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Lagrangian Subspaces of Manifolds

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Abstract

In this chapter, we provide an overview on the Lagrangian subspaces of manifolds, including but not limited to, linear vector spaces, Riemannian manifolds, Finsler manifolds, and so on. There are also some new results developed in this chapter, such as finding the Lagrangians of complex spaces and providing new insights on the formula for measuring length, area, and volume in integral geometry. As an application, the symplectic structure determined by the Kähler form can be used to determine the symplectic form of the complex Holmes-Thompson volumes restricted on complex lines in integral geometry of complex Finsler space. Moreover, we show that the space of oriented lines and the tangent bundle of unit sphere in Minkowski space are symplectomorphic.

Keywords: Lagrangian subspace, differential geometry

1. Introduction

In differential geometry and differential topology, manifolds are the main objects being studied, and Lagrangian submanifolds are submanifolds that carry differential forms with special property, which are usually called symplectic form in real manifolds and Kahler form in complex manifolds.

This book chapter is concerned with explicit canonical symplectic form for real and complex spaces and answer to the questions on the existence of Lagrangian subspace. One can find and explicitly describe the set of Lagrangian subspaces of \mathbb{R}^2 with L^p norm, $1 \leq p < \infty$, as an example of Finsler spaces. Since Holmes-Thompson volumes, as measures, depend on the differential structures of the spaces, the symplectic structure determined by the symplectic form can be used to determine the symplectic form of Holmes-Thompson volumes restricted on lines in integral geometry of L^p spaces, as an application to integral geometry.

Some ingenious ideas in physics and engineering actually originated from mathematics. For example, the relativity theory in physics, to some sense, originated from Riemmanian geometry.

The real Finsler spaces, as generalizations of real Riemannian manifolds, were introduced in Ref. [1] about a century ago and have been studied by many researchers (see, for instance, Refs. [2–4]), and Finsler spaces (see, for instance, Refs. [5, 6]) have become an interest of research for the studies of geometry, including differential geometry and integral geometry, in recent decades. By the way, there are applications of Finsler geometry in physics and engineering, and in particular, Finsler geometry can be applied to engineering dynamical systems, on which one can see Ref. [7]. As a typical Finsler space, L^p space, $1 < p < \infty$, has the main features of a Finsler space. As such, we focus on L^p space, $1 < p < \infty$, in this chapter, but some results can be generalized to general Finsler spaces, on which one can refer to Ref. [8]. The L^p space, $1 < p < \infty$, as a generalization of Euclidean space, has a rich structure in functional analysis (see, for instance, Refs. [9, 10]), and particularly in Banach space. Furthermore, it has broad applications in statistics (see, for instance, Refs. [11, 12]), engineering (see, for instance, Ref. [13, 27]), mechanics (see, for instance, Ref. [14]), computational science (see, for instance, Ref. [15]), biology (see, for instance, Ref. [16]), and other areas. Along this direction, L^p , $0 < p \leq 1$, in the sense of conjugacy to the scenario of L^p , $1 < p < \infty$, also has broad applications, in particular, signal processing in engineering, on which one can see Refs. [17–19].

This chapter is structured as follows: In Section 2, we provide a description on Gelfand transform, which is one of the most fundamental transforms in integral geometry; in Section 3, we introduce density needed for the measure of length of curves; in Section 4, we further study the Lagrangian subspaces of complex L^p spaces; in Section 5, we work on tangent bundle of unit sphere in Minkowski space and its symplectic or Lagrangian structure; in Section 6, we apply the Lagrangian structure to establish the length formula in integral geometry; and in Section 7, we further apply the Lagrangian structure of a Minkowski space to establish the formula for the Holmes-Thompson area in integral geometry.

2. Gelfand transform

Given a double fibration:

$$\mathbb{R}^2 \xleftarrow{\pi_1} F \xrightarrow{\pi_2} \overline{Gr_1(\mathbb{R}^2)} \quad (1)$$

where

$$F = \left\{ ((x, y), l(r, \theta)) : (x, y) \in \mathbb{R}^2, l(r, \theta) \in \overline{Gr_1(\mathbb{R}^2)}, (x, y) \in l(r, \theta) \right\} \\ \simeq \left\{ (x, y, r, \theta) : x \cos(\theta) + y \sin(\theta) = r \right\},$$

π_1 and π_2 are the natural projections of fibers. The Gelfand transform of a 2-density $\varphi = |dr \wedge d\theta|$ is defined as

$$GT(\varphi) = \pi_{1*} \pi_2^* \varphi, \quad (2)$$

which is a 1-density \mathbb{R}^2 .

