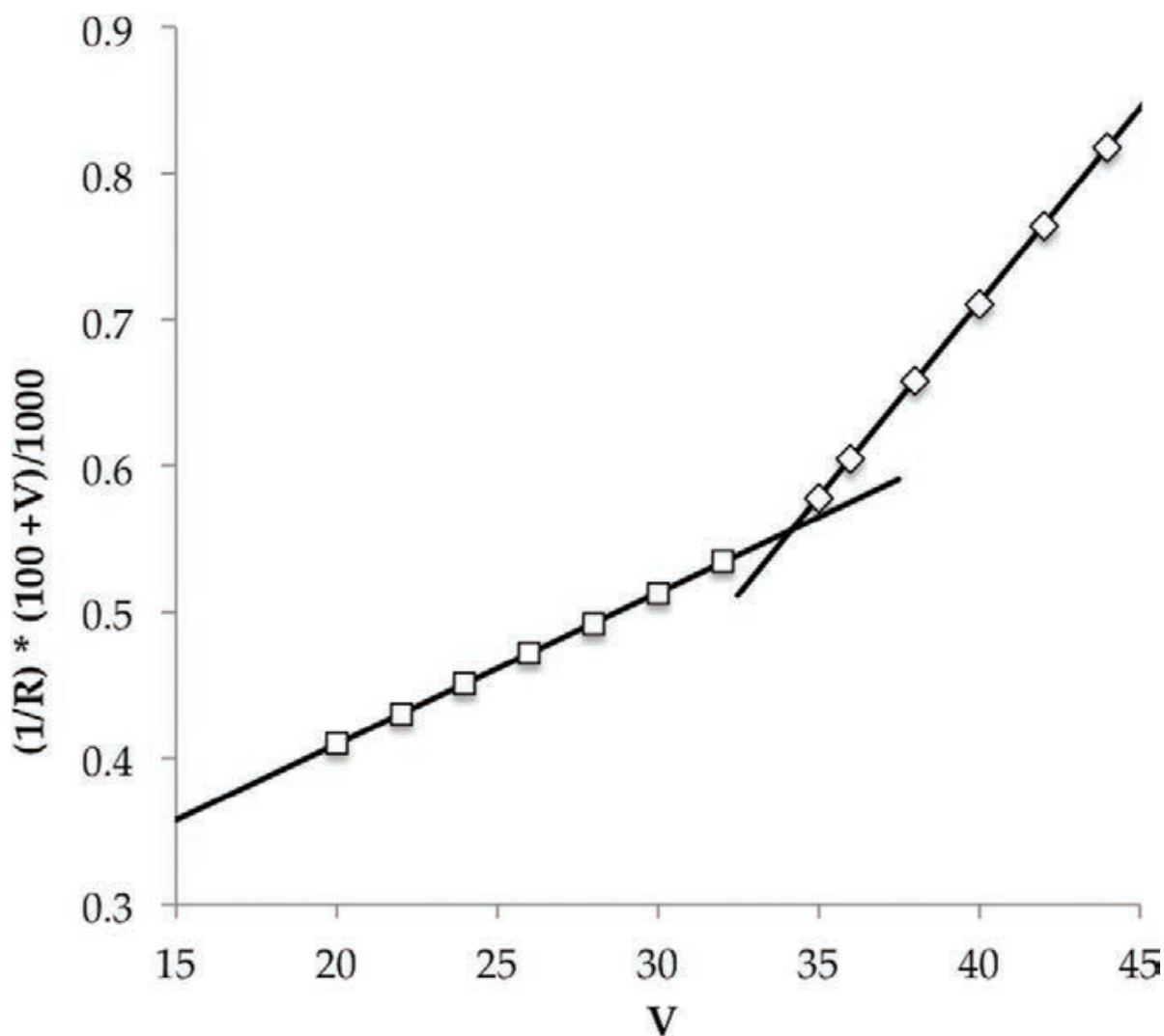


Figure 3. Conductometric titration of acetic acid in the mixture (branches B and C).

Table 6 includes the intermediate results obtained in the calculation of the first endpoint, corresponding to the neutralization of perchloric acid (**Figure 2**), in order to follow the procedures previously detailed. The first endpoint is located at 16.367 mL and therefore 1.637 mmol of HClO_4 . The estimated standard error at the endpoint, using the first-order propagation of variance, is 0.039 mL. The confidence limits are calculated using $t = 2.262$ (9 degrees of freedom) and correspond to 16.455 and 16.279 mL, respectively, for the upper and lower limits, being the confidence interval equal to 0.176 mL. The application of Fieller's theorem leads to the values of 16.455 and 16.278 mL, respectively. Carter et al. [67] give values of 16.455 and 16.279 mL, identical to the first ones indicated.

The second endpoint, corresponding to the complete neutralization of both perchloric and acetic acids, is located at 34.197 mL. If $x_{(I)}$ is the volume in which lines A and B intersect, and $x_{(II)}$ the volume in which lines B and C intersect, the difference $x_{(II)} - x_{(I)}$ (34.1971–16.3665 mL) corresponds to acetic acid in the sample, 17.831 mL. If the above methodology is used for lines, B and C (**Figure 3**) give $x_{(II)} \pm s_d [x_{(II)}]$ equal to 34.197 ± 0.0478 , and 34.305 and 34.089 mL for the confidence limits.

$1/R$	x	$y = (1/R)(100 + x)$	$(100 + x_i)^{-2}$		$1/R$	x	$y = (1/R)(100 + x)$	$(100 + x_i)^{-2}$
6.975	4	0.7254	9.246E-05		3.420	20	0.4104	6.944E-05
6.305	6	0.6683	8.900E-05		3.522	22	0.4297	6.719E-05
5.638	8	0.6089	8.573E-05		3.633	24	0.4505	6.504E-05
5.020	10	0.5522	8.264E-05		3.742	26	0.4715	6.299E-05
4.432	12	0.4964	7.972E-05		3.840	28	0.4915	6.104E-05
3.865	14	0.4406	7.695E-05		3.946	30	0.5130	5.917E-05
					4.052	32	0.5349	5.739E-05
$N_1 =$	6	$[\Sigma W_i]_1 =$	5.065E-04		$N_2 =$	7	$[\Sigma W_i]_2 =$	4.423E-04
$x_1(\text{mean}) =$	8.786	$y_1(\text{media}) =$	0.5881		$x_2(\text{mean}) =$	25.746	$y_2(\text{mean}) =$	0.4690
		$[S(xx)]_1 =$	5.8987E-03				$[S(xx)]_2 =$	7.0636E-03
		$[S(xy)]_1 =$	-1.683E-04				$[S(xy)]_2 =$	7.323E-05
		$[S(yy)]_1 =$	4.804E-06				$[S(yy)]_2 =$	7.594E-07
	$b_1 =$	-0.028535824		$\Delta b =$	0.038903	$b_2 =$	0.010367102	
	$a_1 =$	0.838792976		$\Delta a =$	-0.636705	$a_2 =$	0.202087987	
	$[R^2]_1 =$	0.99988		$x(I) =$	16.3665	$[R^2]_2 =$	0.99974	
	$V[y/x]_1 =$	1.45580E-10				$V[y/x]_2 =$	3.96396E-11	
	$V(b_1) =$	2.468E-08		$V[\text{pooled}] =$	8.67244E-11	$V(b_2) =$	5.612E-09	
	$V(a_1) =$	2.193E-06				$V(a_2) =$	3.809E-06	
	$Cov(a_1, b_1) =$	-2.168E-07				$Cov(a_2, b_2) =$	-1.445E-07	
$V[\Delta a] =$	6.002E-06	$V[\Delta a] =$	9.640E-06	$V[x(I)] =$	0.0015	FIELLER $ax^2 + bx + c = 0$		
$V[\Delta b] =$	3.029E-08	$V[\Delta b] =$	2.698E-08	$s[x(I)] =$	0.0389	$a =$	1.513E-03	$V(u) =$ 16.455
$cov(\Delta a, \Delta b) =$	-3.613E-07	$cov(\Delta a, \Delta b) =$	-4.453E-07	$t(0, 05, 9) =$	2.262	$b =$	-4.953E-02	$V(l) =$ 16.278
		Pooled variances		$t s[x(I)] =$	0.0880	$c =$	4.053E-01	

