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State-Feedback Output Tracking Via a Novel Optimal-Sliding Mode Control

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Abstract

This chapter describes a new framework for the design of a novel suboptimal state-feedback-sliding mode control for output tracking while $\mathcal{H}_2/\mathcal{H}_\infty$ performances of the closed-loop system are under control. In contrast to most of the current sliding surface design schemes, in this new framework, the level of control effort required to maintain sliding is penalized. The proposed method for the design of optimal-sliding mode control is carried out in two stages. In the first stage, a state-feedback gain is derived using a linear matrix inequality (LMI)-based scheme that can assign a number of the closed-loop eigenvalues to a known value while satisfying performance specifications and ensuring that all the closed-loop poles are located in a preselected subregion. The sliding function matrix related to the particular state feedback derived in the first stage is obtained in the second stage by using one of the two different methods developed for this goal. We present a numerical example to demonstrate the remarkable performance of the proposed scheme.

Keywords: optimal $\mathcal{H}_2/\mathcal{H}_\infty$-based sliding mode control, output tracking, partial eigenstructure assignment, regional pole placement

1. Introduction

Sliding mode control (SMC) is now a well-developed method of control and its invariance properties against matched uncertainties have inspired researchers to apply this technique to different applications [1–6]. Traditionally, SMC is designed in two stages. In the first stage, a sliding surface whose sliding motions have suitable dynamics is chosen. Many methods have been proposed in the existing literature for this purpose, for example, eigenstructure assignment, pole placement, optimal quadratic methods, and linear matrix inequality (LMI) methods; see for instance [4, 5, 7, 8]. Then, a controller is designed to induce and maintain the sliding motion.

However, these traditional design methods are unable to limit the available control action required for satisfying the control objective [3], since during the switching-function synthesis, there is no sense of the level of the control action required to persuade and retain sliding [3]. It is worth noting that without having limits on the available control actions, a sliding surface and thereby a control law may always be obtained which is not practically applicable, as it may lead to high level of control efforts.

To tackle this problem, for instance, the authors of [9] propose a scheme to design a sliding surface which minimizes an objective function of the system state and control input, in the meantime. However, since the method in [9] needs to ensure that at least one eigenvalue of the closed-loop system (for single-input systems) is a real value, not necessarily any arbitrary weighting matrices in the objective function may result in a sliding mode control. This reference, therefore, either reselects the weighting matrices or approximates the closed-loop system eigenvalues so that a set of eigenvalues are generated which can be divided into the null-space and range-space dynamics. However, no precise scheme is given on how to reselect the weighting matrices. Further, the approximation of eigenvalues may lead to a loss in optimality and possibly robustness.

For addressing the limitations of [9], Tang and Misawab [10] propose an LQR-like scheme in which a weighting matrix is computed which is closest to the desired one and can result in the desired eigenvalues. Following this, the associated SMC is designed according to the obtained eigenvalues and weighting matrix. Nevertheless, both methods in [9, 10] are suitable to single-input systems. Alternatively, Edwards [3] proposes two new frameworks exploiting two special system coordinate transformations, which are fundamentally different from the aforementioned schemes.

This chapter aims to propose a different way for the sliding surface design while optimizing the control effort associated with the linear part of the control law. This approach is a middle-of-the-road method in that it uses a specific *partial eigenstructure assignment* method to assign m arbitrary stable real eigenvalues while an appropriate sliding motion dynamics will be ensured by enforcing different Lyapunov-type constraints such as the $\mathcal{H}_2/\mathcal{H}_\infty$ and regional pole-placement constraints. The advantages of the proposed approach for the design of sliding surface compared to all the aforementioned references are threefold: (i) it can set the stage for designing SMC while the level of control efforts is taken into account; (ii) it makes it possible to integrate several Lyapunov-type constraints, for example, regional pole-placement constraints, in the SMC design problem; and (iii) the controller can be computed in a numerically very efficient method. The proposed scheme for the design of suboptimal SMC is indeed a two-stage LMI-based approach. In the first stage, while enforcing different Lyapunov-type constraints, for example, the mixed $\mathcal{H}_2/\mathcal{H}_\infty$, a state-feedback gain is derived, using an LMI-based optimization program employing an instrumental matrix variable, which can precisely assign some of the closed-loop eigenvalues to a priori known value. Following this, the sliding surface, associated with the state-feedback gain obtained in the first stage, is determined in the second stage. Two different approaches are presented for deriving the associated switching-function matrix. This chapter indeed examines the problem of designing a state-feedback SMC which utilizes integral action to provide tracking. From the implementation point of view, the simplicity of such a scheme is very advantageous.

The structure of the chapter is as follows. Section 2 is dedicated to the problem statement and preliminaries. Section 3 explains the novel design strategy for the design of $\mathcal{H}_2/\mathcal{H}_\infty$-based SMC. Section 4 discusses two different approaches for deriving the sliding function matrix associated with the linear controller obtained in Section 3. Section 5 summarizes the proposed $\mathcal{H}_2/\mathcal{H}_\infty$-based SMC. In Section 6, we discussed the issue of designing SMC with additional regional pole-placement constraints. Section 7 illustrates this method via an example considering the flight control problem. Section 8 finally concludes the chapter.

2. Problem statement and preliminaries

Consider the following linear-time invariant (LTI) system:

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}[u(t) + f(t)], \tag{1}$$

where $\tilde{x} \in \mathbb{R}^{\tilde{n}}$ and $u \in \mathbb{R}^m$ are the state vector and control input vector, respectively. The matrices in Eq. (1) are constant and of appropriate dimensions. The unknown signal $f(t) \in \mathbb{R}^m$ denotes matched uncertainty in Eq. (1) whose Euclidean norm is bounded by a known function $\rho(t)$. Without the loss of generality, it is assumed that rank $(\tilde{B}) = m$ and the matrix pair $(\tilde{A}, \tilde{B})$ are controllable.

In order to provide the problem with a tracking facility, we exploit an integral action as follows. Defining

$$\dot{\xi}(t) = \mathbb{r}(t) - \tilde{y}(t), \tag{2}$$

where $\mathbb{r}(t)$ is the input reference to be tracked by $\tilde{y}(t) = \tilde{C}\tilde{x}(t) \in \mathbb{R}^p$, and ξ represents the integral of the tracking error, that is, $\mathbb{r}(t) - \tilde{y}(t)$, and introducing $x := \begin{bmatrix} \xi \\ \tilde{x} \end{bmatrix} \in \mathbb{R}^n$, an augmented system can be derived as

$$\dot{x}(t) = Ax(t) + B_2 u(t) + B_{\mathbb{r}}\mathbb{r}(t), \tag{3}$$

with

$$A = \begin{bmatrix} 0 & -\tilde{C} \\ 0 & \tilde{A} \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ \tilde{B} \end{bmatrix}, B_{\mathbb{r}} = \begin{bmatrix} I_p \\ 0 \end{bmatrix}. \tag{4}$$

Note that if the matrix pair $(\tilde{A}, \tilde{B})$ is controllable and the matrix triplet $(\tilde{A}, \tilde{B}, \tilde{C})$ has no zeros at the origin, it can be shown that (A, B_2) is controllable [11].

