Chapter 11

Relaxation Theory for Point Vortices

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Additional information is available at the end of the chapter

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Abstract

We study relaxation dynamics of the mean field of many point vortices from quasi-equilibrium to equilibrium. Maximum entropy production principle implies four consistent equations concerning relaxation-equilibrium states and patch-point vortex models. Point vortex relaxation equation coincides with Brownian point vortex equation in micro-canonical setting. Mathematical analysis to point vortex relaxation equation is done in accordance with the Smoluchowski-Poisson equation.

Keywords: point vortex, quasi-equilibrium, relaxation dynamics, maximum entropy production, global-in-time solution

1. Introduction

The physical object studied in this chapter is non-viscous, noncompressible fluid with high Reynolds number occupied in bounded, simply-connected domain. $\Omega \in \mathbb{R}^2$. Motion of this fluid is described by the Euler-Poisson equation

$$\omega_t + \nabla \cdot u\omega = 0, \quad \Delta\psi = -\omega, \ u = \nabla^{\perp}\psi, \ \psi|_{\partial\Omega} = 0 \tag{1}$$

where

$$\nabla^{\perp} = \begin{pmatrix} \dfrac{\partial}{\partial x_2} \\ -\dfrac{\partial}{\partial x_1} \end{pmatrix}, \ x = (x_1, \ x_2),$$

and u, ω and ψ stand for the velocity, vorticity and stream function, respectively.

In the point vortex model

$$\omega(x,t) = \sum_{i=1}^{N} \alpha_i \delta_{x_i(t)}(dx) \tag{2}$$

system of Eq. (1) is reduced to

$$\alpha_i \frac{dx_i}{dt} = \nabla_{x_i}^{\perp} H_N, \quad i = 1, 2, \cdots, N \tag{3}$$

associated with the Hamiltonian

$$H_N(x_1, \cdots x_N) = \frac{1}{2} \sum_i \alpha_i^2 R(x_i) + \sum_{i<j} \alpha_i \alpha_j G(x_i, x_j), \tag{4}$$

where $G = G(x, x')$ is the Green's function of $-\Delta$ provided with the Dirichlet boundary condition and

$$R(x) = \left[G(x, x') + \frac{1}{2\pi} \log|x - x'| \right]_{x'=x}.$$

Onsager [1] proposed to use statistical mechanics of Gibbs to Eq. (3). In the limit $N \to \infty$ with $\alpha N = 1$, local mean of vortex distribution is given by

$$\overline{\omega}(x) = \int_I \tilde{\alpha}\, \rho^{\tilde{\alpha}}(x) P(d\tilde{\alpha}), x \in \Omega \tag{5}$$

where $\alpha_i = \tilde{\alpha}^i\ \alpha,\ \tilde{\alpha}^i \in I = [-1, 1]$ is the intensity of the i-th vortex, $\rho^{\tilde{\alpha}}(x)$ is the existence probability of the vortex at x with relative intensity $\tilde{\alpha}$, which satisfies

$$\int_\Omega \rho^{\tilde{\alpha}}(x) dx = 1, \quad \forall \tilde{\alpha} \in I,$$

and $P(d\tilde{\alpha})$ is the numerical density of the vortices with the relative intensity $\tilde{\alpha}$. Under $H_N = E =$ constant, $\alpha^2 N \beta_N = \beta =$ constant and $N \to \infty$, mean field equation is derived by several arguments [2–7], that is,

$$-\Delta\overline{\psi} = \int_I \tilde{\alpha} \frac{e^{-\beta\tilde{\alpha}\overline{\psi}}}{\int_\Omega e^{-\beta\tilde{\alpha}\overline{\psi}}} P(d\tilde{\alpha}), \quad \overline{\psi}|_{\partial\Omega} = 0 \tag{6}$$

with

$$\overline{\omega} = -\Delta\overline{\psi}, \quad \rho^{\tilde{\alpha}} = \frac{e^{-\beta\tilde{\alpha}\overline{\psi}}}{\int_\Omega e^{-\beta\tilde{\alpha}\overline{\psi}}}$$

where

$$\rho^{\tilde{\alpha}}(x) = \lim_{N\to\infty}\int_{\Omega^{N-1}} \mu_N^{\beta_N}(dx, dx_2, \cdots dx_N)$$

$$\mu_N^{\beta_N}(dx_1, \cdots dx_N) = \frac{1}{Z(N,\beta_N)} e^{-\beta_N H_N} dx_1 \cdots dx_N$$

$$Z(N,\beta_N) = \int_{\Omega^N} e^{-\beta_N H_N} dx_1 \cdots dx_N.$$

Since Ref. [8], structure of the set of solutions to Eq. (6) has been clarified in accordance with the Hamiltonian given by Eq. (4) (see [9] and the references therein).

