Chapter 4

Generalized Ratio Control of Discrete-Time Systems

Dušan Krokavec and Anna Filasová

Additional information is available at the end of the chapter

http://dx.doi.org/10.5772/67159

Abstract

This chapter exposes the important connection between ratio control and the state control reflecting equality constraint for linear discrete-time systems, which allows significant reduction in computational complexity and efforts. Based on an enhanced bounded real lemma form, to outperform known approaches, the existence of the state feedback for such defined singular task is proven, and the design procedure based on the linear matrix inequalities is provided. The proposed principle, guaranteeing feasibility of the set of inequalities, improves steady-state accuracy of the ratio control and essentially reduces the design effort. The approach is illustrated on simulation examples, where the validity of the proposed method is demonstrated.

Keywords: discrete-time systems, ratio control, state feedback, equality constraint, singular systems, linear matrix inequalities

1. Introduction

The problem of the ratio feedback control is one of the specific topics in the theory of control synthesis. It is well practically motivated by applied realizations but not favorable developed in a state control technique or in combination with the state estimation theory. However, a considerable number of problems in the ratio control design have to deal with systems subjected to constraint conditions, which are other than linear, or directly formulated as singular constrained tasks. In the typical case [1, 2] where the system state reflects certain physical entities, constraints usually prescribe the system state, the region of technological conditions. If the ratio control is not formulated as a task with the equality constraints, the application requires further procedures of controlling the evolution of the set-valued ratio. Notably, a special form of the problems can be defined while the system state variables satisfy constraints and interpreted as descriptor systems [3–6]; but, the system with state equality constraints generally does not satisfy the conditions under which the results of descriptor systems can be used. Moreover, if the design task is interpreted as a singular problem, constraint associated methods have to be developed to design the controller.

In principle, it is possible to design the controller that stabilizes a system and simultaneously forces its closed-loop properties to satisfy given constraints [7, 8]. Following the idea of linear quadratic (LQ) control application, these approaches heavily rely on set-valued calculus as well as on min-max theory [9, 10], which are not simple and lead to rather cumbersome technical and numerical procedures. A more simple technique, using equality constraints formulation for discrete-time multiinput/multioutput (MIMO) systems, is introduced in Refs. [11, 12]. Based on the eigenstructure assignment principle, a slight modification of equality constraint technique is presented in Ref. [13].

Many tasks that arise in state-feedback control formulation can be reduced to standard convex problems that involve matrix inequalities. Generally, optimal solutions of such problems can be computed by using the interior point method [14], which converges in polynomial time with respect to the problem size. A review of the progress made in this field can be found in Refs. [15–17] and the references therein. In the given sense, the stability conditions are expressed in terms of linear matrix inequalities (LMI), which have a notable practical interest due to the existence of numerical LMI solvers [18, 19].

The chapter devotes the design conditions to obtain a closed-loop system in which minimally two state variables are rebind by the prescribed ratio. The generalized ratio control principle is reformulated as the full-state feedback control with one equality constraint. Solving this problem, the technique for an enhanced BRL representation [20, 21] is exploited, to circumvent potentially ill-conditioned singular task concerning the discrete-time systems control design with state equality constraints [22]. Due to application of the enhanced BRL, which decouple the Lyapunov matrix and the system matrices, the design task stays well-conditioned. These conditions impose such control that assures asymptotic stability for time-invariant discrete control under defined equality constraints. The presented way, based on projecting the target state variables into a subset of the system state space, adapts the idea of performing the LQ control principle in the fault tolerant control and the constraint control of discrete-time stochastic systems [23, 24].

The outline of this chapter is as follows. Continuing the introduction outlines in Section 1, the problem formulation is principally presented in Section 2. Section 3 is dedicated to the mathematical backgrounds supporting the problem solution and the exploited discrete-time LMI modifications are given in Section 4. These results are used in Section 5 to examine the linearization problems in bilinear matrix inequalities, so that in Section 5, these results can be given with convex formulation of control design condition, guaranteeing a feasible solution of the generally singular design task. Subsequently, numerical examples to illustrate basic properties of the proposed method are presented in Section 6, and Section 7 is finally devoted to a brief concluding remarks.

Throughout the chapter, the following notations are used: $\boldsymbol{x}^T$ and $\boldsymbol{X}^T$ denote the transpose of the vector $\boldsymbol{x}$ and matrix $\boldsymbol{X}$, respectively, for a square matrix $\boldsymbol{X} < 0$ that $\boldsymbol{X}$ is a symmetric negative definite matrix, the symbol $\boldsymbol{I}_n$ represents the nth order unit matrix, $\boldsymbol{Y}^{\ominus 1}$ denotes the Moore-Penrose pseudoinverse of a nonsquare $\boldsymbol{Y}$, $\|\cdot\|$ represents the Euclidean norm for vectors and the spectral norm for matrices, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}^{n \times r}$ the set of all $n \times r$ real matrices.

2. Problem formulation

Through this chapter, the task is concerned with design of the full-state feedback control to discrete-time linear dynamic systems in such a way that the closed-loop system state variables are constrained in the prescribed ratio. The systems are defined by the set of state equations

$$\boldsymbol{q}(i+1) = \boldsymbol{F}\boldsymbol{q}(i) + \boldsymbol{G}\boldsymbol{u}(i), \tag{1}$$

$$\boldsymbol{y}(i) = \boldsymbol{C}\boldsymbol{q}(i), \tag{2}$$

where $\boldsymbol{q}(i) \in \mathbb{R}^n$ is the vector of the state variables, $\boldsymbol{u}(i) \in \mathbb{R}^r$ is the vector of the input variables, $\boldsymbol{y}(i) \in \mathbb{R}^m$ is the vector of the output variables, and nominal system matrices $\boldsymbol{F} \in \mathbb{R}^{n \times n}$, $\boldsymbol{G} \in \mathbb{R}^{n \times r}$, and $\boldsymbol{C} \in \mathbb{R}^{m \times n}$ are real matrices, and $i \in Z_+$.

The discrete transfer function matrix of dimension $m \times r$, associated with the system Eqs. (1) and (2) is defined as

$$\boldsymbol{H}(z) = \boldsymbol{C}(z\boldsymbol{I} - \boldsymbol{F})^{-1}\boldsymbol{G} = \frac{\tilde{\boldsymbol{y}}(z)}{\tilde{\boldsymbol{u}}(z)} \tag{3}$$

where $\boldsymbol{I}_n \in \mathbb{R}^{n \times n}$ is the identity matrix, $\tilde{\boldsymbol{y}}(z)$ and $\tilde{\boldsymbol{u}}(z)$ stand for the Z transform of m dimensional output vector and r dimensional input vector, respectively, and a complex number z is the transform variable of the Z transform [25].

In practice, the ratio control maintains the relationship between two state variables [26, 27] and is defined for all $i \in Z$ as

$$\frac{q_h(i+1)}{q_k(i+1)} = a_h \Rightarrow q_h(i+1) - a_h q_k(i+1) = 0. \tag{4}$$

Assuming the parameter vector $\boldsymbol{e}_h$, the task can be expressed by using the system state vector $\boldsymbol{q}(i+1)$ as

$$\boldsymbol{e}_h^T \boldsymbol{q}(i+1) = 0, \tag{5}$$

where

$$\boldsymbol{e}_h^T = [0_1 \quad \cdots \quad 1_h \quad \cdots \quad -a_h \quad \cdots \quad 0_n], \tag{6}$$

$$\boldsymbol{q}^T(i+1) = \begin{bmatrix} q_1(i+1) & \cdots & q_h(i+1) & \cdots & q_k(i+1) & \cdots & q_n(i+1) \end{bmatrix}. \tag{7}$$

It is evident that the generalized ratio control can be defined by a composed structure of $\mathbf{e}$, as well as by a structured matrix $\boldsymbol{E}$ [28].

The task formulated above means the design problem that can be generally defined as the stable closed-loop system synthesis using the linear full-state feedback controller of the form

$$\boldsymbol{u}(i) = -\boldsymbol{K}\boldsymbol{q}(i), \tag{8}$$

where $\boldsymbol{K} \in \mathbb{R}^{r \times n}$ is the controller feedback gain matrix, and the design constraint is considered in the general matrix equality form

$$\boldsymbol{E}\boldsymbol{q}(i+1) = 0, \tag{9}$$

with $\boldsymbol{E} \in \mathbb{R}^{p \times n}$, rank $\boldsymbol{E} = p \le r$. In general, the matrix $\boldsymbol{E}$ reflects prescribed fixed ratio of two or more state variables. The equality Eq. (9) evidently implies $\boldsymbol{\Lambda}\boldsymbol{E}\boldsymbol{q}(i+1) = 0$, where $\boldsymbol{\Lambda} \in \mathbb{R}^{p \times p}$ is an arbitrary matrix.

It is considered in the following the discrete-time system is controllable and observable that is, $\text{rank}(z\boldsymbol{I} - \boldsymbol{F}, \boldsymbol{G}) = n \;\; \forall z \in \mathcal{C}$ and $\text{rank}\left(z\boldsymbol{I} - \boldsymbol{F}^T, \boldsymbol{C}^T\right) = n \;\; \forall z \in \mathcal{C}$, respectively [29], and that all state variables are measurable.

