Chapter 4

Statistical Modeling for the Energy-Containing Structure of Turbulent Flows

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Additional information is available at the end of the chapter

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Abstract

The development of statistical theory for the energy-containing structure of turbulent flows, taking the phenomenon of internal intermittency into account, is proposed, and new differential equations for conditional means of turbulent and nonturbulent fluid flow are established. Based on this fact, a new principle of constructing mathematical models is formulated as the method of autonomous statistical modeling of turbulent flows, *ASMTurb* method. Testing of the method is attained on the example of constructing a mathematical model for the conditional means of turbulent fluid flow in a turbulent mixing layer of co-current streams. Test results showed excellent agreements between the predictions of the *ASMTurb* model and known experimental data.

Keywords: turbulence, statistical modeling, intermittency, *ASMTurb* method

1. Introduction

The Reynolds-averaged Navier-Stokes equations (*RANS*) method does not take the intermittency of turbulent and nonturbulent fluid into consideration. As a result, this method allows us to model only the unconditional averages of a turbulent flow and does not provide a description of the conditional averages for each of the intermittent region, taking place in a turbulent stream. At the same time, the intermittency is an inherent property of such flows and that is why the conditional average modeling is necessary, for example see [1–4]. The phenomenon of intermittency (hydrodynamic intermittency) represents an interleaving process of the space-time domains of the flow, hydrodynamic structures of which are different. As is known, such domains contain so-called "turbulent" and "nonturbulent" fluid [1]. In this connection, the turbulent fluid contains a hierarchy of all possible scales and amplitudes of the fluctuations (pulsations) of hydrodynamic values, i.e., the whole spectrum of wavenumbers, while the nonturbulent fluid may contain only the large-scale fluctuations or absolutely does not contain any ones (when the nonturbulent fluid is far away from the mixing layer). The main purpose of this chapter is to justify a new method of statistical modeling of turbulent flows as the *ASMTurb* method, which enables to construct mathematical models of such flows with a high efficiency. The presented *ASMTurb* method, declared in [5], fundamentally differs from the previously proposed (for example, see Refs. [6–8]) in that it is based on the conditional statistical averaging of the Navier-Stokes equations, as applied to each of the intermittent region of turbulent flow, while the generating process of the turbulent fluid begins in a thin superlayer between turbulent and nonturbulent fluid and finishes in separate small areas, involved inside the turbulent flow. The first attempts to substantiate such an approach [5] have been presented previously [9, 10]. However, the deficiency of the mathematical body of statistical hydrodynamics under the intermittency conditions makes such an approach vulnerable. In this regard, we need primarily to develop a mathematical body for statistical modeling of turbulent flows.

2. Development of the statistical modeling theory

A spectacular example of the intermittent turbulent flow is the flow in the mixing layer of co-current streams, **Figure 1**. With that at point $\boldsymbol{x} = \boldsymbol{x}_0$, with the course of time, will be observed an interleaving of the turbulent and nonturbulent fluid. The behavior of the instantaneous longitudinal velocity $u(\boldsymbol{x}, t)$ in the flow range with strong intermittency at the point $\boldsymbol{x}_0$ is shown in **Figure 2**. As it seen, the structure of the turbulent fluid flow is fundamentally different from the structure of the nonturbulent fluid flow (the nonturbulent fluid involvement is shown with the arrows in **Figure 1**). It is evident that the behavior of any other hydrodynamic variable $f(\boldsymbol{x}, t)$ will be the same. It is important to note that the conditional averaging of variable $f(\boldsymbol{x}, t)$ is interpreted as the result of the averaging operation, referring only to as the turbulent ($r = t$) or nonturbulent ($r = n$) fluid, i.e., for the conditional time averaging

$$\overline{f(\boldsymbol{x}, t)}^r = \lim_{\tau_0 \to \infty} \frac{1}{\tau_r} \int_0^{\tau_r} f(\boldsymbol{x}, t; \tau_0) dt, \ \ r = t, \ n \tag{1}$$

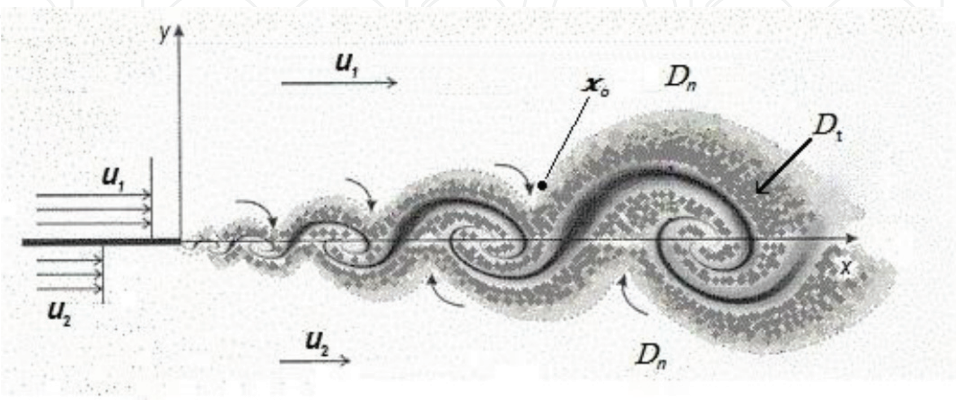

Figure 1. A sketch of the turbulent and nonturbulent fluid in the mixing layer of co-current streams. Here D_t is the region with the turbulent fluid and D_n– is the region with the nonturbulent fluid.

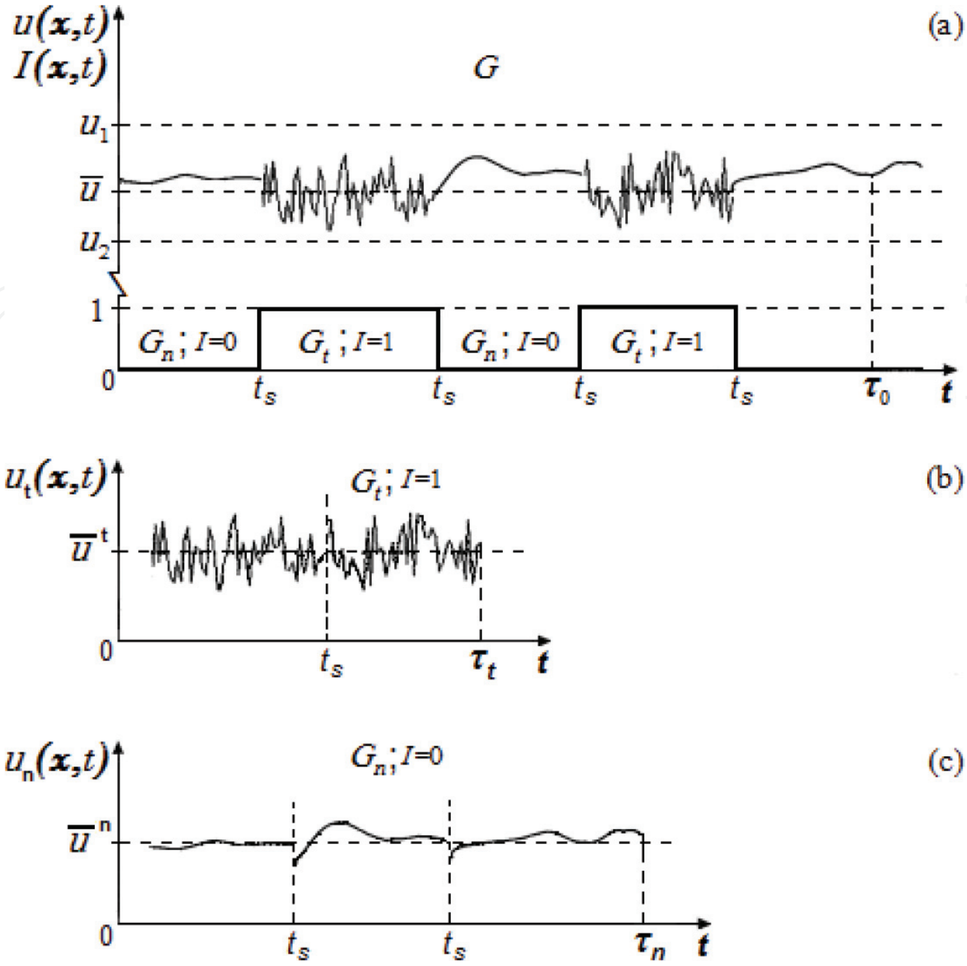

Figure 2. Behavior of the instantaneous longitudinal velocity in different regions, interleaving at the preset point $\boldsymbol{x} = \boldsymbol{x}_0$ in **Figure 1**: (a) unconditional velocity $u = u(\boldsymbol{x}, t)$, $(\boldsymbol{x}, t) \in G$; (b) "cross-linking" of the velocity over the turbulent fluid domain, $u_t = u_t(\boldsymbol{x}, t)|_{\mathrm{I}(\boldsymbol{x},t)=1}$, $(\boldsymbol{x}, t) \in G_t$; (c) "cross-linking" of the velocity over the nonturbulent fluid domain, $u_n = u_n(\boldsymbol{x}, t)|_{\mathrm{I}(\boldsymbol{x},t)=0}$, $(\boldsymbol{x}, t) \in G_n$. Here, $G = G_t + G_n$, $\overline{u}$ is the total time average, $\overline{u}^t$ and $\overline{u}^n$ is the conditional time means, t_s is the time point of observing the interfacial joint between the turbulent and nonturbulent flow domain in which cross-linking is carried out.

where $\overline{f(\boldsymbol{x}, t)}^r \equiv \overline{f_r(\boldsymbol{x}, t)}^r$, $f_t(\boldsymbol{x}, t) = f(\boldsymbol{x}, t)|_{\mathrm{I}=1}$, $f_n(\boldsymbol{x}, t) = f(\boldsymbol{x}, t)|_{\mathrm{I}=0}$, $I = I(\boldsymbol{x}, t)$ is the intermittency function and $\tau_0 = \tau_t + \tau_n$. At that the total average is

$$\overline{f(\boldsymbol{x}, t)} = \gamma(\boldsymbol{x})\overline{f(\boldsymbol{x}, t)}^t + \Big(1 - \gamma(\boldsymbol{x})\Big)\overline{f(\boldsymbol{x}, t)}^n \tag{2}$$

and $\gamma(\boldsymbol{x}) = \overline{I(\boldsymbol{x}, t)}$ is an intermittency factor. At the same time in the theory of statistical modeling are used the statistical characteristics, i.e., instead of the averaging operation of Eq. (1) is required the operation of statistical averaging.