Table 6. Intermediate results obtained in the calculation of the first endpoint (titration of perchloric acid with potassium hydroxide (**Figure 2**)).

However, as it is indicated in the section on “statistical uncertainty of endpoint differences,” the statistical uncertainty of Δx is not a simple combination of uncertainties for $x_{(I)}$ and $x_{(II)}$. The attempt to deduce equations analogous to Eqs. (22) and (25) in order to calculate the confidence limits for Δx , is not applicable since the magnitude analogous to z in Eq. (20) implies, in this case, the product of random variables.

This quantity is not normally distributed, and therefore, no exact confidence limits can be calculated in terms of the Student t distribution. The application of (first-order) propagation of the variance is nonetheless feasible, leading this procedure to an expression for the standard error of Δx of the same type as Eq. (9) for a single endpoint.

The latter methodology is applied to the optimal case detailed by Schwartz and Gelb [65]. The corresponding data are shown in **Figure 4**, and the calculations necessary to locate the equivalence points, first and second, are shown in **Table 7**. The results obtained are: first equivalence point (perchloric acid): $x_{(I)} = 16.358$ mL, $s[x_{(I)}] = 0.035$ mL, $t s [x_{(I)}] = 0.078$ mL, $[I.C.]_I = 0.156$ mL. Second equivalence point (mixture of perchloric and acetic acids): $x_{(II)} = 34.244$ mL, $s[x_{(II)}] = 0.027$ mL, $t s [x_{(II)}] = 0.061$ mL, $[IC]_{II} = 0.122$ mL. This latter is not correct because it does not take into account the covariances described in Section 7. If covariances are incorporated into the calculations, we get for the second point (acetic acid): $x = 17.887$ mL, $s[\Delta x] = 0.040$ mL; $t s [\Delta x] = 0.086$ mL, $[IC]_{\Delta x} = 0.172$ mL. The confidence interval, as expected, is higher than that found for $x_{(II)}$, despite decreasing the value of Student's t by increasing the number of degrees of freedom: $N_1 + N_2 - N_3 - 2 \times 3 = 13$).

Some points near to the endpoint appear to deviate slightly from linearity. However, it is not always clear whether or not to omit these problem points in the analysis, which can be done by

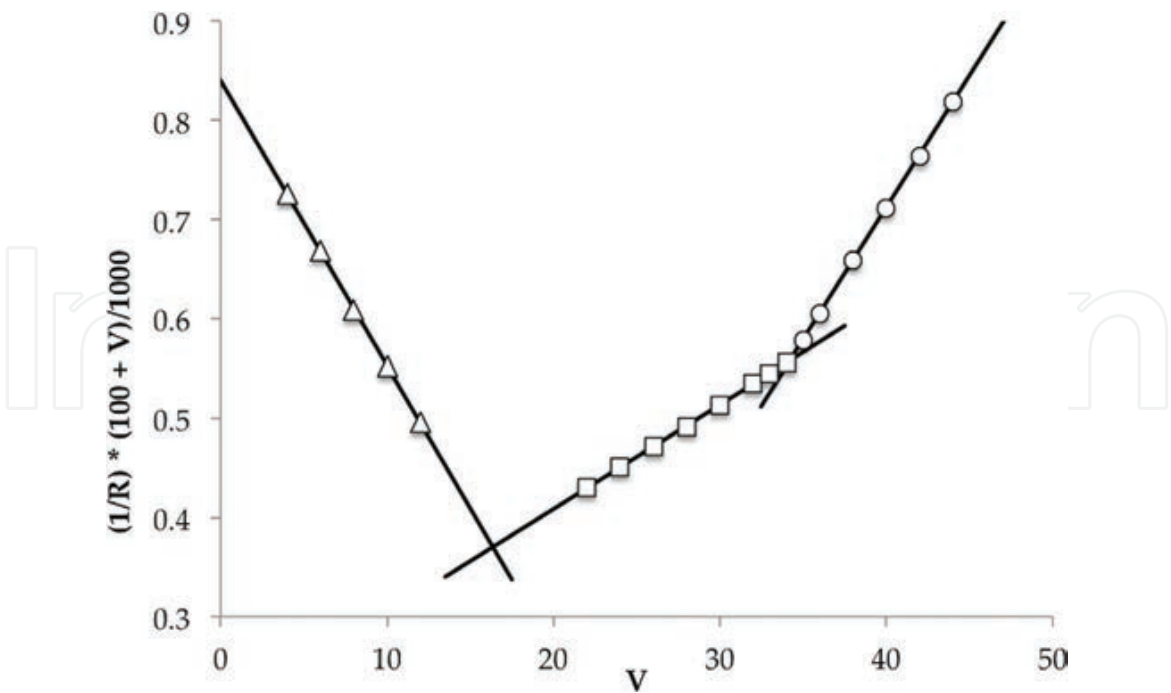


Figure 4. Illustrative example described by Schwartz and Gelb [65] as optimal. Numerical data are shown in **Table 5**. First branch (A), volumes of 4–12 mL, 5 points. Second branch (B), volumes of 22–34 mL, 8 points. Third branch (C), volumes of 35–44 mL, 6 points.