3. Basic preliminaries

Proposition 1. *(Matrix pseudoinverse) Let* $\boldsymbol{\Theta}$ *is a matrix variable and* $\boldsymbol{A}$, $\boldsymbol{B}$, *and* $\boldsymbol{\Pi}$ *are known nonsquare matrices of appropriate dimensions such that*

$$\boldsymbol{A}\boldsymbol{\Theta}\boldsymbol{B} = \boldsymbol{\Pi}. \tag{10}$$

Then all solution to $\boldsymbol{\Theta}$ *means that*

$$\boldsymbol{\Theta} = \boldsymbol{A}^{\ominus 1}\boldsymbol{\Lambda}\boldsymbol{B}^{\ominus 1} + \boldsymbol{\Theta}^{\circ} - \boldsymbol{A}^{\ominus 1}\boldsymbol{A}\boldsymbol{\Theta}^{\circ}\boldsymbol{B}\boldsymbol{B}^{\ominus 1}, \tag{11}$$

where

$$\boldsymbol{A}^{\ominus 1} = \boldsymbol{A}^T\left(\boldsymbol{A}\boldsymbol{A}^T\right)^{-1}, \quad \boldsymbol{B}^{\ominus 1} = \left(\boldsymbol{B}^T\boldsymbol{B}\right)^{-1}\boldsymbol{B}^T, \tag{12}$$

while $\boldsymbol{A}^{\ominus 1}$ *is the left Moore-Penrose pseudoinverse of* $\boldsymbol{A}$, $\boldsymbol{B}^{\ominus 1}$ *is the right Moore-Penrose pseudoinverse of* $\boldsymbol{B}$ *and* $\boldsymbol{\Theta}^{\circ}$ *is an arbitrary matrix of appropriate dimension.*

Proof. (see, e.g., Ref. [15])

Proposition 2. *Let* $\boldsymbol{\Xi} \in \mathbb{R}^{n \times n}$ *is a real square matrix with nonrepeated eigenvalues, satisfying the equality constraint*

$$\boldsymbol{e}^T\boldsymbol{\Xi} = \boldsymbol{0}, \tag{13}$$

then one from its eigenvalues is zero, and the (normalized) vector $\boldsymbol{e}^T$ *is the left raw eigenvector of* $\boldsymbol{\Xi}$ *associated with the zero eigenvalue.*

Proof. If $\Xi \in \mathbb{R}^{n \times n}$ is a real square matrix satisfying the above given eigenvalues limitation, then the eigenvalue decomposition of Ξ takes the following form

$$\Xi = \boldsymbol{N}\boldsymbol{\Sigma}\boldsymbol{M}^T, \tag{14}$$

$$\boldsymbol{N} = [\boldsymbol{n}_1 \quad \cdots \quad \boldsymbol{n}_n], \quad \boldsymbol{M} = [\boldsymbol{m}_1 \quad \cdots \quad \boldsymbol{m}_n], \quad \boldsymbol{M}^T\boldsymbol{N} = \boldsymbol{I}, \quad \boldsymbol{\Sigma} = \mathrm{diag}[z_1 \quad \cdots \quad z_n], \tag{15}$$

where $\boldsymbol{n}_l$ is the right eigenvector and $\boldsymbol{m}_l^T$ is the left eigenvector associated with the eigenvalue z_l of Ξ, and $\{z_l,\ l = 1, 2, \ldots n\}$ is the set of the eigenvalues of Ξ. Then Eq. (13) can be rewritten as follows:

$$\boldsymbol{0} = \boldsymbol{e}^T[\boldsymbol{n}_1 \quad \cdots \quad \boldsymbol{n}_h \quad \cdots \quad \boldsymbol{n}_n]\mathrm{diag}[z_1 \quad \cdots \quad z_h \quad \cdots \quad z_n]\boldsymbol{M}^T. \tag{16}$$

If $\boldsymbol{e}^T = \boldsymbol{m}_h^T$, then orthogonal property Eq. (15) implies

$$\boldsymbol{0} = [0_1 \quad \cdots \quad 1_h \quad \cdots \quad 0_n]\mathrm{diag}[z_1 \quad \cdots \quad z_h \quad \cdots \quad z_n]\boldsymbol{M}^T \tag{17}$$

and it is evident that Eq. (17) can be satisfied only if $z_h = 0$. This concludes the proof. □

Proposition 3. *(Quadratic performance) Given a stable system of the structure* Eqs. (1) and (2)*, then it yields*

$$\sum_{l=0}^{\infty}\left(\boldsymbol{y}^T(l)\boldsymbol{y}(l) - \gamma_\infty^2\boldsymbol{u}^T(l)\boldsymbol{u}(l)\right) > 0, \tag{18}$$

where $\gamma_\infty \in \mathbb{R}$ is the H_∞ norm of the transfer function matrix of the system Eq. (3).

Proof. Since Eq. (3) implies

$$\tilde{\boldsymbol{y}}(z) = \boldsymbol{H}(z)\tilde{\boldsymbol{u}}(z), \tag{19}$$

then, evidently,

$$||\tilde{\boldsymbol{y}}(z)|| \leq ||\boldsymbol{H}(z)||_2||\tilde{\boldsymbol{u}}(z)||, \tag{20}$$

where $\| \boldsymbol{H}(z) \|$ is H_2 norm of the discrete transfer function matrix $\boldsymbol{H}(z)$.

Since the H_∞ norm property states

$$\frac{1}{\sqrt{m}}\|\boldsymbol{H}(z)\|_\infty \leq \|\boldsymbol{H}(z)\|_2 \leq \sqrt{r}\|\boldsymbol{H}(z)\|_\infty, \tag{21}$$

using the notation $\| \boldsymbol{H}(z) \|_\infty = \gamma_\infty$, then Eq. (21) can be naturally rewritten as

$$\frac{1}{\sqrt{m}} \leq 1 < \frac{1}{\gamma_\infty}\frac{\|\tilde{\boldsymbol{y}}(z)\|}{\|\tilde{\boldsymbol{u}}(z)\|} \leq \frac{1}{\gamma_\infty}\|\boldsymbol{H}(z)\|_2 \leq \sqrt{r}. \tag{22}$$

Thus, based on the Parseval's theorem, Eq. (22) gives

$$1 < \frac{\|\tilde{\boldsymbol{y}}(z)\|}{\gamma_\infty \|\tilde{\boldsymbol{u}}(z)\|} = \frac{\sqrt{\sum_{i=0}^{\infty} \boldsymbol{y}^T(i)\boldsymbol{y}(i)}}{\gamma_\infty \sqrt{\sum_{i=0}^{\infty} \boldsymbol{u}^T(i)\boldsymbol{u}(i)}} \tag{23}$$

and using squares of the elements, the inequality Eq. (23) subsequently results in

$$\sum_{i=0}^{\infty} \boldsymbol{y}^T(i)\boldsymbol{y}(i) - \gamma_\infty^2 \sum_{i=0}^{\infty} \boldsymbol{u}^T(i)\boldsymbol{u}(i) > 0. \tag{24}$$

Thus, Eq. (24) implies Eq. (18). This concludes the proof. □

If it is not in contradiction with other design constraints, Eq. (18) can be used as the extension to a Lyapunov function candidate for linear discrete-time systems, since it is positive.

4. Quadratic performances

The above presented assumptions are imposed to obtain LMI structures exploiting H_∞ norm, known as the bounded real lemma LMIs. To simplify proofs of theorems in following, proof sketches of the BRL are presented, since more versions of BRL can be constructed.

Proposition 4. *(Bounded real lemma) The autonomous system* Eqs. (1) and (2) *is stable with the quadratic performance γ_∞, if there exist a symmetric positive definite matrix $\boldsymbol{P} \in \mathbb{R}^{n \times n}$ and a positive scalar $\gamma_\infty \in \mathbb{R}$ such that*

$$\boldsymbol{P} = \boldsymbol{P}^T > 0, \qquad \gamma_\infty > 0, \tag{25}$$

$$\begin{bmatrix} -\boldsymbol{P} & * & * & * \\ \boldsymbol{F}^T\boldsymbol{P} & -\boldsymbol{P} & * & * \\ \boldsymbol{G}^T\boldsymbol{P} & \boldsymbol{0} & -\gamma_\infty \boldsymbol{I}_r & * \\ \boldsymbol{0} & \boldsymbol{C} & \boldsymbol{0} & -\gamma_\infty \boldsymbol{I}_m \end{bmatrix} < 0, \tag{26}$$

where $\boldsymbol{I}_r \in \mathbb{R}^{r \times r}$ and $\boldsymbol{I}_m \in \mathbb{R}^{m \times m}$ are identity matrices, respectively.

Hereafter, $*$ denotes the symmetric item in a symmetric matrix.