To construct the mathematical model, first of all, it is necessary to determine what kind of statistical characteristics are the most suitable for modeling. In the classical *RANS* method, such characteristics are the unconditional means. In the methods, taking the intermittency into consideration, such characteristics are the conditional means of each intermittent region of turbulent flow. But in this case, it requires the development of a theory of statistical hydrodynamics under the conditions of intermittency.

The mathematical body of this theory may be developed from both the theoretical-probabilistic approach, based on the N-th repetition of the turbulent flow experiment [4, 11], and the theoretical-set approach [12, 13], which elementary events can be represented as a some set in the generalized space of the specifically considered turbulent flow. The advantage of the theoretical-set approach is that it can be implemented in the experimental research.

2.1. The mathematical body of statistical hydrodynamics

Getting started to the development of the mathematical body of statistical hydrodynamics in terms of intermittency, first of all we need to create a probability space $(\Omega, \mathcal{F}, P)$ of a random field of any hydrodynamic value as a random process in the generalized physical space of turbulent flow, where Ω – is the sample space, $\mathcal{F}$ – *is* σ-algebra of subsets, P– is the probability measure in $\mathcal{F}$.

2.1.1. The introduction of the sample space into the body of statistics

For the introduction of the sample space Ω, we consider the behavior of the value of $f(\boldsymbol{x}, t)$, measured by the sensor at the point $\boldsymbol{x} = \boldsymbol{x}_0$ of statistically stationary turbulent flow with strong intermittency, i.e., when $\gamma(\boldsymbol{x}) \cong 0.5$, see **Figure 3**. According to **Figure 2**, function $f(\boldsymbol{x}, t)$ forms a random continuous field in the space $G = D \times [0, \tau_0\ ]$. Hence it follows that at the point $\boldsymbol{x}_0 \in D$ we have a continuous random varying function of time $f(\boldsymbol{x}_0, t)$. Let the measurements of $f = f(\boldsymbol{x}_0, t)$ were carried out in a fairly wide range (order to the averaged statistical value of this function was stable) of the observation time $t = [0, \tau_0]$. It allows us to form an ensemble of values Ω as the one-point countable set of elementary events f, if we split the range of values of function $f(\boldsymbol{x}_0, t)$ at

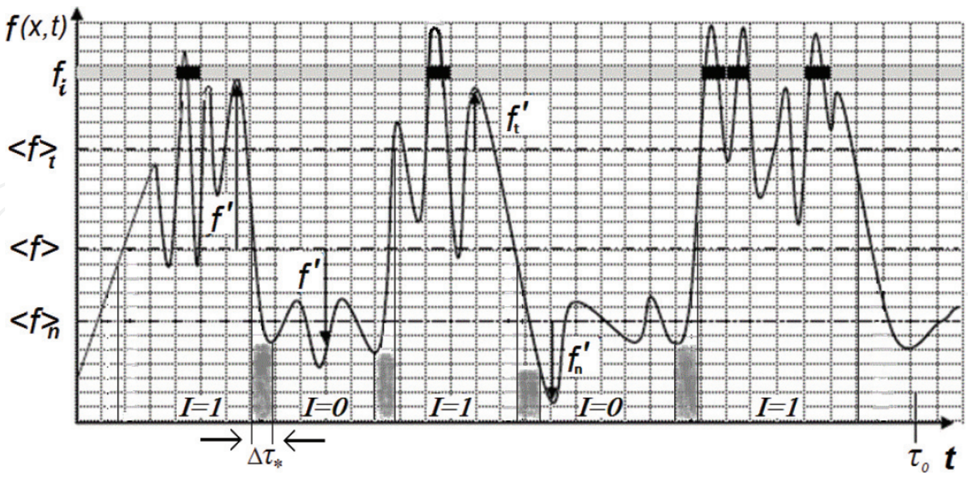

Figure 3. Illustration of statistical averaging of the instantaneous hydrodynamic variable. Here $f = \{f(\boldsymbol{x}, t)\}$ is the range of function $f(\boldsymbol{x}, t)$ at the point $\boldsymbol{x} = \boldsymbol{x}_0$; $\langle f \rangle$ is the total statistical average; $\langle f \rangle_t$ and $\langle f \rangle_n$ is the conditional statistical mean in each of the intermittent media of the turbulent flow; $f' = f - \langle f \rangle$, $f'_t = f_t - \langle f \rangle_t$, and $f'_n = f_n - \langle f \rangle_n$ are fluctuations (pulsations), measured from its own statistical means; τ_0 is the period of averaging time, sufficient to ensure sustainable statistical mean of values f; $\Delta\tau_*$ is the characteristic time of the superlayer observation, $I = I(\boldsymbol{x}, t)$ is the intermittency function of the turbulent fluid domain.

sufficiently small intervals Δf, and the range $[0, \tau_0]$ at sufficiently small intervals Δt, **Figure 2**. Having fixed a certain value of function f_i in each of the selected intervals Δf we come to the Lebesgue integral in terms of the set theory, formed in the physical space. Indeed, in the i-th layer there are N_i sampled values f_i in the form of shaded elementary cells $\Delta f\,\Delta t$, **Figure 2** (the selection of one particular value f_i from these cells plays no special role due to their small value). The total number of cells $N = \tau_0 / \Delta t$ and represented as the ensemble of values f, and also in the limit $\Delta f \to 0$ and $\Delta t \to 0$ this set will be dense, and the numerical value of f will be an element of this set, i.e., an elementary event.

In other words, for every fixed point $\boldsymbol{x} = \boldsymbol{x}_0$, the total number of all sample values f_i forms a random continuous field of values $f \in \Omega$ in the physical space $G = D \times [0, \tau_0]$. As a result, we come to a random process in the Borel space, in which a random variable $f(\boldsymbol{x}, t)$ takes all values of $f = \{f(\boldsymbol{x}, t)\}$, which are the elements of continuous set

$$\Omega = \{f : f_{min} \le f(\boldsymbol{x}, t) < f_{max}, N|_{\boldsymbol{x}=\boldsymbol{x}_0} = \lim_{\tau_0 \to \infty} \tau_0/\Delta t, \ (\boldsymbol{x}, t) \in G\} \tag{3}$$

So, from the physical space G with the hydrodynamic quantity $f(\boldsymbol{x}, t)$ we went to sample space Ω, elements of which are a set of values of $f = \{f(\boldsymbol{x}, t)\}$, i.e.

$$f(\boldsymbol{x}, t) \to f = \{f(\boldsymbol{x}, t)\};\ f(\boldsymbol{x}, t)|_{I=1} \to f_t = \{f(\boldsymbol{x}, t)|_{I=1}\};\ f(\boldsymbol{x}, t)|_{I=0} \to f_n = \{f(\boldsymbol{x}, t)|_{I=0}\} \tag{4}$$

$$\overline{f(\boldsymbol{x}, t)} \to \langle f \rangle;\ \overline{f(\boldsymbol{x}, t)}^{t} \to \langle f \rangle_t,\ \overline{f(\boldsymbol{x}, t)}^{n} \to \langle f \rangle_n \tag{5}$$

$$f'(\boldsymbol{x}, t) \to f' = \{f(\boldsymbol{x}, t) - \overline{f(\boldsymbol{x}, t)}\};\ f'_r(\boldsymbol{x}, t) \to\ f'_r = \{f_r(\boldsymbol{x}, t) - \overline{f(\boldsymbol{x}, t)}^{r}\} \tag{6}$$

Now we need develop the apparatus of statistics together with the operations of statistical averaging of the hydrodynamic quantities. For this we represent the apparatus of statistics based on a formal using of the probability density function (one-point PDF) of some hydrodynamic quantity $f = \{f(\boldsymbol{x}, t)\}$. At that the intermittency function $I = I(\boldsymbol{x}, t)$ will be used to obtain conditional one-point statistics.

2.1.2. The introduction of the algebra of events and PDFs

Let us introduce a one-point probability density function $p(f) = p(f; \boldsymbol{x}, t)$ into the body of statistics. According to the Kolmogorov axioms [12], it can be carried out via the Lebesgue-Stieltjes integral:

$$\boldsymbol{P}(\Omega) = \int_{\Omega} p(f)df \tag{7}$$

where $p(f) = \lim_{N \to \infty} p(f;\ f \in \Omega,\ N)$ and $\int_{\Omega} p(f)df = 1$.

For introduction of the algebra of events, we suppose that the space Ω, defined by Eq. (3), contains two independent subspaces (subsets)

$$\Omega_1 = \Omega|_{\mathcal{J}_1=1}, \quad \Omega_2 = \Omega|_{\mathcal{J}_2=1} \tag{8}$$

i.e., the generalized set $\Omega = \Omega_1 + \Omega_2$, and we have $\mathcal{F}$ as σ-algebra of the subsets. The indicators of these subsets are the characteristic functions

$$\mathcal{J}_1 = \begin{cases} 1 \text{ } if \text{ } f \in \Omega_1 \\ 0 \text{ } otherwise \end{cases}, \quad \mathcal{J}_2 = \begin{cases} 1 \text{ } if \text{ } f \in \Omega_2 \\ 0 \text{ } otherwise \end{cases} \tag{9}$$

at that the set of values f, belonging to the super-layer, is excluded. In the results, we have the Borel algebra subsets of the set Ω with the Kolmogorov axioms, which according to the total probability formula gives

$$\boldsymbol{P}(\Omega) = \sum_{k=1}^{2} \boldsymbol{P}\{\Omega|_{\mathcal{J}_k=1}\}\boldsymbol{P}\{\mathcal{J}_k = 1\} \tag{10}$$

where $\boldsymbol{P}\{\mathcal{J}_k = 0\} = 0$ as an impossible event.