$1/R$	x	$y = (1/R)(100 + x)$	$(100 + x_i)^{-2}$		$1/R$	x	$y = (1/R)(100 + x)$	$(100 + x_i)^{-2}$
6.975	4	0.7254	9.246E-05		3.522	22	0.4297	6.719E-05
6.305	6	0.6683	8.900E-05		3.633	24	0.4505	6.504E-05
5.638	8	0.6089	8.573E-05		3.742	26	0.4715	6.299E-05
5.020	10	0.5522	8.264E-05		3.840	28	0.4915	6.104E-05
4.432	12	0.4964	7.972E-05		3.946	30	0.5130	5.917E-05
					4.052	32	0.5349	5.739E-05
					4.097	33	0.5449	5.653E-05
					4.145	34	0.5554	5.569E-05
$N_1=$	5	$[\Sigma W_i]1=$	4.296E-04		$N_2=$	8	$[\Sigma W_i]2=$	4.850E-04
$x_1(\text{mean})=$	7.852	$y_1(\text{media})=$	0.6145		$x_2(\text{mean})=$	28.362	$y_2(\text{mean})=$	0.4962
		$[S(xx)]1=$	3.4319E-03				$[S(xx)]2=$	8.221E-03
		$[S(xy)]1=$	-9.856E-05				$[S(xy)]2=$	8.624E-05
		$[S(yy)]1=$	2.831E-06				$[S(yy)]2=$	9.048E-07
	$b_1=$	-0.028717584		$\Delta b=$	0.039208	$b_2=$	0.010490769	
	$a_1=$	0.839983924		$\Delta a=$	-0.641362	$a_2=$	0.198621917	
	$[R2]1=$	0.99989		$x(I)=$	16.3578	$[R2]2=$	0.99992	
	$V[y/x]1=$	1.03732E-10				$V[y/x]2=$	1.25234E-11	
	$V(b_1)=$	3.023E-08		$V[\text{pooled}]=$	4.29264E-11	$V(b_2)=$	1.523E-09	
	$V(a_1)=$	2.105E-06				$V(a_2)=$	1.251E-06	
	$cov(a_1, b_1)=$	-2.373E-07				$cov(a_2, b_2)=$	-4.321E-08	
$V[\Delta a]=$	3.356E-06	$V[\Delta a]=$	5.160E-06	$V[x(I)]=$	0.0012	FIELLER	$ax^2 + bx + c = 0$	
$V[\Delta b]=$	3.175E-08	$V[\Delta b]=$	1.773E-08	$s[x(I)]=$	0.0347	$a=$	1.537E-03	$V(u)=$ 16.436
$cov(\Delta a, \Delta b)=$	-2.805E-07	$cov(\Delta a, \Delta b)=$	-2.463E-07	$t(0, 05, 9)=$	2.262	$b=$	-5.029E-02	$V(l)=$ 16.280
		Pooled variances		$t s[x(I)]=$	0.0784	$c=$	4.113E-01	
		$V(u)=$	16.436	$V(l)=$	16.279			

1/R	x	y = (1/R)(100 + x)	(100 + x _i) ⁻²		1/R	x	y = (1/R)(100 + x)	(100 + x _i) ⁻²	
4.280	35	0.5778	5.487E-05	Δb=	0.016147				
4.445	36	0.6045	5.407E-05	Δa=	-0.552936				
4.772	38	0.6585	5.251E-05	x(I)=	34.2444				
5.080	40	0.7112	5.102E-05	Δx=	17.8866				
5.380	42	0.7640	4.959E-05						
5.680	44	0.8179	4.823E-05	V[pooled]=	1.12632E-11				
N ₃ =	6	[ΣW _i]1=	3.103E-04						
x ₃ (mean)=	39.022	y ₃ (media)=	0.6851						
		[S(xx)]1=	3.1199E-03						
		[S(xy)]1=	8.311E-05						
		[S(yy)]1=	2.214E-06						
				V[x(I)]=	0.0007	t(0, 05, 9)=	2.262	V(u)=	34.305
	b3=	0.026637531		s[x(I)]=	0.0269	t s[x(I)]=	0.0610	V(l)=	34.183
	a3=	-0.354313932							
	[R2]3=	0.99998							
	V[y/x]3=	9.37285E-12		V[x(I)]=	1.201E-03				
	V(b3)=	3.004E-09		V[x(II)]=	7.262E-04				
	V(a3)=	4.605E-06		Cov(Δa ₁ , Δa ₂)=	3.871E-03				
	cov(a ₃ , b ₃)=	-1.172E-07		Cov(Δa ₁ , Δb ₂)=	-4.674E-03				
				Cov(Δa ₂ , Δb ₁)=	-2.233E-03				
				Cov(Δb ₁ , Δb ₂)=	2.696E-03				
V[Δa]=	5.761E-12	V[Δa]=	6.659E-06						
V[Δb]=	4.528E-09	V[Δb]=	4.980E-09						
cov(Δa, Δb)=	-1.294E-07	cov(Δa, Δb)=	-1.797E-07	V[Δx]=	1.587E-03	t(0, 05, 13)=	2.160	V(u)=	17.973
		Pooled variances		s[Δx]=	3.984E-02	t s[Δx]=	0.086	V(l)=	17.801

Table 7. Evaluation of endpoints in the titration of a mixture of HClO_4 and CH_3COOH with KOH 0.100 M, optimum case (Figure 4).

trial and error. The optimal point set (**Figure 4**) is one that minimizes, for example, the confidence interval [63].

The weighting factors are very similar so that the values obtained by weighted linear regression and the simple one become equivalent.

8.2. Conductometric titration of hydrochloric acid 0.1 M with sodium hydroxide 0.1 M

The data corresponding to the two branches of the conductometric titration of 0.1 M HCl with 0.1 M NaOH is shown in the upper part of **Table 8** and plot in **Figure 5**. The cut-off point of both lines is (6.414, 0.358) [57, 58, 89].

Table 8 also shows all the operations required to calculate the minimum and maximum values of the confidence interval by the use of hyperbolic confidence bands for the two linear branches. The limit x_l of the confidence interval is obtained by solving Eq. (36), which in this case (**Table 8**) is

$$\begin{aligned} \Theta_l = 1.403 - 0.0637x_l + 1.943 \cdot 0.01034 \sqrt{\frac{1}{8} + \frac{(x_l - 9)^2}{168}} \\ + 0.4908 - 0.0517x_l + 2.353 \cdot 0.0024 \sqrt{\frac{1}{5} + \frac{(x_l - 20.2)^2}{32.8}} = 0 \end{aligned} \quad (54)$$

leading to $x_l = 16.264$ mL. The highest value is obtained by solving (Eq. (37))

$$\begin{aligned} \Theta_u = 1.403 - 0.0637x_u + 1.943 \cdot 0.01034 \sqrt{\frac{1}{8} + \frac{(x_u - 9)^2}{168}} \\ + 0.4908 - 0.0517x_u + 2.353 \cdot 0.0024 \sqrt{\frac{1}{5} + \frac{(x_u - 20.2)^2}{32.8}} = 0 \end{aligned} \quad (55)$$

which leads to $x_u = 16.564$ mL. Both equations $\theta_l = 0$ and $\theta_u = 0$ are resolved by successive approximations. Different values are tested for the lower and upper limits to get a change of sign in θ_l and θ_u .