Proof. (compare, e.g., Refs. [16] and [23]) Defining the Lyapunov function candidate as follows:

$$v(\boldsymbol{q}(i)) = \boldsymbol{q}^T(i)\boldsymbol{P}\boldsymbol{q}(i) + \gamma_\infty^{-1} \sum_{l=0}^{i-1} \left(\boldsymbol{y}^T(l)\boldsymbol{y}(l) - \gamma_\infty^2 \boldsymbol{u}^T(l)\boldsymbol{u}(l)\right) > 0, \tag{27}$$

then Eq. (18) implies that with the H_∞ norm γ_∞ of the transform function matrix Eq. (3), the inequality Eq. (27) is positive. The forward difference of Eq. (27) along a solution of the autonomous system Eq. (1) can be written as

$$\begin{aligned}\Delta v(\boldsymbol{q}(i)) &= v(\boldsymbol{q}(i+1)) - v(\boldsymbol{q}(i)) \\ &= \boldsymbol{q}^T(i+1)\boldsymbol{P}\boldsymbol{q}(i+1) - \boldsymbol{q}^T(i)\boldsymbol{P}\boldsymbol{q}(i) + \gamma_\infty^{-1}\boldsymbol{y}^T(i)\boldsymbol{y}(i) - \gamma_\infty\boldsymbol{u}^T(i)\boldsymbol{u}(i) < 0\end{aligned} \tag{28}$$

and, using the description of the state system Eqs. (1) and (2), the inequality Eq. (28) becomes

$$\begin{aligned}\Delta v(\boldsymbol{q}(i)) = \boldsymbol{q}^T(i)\left(\gamma_\infty^{-1}\boldsymbol{C}^T\boldsymbol{C} - \boldsymbol{P} + \boldsymbol{F}^T\boldsymbol{P}\boldsymbol{F}\right)\boldsymbol{q}(i) + \boldsymbol{u}^T(i)\boldsymbol{G}^T\boldsymbol{P}\boldsymbol{F}\boldsymbol{q}(i) \\ +\boldsymbol{q}^T(i)\boldsymbol{F}^T\boldsymbol{P}\boldsymbol{G}\boldsymbol{u}(i) + \boldsymbol{u}^T(i)\left(\boldsymbol{G}^T\boldsymbol{P}\boldsymbol{G} - \gamma_\infty\boldsymbol{I}_r\right)\boldsymbol{u}(i) < 0.\end{aligned} \tag{29}$$

Thus, introducing the notation

$$\boldsymbol{q}_c^T(i) = \begin{bmatrix}\boldsymbol{q}^T(i) & \boldsymbol{u}^T(i)\end{bmatrix}, \tag{30}$$

it is obtained

$$\Delta v\left(\boldsymbol{q}_c(i)\right) = \boldsymbol{q}_c^T(i)\boldsymbol{P}_c\boldsymbol{q}_c(i) < 0, \tag{31}$$

where

$$\boldsymbol{P}_c = \begin{bmatrix}\boldsymbol{F}^T\boldsymbol{P}\boldsymbol{F} + \gamma_\infty^{-1}\boldsymbol{C}^T\boldsymbol{C} - \boldsymbol{P} & \boldsymbol{F}^T\boldsymbol{P}\boldsymbol{G} \\ \boldsymbol{G}^T\boldsymbol{P}\boldsymbol{F} & \boldsymbol{G}^T\boldsymbol{P}\boldsymbol{G} - \gamma_\infty\boldsymbol{I}_r\end{bmatrix} < 0. \tag{32}$$

Since, using the Schur complement property with respect to the matrix element $\gamma_\infty^{-1}\boldsymbol{C}^T\boldsymbol{C}$, Eq. (32) can be rewritten as

$$\boldsymbol{P}_c = \begin{bmatrix}-\boldsymbol{P} & \boldsymbol{0} & \boldsymbol{C}^T \\ \boldsymbol{0} & -\gamma_\infty\boldsymbol{I}_r & \boldsymbol{0} \\ \boldsymbol{C} & \boldsymbol{0} & -\gamma_\infty\boldsymbol{I}_m\end{bmatrix} + \begin{bmatrix}\boldsymbol{F}^T\boldsymbol{P} \\ \boldsymbol{G}^T\boldsymbol{P} \\ \boldsymbol{0}\end{bmatrix}\boldsymbol{P}^{-1}\begin{bmatrix}\boldsymbol{P}\boldsymbol{F} & \boldsymbol{P}\boldsymbol{G} & \boldsymbol{0}\end{bmatrix} < 0, \tag{33}$$

then, applying the dual Schur complement property, Eq. (33) implies Eq. (26). This concludes the proof. □

Direct application of the second Lyapunov method [30, 31] and BRL in the structure given by Eqs. (25) and (26) for affine uncertain systems as well as in constrained control design is in general ill-conditioned owing to singular design conditions [13]. To circumvent this problem, an enhanced LMI representation of BRL is proposed, where design condition proof is based on another form of LMIs.

Proposition 5. *(Enhanced LMI representation of BRL) The autonomous system* Eqs. (1) and (2) *is stable with the quadratic performance γ_∞, if there exist a symmetric positive definite matrix $\boldsymbol{P} \in \mathbb{R}^{n \times n}$, a regular square matrix $\boldsymbol{Q} \in \mathbb{R}^{n \times n}$, and a positive scalar $\gamma_\infty \in \mathbb{R}$ such that*

$$\boldsymbol{P} = \boldsymbol{P}^T > 0, \quad \gamma_\infty > 0, \tag{34}$$

$$\mathbf{Y} = \begin{bmatrix}\boldsymbol{P} - \boldsymbol{Q} - \boldsymbol{Q}^T & * & * & * \\ \boldsymbol{F}^T\boldsymbol{Q}^T & -\boldsymbol{P} & * & * \\ \boldsymbol{G}^T\boldsymbol{Q}^T & \boldsymbol{0} & -\gamma_\infty\boldsymbol{I}_r & * \\ \boldsymbol{0} & \boldsymbol{C} & \boldsymbol{0} & -\gamma_\infty\boldsymbol{I}_m\end{bmatrix} < 0, \tag{35}$$

where $\boldsymbol{I}_r \in \mathbb{R}^{r \times r}$ and $\boldsymbol{I}_m \in \mathbb{R}^{m \times m}$ are identity matrices.

Proof. Since, Eq. (1) can be rewritten as

$$\boldsymbol{F}\boldsymbol{q}(i) + \boldsymbol{G}\boldsymbol{u}(i) - \boldsymbol{q}(i+1) = \boldsymbol{0}, \tag{36}$$

with an arbitrary square matrix $\boldsymbol{Q} \in \mathbb{R}^{n \times n}$, it yields

$$\boldsymbol{q}^T(i+1)\boldsymbol{Q}(\boldsymbol{F}\boldsymbol{q}(i) + \boldsymbol{G}\boldsymbol{u}(i) - \boldsymbol{q}(i+1)) = \boldsymbol{0}. \tag{37}$$

Now, not substituting Eq. (1) into Eq. (28), but adding Eq. (37) and its transposition to Eq. (28), it can be obtained that

$$\begin{aligned}\Delta v(\boldsymbol{q}(i)) = {} & \boldsymbol{q}^T(i+1)\boldsymbol{P}\boldsymbol{q}(i+1) - \boldsymbol{q}^T(i)\boldsymbol{P}\boldsymbol{q}(i) + \gamma_\infty^{-1}\boldsymbol{y}^T(i)\boldsymbol{y}(i) - \gamma_\infty\boldsymbol{u}^T(i)\boldsymbol{u}(i) \\ & + (\boldsymbol{F}\boldsymbol{q}(i) + \boldsymbol{G}\boldsymbol{u}(i) - \boldsymbol{q}(i+1))^T\boldsymbol{Q}^T\boldsymbol{q}(i+1) \\ & + \boldsymbol{q}^T(i+1)\boldsymbol{Q}(\boldsymbol{F}\boldsymbol{q}(i) + \boldsymbol{G}\boldsymbol{u}(i) - \boldsymbol{q}(i+1)) < 0.\end{aligned} \tag{38}$$

Thus, considering Eq. (2), then Eq. (38) can be rewritten as

$$\boldsymbol{q}^{\circ T}(i)\boldsymbol{P}^{\circ}\boldsymbol{q}^{\circ}(i) < 0, \tag{39}$$

where

$$\boldsymbol{q}^{\circ T}(i) = \begin{bmatrix} \boldsymbol{q}^T(i) & \boldsymbol{q}^T(i+1) & \boldsymbol{u}^T(i) \end{bmatrix} \tag{40}$$

and

$$\boldsymbol{P}^{\circ} = \begin{bmatrix} -\boldsymbol{P} + \gamma_\infty^{-1}\boldsymbol{C}^T\boldsymbol{C} & \boldsymbol{F}^T\boldsymbol{Q}^T & \boldsymbol{0} \\ \boldsymbol{Q}\boldsymbol{F} & \boldsymbol{P} - \boldsymbol{Q} - \boldsymbol{Q}^T & \boldsymbol{Q}\boldsymbol{G} \\ \boldsymbol{0} & \boldsymbol{G}^T\boldsymbol{Q}^T & -\gamma_\infty\boldsymbol{I}_r \end{bmatrix} < 0. \tag{41}$$

Since Eq. (41) can be written as

$$\boldsymbol{P}^{\circ} = \begin{bmatrix} -\boldsymbol{P} & \boldsymbol{F}^T\boldsymbol{Q}^T & \boldsymbol{0} \\ \boldsymbol{Q}\boldsymbol{F} & \boldsymbol{P} - \boldsymbol{Q} - \boldsymbol{Q}^T & \boldsymbol{Q}\boldsymbol{G} \\ \boldsymbol{0} & \boldsymbol{G}^T\boldsymbol{Q}^T & -\gamma_\infty\boldsymbol{I}_r \end{bmatrix} + \gamma_\infty^{-1}\begin{bmatrix} \boldsymbol{C}^T \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}\begin{bmatrix} \boldsymbol{C} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} < 0, \tag{42}$$

then, using the dual Schur complement property, Eq. (43) can be transformed in the form

$$\begin{bmatrix} -\gamma_\infty\boldsymbol{I}_m & \boldsymbol{C} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{C}^T & -\boldsymbol{P} & \boldsymbol{F}^T\boldsymbol{Q}^T & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{Q}\boldsymbol{F} & \boldsymbol{P} - \boldsymbol{Q} - \boldsymbol{Q}^T & \boldsymbol{Q}\boldsymbol{G} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{G}^T\boldsymbol{Q}^T & -\gamma_\infty\boldsymbol{I}_r \end{bmatrix} < 0. \tag{43}$$