2.2. Statistical averaging of hydrodynamic quantities

2.2.1. Applied to the intermittent turbulent flows

For the intermittent turbulent flows, the sample sets, which we designated as $\Omega_t = \Omega|_{\mathcal{J}_1=1}$ and $\Omega_n = \Omega|_{\mathcal{J}_2=1}$, are the set of values of hydrodynamic variable, belonging to the turbulent and nonturbulent fluid of turbulent flow. Indicators of these sets are the functions $\mathcal{J}_1 = \mathcal{J}_t$ and $\mathcal{J}_2 = \mathcal{J}_n$, while $\boldsymbol{P}\{\mathcal{J}_t = 1\} = \gamma_t$ and $\boldsymbol{P}\{\mathcal{J}_n = 1\} = \gamma_n$ are the measures of these sets with the condition $\gamma_t + \gamma_n = 1$, and represent the intermittency factors as the probability of observing the turbulent and nonturbulent fluid at the point $\boldsymbol{x}$ of turbulent flow, i.e., $\gamma_t = \gamma_t(\boldsymbol{x})$ and $\gamma_n = \gamma_n(\boldsymbol{x})$. Now, according to Eq. (10),

$$\boldsymbol{P}(\Omega) = \gamma_t \boldsymbol{P}(\Omega_t) + \gamma_n \boldsymbol{P}(\Omega_n) \tag{11}$$

where $\boldsymbol{P}(\Omega_t)$ и $\boldsymbol{P}(\Omega_n)$ – conditional random set of value f, belonging to the turbulent $\Omega_t = \{f_t\}$ and nonturbulent $\Omega_n = \{f_n\}$ fluid at the point $\boldsymbol{x}$; the values $\gamma_t = \gamma_t(\boldsymbol{x})$ and $\gamma_n = \gamma_n(\boldsymbol{x})$, while the one-point PDF

$$p(f) = \gamma_t p_t(f) + \gamma_n p_n(f) \tag{12}$$

where $p_t(f) = p_t(f; \boldsymbol{x}, t)$, $(\boldsymbol{x}, t) \in G_t$ and $p_n(f) = p_n(f; \boldsymbol{x}, t)$, $(\boldsymbol{x}, t) \in G_n$ represent the conditional one-point PDFs. As it turns out, a PDF may have or not to have an explicit dependence on $\boldsymbol{x}$. In actual fact, if the flow is intermittent, it has a dual structure [1] and in the generalized set we have $\Omega = \Omega_t + \Omega_n$ so that the measures of sample sets γ_t and γ_n are depend on $\boldsymbol{x}$; if the flow is not intermittent (when the phenomenon of intermittency is not considered) it occurs in a "single" space as a set of elementary events $\Omega = \Omega_R$, the measure of which does not depend on $\boldsymbol{x}$. In the case of the explicit dependence, we denote the PDF $p(f)$ in Eq. (7) as

$$P(f) = \lim_{N\to\infty} p\Big(f;\ f \in (\Omega_t + \Omega_n),\ N,\ \boldsymbol{x}\Big) \tag{13}$$

and we call this function as the "*total*" PDF, and the flow—flow of the "*intermittent*" continuous media with turbulent and nonturbulent fluid. In the absence of such dependence, we denote it as

$$p_R(f) = \lim_{N\to\infty} p(f;\ f \in \Omega_R,\ N) \tag{14}$$

and called the "*unconditional*" PDF $p_R(f)$, and the flow—flow of the "*nonintermittent*" continuous medium, which is modeled by the *RANS* method. The explicit dependence of the PDF $P(f)$ Eq. (13) on the coordinates creates certain difficulties in its using in the statistical modeling and also leads to the necessity of introducing in the theory of statistical hydrodynamics the conditional PDF for the hydrodynamic characteristics of turbulent and nonturbulent media.

So, to perform the conditional averaging of the instantaneous characteristics of the flow, we introduce into statistical body the conditional PDF, i.e., the CPDFs:

$$p_t(f) = p(f|_{I=1}),\ p_n(f) = p(f|_{I=0}) \tag{15}$$

with the indicator (characteristic) function of the turbulent fluid

$$I = \begin{cases} 1\ if\ f \in \Omega_t \\ 0\ if\ f \in \Omega_n \end{cases} \tag{16}$$

represents a probability of observing the turbulent flow at the given point $\boldsymbol{x}$, i.e., it is the intermittency factor $\gamma = \gamma(\boldsymbol{x})$. Now the expression for the "*total*" PDF in Eq. (12), by virtue of the fact that $\gamma_n = 1 - \gamma$, is transformed into

$$P(f) = \gamma p_t(f) + (1-\gamma)p_n(f) \tag{17}$$

with the explicit dependence on $\boldsymbol{x}$, while the CPDF $p_t(f)$ and $p_n(f)$ obviously do not depend on $\boldsymbol{x}$.

Now conduct the operations of statistical averaging of hydrodynamic quantities. These operations we will conduct with the help of a formal using of the PDFs, i.e., when a particular form of this function does not necessarily need to know.

2.2.2. Operations of statistical averaging of the hydrodynamic quantity

The statistical averaging of the hydrodynamic quantity $f(\boldsymbol{x},\ t)$ can be performed by a formal using of the PDF. The results of statistical averaging operation are the *conditional* statistical means when $r = t$ for turbulent and $r = n$ for nonturbulent fluid

$$\langle f \rangle_r = \int_{\Omega_r} f p_r(f) df, \qquad r = t,\ n \tag{18}$$

and also the *total* statistical average

$$\langle f \rangle = \int_{\Omega} fP(f)df \tag{19}$$

which by virtue of the expression in Eq. (17) gives

$$\langle f \rangle = \gamma \langle f \rangle_t + (1-\gamma)\langle f \rangle_n \tag{20}$$

At that by definition the value $f_t = f|_{I=1}$ and $f_n = f|_{I=0}$ and for the "pulsations" we have

$$f' = f - \langle f \rangle \ ; \ f'_r = f_r - \langle f_r \rangle_r ; \ f'|_{I=1} = f'_t + \langle f \rangle_t - \langle f \rangle \ ; \ f'|_{I=0} = f'_n + \langle f \rangle_n - \langle f \rangle \tag{21}$$

whence it follows that

$$\langle f_r \rangle_r = \langle f \rangle_r ; \langle f'_r g'_r \rangle_r \neq \langle f' g' \rangle_r \tag{22}$$

As is evident that the total average in Eq. (20) represents the statistical characteristic of a rather complex structure, while the unconditional mean, when in Eq. (18) we have $r = R$, is a characteristic of the "simplified" flow without considering effects of intermittency. At that $\langle f \rangle_R \cong \langle f \rangle$ because the total average $\langle f \rangle$ does not contain the values of f belonging to the superlayer [16].

2.2.3. Operations of statistical averaging of derivative of the hydrodynamic quantity

The statistical averaging of the derivative of hydrodynamic quantity $\xi = \partial f/\partial x$ gives the following. In point of fact, on the one side using the joint CPDF $p_r(f, \xi)$ we have

$$\iint \frac{\partial f}{\partial x} p_r(f, \xi) df d\xi = \frac{\partial}{\partial x} \int \langle f|_{\xi} \rangle_r p_r(\xi) d\xi = \frac{\partial \langle f \rangle_r}{\partial x} \tag{23}$$

because that in accordance with [14]

$$p_r(f, \xi) = p_r(f|_{\xi}) p_r(\xi)$$

and

$$\iint \Big(\partial f p_r(f, \xi)/\partial x - f \partial p_r(f, \xi)/\partial x \Big) df d\xi = \int \Big(\partial \Big(\int f p_r(f|_{\xi}) df \Big) /\partial x \Big) p_r(\xi) d\xi \tag{24}$$

where $\langle f|_{\xi} \rangle_r$ is the conditional mean values of f in turbulent ($r = t$) or nonturbulent ($r = n$) medium for all possible fixed values of ξ. At that $\partial p_r(f, \xi)/\partial x = 0$ due to the fact that the function (f, ξ) does not depend obviously on the coordinate x. On the flip side, we have

$$\iint \frac{\partial f}{\partial x} p_r(f, \xi) df d\xi = \left\langle \frac{\partial f}{\partial x} \right\rangle_r \tag{25}$$

because that

$$\iint \xi p_r(\xi|_f) p_r(f) d\xi df = \int \langle \xi|_f \rangle_r p_r(f) df \tag{26}$$

where $\langle \xi|_f \rangle_r$ is the conditional mean of the gradient $\xi = \partial f/\partial x$ in turbulent or nonturbulent medium, given for all possible fixed values of *f*. As a result, we have

$$\left\langle \frac{\partial f}{\partial x} \right\rangle_r = \frac{\partial \langle f \rangle_r}{\partial x}, \; r = t, n \tag{27}$$

Thus, the operation of conditional statistical averaging of derivatives is permutational. So, we have proved that the permutation of conditional averaging operation has a strict mathematical justification.

It is appropriate to note that in the classical method of *RANS*, the operation of unconditional statistical averaging of derivatives gives the same result. Actually, the *unconditional* joint PDF $p_R(f, \xi)$ of Eq. (14) does not depend on the coordinates obviously and therefore it is correctly Eqs. (23)–(27) with index $r = R$, i.e.,

$$\left\langle \frac{\partial f}{\partial x} \right\rangle_R = \frac{\partial \langle f \rangle_R}{\partial x} \tag{28}$$

that proves the rule of permutation of the operation of unconditional averaging of derivatives in the method of *RANS*.