Consider a linear switching surface as

$$\mathcal{S} = \{x : \ \sigma(t) \triangleq Sx(t) = 0\}, \tag{5}$$

where $S \in \mathbb{R}^{m \times n}$ is the full-rank-sliding matrix to be designed later so that the associated reduced-order-sliding motions have suitable dynamics.

Let us consider the following controller:

$$u(t) = -(SB_2)^{-1}(SA - \Phi S)x(t) + \vartheta(t), \tag{6}$$

where $\Phi \in \mathbb{R}^{m \times m}$ is a stable matrix and $\vartheta(t) \in \mathbb{R}^m$ denotes the nonlinear part of the controller with the following form:

$$\vartheta(t) = -(SB_2)^{-1}\rho(t)\frac{\sigma(t)}{\|\sigma(t)\|} \quad \text{if } \sigma(t) \neq 0, \tag{7}$$

in which the scalar function $\rho(\cdot)$ satisfies $\|\rho(t)\| \geq \|SB_2 f(t)\|$; for example, see [2]. It is worth noting that the term $(SB_2)^{-1}\Phi S x(t)$ in the controller Eq. (6) is to govern the convergence rate of the system state to the sliding manifold in association with the nonlinear controller. Further, $-(SB_2)^{-1}SA$ is the so-called equivalent control necessary to maintain sliding in the absence of uncertainty. Here, similar to [3], it is assumed that $\Phi = \lambda I_m$, where $\lambda < 0$ is a given constant value. Note that unlike in [3], λ can also belong to the spectrum of A. Because we set $\Phi = \lambda I_m$, the control law $u(t)$ in Eq. (6) can be written as

$$u(t) = (SB_2)^{-1}SA_\lambda x(t) + \vartheta(t), \tag{8}$$

where $A_\lambda = \lambda I_n - A$. Now assuming that there is no matched uncertainty in Eq. (3) and letting $\rho(\cdot) \to 0$, we can consider that the controller in Eq. (8) contains only the linear part. Hence,

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + B_2 u(t) + B_r \mathbf{r}(t) + B_1 w(t) \\
z_2(t) &= C_2 x(t) + D_2 u(t) \\
z_\infty(t) &= C_\infty x(t) + D_\infty u(t), \\
u(t) &= (SB_2)^{-1}SA_\lambda x(t),
\end{aligned} \tag{9}$$

where $w(t)$ is a fictitious exogenous disturbance, $z_2(t) \in \mathbb{R}^{q_1}$ and $z_\infty(t) \in \mathbb{R}^{q_2}$ are the $\mathcal{H}_2$ performance output vector and $\mathcal{H}_\infty$ performance output vector of the system, respectively. The matrices in Eq. (9) are constant and of appropriate dimensions. Without the loss of generality, it is also assumed that $m \leq q_i \leq n$, $i = 1, 2$. Now, the objective can be regarded as finding a sliding matrix S so that the resulting reduced-order motion, when restricted to $\mathcal{S}$, is stable and meets $\mathcal{H}_2/\mathcal{H}_\infty$ performance specifications. Indeed, we need to choose S, with a given $\lambda < 0$, so that the obtained reduced-order-sliding mode

- guarantees $\|T_{wz_2}\|_2^2 < \delta$, where $\|T_{wz_2}\|_2$ is the $\mathcal{H}_2$ norm of the closed-loop transfer function from $w(t)$ to $z_2(t)$ and $\delta > 0$ is a predetermined closed-loop $\mathcal{H}_2$ performance, and

- minimizes the $\mathcal{H}_\infty$ performance, subject to the above item.

For this purpose, one may resort to solve a $\mathcal{H}_2/\mathcal{H}_\infty$-state-feedback problem and thereby find the switching matrix associated with the derived optimal state-feedback gain (say F). Broadly speaking, this simple scheme may not necessarily result in any solution, unless the obtained state-feedback gain F can ensure that m of the closed-loop poles are exactly located at λ. Hence, in order to design an $\mathcal{H}_2/\mathcal{H}_\infty$-based SMC, we need to address the following two problems:

1. Blend the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem with the eigenstructure assignment method, that is, design a state-feedback F enforcing $\mathcal{H}_2/\mathcal{H}_\infty$ constraints while ensuring that m poles of the closed-loop system are precisely located at λ.

2. Obtain the sliding matrix S associated with the particular state-feedback F, derived in the previous step.

The abovementioned problems are dealt with in the following two sections.

Remark 1. *Note that the linear part of the control law can be considered as*

$$u(t) = [\, F_\mathbb{r} \ \tilde{F}\,] \begin{bmatrix} \xi(t) \\ \tilde{x}(t) \end{bmatrix} \triangleq Fx(t),$$

(10)

where $\tilde{F} \in \mathbb{R}^{m \times \tilde{n}}$ is the state-feedback gain and $F_\mathbb{r} \in \mathbb{R}^{m \times p}$ is the feed-forward gain due to the reference signal $\mathbb{r}(t)$.

The following lemma is recalled from [12], which will be useful in the sequel of this chapter.

Lemma 1 [12]. *The following two statements are equivalent:*

1. $\Psi + S + S^T < 0$.

2. *The following LMI is feasible with respect to U.*

$$\begin{bmatrix} \Psi + P - (U + U^T) & S^T + U^T \\ S + U & -P \end{bmatrix} < 0,$$

where P is a positive definite matrix.

It should be noted that Lemma 1 provides a necessary and sufficient condition. However, while imposing some constraints (e.g., structural constraints) on U, the sufficiency of the lemma is not violated; that is, always (2) $\Rightarrow$ (1).

3. Partial eigenstructure assignment for the design of $\mathcal{H}_2/\mathcal{H}_\infty$-based SMC

3.1. LMI characterizations

We need to consider the state-feedback synthesis with a combination of $\mathcal{H}_2/\mathcal{H}_\infty$ performance specifications. In what follows, to avoid the conservatism introduced by the so-called *quadratic* approach for the design of feedback gains with respect to $\mathcal{H}_2/\mathcal{H}_\infty$ performance specifications, we need to recall the so-called *extended* LMI methods developed for the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems from, for example, [12, 13]. This form of LMI characterization will also be shown to be very useful for the novel SMC of this chapter, as it provides us with the possibility to design a certain partial eigenstructure assignment scheme which can ensure precise locations for some of the closed-loop system poles.

3.1.1. $\mathcal{H}_2$ LMI characterization

The $\mathcal{H}_2$ control synthesis problem, by assuming the control law as $u(t) = Fx(t)$, can be addressed through the following optimization problem [12]:

$$\text{Minimize } \delta \text{ subject to} \tag{MH2}$$

$$\begin{bmatrix} -(G + G^T) & \star & \star & \star \\ AG + B_2Y + X_i & -X_i & \star & \star \\ C_2G + D_2Y & 0 & -\delta I & \star \\ G & 0 & 0 & -X_i \end{bmatrix} < 0, \tag{11}$$

$$\begin{bmatrix} -Z & \star \\ B_1 & -X_i \end{bmatrix} < 0, \tag{12}$$

$$\text{trace}(Z) < 1, \tag{13}$$

with respect to decision variables X_i, $i = 1, \cdots, \mathcal{N}$, Z, Y, and G, where X_i and Z are s.p.d matrices. $\mathcal{N}$ hereafter denotes the number of constraints and thus the independent Lyapunov variables. As $G + G^T > 0$, G will be invertible and the state feedback is obtained as $F = YG^{-1}$.