To obtain a LMI structure visually comparable with Eq. (26), the following block permutation matrix is defined

$$\boldsymbol{T}_a{}^\circ = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \boldsymbol{I}_n & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I}_n & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{I}_r \\ \boldsymbol{I}_m & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{44}$$

Then, premultiplying the left side of Eq. (43) by $\boldsymbol{T}_a{}^\circ$ and postmultiplying the right side of Eq. (43) by the transposition of $\boldsymbol{T}_a{}^\circ$ lead to the inequality in Eq. (35). This concludes the proof.□

It is evident that Lyapunov matrix $\boldsymbol{P}$ is separated from the matrix parameters of the system $\boldsymbol{F}$, $\boldsymbol{G}$, and $\boldsymbol{C}$, i.e., there are no terms containing the product of $\boldsymbol{P}$ and any of them. By introducing the slack variable matrix $\boldsymbol{Q}$, the product forms are relaxed to new products $\boldsymbol{QF}$ and $\boldsymbol{QG}$, where $\boldsymbol{Q}$ needs not be symmetric and positive definite. This enables a robust BRL, which can be obtained to deal with linear systems with parametric uncertainties, as well as with singular system matrices.

Considering a symmetric positive definite matrix $\boldsymbol{Q} \in \mathbb{R}^{n \times n}$, the following symmetric enhanced LMI representation of BRL is evidently obtained.

Corollary 1. *(Enhanced symmetric LMI representation of BRL) The autonomous system* Eqs. (1) and (2) *is stable with the quadratic performance γ_∞, if there exist symmetric positive definite matrices $\boldsymbol{P}, \boldsymbol{Q} \in \mathbb{R}^{n \times n}$ and a positive scalar $\gamma_\infty \in \mathbb{R}$ such that*

$$\boldsymbol{P} = \boldsymbol{P}^T > 0, \quad \boldsymbol{Q} = \boldsymbol{Q}^T > 0, \quad \gamma_\infty > 0, \tag{45}$$

$$\begin{bmatrix} \boldsymbol{P} - 2\boldsymbol{Q} & * & * & * \\ \boldsymbol{F}^T\boldsymbol{Q} & -\boldsymbol{P} & * & * \\ \boldsymbol{G}^T\boldsymbol{Q} & \mathbf{0} & -\gamma_\infty \boldsymbol{I}_r & * \\ \mathbf{0} & \boldsymbol{C} & \mathbf{0} & -\gamma_\infty \boldsymbol{I}_m \end{bmatrix} < 0, \tag{46}$$

where $\boldsymbol{I}_r \in \mathbb{R}^{r \times r}$, $\boldsymbol{I}_m \in \mathbb{R}^{m \times m}$ are identity matrices.

Note, Corollary 1 provides the identical condition of existence to Proposition 4, if the equality $\boldsymbol{P} = \boldsymbol{Q}$ is set.

5. Control law parameter design

The state-feedback control problem is finding, for an optimized (or prescribed) scalar $\gamma > 0$, the state-feedback gain $\boldsymbol{K}$ such that the control law guarantees an upper bound of γ_∞ of the closed-loop transfer function, while the closed-loop is stable. Note, all the above presented BRL structures applied in the control law synthesis lead to bilinear matrix inequalities and have to be linearized.

Theorem 1. *System* Eqs. (1) and (2) *under control* Eq. (3) *is stable with quadratic performance γ_∞, if there exist a positive definite symmetric matrix $\boldsymbol{R} \in \mathbb{R}^{n \times n}$, a matrix $\boldsymbol{Y} \in \mathbb{R}^{r \times n}$, and a positive scalar $\gamma_\infty \in \mathbb{R}$ such that*

$$R = R^T > 0, \qquad \gamma_\infty > 0, \tag{47}$$

$$\begin{bmatrix} -R & * & * & * \\ RF^T - Y^T G^T & -R & * & * \\ G^T & \mathbf{0} & -\gamma_\infty I_r & * \\ \mathbf{0} & CR & \mathbf{0} & -\gamma_\infty I_m \end{bmatrix} < 0. \tag{48}$$

When these inequalities are satisfied, the control law gain matrix is given as

$$K = YR^{-1}. \tag{49}$$

Proof. Since P is positive definite, the transform matrix T_∞ can be defined as follows:

$$T_\infty = \mathrm{diag}[R \quad R \quad I_r \quad I_m], \quad R = P^{-1}. \tag{50}$$

Then, premultiplying the left side of Eq. (35) and postmultiplying the right side of Eq. (35) by T_∞ gives

$$\begin{bmatrix} -R & FR & G & \mathbf{0} \\ RF^T & -R & \mathbf{0} & RC^T \\ G^T & \mathbf{0} & -\gamma_\infty I_r & \mathbf{0} \\ \mathbf{0} & CR & \mathbf{0} & -\gamma_\infty I_m \end{bmatrix} < 0. \tag{51}$$

Inserting $F \leftarrow F_c = (F - GK)$ into Eq. (51) gives

$$\begin{bmatrix} -R & (F-GK)R & G & \mathbf{0} \\ R(F-GK)^T & -R & \mathbf{0} & RC^T \\ G^T & \mathbf{0} & -\gamma_\infty I_r & \mathbf{0} \\ \mathbf{0} & CR & \mathbf{0} & -\gamma_\infty I_m \end{bmatrix} < 0 \tag{52}$$

and with

$$Y = KR \tag{53}$$

Eq. (53) implies Eq. (48). This concludes the proof. □

Theorem 2. *System* Eqs. (1) and (2) *under control* Eq. (3) *is stable with quadratic performance* γ_∞*, if there exist positive definite symmetric matrices* $S, O \in \mathbb{R}^{n \times n}$*, a matrix* $Y \in \mathbb{R}^{r \times n}$*, and a positive scalar* $\gamma_\infty \in \mathbb{R}$ *such that*

$$S = S^T > 0, \quad O = O^T > 0, \quad \gamma_\infty > 0, \tag{54}$$

$$\begin{bmatrix} O - 2S & * & * & * \\ SF^T - Y^T G^T & -O & * & * \\ G^T & \mathbf{0} & -\gamma_\infty I_r & ast \\ \mathbf{0} & CS & \mathbf{0} & -\gamma_\infty I_m \end{bmatrix} < 0. \tag{55}$$

When these inequalities are satisfied, the control law gain matrix is given as

$$\boldsymbol{K} = \boldsymbol{Y}\boldsymbol{S}^{-1}. \tag{56}$$

Proof. Considering that $\boldsymbol{Q}$ is positive definite, the transform matrix $\boldsymbol{T}_{\infty}^{\circ}$ can be defined as follows:

$$\boldsymbol{T}_{\infty}^{\circ} = \text{diag}[\boldsymbol{S} \quad \boldsymbol{S} \quad \boldsymbol{I}_r \quad \boldsymbol{I}_m], \quad \boldsymbol{S} = \boldsymbol{Q}^{-1}. \tag{57}$$

Therefore, premultiplying the left side of Eq. (46) and postmultiplying the right side of Eq. (46) by the matrix $\boldsymbol{T}_{\infty}^{\circ}$ gives

$$\begin{bmatrix} \boldsymbol{SPS} - 2\boldsymbol{S} & \boldsymbol{FS} & \boldsymbol{G} & \boldsymbol{0} \\ \boldsymbol{SF}^T & -\boldsymbol{SPS} & \boldsymbol{0} & \boldsymbol{SC}^T \\ \boldsymbol{G}^T & \boldsymbol{0} & -\gamma_{\infty}\boldsymbol{I}_r & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{CS} & \boldsymbol{0} & -\gamma_{\infty}\boldsymbol{I}_m \end{bmatrix} < 0. \tag{58}$$

Substituting $\boldsymbol{F} \leftarrow \boldsymbol{F}_c = (\boldsymbol{F} - \boldsymbol{GK})$ into Eq. (58) gives

$$\begin{bmatrix} \boldsymbol{SPS} - 2\boldsymbol{S} & (\boldsymbol{F} - \boldsymbol{GK})\boldsymbol{S} & \boldsymbol{G} & \boldsymbol{0} \\ \boldsymbol{S}(\boldsymbol{F} - \boldsymbol{GK})^T & -\boldsymbol{SPS} & \boldsymbol{0} & \boldsymbol{SC}^T \\ \boldsymbol{G}^T & \boldsymbol{0} & -\gamma_{\infty}\boldsymbol{I}_r & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{CS} & \boldsymbol{0} & -\gamma_{\infty}\boldsymbol{I}_m \end{bmatrix} < 0. \tag{59}$$

and with

$$\boldsymbol{Y} = \boldsymbol{KQ}, \quad \boldsymbol{O} = \boldsymbol{SPS}, \tag{60}$$

Eq. (59) implies Eq. (55). This concludes the proof. □

6. Ratio control design

Using the control law Eq. (3), the closed-loop system equations take the form

$$\boldsymbol{q}(i+1) = (\boldsymbol{F} - \boldsymbol{GK})\boldsymbol{q}(i), \tag{61}$$

$$\boldsymbol{y}(i) = \boldsymbol{Cq}(i). \tag{62}$$

Prescribed by a matrix $\boldsymbol{E} \in \mathbb{R}^{p \times n}$, rank $\boldsymbol{E} = p \leq r$, it is considered the design constraint Eq. (9) for all nonzero natural numbers i. From Proposition 2, it is clear that such kind of design is a singular task, where Eq. (9) gives

$$\boldsymbol{Eq}(i+1) = \boldsymbol{E}(\boldsymbol{F} - \boldsymbol{GK})\boldsymbol{q}(i) = 0, \tag{63}$$

which evidently implies

$$\boldsymbol{E}(\boldsymbol{F} - \boldsymbol{GK}) = \boldsymbol{0}. \tag{64}$$

Evidently, the equality

$$EF = EGK \tag{65}$$

can be satisfied, as well as the closed-loop system matrix $F_c = F - GK$ has to stable (all its eigenvalues are from the unit circle in the complex plane Z).