About the permutation of the operation of derivatives total averaging I must say the following. The operation of total statistical averaging of partial derivatives of type $\xi = \partial f/\partial x$ by Eq. (19) for intermittent continuous media with turbulent and nonturbulent fluid cannot be a permutational. This operation is carried out similarly in Eqs. (23)–(27). Here, however, must keep in mind that now the total PDF $P(f)$ in Eq. (17) obviously depend on the coordinates due to $\gamma = \gamma(x)$. The legitimacy of such a permutation of the operation is easy to establish if we attract Eq. (20) as applied to the partial derivatives. In this case

$$\left\langle \frac{\partial f}{\partial x} \right\rangle = \gamma \left\langle \frac{\partial f}{\partial x} \right\rangle_t + (1-\gamma) \left\langle \frac{\partial f}{\partial x} \right\rangle_n \tag{29}$$

and

$$\frac{\partial \langle f \rangle}{\partial x} = \gamma \frac{\partial \langle f \rangle_t}{\partial x} + (1-\gamma) \frac{\partial \langle f \rangle_n}{\partial x} + (\langle f \rangle_t - \langle f \rangle_n) \frac{\partial \gamma}{\partial x} \tag{30}$$

after comparing of which, with regard to Eq. (27) we get

$$\left\langle \frac{\partial f}{\partial x} \right\rangle = \frac{\partial \langle f \rangle}{\partial x} - (\langle f \rangle_t - \langle f \rangle_n) \frac{\partial \gamma}{\partial x} \tag{31}$$

It follows that the permutation of the operation of total statistical averaging of derivatives is not legitimate, i.e.,

$$\left\langle \frac{\partial f}{\partial x} \right\rangle \neq \frac{\partial \langle f \rangle}{\partial x} \tag{32}$$

With regard to the total statistical averaging of time derivatives, instead of expression (31) we have $\langle \partial f/\partial t \rangle = \partial \langle f \rangle / \partial t$ because of $\partial \gamma / \partial t = 0$, i.e., for statistically stationary turbulent flows such a permutation is possible. The same applies to the conditional averaging of derivatives. So, we showed that the statistical modeling of turbulent flows, in the case of taking into account the effects of intermittency, should be based on Eqs. (20), (27), and (32).

2.2.4. The statistical averaging of hydrodynamics equations

The Navier-Stokes equations for an incompressible fluid together with the continuity equation are accepted as the basis of the hydrodynamic equations system [11]. When the external forces are absent, this system has the following form:

$$\begin{cases} \dfrac{\partial u_i}{\partial t} + \dfrac{\partial}{\partial x_k}\left(u_i u_k + \dfrac{p\delta_{ik}}{\rho} - \sigma_{ik} \right) = 0, \; i = 1, 2, 3 \\ \dfrac{\partial u_k}{\partial x_k} = 0 \end{cases} \tag{33}$$

where $\sigma_{ik} = \mu(\partial u_i / \partial x_k + \partial u_k / \partial x_i)$ is the tensor of viscous stress, μ is the dynamic factor of viscosity, p is the pressure, and ρ is the density. Our primary goal is to conduct an operation of statistical averaging of the SE (33) so as to obtain a system of equations for the conditional mean $\langle u_i \rangle_t$. At the beginning, we will conduct an operation of conditional statistical averaging of the continuity equation in SE (33). For this, we introduce the joint CPDF

$$p_r(u_i, \xi_i) = p_r(u_1, u_2, u_3, \xi_1, \xi_2, \xi_3) \tag{34}$$

with index $r = t$ for turbulent and $r = n$ for nonturbulent fluid, $\xi_1 = \partial u_1 / \partial x_1$, $\xi_2 = \partial u_2 / \partial x_2$, $\xi_3 = \partial u_3 / \partial x_3$, and use the procedure of conditional averaging (23). As a result, we reach the following averaging procedure:

$$\int .. \int \frac{\partial u_k}{\partial x_k} p_r(u_i, \xi_i) du_1, \ldots, d\xi_3 = \int .. \int \frac{\partial u_k p_r(u_i, \xi_i)}{\partial x_k} du_1, \ldots, d\xi_3 = \frac{\partial}{\partial x_k} \int .. \int u_k p_r(u_i, \xi_i) du_1 \ldots, d\xi_3 = 0 \tag{35}$$

because that the function $p_r(u_i, \xi)$ does not depend on x_k, i.e., $\partial p_r(u_i, \xi_i)/\partial x_k = 0$ and $p_r(u_i, \xi_i) \partial u_k / \partial x_k = \partial u_k p_r(u_i, \xi_i) / \partial x_k$. From here toward $k = 1$ in Eq. (35), we deduce

$$\frac{\partial}{\partial x_1} \int u_1 p_r(u_1) du_1 = \frac{\partial \langle u_1 \rangle_r}{\partial x_1} \tag{36}$$

as $p_r(u_i) = \int \ldots \int p_r(u_1, u_2, u_3, \xi_1, \xi_2, \xi_3) du_{i-1}, du_{i+1}..d\xi_3$, $du_0 = 1$, for example, $p_r(u_1) = \int \ldots \int$ $p_r(u_1, u_2, u_3, \xi_1, \xi_2, \xi_3) du_2 \ldots d\xi_3$. The same operation is carried out for $k = 2, 3$ using $p_r(u_2)$ and

$p_r(u_3)$. Now then, the conditionally averaged continuity equation for each of the intermittent media of turbulent flow has the form

$$\frac{\partial\langle u_k\rangle_r}{\partial x_k} = 0,\ r = t, n \tag{37}$$

To conduct the operation of conditional statistical averaging of the Navier-Stokes SE (33), we use the CPDF $p_r = p_r(\ \xi_1, \xi_2)$, where $\xi_1 = \frac{\partial u_i}{\partial t},\ \xi_2 = \frac{\partial}{\partial x_k}\left(u_i u_k + \frac{p\delta_{ik}}{\rho} - \sigma_{ik}\right)$ with the summation over k = 1, 2, 3. Then, according to Eq. (25) for the momentum equation in SE (33) we obtain $\iint \xi_1 p_r(\ \xi_1, \xi_2) d\xi_1 d\xi_2 = \langle \partial u_i/\partial t\rangle_r$. Similarly, it conducted the averaging operation of the value ξ_2:

$$\iint \xi_2 p_r(\ \xi_1, \xi_2) d\xi_1 d\xi_2 = \left\langle \frac{\partial}{\partial x_k}\left(u_i u_k + \frac{p\delta_{ik}}{\rho} - \sigma_{ik}\right)\right\rangle_r \tag{38}$$

Applying the rule of permutation (27) and using the Reynolds development $f_r = \langle f\rangle_r + f_r{}'$, we deduce $\langle u_i u_k\rangle_r = \langle u_i\rangle_r\langle u_k\rangle_r + \langle u'_{ir}u'_{kr}\rangle_r$ in view of Eq. (22). As a result of the above-performed operation of statistical averaging of SE (33) now for the statistically stationary turbulent flow, we have the system of equations with two autonomous subsystems for the flow's conditional means of each of the intermittent media with turbulent and nonturbulent fluid:

$$\begin{cases} \dfrac{\partial\langle u_i\rangle_r}{\partial t} + \dfrac{\partial\langle u_i\rangle_r\langle u_k\rangle_r}{\partial x_k} + \dfrac{\langle u'_{ir}u'_{kr}\rangle_r}{\partial x_k} + \dfrac{\partial\langle(\ p\delta_{ik} - \sigma_{ik})/\rho\rangle_r}{\partial x_k} = 0 \\ \dfrac{\partial\langle u_k\rangle_r}{\partial x_k} = 0,\ r = t,\ n \end{cases} \tag{39}$$

where the fluctuating velocity of the turbulent or nonturbulent fluid flow $u'_{ir} = u_{ir} - \langle u_{ir}\rangle_r$ and $\langle u_i\rangle_r \equiv \langle u_{ir}\rangle_r$, but the one-point covariances $\langle u'_{ir}u'_{kr}\rangle_r \neq \langle u'_i u'_k\rangle_r$ according to Eq. (22). Besides, $\partial\langle u_i\rangle_r/\partial t = 0$ for statistically stationary turbulent flows. Each SS (39) with index $r = t$ or $r = n$ is statistically independent and is determined by the fact that the one-point correlation of the hydrodynamic quantities of turbulent and nonturbulent media is absent, i.e., $\langle f_t f_n\rangle = 0$. These subsystems allow the conditional means modeling of each of the intermittent media with turbulent and nonturbulent fluid independently from the each other.

The derivation of the turbulent kinetic energy budget equation by the *RANS* method is well known [1, 4]. The procedure of the budget equations derivation for conditional means of kinetic energy fluctuations in each of the intermittent medium of the turbulent flow will be the same. In the approximation of a free boundary layer, these equations have the following form:

$$\frac{\partial\langle E_r\rangle_r}{\partial t} + \langle u_k\rangle_r\frac{\partial\langle E_r\rangle_r}{\partial x_k} + \frac{\partial\langle(E_r + p'_r/\rho)v'_r\rangle_r}{\partial x_k} + \langle u'_{ir}u'_{kr}\rangle_r\frac{\partial\langle u_i\rangle_r}{\partial x_k} + \langle\varepsilon_r\rangle_r = 0 \tag{40}$$

Hereinafter $E_r = 0.5(u_r'^2 + v_r'^2 + w_r'^2)$ and $\langle E_r\rangle_r = 0.5(\langle u_r'^2\rangle_r + \langle v_r'^2\rangle_r + \langle w_r'^2\rangle_r)$.

3. The *ASMTurb* method

The new principle of constructing mathematical models of the energy-containing structure of turbulent flows (the large-scale turbulent motion) is as follows: (1) as the main statistical characteristics of modeling are chosen the conditional averages of hydrodynamic quantities of the turbulent and nonturbulent fluid; (2) to describe the conditional means of hydrodynamic quantities are used two statistically independent (autonomous) systems of differential equations; (3) each of the autonomous systems for the conditional averages is closed by its own closure hypothesis; and (4) the total average of hydrodynamic quantities is obtained by the algebraic relations of statistical hydrodynamics, which bind the total and conditional means through the mediation of the intermittency factor. To realize this principle, the mechanism of the turbulent fluid formation in a turbulent flow is proposed. This is achieved by the introduction of the "superlayer" between turbulent and nonturbulent fluid, where shear rate and pressure fluctuations in the turbulent fluid generate the pressure fluctuations in the nonturbulent fluid. This process leads to the so-called "nonlocal" transfer of the impulse and initiates the occurrence of velocity fluctuations (for particulars see in [15, 16]). The formulated principle of constructing mathematical models is called the *ASMTurb* method [5].