In the weighted mean method (**Table 8**), the following equations are solved

$$0.00405(x_l)_1^2 - 0.1332(x_l)_1 + 1.09179 = 0 \quad (56)$$

$$0.00267(x_l)_2^2 - 0.08776(x_l)_2 + 0.720 = 0 \quad (57)$$

being resulted from squaring and reordering the Eqs. (44) and (45), respectively (expressed as a function of the variances of a_1 , b_1 and of the covariance between a_1 and b_1). Once calculated the solutions of the Eqs. (56): 16.630 and 16.487 mL, and (57): 16.206 and 16.339 mL, we have

x		y	x		y		
	2	1.265		17	0.388		
	4	1.141		18	0.441		
	6	1.028		20	0.544		
	8	0.906		22	0.644		
	10	0.777		24	0.752		
	12	0.641					
	14	0.51					
	16	0.372					
N1=	8		N2=	5			
MEAN1=	9	0.83	MEAN2=	20.2	0.5538		
[SXX]1=	168		[SXX]2=	32.8			
$a_1=$	-0.06367	1.40300	$=a_0$	$a_1=$	0.05171	-0.49081	$=a_0$
$s(a_1)=$	0.00080	0.00806	$=s(a_0)$	$s(a_1)=$	0.00036	0.00726	$=s(a_0)$
R2=	0.99906	0.01034	$=s(y/x)$	R2=	0.99986	0.00204	$=s(y/x)$
		$x(I)=$	16.414	$y(I)=$	0.358		
$t(0.05;6)=$	1.943	$t_1 s(y/x)1=$	0.0201	$t(0.05;3)=$	2.353	$t_2 s(y/x)2=$	0.0048
θ	DIFF-1	DIFF-2	θ	DIFF-1	DIFF-2		
16.25	0.00164	0.036126	16.50	-0.02736	0.007436		
16.26	0.00048	0.034979	16.51	-0.02852	0.006289		
16.261	0.00037	0.034864	16.52	V0.02968	0.005141		
16.262	0.00025	0.034749	16.53	-0.03084	0.003994		
16.263	0.00014	0.034634	16.54	-0.03200	0.002846		
16.264	0.0000191	0.034520	16.55	-0.03316	0.001699		
16.2641	0.0000075	0.034508	16.56	-0.03432	0.000552		
16.2642	-0.0000041	0.034497	16.561	-0.03443	0.000437		
16.265	-0.00010	0.034405	16.562	-0.03455	0.000322		
16.266	-0.00021	0.034290	16.563	-0.03467	0.000207		
16.267	-0.00033	0.034175	16.564	-0.03478	0.000093		
16.268	-0.00044	0.034060	16.5648	-0.03487	0.000001		
16.269	-0.00056	0.033946	16.5649	-0.03489	-0.000011		
16.27	-0.00068	0.033831	16.565	-0.03490	-0.000022		
		16.265	16.414	16.565			

Table 8. Hyperbolic confidence intervals for the two lines: successive approximations.

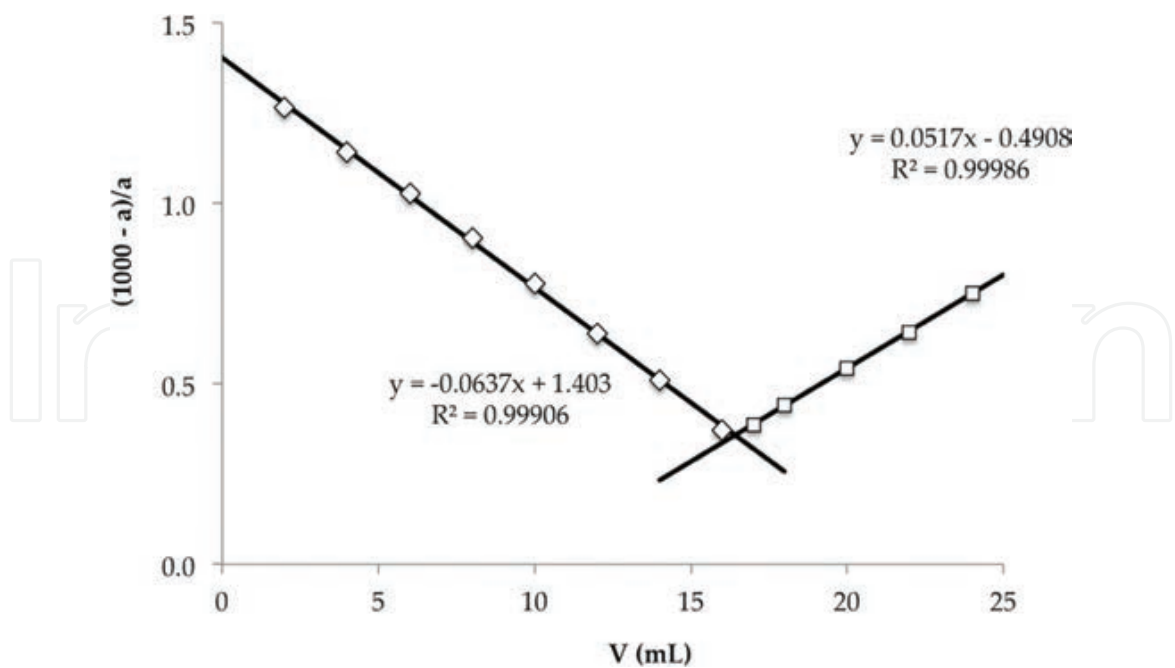


Figure 5. Conductometric titration of hydrochloric acid 0.1 M with sodium hydroxide 0.1 M as a titrant (data are shown in Table 8).

$$x_u = \frac{(8 - 2)16.630 + (5 - 2)16.487}{8 + 5 - 4} = 16.583 \text{ mL} \tag{58}$$

$$x_l = \frac{(8 - 2)16.206 + (5 - 2)16.339}{8 + 5 - 4} = 16.250 \text{ mL} \tag{59}$$

8.3. Experimental measurements: conductometric titration of 100 mL of a mixture of hydrochloric acid and acetic acids with potassium hydroxide 0.100 M

8.3.1. Reagents

Acetic acid (C₂H₄O₂) *M* = 60 g/mol (MERCK > 99.5%; 1.049 g/mL); hydrochloric acid (HCl) 1 M (MERCK, analytical grade); potassium hydroxide (KOH) 1 M (MERCK, analytical grade); potassium hydrogen phthalate (C₈H₅KO₄) *M* = 204.23 g/mol (MERCK > 99.5%).

8.3.2. Instruments

4-decimal point analytical balance (Metler AE200), conductivity meter Crimson (EC-Metro GLP 31), calibrated by standards of 147 μS/cm, 1413 μS/cm, 12.88 mS/cm. Digital burette of 50 mL (Brand) (accuracy: 0.2%, precision: <0.1%, resolution: 0.01 mL, with standard vent valve at 20°C).

8.3.3. Solutions

- Mixture of hydrochloric and acetic acids 0.015 M.
- Potassium hydroxide 0.1 M.

8.3.4. Experimental

About 100 mL of mixture of hydrochloric and acetic acids 0.015 M is transferred to a 250 mL volumetric flask containing 100 mL of distilled water. Then, the mixture is titrated conductometrically with KOH 0.0992 ± 0.0001 M ($n = 3$), (previously standardized with potassium hydrogen phthalate). **Table 9** shows the data [conductance, volume] as well as the product of the conductance by $(100 + x)/100$ to correct the dilution effect of the titrant. The data are plotted in **Figure 6**.