Lemma 1.*The equivalent state-space description of the system* Eqs. (1) and (2) *under control* Eq. (3), *in which closed-loop state variables satisfying the condition* Eq. (9) *is*

$$q(i+1) = (F - GK)q(i), \tag{66}$$

$$y(i) = Cq(i), \tag{67}$$

where

$$K = J + LK^{\circ}, \quad J = (EG)^{\ominus 1}EF, \quad L = I_r - (EG)^T\left(EG(EG)^T\right)^{-1}EG \tag{68}$$

while $L \in \mathbb{R}^{r \times r}$ *is the projection matrix (the orthogonal projector of* EG *onto the null space* $\mathcal{N}_{EG}$ [23]*) and* $K^{\circ} \in \mathbb{R}^{r \times n}$ *is the ratio control gain matrix.*

Proof. Premultiplying the left side of Eq. (65) by the identity matrix, it yields

$$EG(EG)^T\left(EG(EG)^T\right)^{-1}EF = EGK, \tag{69}$$

which implies the particular solution

$$K = (EG)^{\ominus 1}EF, \tag{70}$$

where

$$(EG)^{\ominus 1} = (EG)^T\left(EG(EG)^T\right)^{-1} \tag{71}$$

is the left Moore-Penrose pseudoinverse of EG.

Using the equality Eq. (65), then Eq. (69) can be also written as

$$EG(EG)^T\left(EG(EG)^T\right)^{-1}EGK = EGK, \tag{72}$$

which implies

$$EG\left(I_r - (EG)^T\left(EG(EG)^T\right)^{-1}EG\right)K = \mathbf{0}, \tag{73}$$

$$EG\left(I_r - (EG)^{\ominus 1}EG\right)K = \mathbf{0}, \tag{74}$$

respectively, where $I_r \in \mathbb{R}^{p \times p}$ is the identity matrix. It is evident that Eq. (74) can be satisfied only if

$$I_r - (EG)^{\ominus 1} EG = \mathbf{0}. \tag{75}$$

Thus, Eq. (11) implies all solutions of K as follows

$$K = (EG)^{\ominus 1} EF + \left(I_r - (EG)^{\ominus 1} EG \right) K^{\circ}, \tag{76}$$

where K° is an arbitrary matrix with appropriate dimension, and evidently Eq. (76) gives Eq. (68). This concludes the proof. □

Considering the model involving the given ratio constraint on the closed-loop system state variables Eqs. (66)–(68), the design conditions are presented in the following theorems.

Theorem 3. *System* Eqs. (1) and (2) *under the control* (3), *and satisfying the constraint* Eq. (4) *is stable with the quadratic performance* γ_∞, *if there exist positive definite matrices* **S**, $O \in \mathbb{R}^{n \times n}$, *a matrix* $Y^{\circ} \in \mathbb{R}^{r \times n}$, *and a positive scalar* $\gamma_\infty \in \mathbb{R}$ *such that*

$$S = S^T > 0, \quad O = O^T > 0, \quad \gamma_\infty > 0, \tag{77}$$

$$\begin{bmatrix} O - 2S & * & * & * \\ S(F - GJ)^T - Y^{\circ T} L^T G^T & -O & * & * \\ G^T & \mathbf{0} & -\gamma_\infty I_r & * \\ \mathbf{0} & CS & \mathbf{0} & -\gamma_\infty I_m \end{bmatrix} < 0. \tag{78}$$

When these inequalities are satisfied, the control law gain matrices are given as

$$K^{\circ} = Y^{\circ} S^{-1}, \quad K = J + LK^{\circ}, \tag{79}$$

where **J**, **L** *are defined in* Eq. (68).

Proof. Substituting Eq. (68) into Eq. (59) gives

$$\begin{bmatrix} O - 2S & (F - GL - GLK^{\circ})S & G & \mathbf{0} \\ S(F - GJ - GLK^{\circ})^T & -O & \mathbf{0} & SC^T \\ G^T & \mathbf{0} & -\gamma_\infty I_r & \mathbf{0} \\ \mathbf{0} & CS & \mathbf{0} & -\gamma_\infty I_m \end{bmatrix} < 0. \tag{80}$$

Using the notation

$$Y^{\circ} = K^{\circ} S \tag{81}$$

Eq. (80) implies Eq. (78). This concludes the proof. □

The ratio control does not exclude a forced regime given by the control law

$$\boldsymbol{u}(i) = -\boldsymbol{K}\boldsymbol{q}(i) + \boldsymbol{W}\boldsymbol{w}(i), \tag{82}$$

where $\boldsymbol{w}(i) \in \mathbb{R}^m$ is desired output signal vector and $\boldsymbol{W} \in \mathbb{R}^{m \times m}$ is the signal gain matrix. Using the static decoupling principle, the conditions to design the signal gain matrix $\boldsymbol{W}$ can be proven.

Lemma 2. *If the system* Eqs. (1) and (2) *is square, which is stabilizable by the control policy* Eq. (82) *and* Ref. [32]

$$rank\begin{bmatrix} \boldsymbol{F} & \boldsymbol{G} \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix} = n + m, \tag{83}$$

then the matrix $\boldsymbol{W}$ *takes the form*

$$\boldsymbol{W} = \left(\boldsymbol{C}(\boldsymbol{I}_n - (\boldsymbol{F} - \boldsymbol{G}\boldsymbol{K}))^{-1}\boldsymbol{G}\right)^{-1}, \tag{84}$$

where $\boldsymbol{I}_n \in \mathbb{R}^{n \times n}$ *is the identity matrix.*

Proof. In a steady state, the system equations Eqs. (1) and (2), and the control law Eq. (82) imply

$$\boldsymbol{q}_o = (\boldsymbol{F} - \boldsymbol{G}\boldsymbol{K})\boldsymbol{q}_o + \boldsymbol{G}\boldsymbol{W}\boldsymbol{w}_o, \tag{85}$$

where $\boldsymbol{q}_o$, $\boldsymbol{w}_o$ are the steady-state values of the vectors $\boldsymbol{q}(i)$, $\boldsymbol{w}(i)$, respectively. Since from Eq. (85), it can be derived that

$$\boldsymbol{q}_o = (\boldsymbol{I}_n - (\boldsymbol{F} - \boldsymbol{G}\boldsymbol{K}))^{-1}\boldsymbol{G}\boldsymbol{W}\boldsymbol{w}_o \tag{86}$$

and

$$\boldsymbol{y}_o = \boldsymbol{C}(\boldsymbol{I}_n - (\boldsymbol{F} - \boldsymbol{G}\boldsymbol{K}))^{-1}\boldsymbol{G}\boldsymbol{W}\boldsymbol{w}_o, \tag{87}$$

considering $\boldsymbol{y}_o = \boldsymbol{w}_o$, Eq. (87) implies Eq. (84). This concludes the proof. □

Theorem 4. *If the closed-loop system state variables satisfy the state constraint* Eq. (63), *then the common state variable vector* $\boldsymbol{q}_d(i) = \boldsymbol{E}\boldsymbol{q}(i)$, $\boldsymbol{q}_d(i) \in \mathbb{R}^k$ *attains the steady-state value*

$$\boldsymbol{q}_{dw} = \boldsymbol{E}\boldsymbol{G}\boldsymbol{W}\boldsymbol{w}_o. \tag{88}$$

Proof. Using the control policy Eq. (82), then

$$\boldsymbol{E}\boldsymbol{q}(i+1) = \boldsymbol{E}(\boldsymbol{F} - \boldsymbol{G}\boldsymbol{K})\boldsymbol{q}(i) + \boldsymbol{E}\boldsymbol{G}\boldsymbol{W}\boldsymbol{w}(i). \tag{89}$$