Quasi-equilibria, on the other hand, are observed for several isolated systems with many components [10]. Thus, we have a relatively stationary state, different from the equilibrium, which eventually approaches the latter. Relaxation indicates this time interval, from quasi-equilibrium to equilibrium. To approach relaxation dynamics of many point vortices, patch model

$$\omega(x,t) = \sum_{i=1}^{N_p} \sigma_i 1_{\Omega_i(t)}(x) \tag{7}$$

is used. It describes detailed vortex distribution, where N_p, σ_i and $\Omega_i(t)$ denote the number of patches, the vorticity of the i-th patch and the domain of the i-th patch, respectively. Mean field equations for equilibrium and for relaxation time are derived by the principles of maximum entropy [11, 12] and maximum entropy production [13, 14], respectively. For the latter case, one obtains a system on $p = p(x, \sigma, t)$,

$$\begin{gathered}\frac{\partial p}{\partial t} + \nabla \cdot p\overline{u} = \nabla \cdot D\Big(\nabla p + \beta_p(\sigma - \overline{\omega})p\nabla\overline{\psi}\Big), \quad \beta_p = -\frac{\int_\Omega D\nabla\overline{\omega}\cdot\nabla\overline{\psi}}{\int_\Omega D\big(\int_I \sigma^2 p d\sigma - \overline{\omega}^2\big)|\nabla\overline{\psi}|^2} \\ \overline{\omega} = \int_I \sigma p d\sigma = -\Delta\overline{\psi}, \quad \overline{\psi}|_{\partial\Omega} = 0, \quad \overline{u} = \nabla^\perp\overline{\psi}\end{gathered} \tag{8}$$

with the diffusion coefficient $D = D(x,t) > 0$.

In this chapter, we regard Eq. (2) as a limit of Eq. (7). First, point vortex model valid to the relaxation time is derived from Eq. (8), that is, a system on $\rho^{\tilde{\alpha}} = \rho^{\tilde{\alpha}}(x,t), \tilde{\alpha} \in I$, in the form of

$$\begin{gathered}\frac{\partial\rho^{\tilde{\alpha}}}{\partial t} + \nabla \cdot \rho^{\tilde{\alpha}}\,\overline{u} = \nabla \cdot D(\nabla\rho^{\tilde{\alpha}} + \beta\tilde{\alpha}\rho^{\tilde{\alpha}}\nabla\overline{\psi}), \\ \overline{\omega} = \int_I \tilde{\alpha}\rho^{\tilde{\alpha}} P(d\tilde{\alpha}) = -\Delta\overline{\psi}, \quad \overline{\psi}|_{\partial\Omega} = 0, \quad \overline{u} = \nabla^\perp\overline{\psi} \\ \beta = -\frac{\int_\Omega D\nabla\overline{\omega}\cdot\nabla\overline{\psi}}{\int_\Omega D\int_I \tilde{\alpha}^2\rho^{\tilde{\alpha}} P(d\tilde{\alpha})|\nabla\overline{\psi}|^2}.\end{gathered} \tag{9}$$

Second, the stationary state of Eq. (9) is given by Eq. (6). Third, Eq. (9) coincides with the Brownian point vortex model of Chavanis [15]. Finally, system of Eq. (9) provided with the boundary condition

$$\frac{\partial \rho^{\tilde{\alpha}}}{\partial \nu} + \beta\tilde{\alpha}\rho^{\tilde{\alpha}} \frac{\partial \overline{\psi}}{\partial \nu}\Big|_{\partial\Omega} = 0 \tag{10}$$

satisfies the requirements of isolated system in thermodynamics.

In fact, averaging Eq. (9) implies

$$\begin{gathered}\frac{\partial \overline{\omega}}{\partial t} + \nabla \cdot \overline{\omega}\,\overline{u} = \nabla \cdot D(\nabla \overline{\omega} + \beta \overline{\omega}_2 \nabla \overline{\psi}), \quad \frac{\partial \overline{\omega}}{\partial \nu} + \beta \overline{\omega}_2 \frac{\partial \overline{\psi}}{\partial \nu}\Bigg|_{\partial\Omega} = 0 \\ \overline{\omega} = -\Delta\overline{\psi}, \quad \overline{\psi}|_{\partial\Omega} = 0, \quad \overline{u} = \nabla^{\perp}\overline{\psi}, \quad \beta = -\frac{\int_\Omega D\nabla\overline{\omega}\cdot\nabla\overline{\psi}}{\int_\Omega D\overline{\omega}_2|\nabla\overline{\psi}|^2}\end{gathered} \tag{11}$$

for

$$\overline{\omega} = \int_I \tilde{\alpha}\,\rho^{\tilde{\alpha}} P(d\tilde{\alpha}),\ \overline{\omega}_2 = \int_I \tilde{\alpha}^2 \rho^{\tilde{\alpha}} P(d\tilde{\alpha}). \tag{12}$$