V KOH (mL)	Conductance (mS/cm)	Conductance* (mS/cm)	V KOH (mL)	Conductance (mS/cm)	Conductance* (mS/cm)
0.0	5.64	5.6400	21.1	2.22	2.6884
1.3	5.30	5.3689	21.5	2.25	2.7338
2.1	5.10	5.2071	22.0	2.27	2.7694
4.6	4.44	4.6442	23.0	2.34	2.8782
5.5	4.21	4.4416	24.0	2.39	2.9636
6.1	4.05	4.2971	25.0	2.45	3.0625
7.0	3.82	4.0874	26.1	2.51	3.1651
8.0	3.58	3.8664	27.0	2.56	3.2512
9.0	3.32	3.6188	28.0	2.61	3.3408
10.0	3.07	3.3770	29.1	2.67	3.4470
11.0	2.84	3.1524	30.0	2.72	3.5360
12.1	2.58	2.8922	31.0	2.88	3.7728
13.1	2.36	2.6692	32.0	3.03	3.9996
14.0	2.16	2.4624	33.0	3.19	4.2427
15.0	2.01	2.3115	34.0	3.35	4.4890
15.5	1.97	2.2765	35.0	3.50	4.7250
16.0	1.96	2.2736	36.0	3.63	4.9368
16.5	1.97	2.2951	37.0	3.79	5.1923
17.1	1.99	2.3303	38.0	3.91	5.3958
17.5	2.01	2.3618	39.0	4.04	5.6156
18.0	2.04	2.4072	40.0	4.19	5.8660
18.5	2.06	2.4411	41.0	4.32	6.0912
19.0	2.09	2.4871	42.1	4.46	6.3377
19.5	2.12	2.5334	43.0	4.57	6.5351
20.0	2.15	2.5800	44.0	4.70	6.7680
20.5	2.18	2.6269	45.0	4.82	6.9890

* Conductivity·((100 + V)/100).

Table 9. Conductance and KOH volume data corresponding to the titration of a mixture of hydrochloric and acetic acids with potassium hydroxide (first assay).

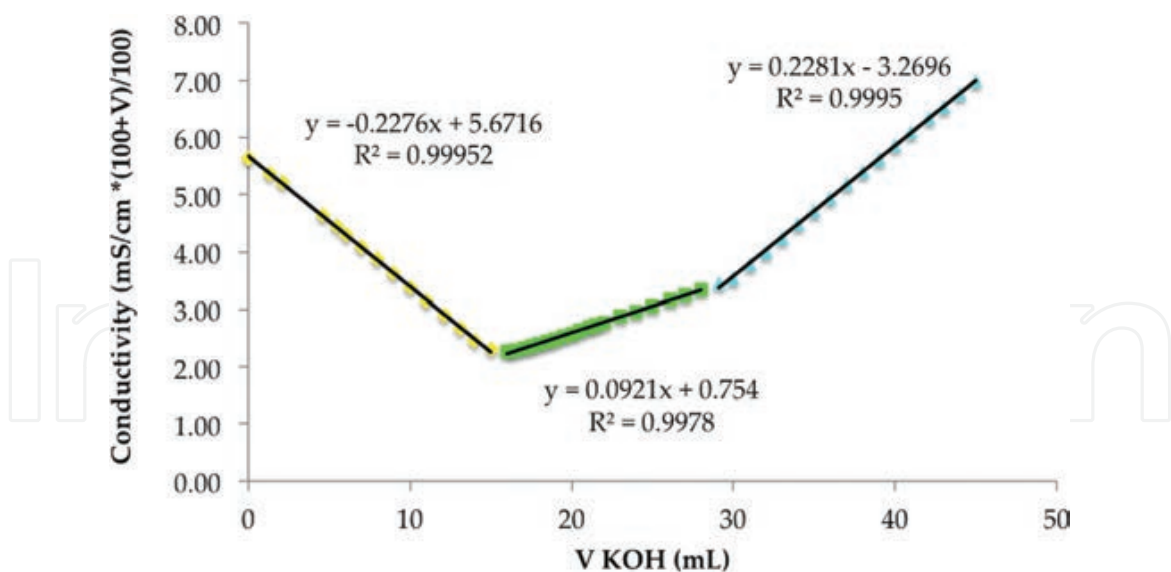


Figure 6. Conductometric titration of a mixture of hydrochloric and acetic acids with potassium hydroxide (data are shown in **Table 9**, first assay). Branch A: V [0–15]. Branch B: V [16–28]. Branch C: V [29.1–45].

The points recorded belong to the three branches of the titration curve; the first (branch A) corresponds to the neutralization of hydrochloric acid, the second (branch B) to the neutralization of acetic acid, and the third (branch C) to the excess of potassium hydroxide.

Figures 7 (hydrochloric acid) and **8** (hydrochloric acid + acetic acid) are the graphs corresponding to the estimation of the endpoints. The points represented in the graph and then



Figure 7. Conductometric titration of hydrochloric acid in the mixture (branches A and B).

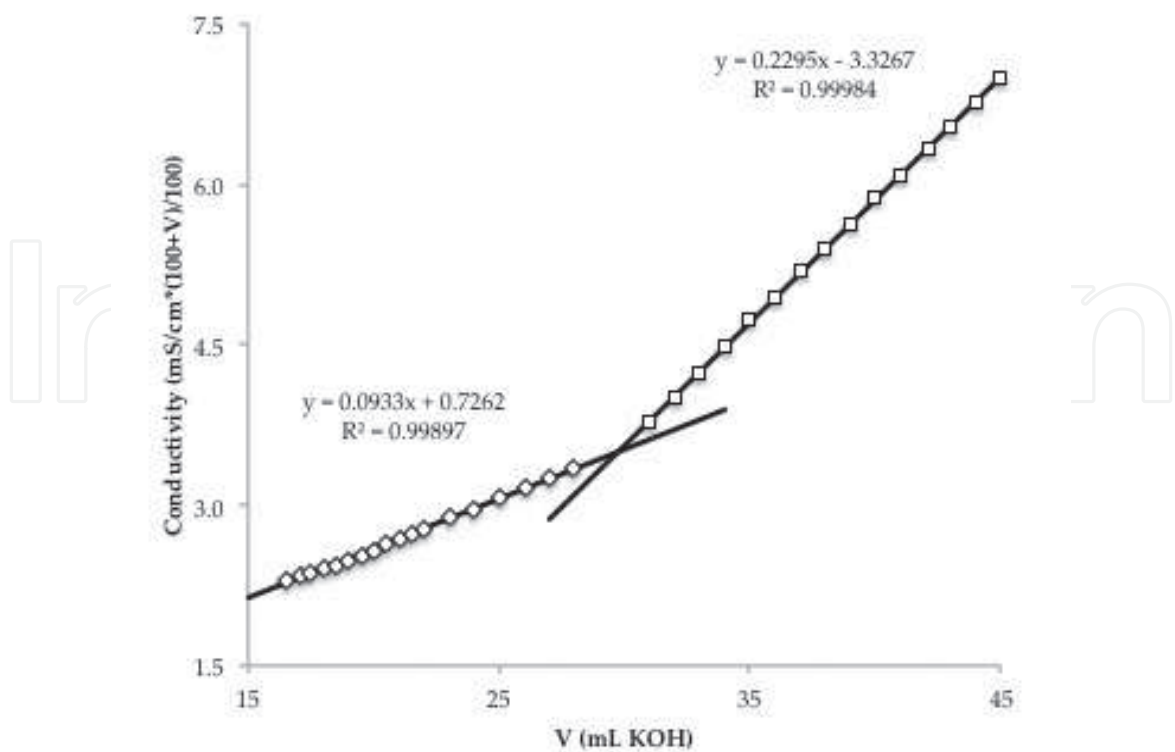


Figure 8. Conductometric titration of acetic acid in the mixture (branches B and C).

used in the calculations are colored yellow (branch A), green (branch B) and blue (branch C) in **Table 9**, thus avoiding proximity to the breakpoints. The values obtained for the intersections of the abscissa are 15.334 mL for hydrochloric acid and 29.743 mL for the sum of hydrochloric and acetic acids. So, acetic acid corresponds to the difference, 14.410 mL. From the data of **Figure 6**, without discarding of points, somewhat different values are obtained: 15.383, 29.582 and 14.189 mL.