3.1.2. $\mathcal{H}_\infty$ LMI characterization

Given scalar $0 < v \ll 1$, the $\mathcal{H}_\infty$ problem, by assuming the control law as $u(t) = Fx(t)$, can be set as the following minimization problem [13].

$$\text{Minimize } \gamma \text{ subject to} \tag{MHI}$$

$$\begin{bmatrix} X_i - (G + G^T) & \star & \star & \star \\ G + v(AG + B_2Y) & -X_i & \star & \star \\ C_\infty G + D_\infty Y & 0 & -v^{-1}I & \star \\ 0 & B_1 & 0 & -\gamma v^{-1}I \end{bmatrix} < 0, \tag{14}$$

with respect to decision variables $X_i > 0$, $i = 1, \cdots, \mathcal{N}$, Y, G, and $\gamma > 0$. Again, the state feedback is obtained as $F = YG^{-1}$.

Remark 2. *It is worth mentioning that the advantage of both LMIs (11) and (14) lies in the fact that the product terms between the matrix A and the Lyapunov matrices X_i disappear which is particularly useful for a wide range of applications such as mixed $\mathcal{H}_2/\mathcal{H}_\infty$-feedback gain design and cases where the system matrices belong to a given polytopic region. Besides, as seen from Eqs. (11) and (14), the controller is not dependant on the Lyapunov matrix, but rather the new introduced matrix variable G.*

3.1.3. Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ state feedback using improved LMI characterizations

An interesting application of the mentioned so-called *extended* LMI characterizations for $\mathcal{H}_2$ and $\mathcal{H}_\infty$ is the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ state-feedback problem. The aim is to design-feedback gains such that they

- ensure the $\mathcal{H}_2$ performance which means that for a prescribed closed-loop $\mathcal{H}_2$ performance $\delta > 0$, we have $\| T_{wz_2} \|_2^2 < \delta$;

- minimize the $\mathcal{H}_\infty$ performance, subject to the above constraint.

This problem can be formulated through an LMI program in decision variables $X_i > 0$, $i = 1, \cdots, \mathcal{N}$, $Z > 0$, Y, G, and $\gamma > 0$:

$$\begin{aligned} &\text{minimize } \ \gamma \\ &\text{subject to Eqs. (11), (12), (13), and (14),} \end{aligned} \qquad \text{(MHH)}$$

where $\delta > 0$ and $0 < \nu \ll 1$ are the given scalars. Notice that another alternative for addressing the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ state-feedback problem is the so-called *quadratic approach* (see, e.g., [14]), which is a well-known scheme to address the nonlinearity involved in the matrix inequalities by using a common Lyapunov matrix for all the involved objectives. However, this scheme introduces a significant conservatism to the problem in most of the practical cases. The other alternatives, such as (MHH), which are more computationally expensive, have been basically considered in the literature in order to reduce the conservatism of the quadratic approach.

Remark 3. *Another alternative for the mixed control problem is to design a feedback gain that minimizes the $\mathcal{H}_2$ norm of one channel while satisfying an $\mathcal{H}_\infty$-norm constraint on the same or another channel; see, for example, [15]. Hence, in this case, the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem, given $\gamma > 0$ and $0 < \nu \ll 1$, can be set as follows:*

$$\begin{aligned} &\text{minimize } \ \delta \\ &\text{subject to (11), (12), (13) and (14),} \end{aligned} \qquad \text{(MHHN)}$$

where $X_i > 0$, $i = 1, \cdots, \mathcal{N}$, $Z > 0$, Y, G, and $\delta > 0$ are decision variables.

3.2. Partial eigenstructure assignment problem

Locating exactly m poles at a specific location can fortunately be performed through the LMI characterization presented in the previous section. Our specific partial assignment of the set of eigenvalues

$$\{ \overbrace{\lambda, \cdots, \lambda}^{m \text{ times}} \}, \qquad (15)$$

by state feedback can be implemented in two steps:

1. compute the base $\begin{bmatrix} M_\lambda \\ N_\lambda \end{bmatrix}$ of null space of $[A - \lambda I \ B_2]$ with conformable partitioning;

2. with arbitrary $\eta_i \in \mathbb{R}^m$, $i = 1, \cdots, m$, the state feedback will be obtained as $F = YG^{-1}$ with

$$Y = N\Sigma_N, \ G = M\Sigma_M, \qquad (16)$$

in which

$$
\begin{aligned}
N &:= [\overbrace{N_\lambda, \cdots, N_\lambda}^{m \text{ times}}, \overbrace{I, \cdots, I}^{(n-m) \text{ times}}], \\
M &:= [\overbrace{M_\lambda, \cdots, M_\lambda}^{m \text{ times}}, \overbrace{I, \cdots, I}^{(n-m) \text{ times}}], \\
\Sigma_N &:= \text{diag}(\eta_1, \cdots, \eta_m, \kappa_1, \cdots, \kappa_{(n-m)}), \\
\Sigma_M &:= \text{diag}(\eta_1, \cdots, \eta_m, \iota_1, \cdots, \iota_{(n-m)})
\end{aligned}
\tag{17}
$$

with $\kappa_i \in \mathbb{R}^n$ and $\iota_i \in \mathbb{R}^n$. Note that only vectors η_k are related to the assignment of the m eigenvalues to λ. In other words, other vectors (κ_k and ι_k) are not exploited in the pole-placement purposes and thereby can be employed to meet other Lyapunov-type constraints.

Now, provided by the LMI characterization, for example, (MHH), the first step of our $\mathcal{H}_2/\mathcal{H}_\infty$-based SMC design can be set as an LMI program in the variables $X_i > 0$, $i = 1, \cdots, \mathcal{N}$, Σ_M, Σ_N, and $\gamma > 0$, by recasting (MHH) as:

$$
\begin{aligned}
&\text{minimize } \gamma \\
&\text{subject to } (11), (12), (13), (14), \text{ and } (16).
\end{aligned}
\tag{MHH2}
$$

However, we have not yet shown that the set of closed-loop eigenvalues encompasses Eq. (15). This is the subject of the following lemma.

Lemma 2. *Set* (15) *is a subset of the closed-loop system eigenvalues, acquired by applying the state feedback $F = YG^{-1}$ with Y and G presented in Eq.* (16), *to the system in Eq.* (3) *in the absence of uncertainty, that is,* $f = 0$.

Proof. Using Eq. (16), we can write

$$
\begin{aligned}
&(A + B_2 F) M_\lambda \eta_i \\
&= \left[A + B_2 (N\Sigma_N)(M\Sigma_M)^{-1} \right] M_\lambda \eta_i \\
&= \left[A + B_2 (N\Sigma_N)(M\Sigma_M)^{-1} \right] (M\Sigma_M) e_i \\
&= [A(M\Sigma_M) + B_2(N\Sigma_N)] e_i \\
&= A M_\lambda \eta_i + B_2 N_\lambda \eta_i \\
&= \lambda M_\lambda \eta_i \qquad\qquad\qquad i = 1, \cdots, m.
\end{aligned}
$$

Note that e_k here denotes the canonical basis of $\mathbb{R}^n$.   □

4. Deriving the switching-function matrix

This subsection proposes two approaches to obtain the sliding matrix S associated with the state feedback F, derived in the previous subsection based on the partial eigenstructure assignment scheme.