3. 1-Density

Lemma 3.1. For any $v = (\alpha, \beta) \in T_{(x,y)}\mathbb{R}^2$,

$$\text{GT}(\varphi)((x,y), v) = 4|v|. \tag{3}$$

Proof. For $v = (\alpha, \beta) \in T_{(x,y)}\mathbb{R}^2$, there exists

$$\tilde{v} = (\alpha, \beta, \alpha \cos(\theta) + \beta \sin(\theta), \theta) \in T_{((x,y), l(r,\theta))}\mathbb{F}, \tag{4}$$

such that $d\pi_1(\tilde{v}) = v$. Therefore, we have

$$\begin{aligned} \text{GT}(\varphi)(v) &= \int_{\pi_1^{-1}((x,y))} \pi_2^* \varphi(\tilde{v}, \bullet) \\ &= \int_{\{(x,y), l(p,\theta): x \cos(\theta) + y \sin(\theta) = r\}} |dr \wedge d\theta|(\tilde{v}, \bullet) \\ &= \int_0^{2\pi} |\alpha \cos(\theta) + \beta \sin(\theta)| d\theta \\ &= \int_0^{2\pi} |v \cdot (\cos(\theta), \sin(\theta))| d\theta \\ &= |v| \int_0^{2\pi} \cos(\theta_0 + \theta) d\theta \text{ where } \alpha = |v| \cos(\theta_0), \beta = |v| \sin(\theta_0) \\ &= 4|v|. \end{aligned} \tag{5}$$

Remark 3.2. By Alvarez's Gelfand transform for Crofton type formulas, we know that

$$\int_{l \in \mathbb{R}^2} \#(\gamma \cap l(r,\theta)) dr d\theta = \int_{\gamma} \text{GT}(\varphi). \tag{6}$$

Thus, we have now proved the Crofton formula: Given a differentiable curve γ in \mathbb{R}^2 , the length of γ can be computed in the following formula:

$$\text{Length}(\gamma) = \frac{1}{4} \int_{l \in \mathbb{R}^2} \#(\gamma \cap l(r,\theta)) dr d\theta. \tag{7}$$

4. Lagrangian subspaces of complex spaces

Some of the results have obtained in Ref. [8], but because the Lagrangian subspaces of complex spaces are essential to establish the generalized volume formula in complex integral geometry, let us give an expository on the Kahler structure of generalized complex spaces.

Theorem 4.1. *The set of Lagrangian subspaces of \mathbb{C}^2 with L^1 norm is $\mathbb{T}^2 \cup \mathbb{T}^1$, where*

$$\mathbb{T}^2 := \{\text{span}((z, 0), (0, w)) : z, w \in U(1)\} \cong U(1) \times U(1) \quad (8)$$

and

$$\mathbb{T}^1 := \{P : P = \{\lambda(z, w) : \lambda \in \mathbb{R}, z, w \in U(1), zw \text{ is a constant in } U(1)\}\} \cong U(1). \quad (9)$$

Proof. First, we can show that

$$P = \{\lambda(z, w) : \lambda \in \mathbb{R}, z, w \in U(1), zw \text{ is a constant in } U(1)\} \quad (10)$$

is identical to some

$$P' := \text{span}((z_1, z_1 e^{i\theta}), (z_2, \frac{z_1^2 \bar{z}_2}{|z_1|^2} e^{i\theta})) \quad (11)$$

where $z_1, z_2 \in \mathbb{C} \setminus \{0\}$. For any $\lambda(e^{i\varphi}, e^{i\psi}) \in P$, let $z_1 = \lambda e^{i\varphi}$, $\theta = \psi - \varphi$, we have $P = \text{span}((z_1, z_1 e^{i\theta}), (z_2, \frac{z_1^2 \bar{z}_2}{|z_1|^2} e^{i\theta})) = P'$ where $z_2 \in \mathbb{C} \setminus \{0\}$.