Table 10 shows in detail all the calculations necessary to estimate the confidence limits of the abscissa of the breakpoint. The first-order variance propagation method [60] leads to the following volumes \pm confidence limits: 15.334 ± 0.0619 (first endpoint), 29.743 ± 0.151 (second endpoint), and 14.410 ± 0.142 (difference). In the second case, the confidence limits cannot refer to the difference (acetic acid), since the covariates involved are not taken into account (as previously explained in Section 7). Three decimal numbers were considered to compare and check calculations.

The application of Fieller’s theorem leads to the same results as those obtained by the law of propagation of errors, not being applicable to the estimation of confidence limits of the difference of volumes. The fundamentals of the first-order variance propagation method and Fieller’s theorem are much stronger than those based on the use of hyperbolic confidence bands, which lead to higher confidence intervals and limits (not applied in this case).

The conductometric titration was carried out in triplicate, on different days, obtaining the results included in **Tables 11** and **12**, and also represented in **Figures 9** and **10**. Again, the data

$N_1=$	13	$[\Sigma W_i]_1=$	13		$N_2=$	18	$[\Sigma W_i]_2=$	18
$x_1(\text{mean})=$	7.985	$y_1(\text{media})=$	3.8527		$x_2(\text{mean})=$	21.350	$y_2(\text{media})=$	2.7175
		$[S(xx)]_1=$	194.9369				$[S(xx)]_2=$	209.5250
		$[S(xy)]_1=$	-44.9940				$[S(xy)]_2=$	19.5426
		$[S(yy)]_1=$	10.3870				$[S(yy)]_2=$	1.8246
	$b_1=$	-0.230813226		$\Delta b=$	0.324084		$b_2=$	0.093270863
	$a_1=$	5.695614832		$\Delta a=$	-4.969405		$a_2=$	0.726209860
	$[R^2]_1=$	0.99983		$x(l)=$	15.3337		$[R^2]_2=$	0.99897
	$V[y/x]_1=$	1.60452E-04					$V[y/x]_2=$	0.000117906
	$V(b_1)=$	8.231E-07		$V[\text{pooled}]=$	0.000135239		$V(b_2)=$	5.627E-07
	$V(a_1)=$	6.482E-05					$V(a_2)=$	2.631E-04
	$\text{cov}(a_1, b_1)=$	-6.572E-06					$\text{cov}(a_2, b_2)=$	-1.201E-05
$V[\Delta a]=$	3.279E-04	$V[\Delta a]=$	3.564E-04	$V[x(l)]=$	0.0007	FIELLER	$ax^2 + bx + c = 0$	
$V[\Delta b]=$	1.386E-06	$V[\Delta b]=$	1.339E-06	$s[x(l)]=$	0.0274	$a=$	1.050E-01	$V(u)=$ 15.390
$\text{cov}(\Delta a, \Delta b)=$	-1.859E-05	$\text{cov}(\Delta a, \Delta b)=$	-1.932E-05	$t(0, 05, 27)=$	2.052	$b=$	-3.221E + 00	$V(l)=$ 15.278
		Pooled variances		$t s[x(l)]=$	0.0562	$c=$	2.469E + 01	
		$V(u)=$	15.390	$V(l)=$	15.278			

Table 10. Evaluation of endpoints in the titration of a mixture of HCl and CH₃COOH with KOH 0.0992 M (data **Table 9**).

V KOH (mL)	Conductance (mS/cm)	Conductance* (mS/cm)	V KOH (mL)	Conductance (mS/cm)	Conductance* (mS/cm)
0.0	5.71	5.7100	22.0	2.27	2.7694
1.0	5.47	5.5247	23.0	2.33	2.8659
2.0	5.17	5.2734	24.0	2.39	2.9636
3.0	4.90	5.0470	25.0	2.45	3.0625
4.0	4.61	4.7944	26.0	2.51	3.1626
5.0	4.34	4.5570	27.0	2.57	3.2639
6.0	4.06	4.3036	28.0	2.62	3.3536
7.0	3.79	4.0553	29.0	2.67	3.4443
8.0	3.52	3.8016	30.0	2.78	3.6140
9.0	3.28	3.5752	31.0	2.94	3.8514
10.0	3.02	3.3220	32.0	3.10	4.0920
11.1	2.75	3.0553	33.0	3.27	4.3491
12.0	2.53	2.8336	34.0	3.42	4.5828
13.0	2.29	2.5877	35.0	3.57	4.8195
14.0	2.10	2.3940	36.0	3.73	5.0728
15.0	1.96	2.2540	37.0	3.88	5.3156
15.5	1.94	2.2453	38.0	4.03	5.5614
16.0	1.95	2.2585	39.0	4.17	5.7963
16.5	1.96	2.2869	40.0	4.30	6.0200
17.0	1.99	2.3248	41.0	4.44	6.2604
17.5	2.01	2.3618	42.0	4.57	6.4894
18.0	2.04	2.4072	43.0	4.70	6.7210
19.0	2.10	2.4990	44.0	4.83	6.9552
20.0	2.16	2.5920	45.0	4.95	7.1775
21.0	2.22	2.6862			

* Conductivity·((100 + V)/100).

Table 11. Conductance and KOH volume data corresponding to the titration of a mixture of hydrochloric and acetic acids with potassium hydroxide (second assay).

used in the detailed calculations are colored in the tables. The results obtained (and intermediate calculations) for the second assessment are shown in **Table 13**: 14.913 ± 0.041 (propagation of errors and Fieller), 29.372 ± 0.120 (approximate method of propagation of errors) and 14.458 ± 0.113 (propagation of errors). In the third assessment: 15.032 ± 0.043 , 29.414 ± 0.146 , and 14.383 ± 0.140 mL.