4.1. Approach 1

The first approach is represented in the following theorem.

Theorem 1. *Let (A, B_2) be a controllable matrix pair. Then*

i. $\forall \lambda \in \mathbb{R}_-$, *there always exists a gain matrix F so that m of the eigenvalues of $A + B_2 F$ are equivalent to λ, and $A + B_2 F$ has m-independent eigenvectors associated with λ.*

ii. *Define $S = [v_1, \cdots, v_m]^T$, where v_i is a left eigenvector of $A + B_2 F$ associated with the eigenvalue λ, then, $S(A_\lambda - B_2 F) = 0$ and SB_2 is invertible.*

Proof. (i) As (A, B_2) is controllable, we can claim that $(\lambda I - A, B_2)$ is also controllable for any $\lambda \in \mathbb{R}_-$. Then, it is easy to see that we can always find F such that the null space of $A_\lambda - B_2 F$ has dimension m, which implies that $A + B_2 F$ has m-independent eigenvectors associated with λ.

(ii) Define $S = [v_1, \cdots, v_m]^T$, it is easy to show that $S(A_\lambda - B_2 F) = 0$. Now, assume

$$SB_2 := \begin{bmatrix} v_1^T \\ \vdots \\ v_m^T \end{bmatrix} B_2 = \Omega,$$

where $\Omega \in \mathbb{R}^{m \times m}$. If Ω is not full rank, then there exists a nonsingular matrix Λ such that the first row of $\Lambda\Omega$ is zero. This is equivalent to

$$\Lambda \begin{bmatrix} v_1^T \\ \vdots \\ v_m^T \end{bmatrix} B_2 := \begin{bmatrix} \tilde{v}_1^T \\ \vdots \\ \tilde{v}_m^T \end{bmatrix} B_2 = \Lambda\Omega,$$

that is, there exists a vector $\tilde{v}_1$ such that $\tilde{v}_1^T B_2 = 0$. On the other hand, we know $\tilde{v}_1^T [A_\lambda - B_2 F] = 0$, and so

$$\text{rank}\left([\lambda I - (A + B_2 F) \ \ B_2] \right) < n.$$

This is clearly in contradiction with the controllability of (A, B_2). In other words, if we can find a left eigenvector of $A + B_2 F$ associated with λ that is orthogonal to B_2, (A, B_2) must be uncontrollable, which is obviously a contradiction.

In brief, by virtue of Theorem 1, the switching-function matrix S, associated with the state feedback F, obtained through solving the LMI problem in (MHH2), can be selected as the set of m linearly independent left eigenvectors of $A + B_2 F$ associated with the (arbitrarily selected) repeated eigenvalue $\lambda \in \mathbb{R}_-$.

4.2. Approach 2

An alternative approach to obtain the sliding matrix is to address the equality

$$(SB_2)^{-1} SA_\lambda = F, \tag{18}$$

utilizing an LMI optimization method as follows. As the matrix S must ensure the invertibility of SB_2, let us suppose $S = B_2^T P$, with P an s.p.d matrix which will be obtained hereafter. The condition in Eq. (18) can be dealt with a simple relaxation method as

$$\text{minimize } \alpha \text{ subject to } \| B_2^T P(A_\lambda - B_2 F) \| < \alpha,$$

where $\alpha > 0$ is a scalar variable and F is a given state-feedback matrix, obtained in the previous subsection, ensuring m of the closed-loop eigenvalues are equal to λ. Simply it can be shown that the above problem is equivalent to the following LMI minimization problem:

$$\text{minimize } \alpha \text{ subject to } \begin{bmatrix} -\alpha I & \star \\ B_2^T P(A_\lambda - B_2 F) & -\alpha I \end{bmatrix} < 0. \tag{19}$$

Hence, the $\mathcal{H}_2/\mathcal{H}_\infty$-based SMC problem is now to find the global solution of the above minimization problem, and then the sliding matrix is obtained as $S = B_2^T P$. In the case of feasibility, this problem will enforce α so that it is an extremely small number associated with the precision of the computational unit.

Notice that this approach is numerically very efficient and attractive compared to the first approach.

5. The summary of $\mathcal{H}_2/\mathcal{H}_\infty$-based SMC design method

Now, we summarize the proposed $\mathcal{H}_2/\mathcal{H}_\infty$-based SMC in the following theorem.

Theorem 2. *Assume that the optimization problem in* (MHH2) *has a solution F for some $\delta > 0$ and $\gamma > 0$. Then, the $\mathcal{H}_2/\mathcal{H}_\infty$ performance constraints $\| T_{wz_2} \|_2^2 < \delta$ and $\| T_{wz_\infty} \|_\infty^2 < \gamma$ are ensured, and after the reaching time t_s, the resulting reduced $n - m$-order-sliding mode dynamics, obtained by applying the following control law:*

$$u(t) = Fx(t) + \vartheta(t), \tag{20}$$

where $\vartheta(t)$ is introduced in Eq. (7), *to system* (3), *is asymptotically stable.*

Proof. Consider a change of coordinates $x \mapsto T_r x$. In this new coordinate system, the new matrix pair $(\overline{A}, \overline{B}_2)$ is of the form

$$\overline{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \overline{B}_2 = \begin{bmatrix} 0 \\ B_p \end{bmatrix} \tag{21}$$

where the square matrix $B_p \in \mathbb{R}^{m \times m}$ has full rank and more importantly is nonsingular; see [1]. Suppose also that $\overline{F}$ is the state feedback in the new coordinate that ensures the closed-loop stability, assigns m poles of the closed-loop system at λ, and satisfies the $\mathcal{H}_2/\mathcal{H}_\infty$ performance constraints $\| T_{wz_2} \|_2^2 < \delta$ and $\| T_{wz_\infty} \|_\infty^2 < \gamma$. Now, let the switching-function matrix in the original coordinates be parameterized such that [1]

$$S = S_2[-\mathcal{M} \quad I_m]T_r, \tag{22}$$

where $S_2 \in \mathbb{R}^{m \times m}$. Notice that theoretically the choice of S_2 may not influence the sliding motion [1]. According to the discussion given in the previous section, it can be readily shown that there exists a matrix $\mathcal{M}$ such that $\overline{F} = (\overline{S}\,\overline{B}_2)^{-1}\overline{S}(\overline{A} - \lambda I_n)$, where $\overline{S} = S_2[-\mathcal{M} \quad I_m]$ denotes the switching-function matrix in the new coordinate. Let $(\overline{x}_1, \overline{x}_2)$ be the partition of the system states associated with the certain system coordinates in Eq. (21), then it can be shown that while the system states are confined to the sliding manifold, that is, $\sigma = 0$, the reduced-order-sliding mode dynamics are governed by the stable reduced-order system matrix $A_{11} + A_{12}\mathcal{M}$.