We can get $\kappa_1(z_1, 0), (0, z_2) = 0$. On the other hand, for any

$$(z, w) = \lambda_1(z_1, z_1) + \lambda_2(z_2, \frac{z_1^2 \bar{z}_2}{|z_1|^2}) \in \text{span}((z_1, z_1), (z_2, \frac{z_1^2 \bar{z}_2}{|z_1|^2})), \quad (12)$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$,

$$\begin{aligned} |w|^2 &= (\lambda_1 z_1 + \lambda_2 \frac{z_1^2 \bar{z}_2}{|z_1|^2})(\lambda_1 \bar{z}_1 + \lambda_2 \frac{\bar{z}_1^2 z_2}{|z_1|^2}) \\ &= \lambda_1^2 z_1 \bar{z}_1 + \lambda_1 \lambda_2 \bar{z}_1 z_2 + \lambda_2 \lambda_1 z_1 \bar{z}_2 + \lambda_2^2 \bar{z}_2 z_2 \\ &= (\lambda_1 z_1 + \lambda_2 z_2)(\lambda_1 \bar{z}_1 + \lambda_2 \bar{z}_2) \\ &= |z|^2, \end{aligned} \quad (13)$$

that implies $|\frac{w}{z}| = 1$. Therefore, we have

$$\begin{aligned} \kappa_{(z,w)}((z_1, z_1), (z_2, \frac{z_1^2 \bar{z}_2}{|z_1|^2})) &= \frac{3}{2} (\text{Im}(z_2 \bar{z}_1) + \frac{3}{2} \text{Im}(\frac{z_1^2 \bar{z}_2}{|z_1|^2} \bar{z}_1)) \\ &\quad - \frac{1}{2} \text{Im}(\frac{z}{w} |\frac{w}{z}| (\frac{z_1^2 \bar{z}_2}{|z_1|^2} \bar{z}_1 - z_1 \bar{z}_2)) \\ &= \frac{3}{2} (\text{Im}(z_2 \bar{z}_1) + \text{Im}(z_1 \bar{z}_2)) \\ &= 0. \end{aligned} \quad (14)$$

So κ vanishes on $\text{span}((z_1, z_1), (z_2, \frac{z_1^2 \bar{z}_2}{|z_1|^2}))$ for any $z_1, z_2 \in \mathbb{C} \setminus \{0\}$, $\text{Im}(\frac{z_1}{z_1}) \neq 0$.

Conversely, suppose that κ vanishes on a plane P spanned by (z_1, w_1) and (z_2, w_2) . We know that

$$\left(1 + \frac{1}{2} \left| \frac{w}{z} \right| \right) \text{Im}(z_2 \bar{z}_1) + \left(1 + \frac{1}{2} \left| \frac{z}{w} \right| \right) \text{Im}(w_2 \bar{w}_1) + \frac{1}{2} \text{Im} \left(\frac{z}{w} \left| \frac{w}{z} \right| (w_2 \bar{z}_1 - w_1 \bar{z}_2) \right) = 0 \quad (15)$$

holds for any $(z, w) \in \text{span}((z_1, w_1), (z_2, w_2))$. In the following argument, we divide it into three cases to discuss in terms of $\left| \frac{w}{z} \right|$ and $\frac{w}{z} \left| \frac{w}{z} \right|$.

The first case is that $\left| \frac{w}{z} \right| = \lambda$ for some fixed $\lambda > 0$. Let $(z, w) = \lambda_1(z_1, w_1) + \lambda_2(z_2, w_2)$ for any $\lambda_1, \lambda_2 \in \mathbb{R}$, then $|\lambda_1 w_1 + \lambda_2 w_2| = \lambda |\lambda_1 z_1 + \lambda_2 z_2|$, that implies $|w_1| = \lambda |z_1|$, $|w_2| = \lambda |z_2|$ and $\text{Re}(w_1 \bar{w}_2) = \lambda^2 \text{Re}(z_1 \bar{z}_2)$. It follows that $w_1 = \lambda e^{i\theta} z_1$, $w_2 = \lambda e^{i\theta} z_2$, or $w_1 = \lambda e^{i\theta} z_1$, $w_2 = \lambda e^{i\theta} \frac{z_1 \bar{z}_2}{|z_1|^2}$ for some $\theta \in [0, 2\pi)$.