Then, we obtain mass and energy conservations

$$\frac{d}{dt}\int_\Omega \overline{\omega} = 0, \quad (\overline{\omega}_t, \overline{\psi}) = \frac{1}{2}\frac{d}{dt}\left(\overline{\omega}, (-\Delta)^{-1}\overline{\omega}\right) = 0 \tag{13}$$

where (,) stands for the L^2 inner product. Assuming $\rho^{\tilde{\alpha}} > 0$, we write the first equation of (9) as

$$\frac{\partial \rho^{\tilde{\alpha}}}{\partial t} + \nabla \cdot \rho^{\tilde{\alpha}}\overline{u} = \nabla \cdot D\rho^{\tilde{\alpha}}\nabla(\log\rho^{\tilde{\alpha}} + \beta\tilde{\alpha}\overline{\psi}). \tag{14}$$

Then, it follows that

$$\frac{d}{dt}\int_\Omega \Phi(\rho^{\tilde{\alpha}})dx + \beta\tilde{\alpha}(\rho_t^{\tilde{\alpha}}, \overline{\psi}) = -\int_\Omega D\rho^{\tilde{\alpha}}|\nabla(\log\rho^{\tilde{\alpha}} + \beta\tilde{\alpha}\overline{\psi})|^2 \tag{15}$$

from Eq. (10), where

$$\Phi(s) = s(\log s - 1) + 1 \geq 0, \qquad s > 0.$$

Hence, it follows that

$$\frac{d}{dt}\int_\Omega\left(\int_I \Phi(\rho^{\tilde{\alpha}})P(d\tilde{\alpha})\right) = -\int_\Omega\left(\int_I D\rho^{\tilde{\alpha}}|\nabla(\log\rho^{\tilde{\alpha}} + \beta\tilde{\alpha}\overline{\psi})|^2 P(d\tilde{\alpha})\right) \leq 0 \tag{16}$$

from Eq. (13), that is, entropy increasing.

2. Vorticity patch model

In Eq. (7), the vorticity σ_i is uniform in a region with constant area $\Omega_i(t)$, called vorticity patch. A patch takes a variety of forms as the time t varies. We collect all the vorticity patches in a small region, called cell. Cell area Δ thus takes the relation $|\Omega_i| \ll \Delta \ll |\Omega|$. The probability that the average vorticity at x is σ is denoted by $p(x, \sigma, t)dx$ which satisfies

$$\int p(x, \sigma, t) d\sigma = 1. \tag{17}$$

Let

$$\int_\Omega p(x, \sigma, t) dx = M(\sigma) \tag{18}$$

be independent of t. Since

$$|\Omega| = \iint p(x, \sigma, t) dx d\sigma = \int M(\sigma) d\sigma \tag{19}$$

equality (18) means conservation of total area of patches of the vorticity σ. Then, the macroscopic vorticity is defined by

$$\overline{\omega}(x, t) = \int \sigma p(x, \sigma, t) d\sigma, \tag{20}$$

which is associated with the stream function $\overline{\psi} = \overline{\psi}(x, t)$ and the velocity $\overline{u} = \overline{u}(x, t)$ through

$$\overline{\omega} = -\Delta\overline{\psi}, \quad \overline{\psi}|_{\partial\Omega} = 0, \quad \overline{u} = \nabla^\perp\overline{\psi}. \tag{21}$$

To formulate equilibrium, we apply the principle of maximum entropy [11, 12], seeking the maximal state of

$$S(p) = -\iint p(x, \sigma) \log p(x, \sigma) dx d\sigma \tag{22}$$

under the constraint Eqs. (17), (18) and

$$E = \frac{1}{2} \int_\Omega \overline{\omega}\overline{\psi}. \tag{23}$$

With the Lagrange multipliers $\left(\beta_p, c(\sigma), \zeta(x)\right)$, it follows that

$$\delta S - \beta_p \delta E - \int c(\sigma) \delta M(\sigma) d\sigma - \int_\Omega \zeta(x) \left(\delta \int p d\sigma\right) dx = 0, \tag{24}$$

which is reduced to

$$p(x,\sigma) = e^{-c(\sigma)-\left(\zeta(x)+1\right)-\beta_p\sigma\overline{\psi}}. \tag{25}$$