Moreover, the dynamics of σ can be derived by taking the time derivative of Eq. (5), substituting in the state equation (3), and using controllers (8) and (7), that is,

$$\dot{\sigma}(t) = \lambda\sigma(t) - \rho(t)\frac{\sigma(t)}{\|\sigma(t)\|} + SB_2 f(t). \tag{23}$$

Finally, it follows from $\|SB_2 f(t)\| \leq \rho(t)$ that the reachability condition $\frac{\sigma^T\dot{\sigma}}{\|\sigma\|} < 0$ holds.   □

6. Design of SMC with additional regional pole-placement constraints

Note that the proposed method here offers the advantage of introducing additional convex constraints on the closed-loop dynamics. By locating the closed-loop system poles in a preselected region, an adequate transient response for system trajectories can be guaranteed [14]. Therefore, the objective is to augment the optimization problems previously described by pole-clustering constraints. Note also that as it is already ensured that m of the closed-loop eigenvalues are exactly assigned to a given negative real value (λ), the remaining eigenvalues in fact belong to the spectrum of the reduced $n - m$-order-sliding motion. As a result, a satisfactory transient response for the sliding motion can be achieved by clustering the poles governing the sliding motion.

Let us have a brief introduction to the LMI region. Simply, an LMI region is a subset $\mathcal{D}$ of the complex plane as

$$\mathcal{D} := \{z \in \mathbb{C} : f_{\mathcal{D}}(z) \triangleq \Xi + z\Pi + \overline{z}\Pi^T < 0\} \tag{24}$$

in which $\Xi = \Xi^T \in \mathbb{R}^{\xi \times \xi}$ and $\Pi \in \mathbb{R}^{\xi \times \xi}$ are real matrices. $f_{\mathcal{D}}(z)$ is also called the characteristic equation of the region $\mathcal{D}$.

Definition 1 [16]. *A real matrix $\mathcal{A}$ is said to be $\mathcal{D}$-stable if all its eigenvalues lie within the LMI region $\mathcal{D}$.*

Lemma 3 [16]. *A real matrix $\mathcal{A}$ is said to be $\mathcal{D}$-stable if a symmetric matrix $X_{\mathcal{D}} > 0$ exists so that*

$$\Xi \otimes X_{\mathcal{D}} + \Pi \otimes (X_{\mathcal{D}}\mathcal{A}) + \Pi^T \otimes (\mathcal{A}^T X_{\mathcal{D}}) < 0, \tag{25}$$

where $\otimes$ denotes the Kronecker product.

However, the synthesis problem obtained by imposing the pole-clustering constraints presented in, for example, [14] or [16] to the synthesis problem in (MHH2) would not result in a convex problem. Alternatively, the regional pole-clustering constraints can be reformulated so that the product term between the Lyapunov matrix X_i and the system matrix A is removed.

An instrumental theorem is represented first and the main theorem will be presented later in Theorem 4.

Theorem 3. *Let $\mathcal{A}$ be a real matrix. The following statements are equivalent, with s.p.d X, $\mathcal{G}$ and given real matrices $0 < \Xi \in \mathbb{R}^{\xi \times \xi}$ and $\Pi \in \mathbb{R}^{\xi \times \xi}$.*

1. *$\mathcal{A}$ is $\mathcal{D}$-stable, where $\mathcal{D}$ is given in Eq. (24).*

2. *$\exists\, X$ such that*

$$\begin{bmatrix} \Xi \otimes X + \Pi \otimes (X\mathcal{A}) + \Pi^T \otimes (X\mathcal{A})^T & \star \\ 0 & -X \end{bmatrix} < 0.$$

3. *$\exists\, X > 0$ and $\mathcal{G}$ such that*

$$\begin{bmatrix} -(\mathcal{G} + \mathcal{G}^T) & \star & \star & \star \\ (\Pi \otimes \mathcal{A})\mathcal{G} + I_\xi \otimes X & -I_\xi \otimes X & \star & \star \\ \mathcal{G} & 0 & -I_\xi \otimes X & \star \\ \mathcal{G} & 0 & 0 & -\Xi^{-1} \otimes X \end{bmatrix} < 0. \tag{26}$$

Proof. Refer to the Appendix.   □

While Eq. (26) can be seen as a necessary and sufficient condition for $\mathcal{D}$-stability, it is not very useful in terms of control synthesis purposes. Further, since $\Xi = 0$, the result of Theorem 3 cannot cover the standard continuous-time systems stability. However, if we let $\mathcal{G} = I_\xi \otimes G$ in Eq. (26), a sufficient condition is achieved which is beneficial for the control synthesis purposes.

Theorem 4. *Let $\mathcal{A}$, $0 \le \Xi \in \mathbb{R}^{\xi \times \xi}$, and $\Pi \in \mathbb{R}^{\xi \times \xi}$ be real matrices. $\mathcal{A}$ is $\mathcal{D}$-stable if*

$$\begin{bmatrix} -I_\xi \otimes (G + G^T) & \star & \star & \star \\ \Pi \otimes (\mathcal{A}G) + I_\xi \otimes X & -I_\xi \otimes X & \star & \star \\ I_\xi \otimes G & 0 & -I_\xi \otimes X & \star \\ \Xi^{\frac{1}{2}} \otimes G & 0 & 0 & -I_\xi \otimes X \end{bmatrix} < 0, \tag{27}$$

where G is a general matrix and X is an s.p.d matrix.

Proof. The proof can be performed similar to the proof of Theorem 3 by letting $\mathcal{G} = I_\xi \otimes G$.   □

Clearly, the above theorem does not require $\Xi > 0$, but $\Xi \ge 0$. This is indeed a generalization of the extended Lyapunov theorem presented in Theorem 3.1 of [12], and the usual stability region can be obtained by letting $\Xi = 0$ and $\Pi = 1$ in Eq. (27):

$$\begin{bmatrix} -(G+G^T) & \star & \star \\ \mathcal{A}G+X & -X & \star \\ G & 0 & -X \end{bmatrix} < 0. \tag{28}$$

Moreover, the equivalence of Eq. (28) to the standard Lyapunov stability inequality for continuous-time linear systems is presented in [12].

Specifically, let us confine the closed-loop poles to the region r}E512(α, r, θ) (see [14]) which can ensure a minimum decay rate α, a minimum damping ratio $\zeta = \cos\theta$, and a maximum undamped natural frequency $\omega_d = r\sin\theta$. The LMI region for an αstability, that is, $\mathrm{Re}(z) < -\alpha$, can be obtained through Eq. (27), with $\Xi = 2\alpha$, $\Pi = 1$, $\mathcal{A}A + B_2F$, and XX_i. Moreover, by letting $\Xi = 0$ and $\Pi = \begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix}$, the LMI region for a conic sector r}E512$(0,0,\theta)$ is achieved. Eventually, a disk centered at the origin with radius r corresponds to

$$\Xi = \begin{bmatrix} -r & 0 \\ 0 & -r \end{bmatrix}, \Pi = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{29}$$

However, for this special pole-clustering constraint, as Ξ is not a semi-positive definite matrix, the LMI region cannot be obtained through Eq. (27). We can alternatively state the following theorem.