Since $\boldsymbol{K}$ satisfies Eq. (65), then Eq. (89) implies

$$\boldsymbol{E}\boldsymbol{q}(i+1) = \boldsymbol{E}\boldsymbol{G}\boldsymbol{W}\boldsymbol{w}(i) \tag{90}$$

and it is evident that the tied state variable $q_d(i)$ of the closed-loop system in a steady state is proportional to the steady state of the desired signal $\boldsymbol{w}_o$ and takes the value Eq. (88). This concludes the proof. □

http://dx.doi.org/10.5772/67159

7. Illustrative examples

To demonstrate properties of proposed approach, the classical example for a helicopter control [33] is taken, where the discrete-time state-space representation Eqs. (1) and (2) for the sampling period $\Delta t = 0.05s$ consists of the following parameters

$$\boldsymbol{F} = \begin{bmatrix} 0.9982 & 0.0013 & 0.0004 & -0.0229 \\ 0.0023 & 0.9507 & -0.0048 & -0.1962 \\ 0.0049 & 0.0176 & 0.9670 & 0.0679 \\ 0.0001 & 0.0004 & 0.0492 & 1.0017 \end{bmatrix}, \quad \boldsymbol{G} = \begin{bmatrix} 0.0221 & 0.0086 \\ 0.1733 & -0.3705 \\ -0.2697 & 0.2173 \\ -0.0068 & 0.0055 \end{bmatrix},$$

$$\boldsymbol{C} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{91}$$

The state constraint, defining the ratio control of two state system variables, is specified as

$$\frac{q_4(t)}{q_1(t)} = 1.5 \quad \Rightarrow \boldsymbol{E} = [\,-1.5 \quad 0 \quad 0 \quad 1\,] \tag{92}$$

and subsequently it yields

$$(\boldsymbol{EG})^{\ominus 1} = \begin{bmatrix} -24.1737 \\ -4.4828 \end{bmatrix}, \quad \boldsymbol{L} = \begin{bmatrix} 0.0332 & -0.1793 \\ -0.1793 & 0.9668 \end{bmatrix}, \tag{93}$$

$$\boldsymbol{J} = \begin{bmatrix} 36.1914 & 0.0372 & -1.1753 & -25.0447 \\ 6.7113 & 0.0069 & -0.2179 & -4.6443 \end{bmatrix}. \tag{94}$$

Solving Eqs. (77) and (78) using self-dual-minimization (SeDuMi) package for Matlab [19], the feedback gain matrix design problem in the constrained control is feasible with the results

$$\boldsymbol{O} = \begin{bmatrix} 2.9027 & 0.2117 & 0.1103 & -1.7595 \\ 0.2117 & 1.3174 & -0.1751 & -0.1245 \\ 0.1103 & -0.1751 & 0.4162 & 0.0060 \\ -1.7595 & -0.1245 & 0.0060 & 3.2464 \end{bmatrix},$$

$$\boldsymbol{S} = \begin{bmatrix} 2.4910 & 0.1375 & 0.0792 & -1.4957 \\ 0.1375 & 1.0779 & -0.0910 & -0.0030 \\ 0.0792 & -0.0910 & 0.3735 & -0.0348 \\ -1.4957 & -0.0030 & -0.0348 & 3.0926 \end{bmatrix}, \tag{95}$$

$$\boldsymbol{Y}^{\circ} = \begin{bmatrix} -2.2113 & 0.2435 & -0.0819 & 1.4281 \\ 11.9245 & -1.3129 & 0.4416 & -7.7011 \end{bmatrix}, \quad \gamma_{\infty} = 8.5565. \tag{96}$$

Inserting $\boldsymbol{Y}^{\circ}$ and $\boldsymbol{S}$ into Eq. (79), the gain matrix $\boldsymbol{K}^{\circ}$ is computed as

$$\boldsymbol{K}^{\circ}=\begin{bmatrix}-0.8887 & 0.3441 & 0.0562 & 0.0329\\ 4.7926 & -1.8555 & -0.3028 & -0.1775\end{bmatrix} \tag{97}$$

and Eq. (79) implies the full-state feedback gain matrix values

$$\boldsymbol{K}=\begin{bmatrix}35.3027 & 0.3813 & -1.1191 & -25.0117\\ 11.5040 & -1.8486 & -0.5208 & -4.8217\end{bmatrix}. \tag{98}$$

It can be easily verified that the closed-loop system matrix takes the format

$$\boldsymbol{F}_c=\boldsymbol{F}-\boldsymbol{G}\boldsymbol{K}=\begin{bmatrix}0.1179 & 0.0088 & 0.0296 & 0.5722\\ -1.8528 & 0.1997 & -0.0038 & 2.3515\\ 7.0258 & 0.5223 & 0.7783 & -5.6297\\ 0.1768 & 0.0132 & 0.0444 & 0.8583\end{bmatrix}, \tag{99}$$

while the ratio control law rises up the stable closed-loop system with the closed-loop system matrix eigenvalues spectrum

$$\rho(\boldsymbol{F}_c)=\{\,0.9527,\quad 0.7566,\quad 0.0000,\quad 0.2449\,\}. \tag{100}$$

Note that one from the resulting eigenvalue of $\boldsymbol{F}_c$ is zero (rank($\boldsymbol{E}$) = 1)), because Proposition 2 prescribes this constrained design task as a singular problem. Using the connection between the eigenvector matrix $\boldsymbol{N}$ and $\boldsymbol{M}$ as given by Eq. (17), it is possible to show that this instance is documented also by the structure of $\boldsymbol{M}$, while

$$\boldsymbol{N}=\begin{bmatrix}-0.3109 & -0.1105 & -0.0800 & -0.0184\\ -0.6937 & -0.3384 & -0.4690 & -0.7382\\ 0.4522 & 0.9197 & 0.8793 & 0.6738\\ -0.4664 & -0.1657 & -0.0218 & -0.0276\end{bmatrix},$$
$$\boldsymbol{M}=\begin{bmatrix}-3.4197 & -0.3938 & -0.5157 & 0.2213\\ 10.2685 & 1.3777 & 1.4844 & -7.4555\\ -15.2705 & 0.0000 & 0.0000 & 10.1803\\ 8.2076 & -1.6162 & -0.1958 & -3.2577\end{bmatrix}, \tag{101}$$

where the structure of the third row of $\boldsymbol{M}$ correspondents to the structure of the constraint vector $\boldsymbol{E}$, while $a_4=\boldsymbol{m}_3^T(1)/\boldsymbol{m}_3^T(4)=-1.5$.

To illustrate the closed-loop system property in the forced mode, the signal gain matrix $\boldsymbol{W}$ is computed by using Eq. (84) as follows

$$\boldsymbol{W}=\begin{bmatrix}1.4575 & 35.9137\\ -1.7651 & 11.6521\end{bmatrix}. \tag{102}$$

Therefore, according to Theorem 4, the constraint given on the states of the system under study is satisfied with zero offset in the autonomous regime and with offset value equal $\boldsymbol{q}_{dw}$ in the forced mode, i.e.,

$$q_d = 0, \qquad q_{dw} = \boldsymbol{EGWw}_o = 3.0001, \tag{103}$$

while

$$\boldsymbol{w}(i) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \text{ for all } i. \tag{104}$$

The simulation results of the closed-loop system response in the autonomous and forced mode are presented, where **Figure 1** is concerned with the system state variables response in the autonomous regime and **Figure 2** with the system state variables response in the forced mode. It is evident that the condition Eq. (9) is satisfied at all time instant, except initial time instant in the above given way (see the time response of the additive of variable, which is included as $q_d(i)$ in the figures).

For comparison, an example is given for default design of state feedback gain matrix using BRL structure of LMIs. Solving Eqs. (54) and (55), the task is feasible with the Lyapunov matrix variables

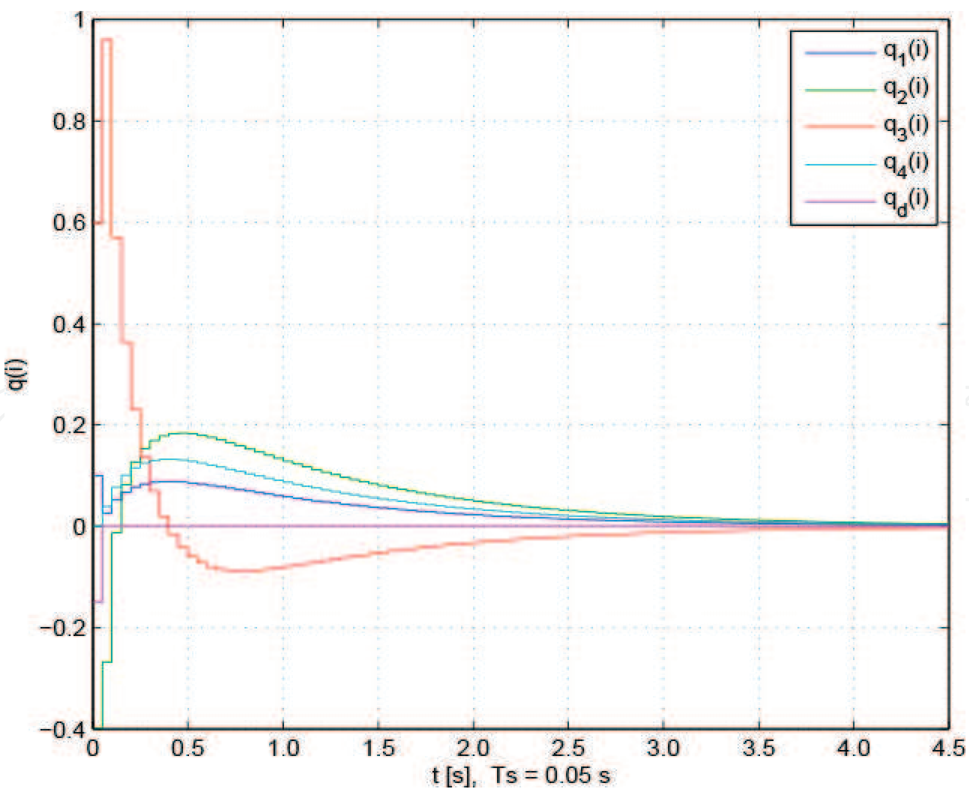

Figure 1. State response in autonomous regime.