3.1. Mathematical tools of the *ASMTurb* method

3.1.1. General system of equations for conditional means

According to the *ASMTurb* method, we have two autonomous subsystems (SS) of the difference equations corresponding to Eqs. (39) and (40) in the form of

$$\begin{cases} \dfrac{\partial\langle u_i\rangle_t}{\partial t}+\dfrac{\partial\langle u_i\rangle_t\langle u_k\rangle_t}{\partial x_k}+\dfrac{\partial\langle u'_{it}u'_{kt}\rangle_t}{\partial x_k}+\dfrac{\partial\langle(p_t\delta_{ik}-\sigma_{tik})/\rho\rangle_t}{\partial x_k}=0 \\ \dfrac{\partial\langle u_k\rangle_t}{\partial x_k}=0 \\ \dfrac{\partial\langle E_t\rangle_t}{\partial t}+\langle u_k\rangle_t\dfrac{\partial\langle E_t\rangle_t}{\partial x_k}+\dfrac{\partial\langle(E_t+p'_t/\rho)v'_t\rangle_t}{\partial x_k}+\langle u'_{it}u'_{kt}\rangle_t\dfrac{\partial\langle u_i\rangle_t}{\partial x_k}+\langle\varepsilon_t\rangle_t=0 \end{cases} \tag{41}$$

and

$$\begin{cases} \dfrac{\partial\langle u_i\rangle_n}{\partial t}+\dfrac{\partial\langle u_i\rangle_n\langle u_k\rangle_n}{\partial x_k}+\dfrac{\partial\langle u'_{in}u'_{kn}\rangle_n}{\partial x_k}+\dfrac{\partial\langle(p_n\delta_{ik}-\sigma_{nik})/\rho\rangle_n}{\partial x_k}=0 \\ \dfrac{\partial\langle u_k\rangle_n}{\partial x_k}=0 \\ \dfrac{\partial\langle E_n\rangle_n}{\partial t}+\langle u_k\rangle_n\dfrac{\partial\langle E_n\rangle_n}{\partial x_k}+\dfrac{\partial\langle(E_n+p'_n/\rho)v'_n\rangle_n}{\partial x_k}+\langle u'_{in}u'_{kn}\rangle_n\dfrac{\partial\langle u_i\rangle_n}{\partial x_k}+\langle\varepsilon_n\rangle_n=0 \end{cases} \tag{42}$$

that describe the conditional mean flow characteristics of each of the intermittent media with the turbulent (r = t) and nonturbulent (r = n) fluid. Let us note that each of the SS (41) and SS (42) is statistically independent, in terms of the one-point correlations $\langle f_t f_n\rangle=0$, so after the completion of these subsystems using the corresponding expressions for $\langle u'_{ir}u'_{kr}\rangle_r$, $\langle (E_r + p'_r/\rho)v'_r\rangle_r$ and $\langle \varepsilon_r\rangle_r$ as the closure hypothesis we obtain mathematical models for the flow of the turbulent and nonturbulent fluid.

3.1.2. The closure hypothesis

The closure hypothesis for SS (41) and SS (42) we will choose in the form of a simple expression gradient relation [16]:

$$-\langle u'_r v'_r\rangle_r = \nu_r \frac{\partial\langle u\rangle_r}{\partial y} \tag{43}$$

$$\left\langle \left(E_r + \frac{p'_r}{\rho}\right) v'_r \right\rangle_r = -\nu_r \frac{\partial\langle E_r\rangle_r}{\partial y}\ , \langle \varepsilon_r\rangle_r = c_* \frac{\nu_r\langle E_r\rangle_r}{L_r^2} \tag{44}$$

where ν_r is the coefficient of turbulent viscosity, expressed by the "second" Prandtl formula

$$\nu_r = k_r(u_1 - u_2)x \tag{45}$$

It is clear that the use of Eq. (45) allows us to solve our "dynamic task" (i.e., the continuity and momentum equations in SS (41) and (42)) without distinction of "fluctuating task" (i.e., turbulent-kinetic-energy budget equations in SS (41) and (42)). This approach greatly simplifies the modeling process.

3.1.3. Modeling of the total averages

To calculate the total statistical averages, we will use the statistical ratio (20). For example, for the velocity components

$$\langle u_i\rangle = \gamma\langle u_i\rangle_t + (1-\gamma)\langle u_i\rangle_n \tag{46}$$

To determine the total averages for correlations of velocity pulsations (the covariances), we will use the ratios of the type

$$\langle u'v'\rangle = \gamma\langle u'_t v'_t\rangle_t + (1-\gamma)\langle u'_n v'_n\rangle_n + \gamma(1-\gamma)(\langle u\rangle_t - \langle u\rangle_n)(\langle v\rangle_t - \langle v\rangle_n) \tag{47}$$

This equation can be obtained according to our theory. In actual fact, for the velocity pulsations we have $u_i'|_{\mathrm{I}=1} = \langle u_{i_t}\rangle_t + u'_{i_t} - \langle u_i\rangle$ and $u_i'|_{\mathrm{I}=0} = \langle u_{i_n}\rangle_n + u'_{i_n} - \langle u_i\rangle$ according to Eqs. (21) and (22) whence it follows from Eq. (47), since $\langle u'v'\rangle = \gamma\langle u'v'\rangle_t + (1-\gamma)\langle u'v'\rangle_n$ and $\langle u'_r\langle v_r\rangle_r\rangle_r = 0$, $\langle\langle u\rangle\rangle_t = \langle u\rangle$ and so on. Eq. (47) aligns with the expression in [4, 17].

The fluctuating structure modeling is determined by the separate terms of equations for kinetic energy of the velocity fluctuations in each of the intermittent media, i.e., the turbulent kinetic energy budget equations in SS (41) and (42). In addition, the expression for the total average of turbulent energy is the same as Eq. (47), viz.,

$$\langle E\rangle = \gamma\langle E_t\rangle_t + (1-\gamma)\langle E_n\rangle_n + Ed \tag{48}$$

where

$$Ed = 0.5\gamma(1-\gamma)[(\langle u\rangle_t - \langle u\rangle_n)^2 + (\langle v\rangle_t - \langle v\rangle_n)^2 + (\langle w\rangle_t - \langle w\rangle_n)^2] \tag{49}$$

Let us note that Eq. (48) can also be obtained as Eq. (47). According to Eq. (47), wherein $u' = v'$, we have

$$\langle u'^2\rangle = \gamma\langle u_t'^2\rangle_t + (1-\gamma)\langle u_n'^2\rangle_n + u_d,\ u_d = \gamma(1-\gamma)(\langle u\rangle_t - \langle u\rangle_n)^2 \tag{50}$$

To calculate the total averages, as is evident from the foregoing, distribution of the intermittency factor γ is required. To model the intermittency factor γ we will use the expression in [16]:

$$\gamma \cong \langle \varepsilon_t\rangle_t / \langle \varepsilon\rangle \tag{51}$$

4. Testing of the *ASMTurb* method

The *ASMTurb* method has been tested in [15, 16] on the example of constructing the mathematical models for self-similar turbulent shear flows such as: I, the two-stream plane mixing layer; II, the outer region of the boundary layer on the wall; III, the far wake behind a cross-streamlined cylinder; and IV, the axisymmetric submerged jet. Test results were presented in the form calculating the main conditional and total statistical averages applied to a self-similar region of turbulent flows. A comparison was performed between the predictions and known experimental data for the energy-containing structure of turbulent flow, and excellent agreements were noted. By this means, it was shown that these *ASMTurb* models are more accurate and more detailed than the RANS models.

In view of the fact that construction of each mathematical model requires a significant volume, here we will present without details only testing results the *ASMTurb* method on the example of constructing a mathematical model for the turbulent fluid flow in a self-similar mixing layer. It is doing because all turbulence processes existing only into turbulent fluid. Calculations of the main "dynamic" and "fluctuating" characteristics we will compare with known experimental data. More detailed of this model see in [16].

4.1. Construction of the model for two-stream plane mixing layer

The mathematical *ASMTurb* model for two-stream mixing layer (see [18, 19], etc.), formed as a result of turbulent mixing of two co-current streams with identical fluid and ρ = Const, moving with different velocities $u_1 = u_{\max}$ and $u_1 = u_{\min}$, includes two subsystems SS (41) and SS (42) for conditional means of each of the intermittent media of the turbulent and nonturbulent fluid. In this case, first of all, we use the SS (41) that was written in approximation of a free boundary layer and reduced to a nondimensional form after the introduction of nondimensional variables. The task of modeling only the velocity field of turbulent flow has been called as "dynamic task."

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4.1.1. Modeling of the turbulent fluid flow

So, the dynamic task for modeling the velocity field of the turbulent fluid is reduced to solving the following system of equations

$$\begin{cases} \langle u\rangle_t \dfrac{\partial\langle u\rangle_t}{\partial x} + \langle v\rangle_t \dfrac{\partial\langle u\rangle_t}{\partial y} + \dfrac{\partial\langle u_t' v_t'\rangle_t}{\partial y} = 0 \\ \dfrac{\partial\langle u\rangle_t}{\partial x} + \dfrac{\partial\langle v\rangle_t}{\partial y} = 0 \end{cases} \tag{52}$$

with boundary conditions, which initially assuming as asymptotical, namely

$$\langle u\rangle_t = \begin{cases} u_1, \ y \to \infty \\ u_2, \ y \to -\infty \end{cases}, \ (\boldsymbol{x}, t) \in G_t \tag{53}$$

At that the closure hypothesis in Eqs. (43) and (45) take the form of

$$-\langle u_t' v_t'\rangle_t = \nu_t \frac{\partial\langle u\rangle_t}{\partial y}, \ \nu_t = k_t(u_1 - u_2)x \tag{54}$$

where ν_t is the coefficient of turbulent viscosity, $k_t = k_t(m)$, $m = u_2/u_1$). For transformation of the SE (52) to the self-similar mode in order to deduce the self-similar solution of our task, let us introduce dimensionless variables

$$\frac{\langle u\rangle_t}{u_1} = F_t'(\eta), \ \eta = \frac{y}{x} \tag{55}$$

where $F_t' = \partial F_t/\partial\eta$ with transformation $\partial/\partial x = -\eta/xd/d\eta$, $\partial/\partial y = 1/xd/d\eta$. The boundary conditions (53) take the form of

$$F_t'(\eta) \cong \begin{cases} 1, \ \eta \to \infty \\ m, \ \eta \to -\infty \end{cases} \tag{56}$$