V KOH (mL)	Conductance (mS/cm)	Conductance* (mS/cm)	V KOH (mL)	Conductance (mS/cm)	Conductance* (mS/cm)
0.0	5.81	5.8100	22.0	2.29	2.7938
1.0	5.51	5.5651	23.0	2.34	2.8782
2.0	5.22	5.3244	24.0	2.40	2.9760
3.0	4.94	5.0882	25.0	2.45	3.0625
4.0	4.66	4.8464	26.0	2.52	3.1752
5.0	4.38	4.5990	27.0	2.57	3.2639
6.0	4.11	4.3566	28.0	2.63	3.3664
7.0	3.85	4.1195	29.0	2.68	3.4572
8.0	3.59	3.8772	30.0	2.75	3.5750
9.0	3.32	3.6188	31.0	2.92	3.8252
10.0	3.07	3.3770	32.0	3.09	4.0788
11.0	2.82	3.1302	33.0	3.26	4.3358
12.0	2.58	2.8896	34.0	3.42	4.5828
13.0	2.33	2.6329	35.0	3.56	4.8060
14.0	2.13	2.4282	36.0	3.71	5.0456
14.5	2.05	2.3473	37.0	3.86	5.2882
15.0	1.99	2.2862	38.0	4.00	5.5200
16.0	1.96	2.2748	39.0	4.14	5.7546
16.5	1.98	2.3020	40.0	4.28	5.9920
17.0	2.00	2.3377	41.0	4.40	6.2040
17.5	2.03	2.3853	42.0	4.54	6.4468
18.0	2.05	2.4190	43.0	4.67	6.6781
19.0	2.11	2.5109	44.0	4.80	6.9120
20.0	2.17	2.6040	45.0	4.92	7.1340
21.0	2.23	2.6983			

* Conductivity·((100 + V)/100).

Table 12. Conductance and KOH volume data corresponding to the titration of a mixture of hydrochloric and acetic acids with potassium hydroxide (third assay).

If the series corresponding to the first equivalence point are analyzed: 15.334, 14.913 and 15.032, one of the data seems to be very distant from the other two, but the values of Q of Dixon 0.717 and of G of Grubbs 1.110 are lower than tabulated values for $P = 0.05$, that is, $Q_{\text{tab}} = 1.155$ and $G_{\text{tab}} = 1.15$ (although the G_{exp} and G_{tab} values are practically the same). The mean \pm confidence limits of the values are 15.093 ± 0.217 mL for hydrochloric acid (first endpoint) and 14.417 ± 0.038 mL for acetic acid (difference), which leads to molarity values of the solutions of hydrochloric and acetic acids of 0.01497 ± 0.00022 M and 0.01430 ± 0.00004 M. If the most distant values were

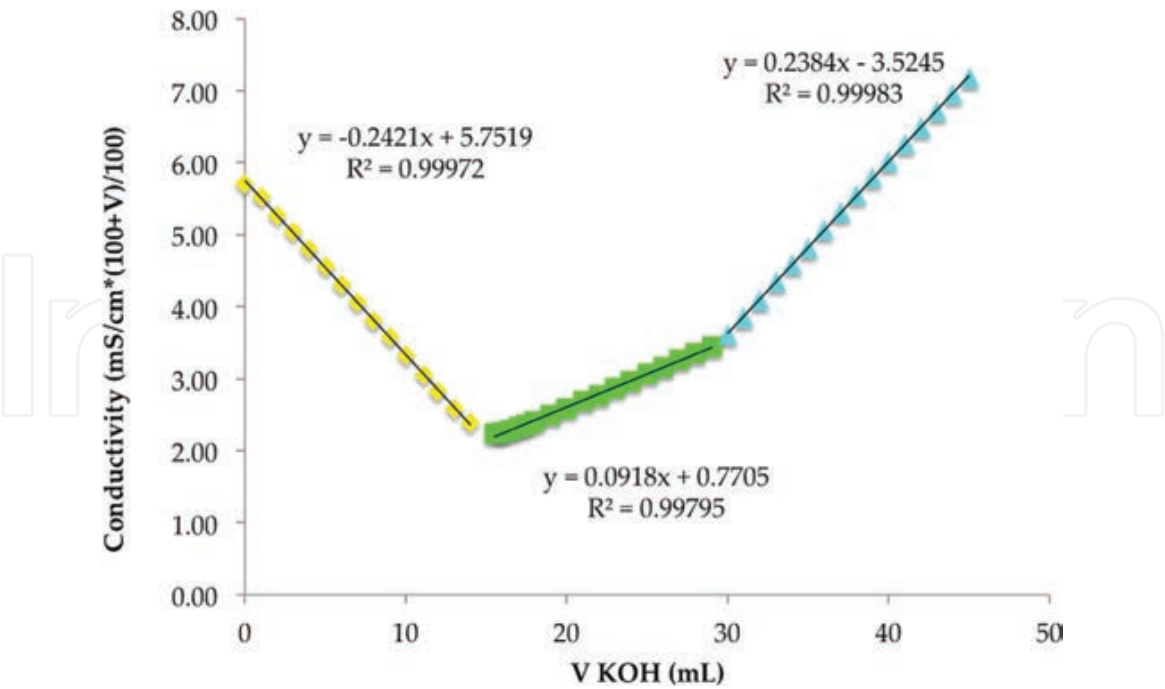


Figure 9. Conductometric titration of a mixture of hydrochloric and acetic acids with potassium hydroxide (data are shown in Table 10, second assay).

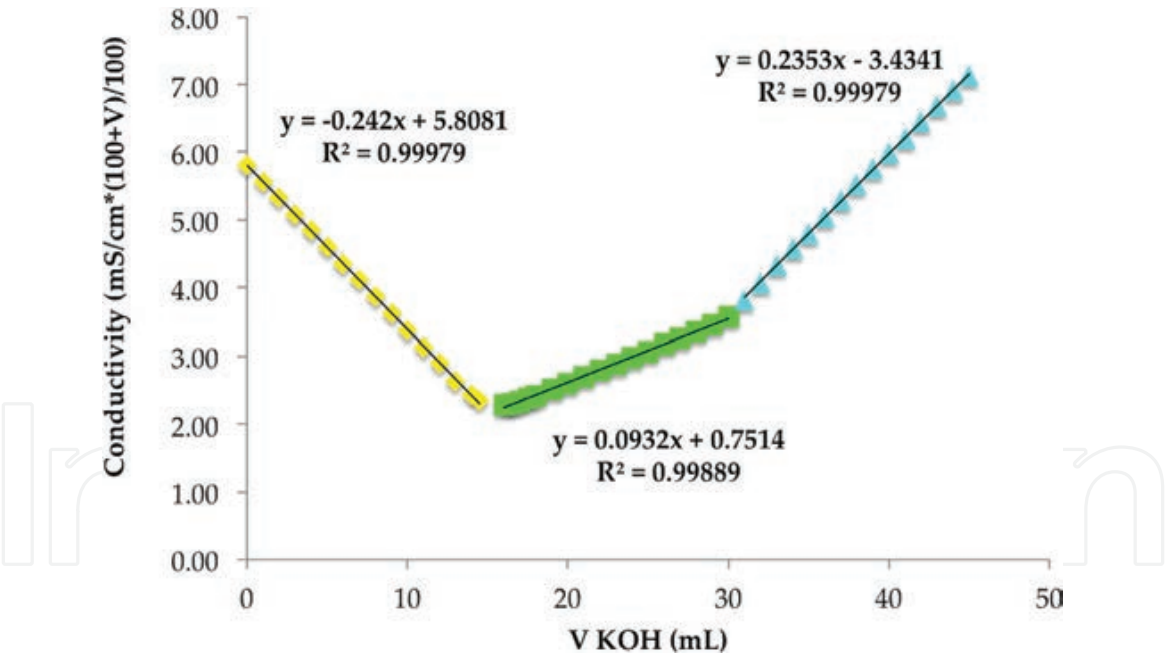


Figure 10. Conductometric titration of a mixture of hydrochloric and acetic acids with potassium hydroxide (data are shown in Table 9, third assay).

discarded, the results obtained would be very close to 14.973 ± 0.084 M and 14.421 ± 0.053 M, although the accuracy would improve considerably in the first case.