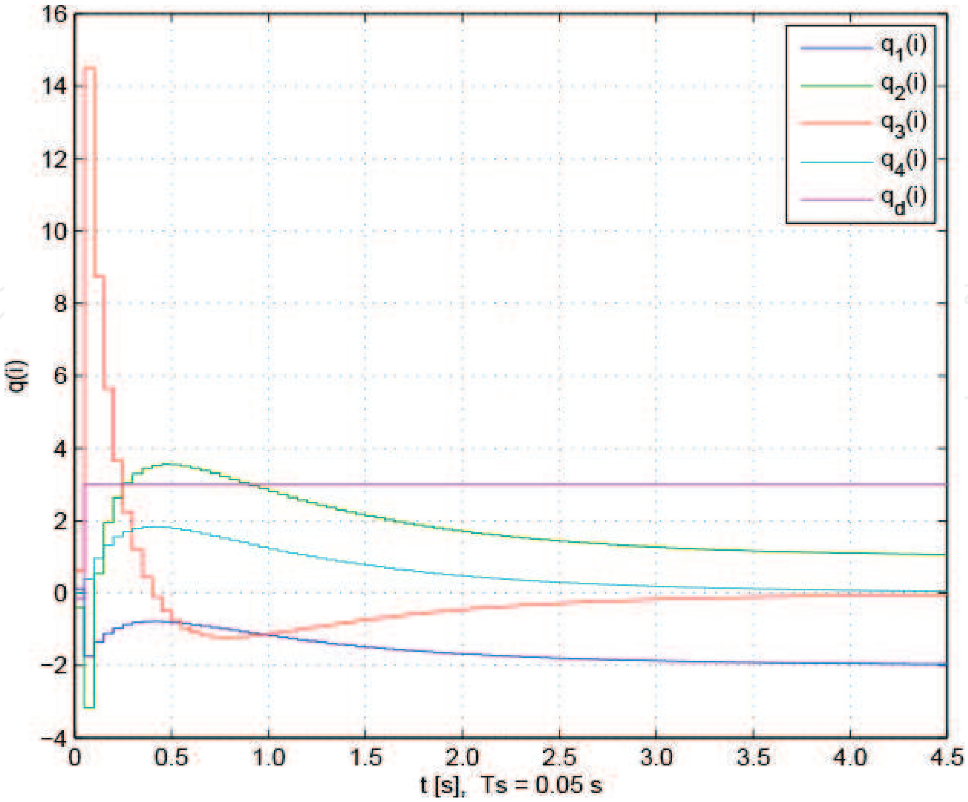

Figure 2. State response in forced mode.

$$
O = \begin{bmatrix} 0.1438 & -0.1090 & -0.1619 & -0.2191 \\ -0.1090 & 1.5603 & -0.2198 & 0.2945 \\ -0.1619 & -0.2198 & 1.6006 & -0.4711 \\ -0.2191 & 0.2945 & -0.4711 & 1.8586 \end{bmatrix},
$$

$$
S = \begin{bmatrix} 0.1338 & -0.0840 & -0.1490 & -0.1928 \\ -0.0840 & 1.2736 & -0.2314 & 0.2439 \\ -0.1490 & -0.2314 & 1.6729 & -0.5520 \\ -0.1928 & 0.2439 & -0.5520 & 1.8296 \end{bmatrix}, \tag{105}
$$

and parameter matrix variable

$$
Y = \begin{bmatrix} 0.6210 & -0.8607 & -2.6800 & -0.7582 \\ 0.4017 & -2.6793 & -0.3804 & 0.1788 \end{bmatrix}, \quad \gamma_\infty = 3.1301. \tag{106}
$$

Therefore, using Eq. (56), the nominal control law gain matrix K is computed as

$$
K = \begin{bmatrix} 0.8951 & -0.8107 & -1.8928 & -0.7830 \\ 2.4671 & -2.0742 & -0.0947 & 0.6056 \end{bmatrix}, \tag{107}
$$

the closed-loop system matrix takes the form

$$F_c = F - GK = \begin{bmatrix} 0.9571 & 0.0371 & 0.0431 & -0.0108 \\ 0.7613 & 0.3227 & 0.2881 & 0.1639 \\ -0.2898 & 0.2498 & 0.4771 & -0.2749 \\ -0.0073 & 0.0063 & 0.0368 & 0.9931 \end{bmatrix}, \tag{108}$$

while the closed-loop system matrix eigenvalues spectrum is

$$\rho(F_c) = \{0.1207, \quad 0.6570, \quad 0.9733, \quad 0.9990\}. \tag{109}$$

To apply in the forced mode, the signal gain matrix W is now computed by using Eq. (84) as follows:

$$W = \begin{bmatrix} -0.8296 & 0.9567 \\ -2.2360 & 2.4922 \end{bmatrix}. \tag{110}$$

The simulation results of the nominal closed-loop system response are illustrated in **Figures 3** and **4**, where **Figure 3** is concerned with the system state variables response in the autonomous regime and **Figure 4** with the system state variables response in the forced mode.

Since these two control structures are of interest in the context of full-state control design, matching the presented results, it is evident that the system dynamics in both cases are comparable.

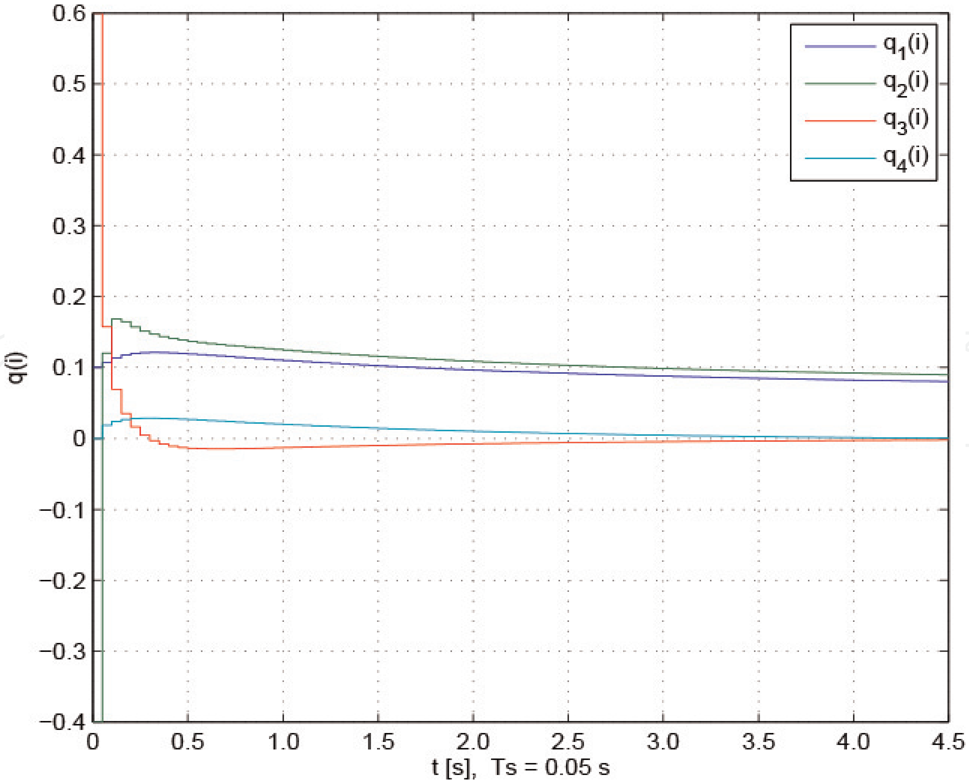

Figure 3. State response in autonomous regime.

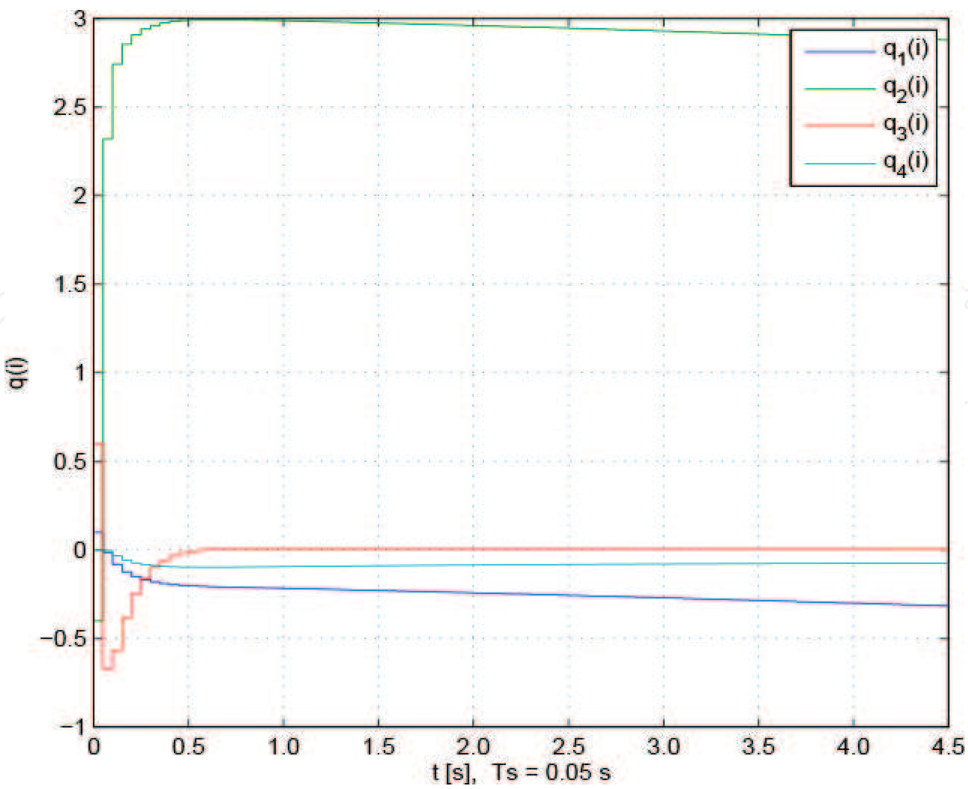

Figure 4. State response in forced mode.