The nondimensional transverse velocity is defined after integrating the continuity equation in SE (52):

$$\frac{\langle v\rangle_t}{u_1} = \eta F_t' - F_t \tag{57}$$

the while correlation in Eq. (52) is

$$-\frac{\langle u_t' v_t'\rangle_t}{u_1^2} = k_t(1 - m)F_t'' \tag{58}$$

As a consequence, the momentum equation in SE (52) takes the form of ordinary differential equation

$$F_t''' + 2{\sigma_t}^2 F_t F_t'' = 0 \tag{59}$$

where σ_t is a first empirical parameter of the model, the value of which is determined by the condition of the best agreement of calculated and measurements of the longitudinal velocity. We now represent a function $F_t(\eta)$ as a power series in the small parameter $(m-1)$:

$$F_t = \sum_{i=0}^{\infty}(m-1)^i F_{it} = F_{0t} + (m-1)F_{1t} + (m-1)^2 F_{2t} + \ldots \tag{60}$$

If we substitute this expression into Eq. (59) and compare the components at the same powers of parametric value $(m-1)^i$, we obtain a system of sequentially interconnected ordinary differential equations (here we confine ourselves to the second approximation of our task):

$$\begin{cases} F_{0t}''' + 2{\sigma_t}^2 F_{0t} F_{0t}'' = 0, \ i = 0 \\ F_{1t}''' + 2{\sigma_t}^2 (F_{0t} F_{1t}'' + F_{1t} F_{0t}'') = 0, \ i = 1 \\ F_{2t}''' + 2{\sigma_t}^2 (F_{0t} F_{2t}'' + F_{1t} F_{1t}'' + F_{2t} F_{0t}'') = 0, \ i = 2 \end{cases} \tag{61}$$

From the boundary conditions (56) it follows that

$$\begin{cases} F_{0t}' = 1\,, \ \ F_{1t}' = 0, \ \ F_{2t}' = 0 \ as \ \eta \to \infty \\ F_{0t}' = 1\,, \ \ F_{1t}' = 1, \ \ F_{2t}' = 0 \ as \ \eta \to -\infty \end{cases} \tag{62}$$

where we get after integration

$$F_{0t}' = 1, \ F_{0t} = \ \eta - \eta_{0t} \tag{63}$$

where $\eta_{0t} = \eta_{0t}(m)$ is the displacement of the symmetry plane of the mixing layer $\eta = 0$. Now the SE (61) takes the form of

$$\begin{cases} \tilde{F}_{1t}''' + 2\varphi \tilde{F}_{1t}'' = 0 \\ \tilde{F}_{2t}''' + 2\varphi \tilde{F}_{2t}'' = -2\tilde{F}_{1t}\tilde{F}_{1t}'' \end{cases} \tag{64}$$

where $\tilde{F}_t' = \partial\tilde{F}/\partial\varphi$, $\tilde{F}_t'' = \partial^2\tilde{F}/\partial\varphi^2$, etc. and

$$\tilde{F}_t(\varphi) = \sigma_t F_t(\eta), \ \varphi = \ \sigma_t(\eta - \eta_{0t}) \tag{65}$$

To determine the value η_{0t} will use the known Karman's condition, namely

$$\langle v(\infty)\rangle_t + m\langle v(-\infty)\rangle_t = 0 \tag{66}$$

The boundary conditions (56) because of $\langle u\rangle_t/u_1 = F_t'(\eta) = \tilde{F}_t{}'(\varphi)$ are converted in accordance with the boundary conditions (62) to the form

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$$\begin{cases} \tilde{F}'_{0t} = 1, \ \ \tilde{F}'_{1t} = 0, \ \ \tilde{F}'_{2t} = 0 \ as \ \varphi \to \infty \\ \tilde{F}'_{0t} = 1, \ \ \tilde{F}'_{1t} = 1, \ \ \tilde{F}'_{2t} = 0 \ as \ \varphi \to -\infty \end{cases} \tag{67}$$

The solution of the dynamic task in the first approximation is easy to obtain in an analytical form [16]. At that according to momentum equation in SE (64) we have

$$\frac{\langle u \rangle_t}{u_1} = \tilde{F}'_t = 1 + \frac{m-1}{2}(1 - erf\varphi) \tag{68}$$

while the transverse velocity in Eq. (57) and correlation in Eq. (58) take the following form

$$\frac{\langle v \rangle_t}{u_1} = \frac{(\varphi + \sigma\eta_{0t})\tilde{F}'_t - \tilde{F}_t}{\sigma_t} \tag{69}$$

$$-\frac{\langle u'_t v'_t \rangle_t}{\Delta U^2} = \frac{\tilde{F}''_t}{2\sigma_t(1-m)^2} \tag{70}$$

where $\Delta U = u_1 - u_2$ and the flow function (65) is

$$\tilde{F}_t = \varphi + \frac{m-1}{2}\left(\varphi - \varphi erf\varphi - \frac{1}{\sqrt{\pi}}e^{-\varphi^2} + 2c_{1t}\right) \tag{71}$$

Now, we can calculate both the longitudinal velocity profile by Eq. (68) and the correlation profile by Eq. (70) to evaluate the accuracy of our model in the first approximation. These calculations showed that the velocity profile in Eq. (68) coincides with the known experimental data at σ_t = 18.0, while the correlation profile of fluctuating velocities in Eq. (70) greatly overestimated (see **Figure 4b** where according to (47) we have to have $\langle u'v' \rangle = \langle u'_t v'_t \rangle_t$ as γ = 1). Therefore, for specification of our model, we must consider the second approximation of our task.

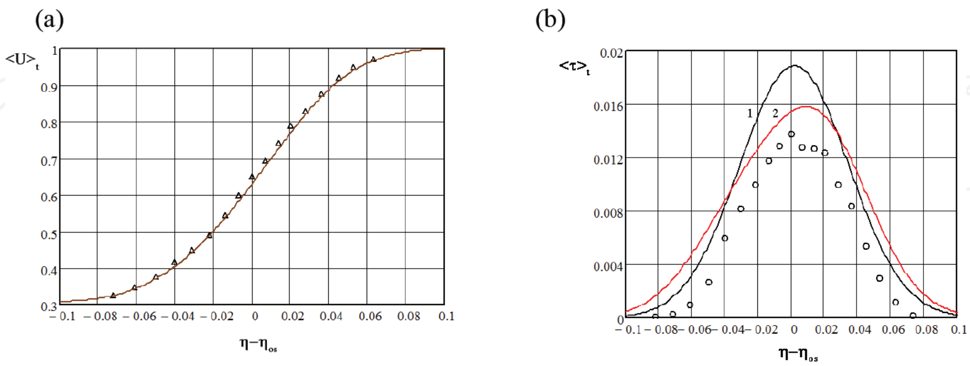

Figure 4. (a) The self-similar profile of the normalized conditional mean longitudinal velocity $\langle U \rangle_t = \langle u \rangle_t / u_1$ over the turbulent fluid. (b) Profiles of the normalized conditional mean shear stress $\langle \tau \rangle_t = -\langle u'_t v'_t \rangle_t / \Delta U^2$: 1, calculation $\langle \tau \rangle_t$ corresponding to the solution of the dynamic task in the first approximation, $\sigma_{t1} = 18.0$,; and 2, calculation $\langle \tau \rangle_t$ corresponding to the solution of the dynamic task in the second approximation, $\sigma_t = 21.5$. Symbol o is the measurements of the total average $\langle \tau \rangle = -\langle u'v' \rangle / \Delta U^2$ (measurements $\langle \tau \rangle_t$ of [20] are absent). From now on the curves—our calculations, symbols—experimental data [20] in the self-similar mixing layer at the parameter m = 0.305.

The solution of the dynamic task in the second approximation was found in such a manner. The solution of the second equation in SE (64)

$$\tilde{F}_{2t}''' = e^{-\varphi^2}(c_0 - \int \tilde{F}_{1t}\tilde{F}_{2t}'' e^{-\varphi^2} d\varphi) \tag{72}$$

was found by numerical calculation. At that according to Eq. (60), function $\tilde{F}_t$ in the second approximation contains two constants of integration c_{1t} и c_{2t}. To determine these constants, the integral relation was involved (for example, see [16]):

$$\lim_{\varphi_1,\, \varphi_2 \to \infty} \int_{-\varphi_2}^{\varphi_1} (\tilde{F}_t''' + 2\, \tilde{F}_t\, \tilde{F}_t'') d\varphi = 0 \tag{73}$$

Hence, the values of the constants are defined with the help of numerical calculation, $c_{1t} \cong 0.4,\ c_{2t} \cong -0.1$. To determine the value $\eta_{0t} = \eta_{0t}(m)$ in the expression of the dimensionless coordinates $\varphi = \sigma_t(\eta - \eta_{0t})$ we will use Eqs. (69) and (66).

The results of calculations of conditional means of this dynamic task for the mixing layer with the parameter $m = 0.305$ in comparison with the experimental data of [20] are shown in **Figure 4**. In this case, according to our model the calculated value $\eta_{0t} = -0.0181$ when $\sigma_t = 21.5$ (in [20] empirical value $\eta_{0S} = -0.02$, i.e., we have a good accuracy for $\eta_{0t} \cong \eta_{0S}$). Hereinafter curves – our calculations, symbols —measurements are mentioned [20].