In the sub-case of $w_1 = \lambda e^{i\theta} z_1$, $w_2 = \lambda e^{i\theta} z_2$ for some $\theta \in [0, 2\pi)$, by Eq. (15) we have

$$\left(1 + \frac{\lambda}{2}\right) \text{Im}(z_2 \bar{z}_1) + \left(1 + \frac{1}{2\lambda}\right) \lambda^2 \text{Im}(z_2 \bar{z}_1) + \lambda \text{Im}(z_2 \bar{z}_1) = (1 + \lambda)^2 \text{Im}(z_2 \bar{z}_1) = 0, \quad (16)$$

which implies $\text{Im}(z_2 \bar{z}_1) = 0$ and furthermore $\text{Im}(w_2 \bar{w}_1) = 0$. That means (z_1, w_1) and (z_2, w_2) are colinear. So this case cannot occur.

However, for the other sub-case of $w_1 = \lambda e^{i\theta} z_1$, $w_2 = \lambda e^{i\theta} \frac{z_1 \bar{z}_2}{|z_1|^2}$ for some $\theta \in [0, 2\pi)$, by Eq. (15) we have

$$\left(1 + \frac{\lambda}{2}\right) \text{Im}(z_2 \bar{z}_1) + \left(1 + \frac{1}{2\lambda}\right) \lambda^2 \text{Im}(z_1 \bar{z}_2) = (1 - \lambda^2) \text{Im}(z_2 \bar{z}_1) = 0. \quad (17)$$

Then $\lambda = 1$ or $\text{Im}(z_2 \bar{z}_1) = 0$, but (z_1, w_1) and (z_2, w_2) cannot be colinear. So, we have $\lambda = 1$ which gives

$$P = \text{span}((z_1, z_1 e^{i\theta}), (z_2, \frac{z_1 \bar{z}_2}{|z_1|^2} e^{i\theta})), \quad (18)$$

where $z_1, z_2 \in \mathbb{C} \setminus \{0\}$ and $\text{Im}(z_1 \bar{z}_2) \neq 0$ for some $\theta \in [0, 2\pi)$. This finishes the first case.

The second case is $\frac{w}{z} \left| \frac{w}{z} \right| = e^{i\theta}$ for some fixed $\theta \in [0, 2\pi)$. Let $w_1 = \lambda_1 e^{i\theta} z_1$, $w_2 = \lambda_2 e^{i\theta} z_2$ for some $\lambda_1, \lambda_2 > 0$. Then it follows from (15) that

$$\begin{aligned} & \left(1 + \frac{\lambda_1}{2}\right) \text{Im}(z_2 \bar{z}_1) + \left(1 + \frac{1}{2\lambda_1}\right) \lambda_1 \lambda_2 \text{Im}(z_2 \bar{z}_1) + \frac{1}{2} (\lambda_1 + \lambda_2) \text{Im}(z_2 \bar{z}_1) \\ &= \left(1 + \frac{\lambda_2}{2}\right) \text{Im}(z_2 \bar{z}_1) + \left(1 + \frac{1}{2\lambda_2}\right) \lambda_1 \lambda_2 \text{Im}(z_2 \bar{z}_1) + \frac{1}{2} (\lambda_1 + \lambda_2) \text{Im}(z_2 \bar{z}_1) \\ &= (1 + \lambda_1)(1 + \lambda_2) \text{Im}(z_2 \bar{z}_1) \\ &= 0 \end{aligned} \quad (19)$$

at the points (z_1, w_1) and (z_2, w_2) , which implies $\text{Im}(z_2 \bar{z}_1) = 0$ and furthermore $\text{Im}(w_2 \bar{w}_1) = 0$. Thus, z_1 and z_2 , w_1 , and w_2 are colinear, which implies that P equals a plane spanned by one vector from $\{(z_1, 0), (z_2, 0)\}$ and the other from $\{(0, w_1), (0, w_2)\}$. Thus $P \in \mathbb{T}^2$.

The last case is the negative to the first one and the second one. It gives $\text{Im}(z_2 \bar{z}_1) = \text{Im}(w_2 \bar{w}_1) = 0$ and $w_2 \bar{z}_1 - w_1 \bar{z}_2 = 0$ because of the linear independence, but the former implies the latter by

linear transformation, so it is brought down to $Im(z_2\bar{z}_1) = Im(w_2\bar{w}_1) = 0$. Thus, we have $P \in \mathbb{T}^2$ by the second case, and that concludes the proof.

5. Tangent bundle of uni-sphere in Minkowski space and symplectic or Lagrangian structure

In this section, we show that the space of oriented lines and the tangent bundle of unit sphere in Minkowski space are symplectomorphic.