Theorem 5. *Let $\mathcal{A}$ be a real matrix. The following conditions are equivalent:*

1. *The eigenvalues of $\mathcal{A}$ lie in a disk centered at the origin with radius r.*

2. *There exists a symmetric matrix $X > 0$ such that*

$$\frac{1}{r}\mathcal{A}X\mathcal{A}^T - rX < 0. \tag{30}$$

3. *There exists a symmetric matrix $X > 0$ such that*

$$\begin{bmatrix} -rX & \star \\ X\mathcal{A}^T & -rX \end{bmatrix} < 0. \tag{31}$$

4. *There exist a symmetric matrix $X > 0$, and a matrix G such that*

$$\begin{bmatrix} -rX & \star \\ G^T\mathcal{A}^T & -(G+G^T)+\frac{1}{r}X \end{bmatrix} < 0. \tag{32}$$

Proof. Refer to the Appendix.   □

Notice that the above theorem with $r = 1$ reduces to the standard Lyapunov stability inequality for discrete-time linear systems and its extended (robust) version; for example, see [17]. Now, the extended LMI region for a disk centered at the origin with radius r is as follows:

$$\begin{bmatrix} -rX_i & \star \\ (AG+B_2Y)^T & -(G+G^T)+\frac{1}{r}X_i \end{bmatrix} < 0, \tag{33}$$

which is obtained by replacing $\mathcal{A}:=A + B_2F$, $X:=X_i$ in Eq. (32) and introducing $Y = FG$.

Remark 4. *Exploiting a common G may also lead to conservatism compared with the methods, for example, in [18]. However, the methods in the aforementioned references are not beneficial for the control synthesis aims, unless gain-scheduled controllers [19] are considered. Moreover, by employing two instrumental variables, a different sufficient condition for robust $\mathcal{D}$ stability has been developed in [20] which is not applicable to the continuous-time control synthesis purposes. Nevertheless, the approach here can achieve less conservative results through employing non-common Lyapunov variables for every involved specification.*

7. Numerical examples

This section evaluates the effectiveness of the proposed theory using a numerical example. Consider a two-input, two-output, fourth-order plant describing the motion of a Boeing B-747 aircraft obtained by linearization around an operating condition of 20,000 ft. altitude with a speed of Mach 0.8 [21]. The system matrices are as follows:

$$\tilde{A} = \begin{bmatrix} -0.1196 & 0.0004 & -1.0001 & 0.0383 \\ -4.1195 & -0.9743 & 0.2919 & -0.0004 \\ 1.6204 & -0.0161 & -0.2320 & -0.0001 \\ 0.0007 & 1.0054 & 0.0003 & 0.0003 \end{bmatrix},$$

$$\tilde{B} = \begin{bmatrix} -0.0004 & 0.0126 \\ 0.3103 & 0.1832 \\ 0.0124 & -0.9219 \\ -0.0001 & -0.0002 \end{bmatrix}, \tilde{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and the system state, output, and input vectors are

$$\tilde{x}(t) = \begin{bmatrix} \beta(t) & p(t) & r(t) & \phi(t) \end{bmatrix}^T,$$
$$\tilde{y}(t) = \begin{bmatrix} \beta(t) & \phi(t) \end{bmatrix}^T,$$
$$u(t) = \begin{bmatrix} \delta_a(t) & \delta_r(t) \end{bmatrix}^T.$$

where $\beta(t)$, $p(t)$, $r(t)$, $\phi(t)$, $\delta_a(t)$, and $\delta_r(t)$ denote the sideslip angle, the roll rate, the yaw rate, the roll angle, the aileron deflection, and the rudder deflection, respectively.

We also let

$$C_2 = \begin{bmatrix} \text{diag}(0.1, 0.1, 10, 10, 1, 1) \\ 0_{2 \times 6} \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 0_{6 \times 2} \\ \text{diag}(1, 1) \end{bmatrix},$$

$$B_1 = I_6.$$

Note that the last two nonzero terms of C_2 are associated with the integral action and are less heavily weighted. In addition, the third and fourth terms of C_2 have strongly been weighted in comparison with the fifth and sixth terms to provide an adequate quick closed-loop response

in terms of the angular acceleration in roll and yaw. We also aim to assign the closed-loop poles in the half-plane $x < -\alpha < -0.1$.

We solve the minimization problem in (MH2), with $\lambda = -3$, and the state-feedback gain is obtained as

$$F = \begin{bmatrix} 0.7166 & 68.7237 & 16.0446 & -19.6616 & -4.0591 & -62.2050 \\ 34.5978 & 24.6097 & -23.0976 & 0.7777 & 7.4387 & -5.7251 \end{bmatrix}. \tag{34}$$

Employing the first proposed approach in Section 4, the associated sliding function matrix for the augmented system is

$$S = \begin{bmatrix} -0.7351 & -0.6091 & 0.2907 & 0.0151 & -0.0621 & -0.0066 \\ -0.4278 & 0.6381 & 0.2355 & -0.1309 & -0.0638 & -0.5772 \end{bmatrix}. \tag{35}$$

The sliding motion is governed by the set of poles $\{-2.3205 \pm 3.0365i, -1.9203 \pm 1.2377i\}$, and the associated true value of $\mathcal{H}_2$ cost from w to z is 28.0959. Assuming the matched
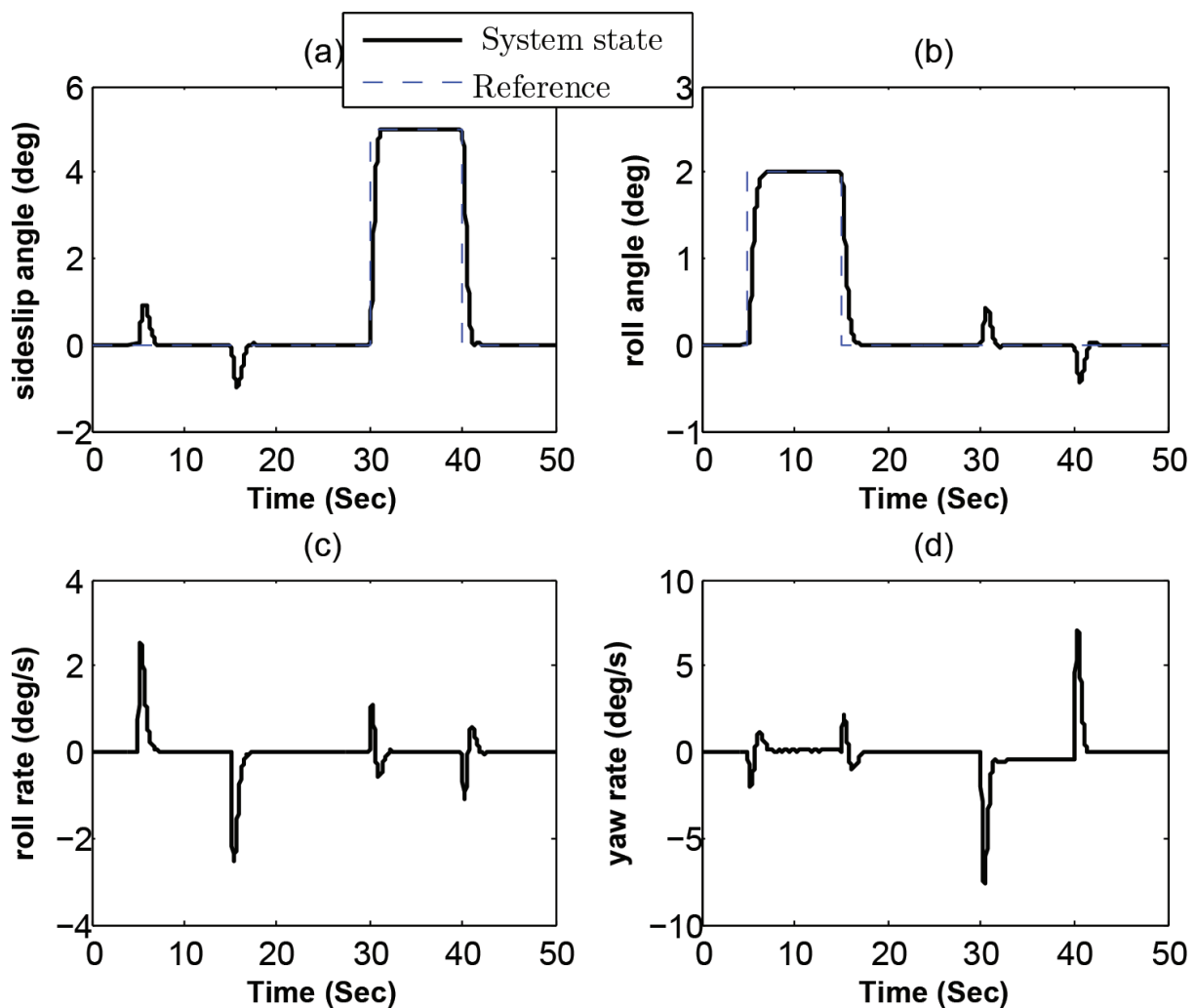