Solution of the "fluctuating" task was found in such a manner. The equation of kinetic energy of the velocity fluctuations in SS (41) for the statistically stationary flow of the turbulent fluid, now in the approximation of a free boundary layer, has the following form:

$$\underbrace{\langle u \rangle_t \frac{\partial \langle E_t \rangle_t}{\partial x} + \langle v \rangle_t \frac{\partial \langle E_t \rangle_t}{\partial y}}_{Conv_t} + \underbrace{\frac{\partial \langle (E_t + p_t'/\rho) v_t' \rangle_t}{\partial y}}_{Turb\ D_t} + \underbrace{\langle u_{it}' u_{kt}' \rangle_t \frac{\partial \langle u \rangle_t}{\partial y}}_{Prod_t} + \underbrace{\langle \varepsilon_t \rangle_t}_{Diss_t} = 0 \tag{74}$$

For completion of Eq. (74) was used the known expressions (43)–(45) with index $r = t$. Transformation of Eq. (74) taking into account to an automodel form gives

$$\frac{d^2 \langle E_t \rangle_{*t}}{d\varphi^2} + 2\, \tilde{F}_t \frac{d \langle E_t \rangle_{*t}}{d\varphi} - 2\nu_{Et} \langle E_t \rangle_{*t} = -\frac{\tilde{F}_t''^2}{(1-m)^2} \tag{75}$$

Here $\langle E_t \rangle_{*t} \equiv \langle E_t \rangle_t / \Delta U^2$, $\Delta U = u_1 - u_2$, $L_t = a_{0t} x$; the second empirical parameter of the model is

$$\nu_{Et} = \frac{c_*}{2(a_{0t}\sigma_t)^2} \tag{76}$$

and is determined by the condition of the best agreement of calculated and experimental data of turbulent kinetic energy. The separate components in Eq. (75) correspond to Eq. (74) and have a definite physical meaning:

$$Conv_t = \tilde{F}_t d\langle E_t\rangle_{*t}/d\varphi \rightarrow \langle u\rangle_t \partial\langle E_t\rangle_t/\partial x + \langle v\rangle_t \partial\langle E_t\rangle_t/\partial y\text{—convective transfer} \tag{77}$$

$$TurbD_t = 0.5d^2\langle E_t\rangle_{*t}/d\varphi^2 \rightarrow \partial\langle (E_t + p'_t/\rho)v'_t\rangle_t/\partial y\text{—}\textit{diffusion through the velocity fluctuations} \tag{78}$$

$$Prod_t = \tilde{F}_t''^2/2(1-m)^2 \rightarrow \langle u'_{it}u'_{kt}\rangle_t \partial\langle u\rangle_t/\partial y\text{—}\textit{production of the energy fluctuations} \tag{79}$$

$$Diss_t = -\nu_{Et}\langle E_t\rangle_{*t} \rightarrow \langle \varepsilon_t\rangle_{*t}\text{—}\textit{dissipation rate of the energy fluctuations} \tag{80}$$

Eq. (75) was solved with boundary conditions in the form

$$\frac{d\langle E\rangle_{*t}}{d\varphi} = 0\,,\ \varphi = \begin{cases} 1.65 \\ -1.65 \end{cases} \tag{81}$$

To calculate separate components of intensity (variance) of fluctuating velocity, we will use approximate ratios:

$$\langle u_t'^2\rangle_t \cong \langle E_t\rangle_t,\ \langle v_t'^2\rangle_t \cong \langle w_t'^2\rangle_t \tag{82}$$

Eq. (75) was solved by the numerical method (mathematical package *MathCad* was used). Interestingly, the solution of Eq. (75) with using asymptotic boundary conditions $\langle E_t\rangle_{*t} \rightarrow 0$ *as* $\varphi \rightarrow \pm\infty$ gives the bad calculation data. In this regard for the flow of the turbulent fluid have been used the hard boundary conditions in the form (for $m = 0.305$)

$$\frac{\langle u\rangle_t}{u_1} = \begin{cases} 0.99,\ \varphi_1 = 1.65 \\ 0.32,\ \varphi_2 = -1.65 \end{cases} \tag{83}$$

and

$$\frac{d\langle E\rangle_{*t}}{d\varphi} = \begin{cases} 0,\ \varphi_1 = 1.65 \\ 0,\ \varphi_2 = -1.65 \end{cases} \tag{84}$$

The results of our calculations of conditional means of this *"fluctuating" task* are presented in **Figure 5**. **Figure 5a** shows the calculation $\langle u_t'^2\rangle_t/\Delta U^2$ corresponding to Eqs. (75) and (82). **Figure 5b** shows the turbulent kinetic energy budget according to Eqs. (77)–(80). At that value of the parameter $\nu_{Et} = 2$. It is worth noting that only Eq. (75) gives the hard edges $-0.075 \leq \eta - \eta_{0S} \leq 0.079$ (the same (84)) for the flow of the turbulent fluid due to the fact that the solution of Eq. (75) loses its physical sense outside these boundaries (see **Figure 5a**). So, we got the hard edges only to the flow of the turbulent fluid.

4.1.2. Modeling of the nonturbulent fluid flow

Solution of the dynamic task for the flow of a nonturbulent fluid was defined in such a manner. Modeling of this flow was carried out according to the SS (42) and was related to modeling of the flow of a turbulent fluid by means of statistical ratios in the central field of the mixing layer. It appeared that division of this subsystem into two with high velocity and low velocity

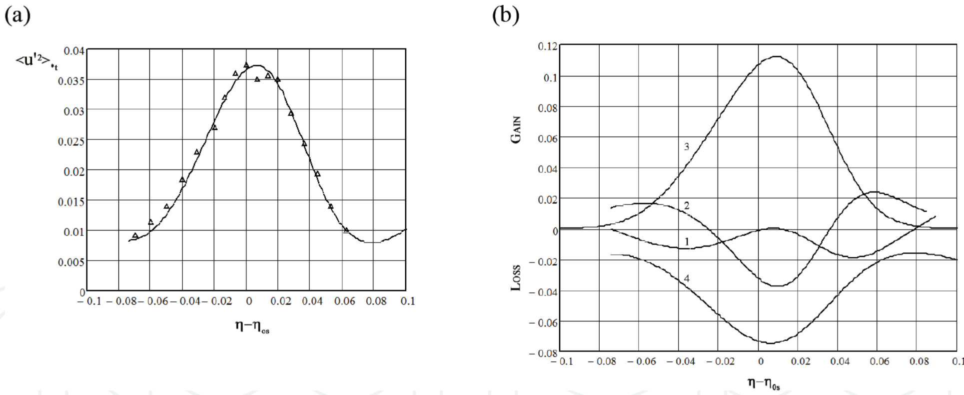

Figure 5. (a) The profile of the normalized conditional mean intensity of longitudinal velocity fluctuations $\langle u_t'^2 \rangle_{*t} = \langle u_t'^2 \rangle_t / \Delta U^2$. (b) The turbulent kinetic energy budget over the turbulent fluid: 1, Conv$_t$; 2, TurbD_t, 3, Prod$_t$; 4, Diss$_t$. The calculated parameter $\nu_{Et} = 2$. Measurements in [20] are absent.

regions $G_{n1} + G_{n2} = G_n$ gives more precise results of modeling. The flow of a nonturbulent fluid in one of these regions has not only its own parameters ($\sigma_{n1} = 51.44$, $\eta_{0n1} = -0.015$; $\sigma_{n2} = 36.4$, $\eta_{0n2} = -0.016$) but also boundary conditions: asymptotic ones in external regions and hard ones inside the mixing layer. A butting of the obtained solutions was carried out on the line $\eta - \eta_{0s} \cong 0.009$ where the condition $\langle u \rangle_t \cong \langle u \rangle_{n1} \cong \langle u \rangle_{n2}$ is satisfied. Solution of the fluctuating task for the flow of a nonturbulent fluid was defined in such a manner. In this case, the solution of the fluctuating kinetic-energy budget equation in SS (42) was found the same as task for the flow of turbulent fluid. Here, however, boundary conditions were given as asymptotic ones. The results of the modeling are presented in **Figures 7** and **8a**.

4.1.3. Modeling of the total averages

The total averages calculation is required a distribution of the intermittency factor γ. Modeling of this factor can be performed with the help of the statistical ratio (51) in view of the dissipation rate $\langle \varepsilon \rangle = \gamma \langle \varepsilon_t \rangle_t + (1-\gamma)\langle \varepsilon_n \rangle_n$ and $\langle \varepsilon \rangle_n \cong 0$. In this case, the value $\langle \varepsilon_t \rangle_t$ is in the process of modeling of turbulent fluid by Eq. (74). To calculate the total average of the dissipation rate $\langle \varepsilon \rangle$, we propose to use the assumption on its equality to the unconditional mean, which is found from the *RANS* model constructed for the mixing layer. At that the empirical constants $\sigma_R = 29.0$ and $\eta_{0R} = -0.0134$ are chosen only from the condition of agreement of the intermittency factor γ calculation with the experimental data. **Figure 6** presents the

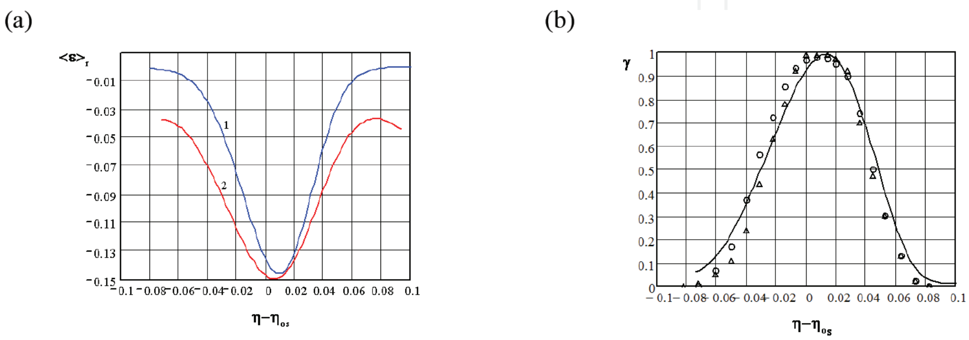

Figure 6. (a) Profiles of the dissipation rate of the energy fluctuations $\varepsilon_r = \text{Diss}_r$: 1, $\text{Diss}_R = -\nu_{ER}\langle E_R \rangle_{*R}$ and 2, $\text{Diss}_t = -\nu_{Et}\langle E_t \rangle_{*t}$ at the calculated parameter $\nu_{ER} = \nu_{Et} = 2$. (b) The profile of the intermittency factor γ of the turbulent fluid.