Let us consider a Minkowski plane (\mathbb{R}^2, F) first, where F is a Finsler metric. The natural symplectic form on $T^*\mathbb{R}^2$ is $dx \wedge d\bar{\xi} + dy \wedge d\bar{\eta}$, and then the natural symplectic form on $T\mathbb{R}^2$ induce by the Finsler metric F is

$$\begin{aligned} \omega &:= dx \wedge d\frac{\partial F}{\partial \xi} + dy \wedge d\frac{\partial F}{\partial \eta} \\ &= \frac{\partial^2 F}{\partial \xi^2} dx \wedge d\xi + \frac{\partial^2 F}{\partial \xi \partial \eta} (dx \wedge d\eta + dy \wedge d\xi) + \frac{\partial^2 F}{\partial \eta^2} dy \wedge d\eta. \end{aligned} \quad (20)$$

Define a projection $\pi : T\mathbb{R}^2 \rightarrow \overline{Gr_1(\mathbb{R}^2)}$ by

$$\pi((x,y); (\xi, \eta)) = ((x,y) - dF(\xi, \eta)((x,y))(\xi, \eta); (\xi, \eta)). \quad (21)$$

Let S_F be the unit circle in the Minkowski plane and TS_F be its tangent bundle. It is a fact that $TS_F \cong \overline{Gr_1(\mathbb{R}^2)}$. On the other hand, since TS_F is embedded in $T\mathbb{R}^2$, it inherits a natural symplectic form $\omega_0 := \omega|_{TS_F}$ from $T\mathbb{R}^2$.

Theorem 5.1. $\pi^*\omega_0 = \omega|_{S^*\mathbb{R}^2}$.

Proof. Applying the equality

$$\frac{\partial F}{\partial \xi} d\xi + \frac{\partial F}{\partial \eta} d\eta = 0, \quad (22)$$

we obtain

$$\begin{aligned} \pi^*\omega_0 &= \frac{\partial^2 F}{\partial \xi^2} d(x - dF(\xi, \eta)((x,y))\xi) \wedge d\xi + \frac{\partial^2 F}{\partial \xi \partial \eta} (d(x - dF(\xi, \eta)((x,y))\xi) \wedge d\eta \\ &\quad + d(y - dF(\xi, \eta)((x,y))\eta) \wedge d\xi) + \frac{\partial^2 F}{\partial \eta^2} d(y - dF(\xi, \eta)((x,y))\eta) \wedge d\eta \\ &= \frac{\partial^2 F}{\partial \xi^2} dx \wedge d\xi + \frac{\partial^2 F}{\partial \xi \partial \eta} (dx \wedge d\eta + dy \wedge d\xi) + \frac{\partial^2 F}{\partial \eta^2} dy \wedge d\eta \\ &\quad - d(dF(\xi, \eta)((x,y))) \wedge \left(\frac{\partial^2 F}{\partial \xi^2} \xi d\xi + \frac{\partial^2 F}{\partial \eta^2} \eta d\eta + \frac{\partial^2 F}{\partial \xi \partial \eta} (\xi d\eta + \eta d\xi) \right). \end{aligned} \quad (23)$$

By the positive homogeneity of F , one can get the useful fact that $F(\xi, \eta) = \xi \frac{\partial F}{\partial \xi} + \eta \frac{\partial F}{\partial \eta}$. Therefore,

$$\xi \frac{\partial F}{\partial \xi} + \eta \frac{\partial F}{\partial \eta} = 1. \quad (24)$$

By differentiating (24), we get

$$\frac{\partial^2 F}{\partial \xi^2} \xi d\xi + \frac{\partial^2 F}{\partial \eta^2} \eta d\eta + \frac{\partial^2 F}{\partial \xi \partial \eta} (\xi d\eta + \eta d\xi) + \frac{\partial F}{\partial \xi} d\xi + \frac{\partial F}{\partial \eta} d\eta = 0. \quad (25)$$

Applying (22) again, we have

$$\frac{\partial^2 F}{\partial \xi^2} \xi d\xi + \frac{\partial^2 F}{\partial \eta^2} \eta d\eta + \frac{\partial^2 F}{\partial \xi \partial \eta} (\xi d\eta + \eta d\xi) = 0. \quad (26)$$

Thus, the claim follows.

Remark 5.2. For a n -dimensional Minkowski space (\mathbb{R}^n, F) , we just need to add more indices, then the theorem above is also true for (\mathbb{R}^n, F) .