It is worth noting the fact that when the covariance between the intercept and slope of the straight lines obtained by the least squares method is not taken into account, the propagation

$N_1=$	13	$[\Sigma W_i]1=$	13	$N_2=$	15	$[\Sigma W_i]2=$	15
$x_1(\text{mean})=$	7.008	$y_1(\text{mean})=$	4.0562	$x_2(\text{mean})=$	22.200	$y_2(\text{mean})=$	2.8029
		$[S(xx)]1=$	182.8092			$[S(xx)]2=$	244.9000
		$[S(xy)]1=$	-44.7709			$[S(xy)]2=$	22.9477
		$[S(yy)]1=$	10.9651			$[S(yy)]2=$	2.1514
$b_1=$	-0.24491	$\Delta b=$	0.33861	$b_2=$			0.09370
$a_1=$	5.77243	$\Delta a=$	-5.04972	$a_2=$			0.72272
$[R2]1=$	0.99996	$x(I)=$	14.9132	$[R2]2=$			0.99948
$V[y/x]1=$	4.175E-05			$V[y/x]2=$			8.57549E-05
$V(b_1)=$	2.284E-07	$V[\text{pooled}]=$	6.5584E-05	$V(b_2)=$			3.502E-07
$V(a_1)=$	1.443E-05			$V(a_2)=$			1.783E-04
$cov(a_1, b_1)=$	-1.600E-06			$cov(a_2, b_2)=$			-7.774E-06
$V[\Delta a]=$	1.927E-04	$V[\Delta a]=$	1.590E-04	$V[x(I)]=$	0.0004	FIELLER	$ax^2 + bx + c = 0$
$V[\Delta b]=$	5.785E-07	$V[\Delta b]=$	6.266E-07	$s[x(I)]=$	0.0200	$a=$	1.147E-01
$cov(\Delta a, \Delta b)=$	-9.374E-06	$cov(\Delta a, \Delta b)=$	-8.459E-06	$t(0, 05, 24)=$	2.064	$b=$	-3.420E+00
		Pooled variances		$t s[x(I)]=$	0.0414	$c=$	2.550E+01
				$Vol (u)=$	14.955	$Vol (u)=$	14.955
				$Vol (l)=$	14.872	$Vol (l)=$	14.872
$N_3=$	15	$[\Sigma W_i]1=$	15				
$x_3(\text{media})=$	37.000	$y_3(\text{media})=$	5.3001	$\Delta b=$		0.145555	
		$[S(xx)]1=$	280.0000	$\Delta a=$		-4.275169	
		$[S(xy)]1=$	66.9920	$x(II)=$		29.3716	
		$[S(yy)]1=$	16.0306	$\Delta x=$		14.4584	
$b_3=$	0.2393			$V[\text{pooled}]=$		0.000132481	
$a_3=$	-3.5525						
$[R2]3=$	0.9999						
$V[y/x]3=$	1.792E-04	$V[x(II)]=$	0.0034	$t(0, 05, 26)=$		2.056	
$V(b_3)=$	6.400E-07	$s[x(II)]=$	0.0587	$t s[x(II)]=$		0.1207	
$V(a_3)=$	8.881E-04			$Vol (u)=$		29.492	
$cov(a_3, b_3)=$	-2.368E-05			$Vol (l)=$		29.251	
				$V[x(I)]=$	4.017E-04		
				$V[x(II)]=$	3.447E-03		
				$Cov(\Delta a_1, \Delta a_2)=$	7.014E-03		
				$Cov(\Delta a_1, \Delta b_2)=$	-9.265E-03		
				$Cov(\Delta a_2, \Delta b_1)=$	-4.704E-03		
				$Cov(\Delta b_1, \Delta b_2)=$	6.224E-03		
$V[\Delta a]=$	1.583E-07	$V[\Delta a]=$	9.320E-04	$V[\Delta x]=$	3.116E-03	$t(0, 05, 37)=$	2.026

$N_1=$	13	$[\Sigma W_i]1=$	13	$N_2=$	15	$[\Sigma W_i]2=$	15
$V[\Delta b]=$	9.902E-07	$V[\Delta b]=$	1.014E-06	$s[\Delta x]=$	5.583E-02	$t s[\Delta x]=$	0.113
$cov(\Delta a, \Delta b)=$	-2.893E-05	$cov(\Delta a, \Delta b)=$	-2.952E-05			$Vol (u)=$	14.571
		Pooled variances				$Vol (l)=$	14.345

Table 13. Evaluation of endpoints in the titration of a mixture of HCl and CH₃COOH with KOH 0.0992 M (data Table 11).

of the error leads to values of much larger confidence limits, 0.429 in the example of Massart (1997) versus 0.104, or 0.648 by Liteanu and Rica [58] versus only 0.113, in this book chapter, for the same data. As in many monographs, the covariance in the propagation of errors is not taken into account, and this is perhaps the reason why the estimates of the uncertainties of the intersection abscissa in the analytical literature do not abound.

9. Final comments

The advance of instrumental methods of endpoint detection increases the importance and the worth of titrimetric analysis. Physicochemical methods are intensively developed nowadays. However, titration continues to maintain its importance for chemical analysis. Plotting two straight line graphs from experimental data i.e., the conductivity versus volume added and determining the corresponding intersection point of the two branches allow locating the endpoint in a conductometric titration. The estimation of uncertainty of endpoint from linear segmented titration curves may be easily carried out by first-order propagation of variance, that is, by applying random error propagation law. The weighted linear regression procedure as being applied to the two branches of the conductometric titration curves leads to results similar to those obtained by the unweighted (single) linear regression procedure. The weighting factors are very similar to each other.

The covariance of measurements can be as important as the variance and both contribute significantly to the total analytical error. In particular, the strong correlation existing between the estimated slope and intercept of a straight line obtained by the least squares method must not be ignored. The inclusion of the covariance term on this respect is of vital importance, being usually a subtractive character lowering, in this case, the confidence limits of the abscissa of the intersection point. Perhaps this omission, which leads to too greater uncertainties, may be the cause for a small number of times that uncertainty is reported in this context.

The algebra associated with the Fieller’s theorem is simple, and no problem is observed with its derivation in this particular case of intersecting straight lines. However, the statistical uncertainty of endpoint differences is a complex problem. Attempt to derive the confidence limits by applying Fieller’s theorem fails in this case, being necessary to resort to the first-order propagation of variance (random error propagation law).

Nevertheless, the algebra associated in this case is simple but cumbersome, as some terms in covariance need to be derived. As a matter of fact, greater accuracy and firmer statistical justification make first-order propagation of variance (random error propagation law) and Fieller's theorem methods preferable to methods based on intersecting confidence bands.

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