8. Concluding Remarks

In this chapter, an extended method is presented, based on the classical memoryless feedback H_∞ control principle of discrete-time systems, if the ratio control is reformulated by an equality constraint setting on associated state variables. The asymptotic stability of the control scheme is guaranteed in the sense of the enhanced representation of BRL, while resulting LMIs are linear with respect to the system state variables, and does not involve products of the Lyapunov matrix and the system matrix parameters, which provides one way of solving the singular LMI problem. Moreover, formulated as a stabilization problem with the full-state feedback controller, the control gain matrix takes no special structure. The formulation allows to find a solution without restrictive assumptions and additional specifications on the design parameters. It is clear from Theorem 4 that the control law strictly solves the problem even in the unforced mode. The validity of the proposed method is demonstrated by numerical examples.

Acknowledgements

The work presented in this chapter was supported by VEGA, the Grant Agency of the Ministry of Education and the Academy of Science of Slovak Republic, under Grant No. 1/0608/17. These supports are very gratefully acknowledged.

Author details

Dušan Krokavec* and Anna Filasová

*Address all correspondence to: dusan.krokavec@tuke.sk

Department of Cybernetics Artificial Intelligence, Faculty of Electrical Engineering Informatics, Technical University of Košice, Košice, Slovakia

References

[1] Benzaouia A., Gurgat C. Regulator problem for linear discrete-time systems with nonsymmetrical constrained control. International Journal of Control. 1988. **48**(6):2441–2451. DOI: 10.1109/CDC.1991.261705.

[2] Castelan E.B., Hennet J.C. Eigenstructure assignment for state constrained linear continuous time systems. Automatica. 1992. **28**(3):605–611. DOI: 10.1016/0005-1098(92) 90185-I.

[3] Hahn H. Linear systems controlled by stabilized constraint relations. In: Proceedings of the 31st IEEE Conference on Decision and Control; 16–18 December 1992; Tucson, USA. pp. 840–848.

[4] Tarbouriech S., Castelan E.B. An eigenstructure assignment approach for constrained linear continuous-time singular systems. Systems & Control Letters. 1995. **24**(5):333–343. DOI: 10.1016/0167-6911(94)00046-X.

[5] Kaczorek T. Externally and internally positive singular discrete-time linear systems. International Journal of Applied Mathematics and Computer Science. 2002. **12**(2):197–202.

[6] Filasová A., and Krokavec D. Enhanced approach to PD control design for linear time-invariant descriptor systems. Journal of Physics: Conference Series. 2017. **783**. 12p. (13th European Workshop on Advanced Control and Diagnosis (ACD 2016)). DOI:10.1088/1742-6596/783/1/012037

[7] Yu T.J., Lin C.F., Müller P.C. Design of LQ regulator for linear systems with algebraic-equation constraints. In: Proceedings of the 35th IEEE Conference on Decision and Control; 13 December 1996; Kobe, Japan. pp. 4146–4151.

[8] Oloomi H., Shafai B. Constrained stabilization problem and transient mismatch phenomenon in singularity perturbed systems. International Journal of Control. 1997. **67**(2):435–454. DOI: 10.1080/002071797224199.

[9] Petersen I.R. Minimax LQG control. International Journal of Applied Mathematics and Computer Science. 2006. **16**(3):309–323. http://eudml.org/doc/207795.

[10] Xue Y., Wei Y., Duan G. Eigenstructure assignment for linear systems with constrained input via state feedback. A parametric approach. In: Proceedings of the 25th Chinese Control Conference; 7–11 August 2006; Harbin, China. pp. 108–113.

[11] Ko S., Bitmead R.R. State estimation for linear systems with state equality constraints. Automatica. 2007. **43**(9):1363–1368. DOI: 10.1016/j.automatica.2007.01.017.

[12] Ko S., Bitmead R.R. Optimal control for linear systems with state equality constraints. Automatica. 2007. **43**(9):1573–1582. DOI: 10.1016/j.automatica.2007.01.024.

[13] Filasová A., Krokavec D. Observer state feedback control of discrete-time systems with state equality constraints. Archives of Control Sciences. 2010. **10**(3):253–266. DOI: 10.2478/v10170-010-0016-5.

[14] Nesterov Y., Nemirovsky A. Interior Point Polynomial Methods in Convex Programming. Theory and Applications. Philadelphia: SIAM; 1994. 407 p. DOI: 10.1137/1.9781611970791.fm

[15] Boyd D., El Ghaoui L., Peron E., Balakrishnan V. Linear Matrix Inequalities in System and Control Theory. Philadelphia: SIAM; 1994. 205 p. DOI: 10. 1137/1.9781611970777.

[16] Skelton R.E., Iwasaki T., Grigoriadis K. A Unified Algebraic Approach to Linear Control Design. London: Taylor & Francis; 1998. 285 p. DOI: 10.1002/rnc.694.

[17] Herrmann G., Turner M.C., Postlethwaite I. Linear matrix inequalities in control. In: Turner M.C., Bates D.G., editors. Mathematical Methods for Robust and Nonlinear Control. Berlin: Springer-Verlag; 2007. pp. 123–142. DOI: 10.1007/978-1-84800-025-4-4.

[18] Gahinet P., Nemirovski A., Laub A.J., Chilali M. LMI Control Toolbox User's Guide. Natick: The MathWorks; 1995. 356 p.

[19] Peaucelle D., Henrion D., Labit Y., Taitz K. User's Guide for SeDuMi Interface 1.04. Toulouse: LAAS-CNRS; 2002. 36 p.

[20] Oliveira de M.C., Bernussou J., Geromel J.C. A new discrete-time robust stability condition. Systems & Control Letters. 1999. **37**(4):261–265. DOI: 10.1016/S0167-6911(99)00035-3.

[21] Wu A.I., Duan G.R. Enhanced LMI representations for H_2 performance of polytopic uncertain systems. Continuous-time case. International Journal of Automation and Computing. 2006. **3**(3):304–308. http://www.ijac.net/EN/Y2006/V3/I3/304.

[22] Filasová A., Krokavec D. H∞ control of discrete-time linear systems constrained in state by equality constraints. International Journal of Applied Mathematics and Computer Science. 2012. **22**(3):551–560. DOI: 10.2478/v10006-012-0042-5.

[23] Krokavec D., Filasová A. Constrained control of discrete-time stochastic systems. IFAC Proceedings Volumes. 2008. **41**(2):15315–15320. DOI: 10.3182/20080706-5-KR-1001.02590.

[24] Krokavec D., Filasová A. Control reconfiguration based on the constrained LQ control algorithms. IFAC Proceedings Volumes. 2009. **42**(8):143–148. DOI: 10.3182/20090630 -4-ES-2003.00024.

[25] Ogata K. Discrete-Time Control Systems. Upper Saddle River: Prentice-Hall; 1995. 760 p.

[26] Debiane L., Ivorra B., Mohammadi B., Nicoud F., Ernz A., Poinsot T., Pitsch H. Temperature and pollution control in flames. In: Proceedings of the Summer Program 2004; 2004; University of Montpellier, France, pp. 1–9.

[27] Cakmakci M., Ulsoy A.G. Modular discrete optimal MIMO controller for a VCT engine. In: Proceedings of the 2009 American Control Conference; 10–12 June 2009; St. Louis, USA, pp. 1359–1364.

[28] Krokavec D., Filasová A. Performance of reconfiguration structures based on the constrained control. IFAC Proceedings Volumes. 2008. **41**(2):1243–1248.

[29] Heij C., Ran A., van Schagen F. Introduction to Mathematical Systems Theory. Linear Systems, Identification and Control. Basel: Birkhäuser Verlag; 2007. 168 p. DOI: 10.1007/978-3-7643-7549-2.

[30] Gajic Z., Qureshi M.T.J. Lyapunov Matrix Equation in System Stability and Control. San Diego: Academic Press; 1995. 271 p. DOI: 10.1137/1038139.

[31] Mason O., Shorten R. On common quadratic Lyapunov functions for stable discrete-time LTI systems. IMA Journal of Applied Mathematics. 2004. **69**(3):271–283. DOI: 10.1093/imamat/69.3.271.

[32] Wang Q.G. Decoupling Control. Berlin: Springer-Verlag; 2003. 369 p. DOI: 10.1007/3-540 - 46151-5.

[33] Wen C.C., Cheng C.C. Design of sliding surface for mismatched uncertain systems to achieve asymptotical stability. Journal of the Franklin Institute. 2008. **345**(8):926–941. DOI: 10.1016/j.jfranklin.2008.06.003.

IntechOpen