Therefore, letting F be a Finsler metric on \mathbb{R}^n and S_F be the unit sphere in the Minkowski space (\mathbb{R}^n, F) , we obtain the following general theorem:

Theorem 5.3. *The symplectic form on the space of lines in a Minkowski space (\mathbb{R}^n, F) is the canonical symplectic form on the tangent bundle TS_F as imbedded in $T\mathbb{R}^n$.*

We have the following remarks:

Remark 5.4. Theorem 5.3 provides a perspective that we can transform calculus on $\overline{Gr_1(\mathbb{R}^2)}$ to ones on TS_F .

and

Remark 5.5. We can analyze the differential structure of the Minkowski space by considering its symplectic form or Lagrangian structure. The Lagrangian structure of tangent spaces of Minkowski space gives the symplectic structure of the space of geodesics in the Minkowski space, and in general, the measures on a space or manifold in integral geometry depend on the differential structures of the space or manifold. Holmes-Thompson volumes are defined based on Lagrangian structure (see, for instance, Refs. [12, 20]), so, as an application, the symplectic structure determined by the symplectic form can be used to determine the symplectic form of the Holmes-Thompson volumes restricted on lines in integral geometry of Minkowski space, about which one can see Refs. [21–23].

Another remark from the proof of Theorem 5.1 is that

Remark 5.6. A combination of (26) and Gelfand transform (see Ref. [6]) may be used to provide a short proof of the general Crofton formula for Minkowski space.

6. Application to generalized length and related

For any rectifiable curve γ in the Euclidean plane, the classic Crofton formula is

$$\text{Length}(\gamma) = \frac{1}{4} \int_0^\infty \int_0^{2\pi} \#(\gamma \cap l(r, \theta)) d\theta dr, \quad (27)$$

where θ is the angle from the x -axis to the normal of the oriented line l and r is the distance from the origin to l . Let us denote the affine 1-Grassmannians consisting of lines in \mathbb{R}^2 by $\overline{Gr_1(\mathbb{R}^2)}$.

As for Minkowski plane, it is a normed two dimensional space with a norm $F(\cdot) = \|\cdot\|$, in which the unit disk is convex and F has some smoothness.

Two significant and useful tools that are used to obtain the Crofton formula for Minkowski plane are the cosine transform and Gelfand transform. Let us explain them one by one first and see the connections between them later. A important fact or result from spherical harmonics about cosine transform is that there is some even function on S^1 such that

$$F(\cdot) = \int_{S^1} |\langle \xi, \cdot \rangle| g(\theta) d\theta, \quad (28)$$

if F is an even C^4 function on S^1 . A great reference for this would be [24] by Groemer. As for Gelfand transform, it is the transform of differential forms and densities on double fibrations, for instance, $\mathbb{R}^2 \xrightarrow{\pi_1} \mathcal{I} \xrightarrow{\pi_2} \overline{Gr_1(\mathbb{R}^2)}$, where $\mathcal{I} := \{(x, l) \in \mathbb{R}^2 \times \overline{Gr_1(\mathbb{R}^2)} : x \in l\}$ is the incidence relations and π_1 and π_2 are projections. A formula one can take as an example of the fundamental theorem of Gelfand transform is the following:

$$\int_\gamma \pi_{1*} \pi_2^* |\Omega| = \int_{l \in \overline{Gr_1(\mathbb{R}^2)}} \#(\gamma \cap l) |\Omega|, \quad (29)$$

where $\Omega := g(\theta) d\theta \wedge dr$. However, here we provide a direct proof for this fundamental theorem of Gelfand transform.

Proof. First, consider the case of $\Omega = d\theta \wedge dr$. For any $v \in T_x \gamma$, since there is some $v' \in T_{x'} \mathcal{I}$, such that $\pi_{1*}(v') = v$, then

$$\begin{aligned} (\pi_{1*} \pi_2^* |\Omega|)_x(v) &= \left(\int_{\pi_1^{-1}(x)} \pi_2^* |\Omega| \right)_x(v) \\ &= \int_{x' \in \pi_1^{-1}(x)} (\pi_2^* |\Omega|)_{x'}(v') \\ &= \int_{S^1} (\pi_2^* |d\theta \wedge dr|)(v') \\ &= \int_{S^1} |dr(\pi_{2*}(v'))| d\theta \\ &= \int_{S^1} |\langle v, \theta \rangle| d\theta \\ &= 4|v|. \end{aligned} \quad (30)$$

Thus, we have

$$\int_{\gamma} \pi_{1*} \pi_2^* |\Omega| = 4 \text{Length}(\gamma) = \int_{l \in Gr_1(\mathbb{R}^2)} \frac{\#(\gamma \cap l) |\Omega|}{4} \quad (31)$$

by using the classic Crofton formula.