Figure 1. System states.

uncertainty term in Eq. (1) as $f(t) = \begin{bmatrix} 0.2\sin(t)\beta(t) \\ 0.3\sin(t)\phi(t) \end{bmatrix}$, using the proposed SMC with the obtained linear gain F in Eq. (34) and the associated switching-function matrix F in Eq. (35), and letting the switching gain $\rho = 1$, and considering a step of 5° for β during 30–40 s as well as a step of 2° for φ during 5–15 s, **Figures 1–3** show the tracking responses of the system. Note that the discontinuity in the nonlinear control term $\vartheta(t)$ in Eq. (7) is smoothed by using a sigmoidal approximation [11] as

$$\vartheta_\varepsilon(t) = -(SB_2)^{-1}\rho(t)\frac{\sigma(t)}{\varepsilon + \|\sigma(t)\|} \tag{36}$$

with the scalar $\varepsilon = 0.01$ and $\rho(t) = 1$, where this can remove the discontinuity at $\sigma = 0$ and introduce the possibility to accommodate the actuator rate limits.
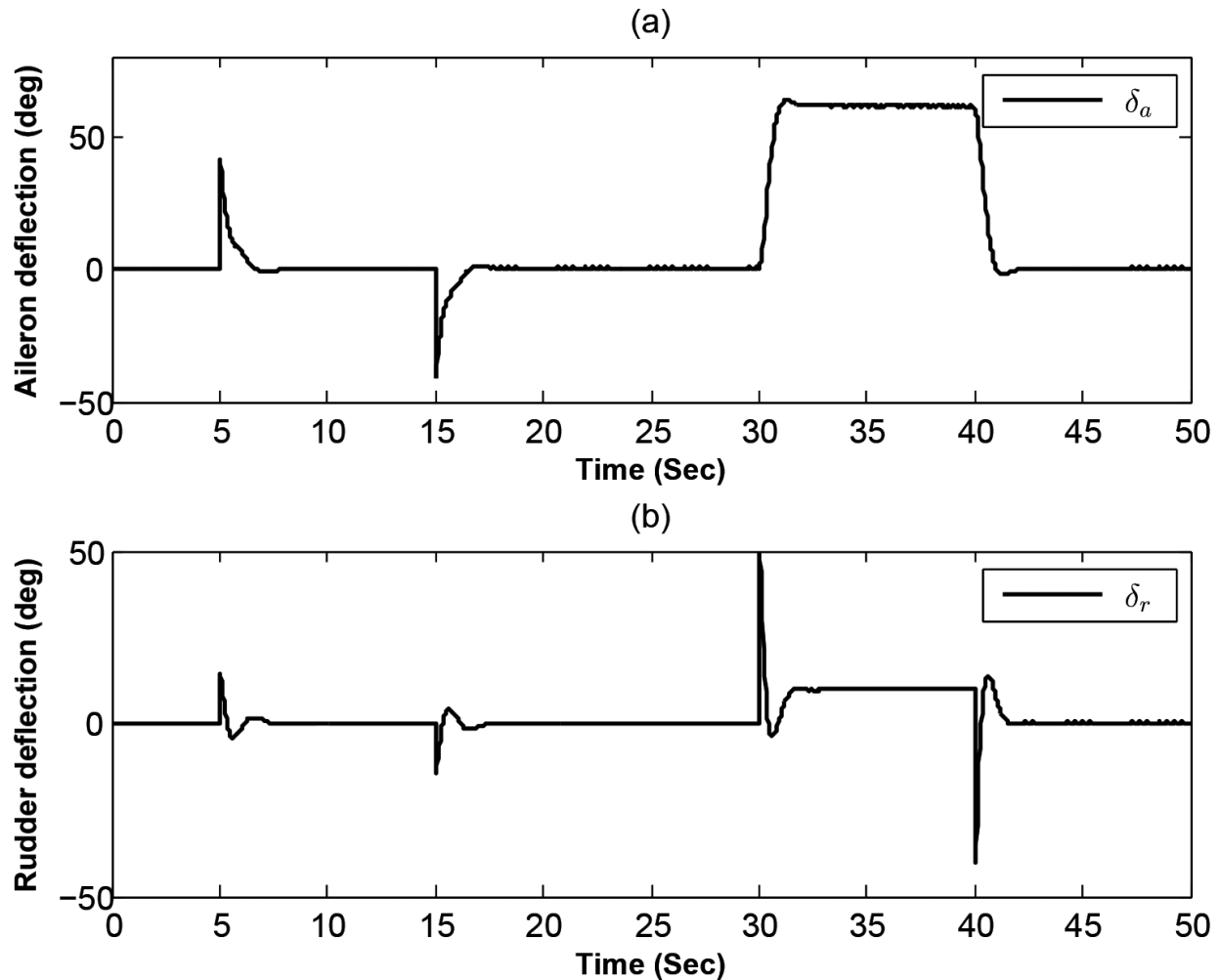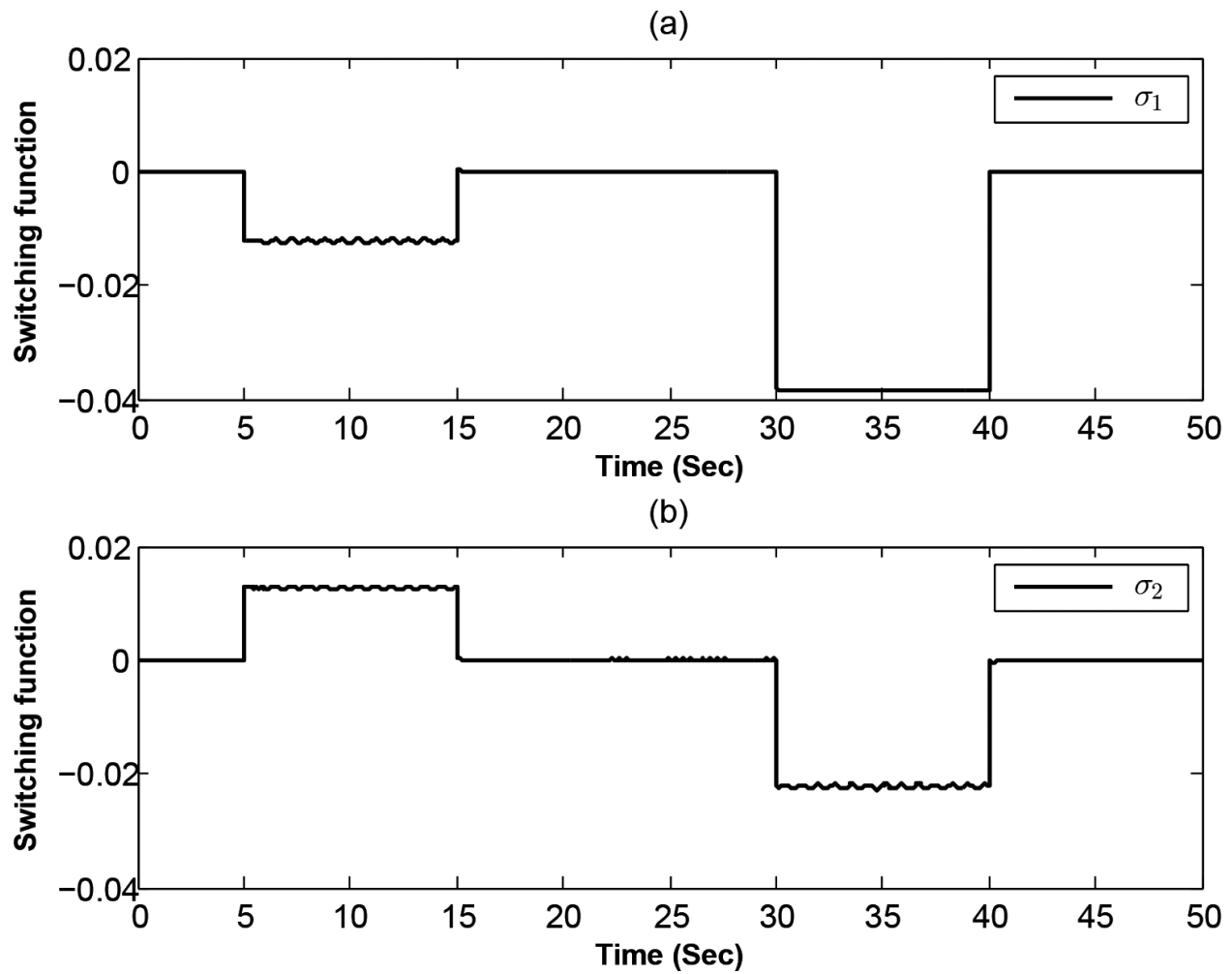