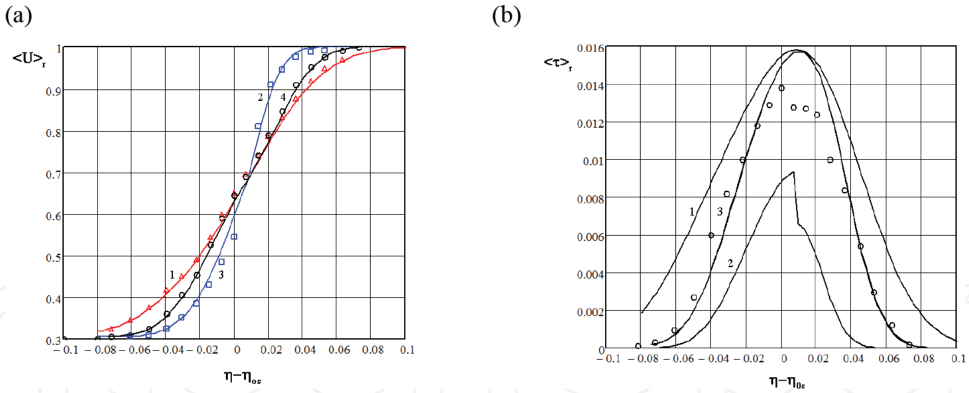

Figure 7. (a) Profiles of normalized conditional and total average longitudinal velocity $\langle U\rangle_r = \langle u\rangle_r/u_1$: 1-Δ-$\langle U\rangle_t$; 2-□-$\langle U\rangle_{n1}$; 3-□-$\langle U\rangle_{n2}$; 4-o-$\langle U\rangle$. (b) Profiles of normalized conditional and total average shear stress $\langle\tau\rangle_r = -\langle u_r' v_r'\rangle_r/\Delta U^2$: 1–$\langle\tau\rangle_t$; 2–$\langle\tau\rangle_n$; 3–o–$\langle\tau\rangle$ (measurements of $\langle\tau\rangle_t$ and $\langle\tau\rangle_n$ in [20] are absent).

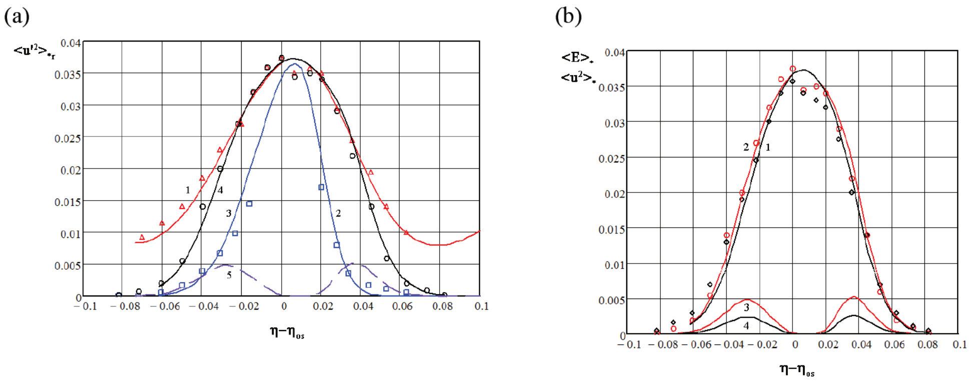

Figure 8. (a) Profiles of normalized conditional and the total average intensity of longitudinal velocity fluctuations $\langle u'^2\rangle_{*r} = \langle u_r'^2\rangle_r/\Delta U^2$: 1-Δ-$\langle u'^2\rangle_{*t}$; 2-□-$\langle u'^2\rangle_{*n1}$, 3-□-$\langle u'^2\rangle_{*n2}$; 4-o-$\langle u'^2\rangle_*$; 5 – $u_d/(1-m)^2$. (b) Profiles of normalized total average turbulent-kinetic-energy $\langle E\rangle_* = \langle E\rangle/\Delta U^2$ and intensity of longitudinal velocity fluctuations $\langle u'^2\rangle_* = \langle u'^2\rangle/\Delta U^2$: 1-o-$\langle E\rangle_*$; 2–Δ–$\langle u'^2\rangle_*$; 3 – $u_d/(1-m)^2$. 4 – $E_d/(1-m)^2$.

calculation. As *RANS* models give a good result only in the regions with insignificant intermittency, such a method for determination of the intermittency factor should be considered only as an approximate one. The results of the modeling of the total averages are presented in **Figures 7** and **8**. The some results of the unconditional means, obtained by the *RANS* model, are presented in **Figure 9**. As it seen that the *RANS* model does not give good results.

5. Conclusion

The new differential equations for the conditional means of turbulent flow are the theory result of this chapter. On the basis of these equations, the method of autonomous statistical modeling *ASMTurb* of such flow was justified. The main feature of this method is that it allows us to construct the mathematical models for the conditional means of each of the intermittent media taking place into a turbulent stream autonomously, i.e., independently. The main advantage of this method is that the system of differential equations for the conditional means does not contain the source terms. According to this method, the process of transformation the nonturbulent fluid in the turbulent fluid (as a generator of the turbulent fluid) occurs in the superlayer. Even more, the *ASMTurb* method allows us to construct the model only for the turbulent fluid flow, without considering the nonturbulent fluid flow. As far as all the mixing turbulent processes (and, as consequence, the processing modeling of turbulent heat and mass transfer) take place only into the turbulent fluid, this peculiarity essentially simplifies the modeling of such processes. Especially it refers to the turbulent combustion processes, in which modeling is attended by difficulties. It is important to note that *ASMTurb* SS (41) and (42) for conditional means of the turbulent and nonturbulent fluid differ from the known ones (for example [7]). It should be emphasized that the presented model contains only two empirical parameters σ_t and ν_{Et}. With regard to these parameters, it must be said that their appearance is due to the fact that we do not know neither the expansion rate of the turbulent fluid downstream nor the maximum value of the turbulent energy generated by the shear rate.

We now make several important remarks.

On the operation of conditional statistical averaging. Sometimes the value $\langle Y_1|Y_2\rangle$ is also called "conditional" mean that makes some confusion in comparison with the conditional means $\langle Y_1|Y_2\rangle_r$, $r = t$ or $r = n$. Indeed, variable $\langle Y_1|Y_2\rangle = \gamma\langle Y_1|Y_2\rangle_t + (1-\gamma)\langle Y_1|Y_2\rangle_n$ where $\langle Y_1|Y_2\rangle_t$ and $\langle Y_1|Y_2\rangle_n$ are the conditional means of the characteristics for the turbulent and nonturbulent fluid, respectively. So, the value of $\langle Y_1|Y_2\rangle$ actually is the *total* average of the random variable Y_1, obtained under the condition of the variable Y_2.

On the source terms. The known equations for conditional means contain the source terms, which are intended to describe the increase in volume of the turbulent fluid downstream. Here, it is interesting to discover the reasons of such source terms appearance. For this, we consider the procedure of statistical "unconditional" averaging of the continuous equation, premultiplied by the intermittency function

$$\frac{\partial u_k}{\partial x_k} = 0 \to I\frac{\partial u_k}{\partial x_k} \to \left\langle I\frac{\partial u_k}{\partial x_k}\right\rangle = \gamma\left\langle\frac{\partial u_k}{\partial x_k}\right\rangle_t = 0 \tag{85}$$

However, the permutation of averaging and differentiation operations, used in the approach, gives

$$\left\langle I\frac{\partial u_k}{\partial x_k}\right\rangle = \left\langle\frac{\partial I u_k}{\partial x_k} - u_k\frac{\partial I}{\partial x_k}\right\rangle \to \frac{\partial\langle I u_k\rangle}{\partial x_k} - \left\langle u_k\frac{\partial I}{\partial x_k}\right\rangle \to \frac{\partial\gamma\langle u_k\rangle_t}{\partial x_k} - \left\langle u_k\frac{\partial I}{\partial x_k}\right\rangle = 0 \tag{86}$$

i.e., gives rise to the appearance of the source terms of a singular type. It stands to reason that the appearance of such source term is only due to the accepted commutation of the averaging operation of the partial derivates and has no physical justification.

On the mathematical model for the turbulent fluid flow. The *ASMTurb* method allows us to construct a model for the turbulent fluid flow without considering the nonturbulent fluid flow. As far as all mixing turbulent processes take place only in the turbulent fluid, this peculiarity essentially simplifies the modeling. Even more, this approach allows us to take into account the source term, using one of the semi-empirical parameters of the mathematical model. To solve the "pulsation" task we use the turbulent-kinetic-energy budget equation. To distribute the intensity of the longitudinal velocity pulsations we use the ratio $\langle u_r'^2\rangle_r \cong \langle E_r\rangle_r$.

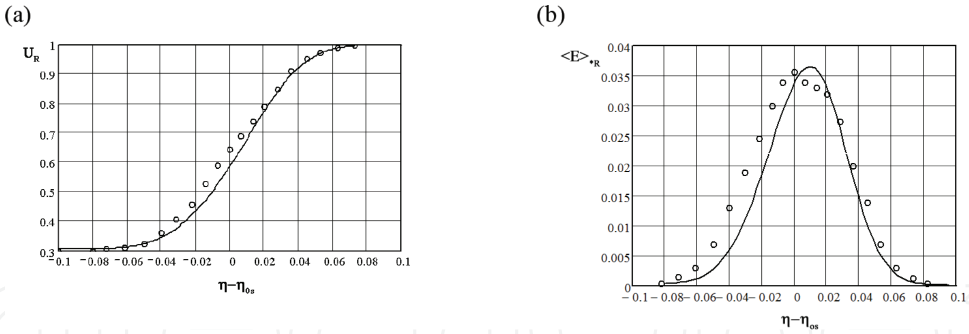

Figure 9. (a) Unconditional mean longitudinal velocity *RANS* $\langle U \rangle_R = \langle u \rangle_R / u_1$. (b) Profiles of the normalized unconditional mean *RANS* turbulent kinetic energy $\langle E \rangle_{*R} = \langle E \rangle_R / \Delta U^2$.

What gives the ASMTurb method. The results of testing *the ASMTurb method* showed a "surprising" precision for the turbulent flows modeling—calculations of the conditional and total averages of statistical characteristics practically completely agreed with the known measurements [20] (see **Figures 7** and **8** where curves—our calculation, symbols—experimental data are mentioned[20]).

What gives the RANS method. The some results of the unconditional means, obtained by the *RANS* model, are presented in **Figure 9**. As can be seen, the *RANS* model does not gives good results.

So, the *ASMTurb* differential equations for the conditional averaged characteristics of the turbulent and nonturbulent fluid flows coincide with each other in external view. Moreover, the *RANS* differential equations have the same external view. However, the boundary conditions and closure hypothesis for the turbulent and nonturbulent fluid flows in the *ASMTurb* models may be different. It is this circumstance allows us to construct highly efficient *ASMTurb* models of turbulent flows. The *RANS* method does not have this property and thus a searching for the "satisfactory" closure hypotheses for the *RANS* models will not give good results.

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