For the general case of $\Omega = f(\theta) d\theta \wedge dr$, we just need to substitute $d\theta$ by $g(\theta) d\theta$ in the equalities in the first case.

Furthermore, we can also see, from the above proof and eq:exist, that

$$\int_{\gamma} \pi_{1*} \pi_2^* |\Omega| = \int_a^b (\pi_{1*} \pi_2^* |\Omega|)(\gamma'(t)) dt = \int_a^b 4F(\gamma'(t)) dt = 4 \text{Length}(\gamma), \quad (32)$$

for any curve $\gamma(t) : [a, b] \rightarrow \mathbb{R}^2$ differentiable almost everywhere in the Minkowski space. Therefore, by using (29), we obtain that

$$\text{Length}(\gamma) = \frac{1}{4} \int_{l \in Gr_1(\mathbb{R}^2)} \frac{\#(\gamma \cap l) |g(\theta) d\theta \wedge dr|}{4} \quad (33)$$

for Minkowski plane.

The Holmes-Thompson area $HT^2(U)$ of a measurable set U in a Minkowski plane is defined as $HT^2(U) := \frac{1}{\pi} \int_{D^*U} |\omega_0|^2$, where ω_0 is the natural symplectic form on the cotangent bundle of \mathbb{R}^2 and $D^*U := \{(x, \xi) \in T^*\mathbb{R}^2 : F^*(\xi) \leq 1\}$. To study it from the perspective of integral geometry, we need to introduce a symplectic form ω to the space of affine lines $\overline{Gr_1(\mathbb{R}^2)}$ and construct an invariant measure based on ω .

7. Application to HT area and related

Now let us see the Crofton formula for Minkowski plane, which is

$$\text{Length}(\gamma) = \frac{1}{4} \int_{Gr_1(\mathbb{R}^2)} \frac{\#(\gamma \cap l) |\omega|}{4} \quad (34)$$

To prove this, it is sufficient to show that it holds for any straight line segment

$$L : [0, \|p_2 - p_1\|] \rightarrow \mathbb{R}^2, L(t) = p_1 + \frac{p_2 - p_1}{\|p_2 - p_1\|} t, \quad (35)$$

starting at p_1 and ending at p_2 in \mathbb{R}^2 . First, using the diffeomorphism between the circle bundle and co-circle bundle, which is

$$\begin{aligned} \varphi_F : S\mathbb{R}^2 &\rightarrow S^*\mathbb{R}^2 \\ \varphi_F(x, \xi) &= (x, dF_\xi), \end{aligned} \tag{36}$$

we can obtain a fact that

$$\begin{aligned} \int_{L \times \left\{ \frac{p_2 - p_1}{\|p_2 - p_1\|} \right\}} \varphi_F^* \alpha_0 &= \int_{\varphi_F(L \times \left\{ \frac{p_2 - p_1}{\|p_2 - p_1\|} \right\})} \alpha_0 \\ &= \int_0^{\|p_2 - p_1\|} \alpha_0 dF_{\frac{p_2 - p_1}{\|p_2 - p_1\|}} \left(\left(\frac{p_2 - p_1}{\|p_2 - p_1\|}, 0 \right) \right) dt \\ &= \int_0^{\|p_2 - p_1\|} dF_{\frac{p_2 - p_1}{\|p_2 - p_1\|}} \left(\frac{p_2 - p_1}{\|p_2 - p_1\|} \right) dt, \end{aligned} \tag{37}$$

where α_0 is the tautological one-form, precisely $\alpha_{0\xi}(X) := \xi(\pi_{0*}X)$ for any $X \in T_\xi T^*\mathbb{R}^2$, and $d\alpha_0 = \omega_0$. Applying the basic equality that $dF_\xi(\xi) = 1$, which is derived from the positive homogeneity of F , for all $\xi \in S\mathbb{R}^2$, the above quantity becomes $\int_0^{\|p_2 - p_1\|} 1 dt$, which equals $\|p_2 - p_1\|$.