Figure 2. Control efforts.

Figure 3. Switching function.

8. Conclusions

The focus of this chapter was on the development of novel framework for designing a sliding surface for a given system while enforcing a number of Lyapunov-type constraints such as the $\mathcal{H}_2/\mathcal{H}_\infty$ and/or regional pole clustering. We specifically considered the problem of output tracking using a suboptimal state-feedback SMC. In doing so, in the first stage, through a convex optimization approach, a state-feedback gain is designed while assigning a certain number (m) of the closed-loop system eigenvalues to a predetermined value, as well as satisfying $\mathcal{H}_2/\mathcal{H}_\infty$-norm constraints. The advantages of the proposed scheme are threefold: (i) it can set the stage for designing SMC while the level of control efforts is taken into account; (ii) it makes it possible to integrate a number of Lyapunov-type constraints, for example, regional pole-placement constraints, into the SMC design problem; and (iii) the controller can be computed in a numerically very efficient method. The achieved results confirmed the effectiveness of the proposed approach.

A. Proof of Theorem 3

Notice that the equivalence between Eqs. (1) and (2) can be obtained from Lemma 3. We will show the equivalence between Eqs. (2) and (3) here. The use of Lemma 1 with $\Psi = \Xi \otimes \overline{X}$, $U = \overline{\mathcal{G}}$ and $S = \Pi \otimes (\overline{X}\mathcal{A})$, with $\overline{X} = X^{-1}$, yields

$$\begin{bmatrix} P - (\overline{\mathcal{G}} + \overline{\mathcal{G}}^T) + \Xi \otimes \overline{X} & \star \\ \Pi \otimes (\overline{X}\mathcal{A}) + \overline{\mathcal{G}} & -P \end{bmatrix} < 0,$$

or equivalently,

$$\begin{bmatrix} P - (\overline{\mathcal{G}} + \overline{\mathcal{G}}^T) + \Xi \otimes \overline{X} & \star \\ \Pi \otimes \mathcal{A} + (I_\xi \otimes X)\overline{\mathcal{G}} & -(I_\xi \otimes X)P(I_\xi \otimes X) \end{bmatrix} < 0, \tag{37}$$

By performing the congruence transformation $\begin{bmatrix} \mathcal{G} & 0 \\ 0 & I \end{bmatrix}$, with $\mathcal{G} = \overline{\mathcal{G}}^{-1}$, and using the Schur complement, Eq. (37) becomes

$$\begin{bmatrix} -(\mathcal{G} + \mathcal{G}^T) & \star & \star & \star \\ (\Pi \otimes \mathcal{A})\mathcal{G} + I_\xi \otimes X & -(I_\xi \otimes X)P(I_\xi \otimes X) & \star & \star \\ \mathcal{G} & 0 & -P^{-1} & \star \\ \mathcal{G} & 0 & 0 & -\Xi^{-1} \otimes X \end{bmatrix} < 0.$$

The above inequality finally linearizes to Eq. (26) with the choice $P = I_\xi \otimes X^{-1}$.

B. Proof of Theorem 5

The equivalence between Eqs. (1) and (3) is shown in, for example, [14]. Moreover, the equivalence between Eqs. (2) and (3) is simply obtained through applying the Schur complement with respect to the block $(2, 2)$ in Eq. (31). The proof can be followed by noticing that if one applies the Schur complement with respect to the block $(1, 1)$ in Eq. (32), Eq. (30) is recovered by choosing $G = G^T = \frac{1}{r}X > 0$, hence Eq. (2) implies Eq. (4). Also, by left and right multiplying Eq. (32) by $\begin{bmatrix} I & \mathcal{A} \end{bmatrix}$ and $\begin{bmatrix} I & \mathcal{A} \end{bmatrix}^T$, respectively, one can achieve Eq. (30). Hence, Eq. (4) implies Eq. (2), and the proof is completed.

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