Let $R := \{ \xi_x \in S^*\mathbb{R}^2 : x \in \overline{p_1 p_2} \}$ and $T = \{ l \in \overline{Gr_1(\mathbb{R}^2)} : l \cap \overline{p_1 p_2} \neq \emptyset \}$, and p' is the projection (composition) from $S^*\mathbb{R}^2$ to $\overline{Gr_1(\mathbb{R}^2)}$.

Apply the above fact and $p'^* \omega = \omega_0$,

$$\begin{aligned} \int_T |\omega| &= \int_{p'(R)} |\omega| = \int_R |p'^* \omega| = \int_R |\omega_0| \\ &= \left| \int_{R^+} \omega_0 \right| + \left| \int_{R^-} \omega_0 \right| \\ &= \left| \int_{\partial R^+} \alpha_0 \right| + \left| \int_{\partial R^-} \alpha_0 \right| \\ &= 4 \|p_2 - p_1\|. \end{aligned} \tag{38}$$

Thus, we have shown the Crofton formula for Minkowski plane.

Furthermore, combining with (33), we have

$$\frac{1}{4} \int_{l \in Gr_1(\mathbb{R}^2)} \#(\gamma \cap l) |\Omega| = \frac{1}{4} \int_{Gr_1(\mathbb{R}^2)} \#(\gamma \cap l) |\omega|, \tag{39}$$

where $\Omega = g(\theta) d\theta \wedge dr$. Then, by the injectivity of cosine transform in Ref. [24], $|\Omega| = |\omega|$.

To obtain the HT area, one can define a map

$$\begin{aligned} \pi : \overline{Gr_1(\mathbb{R}^2)} \times \overline{Gr_1(\mathbb{R}^2)} \setminus \Delta &\rightarrow \mathbb{R}^2 \\ \pi(l, l') &= l \cap l', \end{aligned} \tag{40}$$

extended from Alvarez's construction of taking intersections. The following theorem can be obtained.

Theorem 7.1. For any bounded measurable subset U of a Minkowski plane, we have

$$HT^2(U) = \frac{1}{2\pi} \int_{x \in \mathbb{R}^2} \chi(x \cap U) |\pi_* \Omega^2|. \quad (41)$$

Proof. On the one hand,

$$\frac{1}{\pi} \int_{D^*U} \omega_0^2 = \frac{1}{\pi} \int_{\partial D^*U} \omega_0^2 = \frac{1}{\pi} \int_{S^*U} \alpha_0 \wedge \omega_0. \quad (42)$$

On the other hand,

$$\begin{aligned} \frac{1}{\pi} \int_{x \in \mathbb{R}^2} \chi(x \cap U) \pi_* \Omega^2 &= \frac{1}{\pi} \int \left\{ \overline{Gr_1(\mathbb{R}^2)} \times \overline{Gr_1(\mathbb{R}^2)} \setminus \Delta : \text{inl}' \in U \right\} \Omega^2 \\ &= \frac{1}{\pi} \int \left\{ \overline{Gr_1(\mathbb{R}^2)} \times \overline{Gr_1(\mathbb{R}^2)} \setminus \Delta : \text{inl}' \in U \right\} \omega^2 \\ &= \frac{1}{\pi} \int_{\mathbb{T}^*U \setminus \{(x, \xi, \xi) : \xi \in S_x^*U\}} p'^* \omega^2 \\ &= \frac{1}{\pi} \int_{\mathbb{T}^*U \setminus \{(x, \xi, \xi) : \xi \in S_x^*U\}} \omega_0^2 \\ &= \frac{2}{\pi} \int_{\{(x, \xi, \xi) : \xi \in S_x^*U\}} \alpha_0 \wedge \omega_0 \\ &= \frac{2}{\pi} \int_{S^*U} \alpha_0 \wedge \omega_0, \end{aligned} \quad (43)$$

where

$$\mathbb{T}^*U := \{(x, \xi, \xi') : \xi, \xi' \in S_x^*U\}. \quad (44)$$

So the claim follows.

Remark 7.2. Lagrangian structure provides the underlying differential structure needed to measure the Holme-Thompson area in integral geometry and therefore is essential and doundamental in integral geometry. For Finsler manifolds, real or complex, it is necessary to analyze the Lagrangian structure of the Finsler manifolds, in the forms of symplectic structure and Kahler structure, and many Finsler manifolds may not have a Lagrangian structure, about which one can refer to Ref. [25]. However, for smooth projective Finsler spaces, the integral geometry formulas have been studied in Ref. [26], for instance.

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