Here, β_p and $c(\sigma)$ may be called inverse temperature and chemical potential, respectively. We put $c(0) = 0$ because of the degree of freedom of $c(\sigma)$ admitted by Eq. (19). Then, it follows that

$$p(x,\sigma) = p(x,0)e^{-c(\sigma)-\beta_p\sigma\overline{\psi}} \tag{26}$$

and hence, Eq. (17) implies

$$p(x,\sigma) = \frac{e^{-c(\sigma)-\beta_p\sigma\overline{\psi}}}{\int e^{-c(\sigma')-\beta_p\sigma'\overline{\psi}}d\sigma'}. \tag{27}$$

From Eqs. (18) and (26), similarly, it follows that

$$c(\sigma) = \log\left(\frac{\int_\Omega p(x,0)e^{-\beta_p\sigma\overline{\psi}}dx}{\int_\Omega p(x,\sigma)dx}\right). \tag{28}$$

The equilibrium mean field equation of vorticity patch model is thus given by Eqs. (20), (21), (27) and (28), which is reduced to

$$\begin{aligned} -\Delta\overline{\psi} &= \int \sigma M(\sigma)\frac{p(x,0)e^{-\beta_p\sigma\overline{\psi}}}{\int_\Omega p(x,0)e^{-\beta_p\sigma\overline{\psi}}}d\sigma, \quad \overline{\psi}|_{\partial\Omega} = 0 \\ \overline{\omega} &= \int_I \sigma p d\sigma = -\Delta\overline{\psi}, \quad \int_\Omega p(x,\sigma,t)dx = M(\sigma). \end{aligned} \tag{29}$$

One may use the principle of maximum entropy production to describe near from equilibrium dynamics [13, 14]. We apply the transport equation

$$\frac{\partial p}{\partial t} + \nabla\cdot(p\overline{u}) = -\nabla\cdot J, \quad J\cdot\nu|_{\partial\Omega} = 0 \tag{30}$$

with the diffusion flux $J = J(x,\sigma,t)$ of $p = p(x,\sigma,t)$, where ν denotes the outer unit normal vector. We obtain the total patch area conservation for each σ,

$$\frac{\partial M}{\partial t} = \frac{\partial}{\partial t}\int_\Omega p(x,\sigma,t) = 0 \tag{31}$$

because $\overline{u}\cdot\nu|_{\partial\Omega} = 0$ follows from Eq. (21). Eq. (30) implies

$$\frac{\partial\overline{\omega}}{\partial t} + \nabla\cdot(\overline{\omega}\,\overline{u} + J_\omega) = 0, \tag{32}$$

where $J_\omega = \int \sigma J(x,\sigma,t)d\sigma$ stands for the local mean vorticity flux. Since $J_\omega\cdot\nu = 0$ on $\partial\Omega$, Eq. (32) implies conservation of circulation $\Gamma = \int_\Omega \overline{\omega}$. Furthermore, J_ω is associated with the detailed fluctuation of (ω, u) from $(\overline{\omega}, \overline{u})$ by Eq. (1).

Here, we ignore the diffusion energy $E_d = \frac{1}{2} \iint \frac{J^2}{p} d\sigma dx$ to take

$$E = \frac{1}{2} \int_\Omega \overline{\omega}\overline{\psi} \tag{33}$$

as the total energy of this system. Using maximum entropy production principle, we chose the flux J to maximize entropy production rate $\dot{S}$ under the constraint

$$\dot{E} = 0, \quad \int J d\sigma = 0, \quad \int \frac{J^2}{2p} d\sigma \leq C(x,t) \tag{34}$$

where

$$S(p) = -\iint p(x,\sigma,t) \log p(x,\sigma,t) d\sigma dx.$$

Using Lagrange multipliers $(\beta_p, D, \zeta) = \left(\beta_p(t), D(x,t), \zeta(x,t)\right)$, we obtain

$$\delta\dot{S} - \beta_p \delta\dot{E} - \int_\Omega D^{-1} \left(\delta \int \frac{J^2}{2p} d\sigma\right) dx - \int_\Omega \zeta \left(\delta \int J d\sigma\right) dx = 0. \tag{35}$$

Since

$$\begin{aligned} \dot{E} &= \frac{d}{dt} E = \int_\Omega \overline{\psi} \frac{\partial \overline{\omega}}{\partial t} = \int_\Omega J_\omega \cdot \nabla \overline{\psi} = \iint \sigma J \cdot \nabla \overline{\psi} d\sigma dx \\ \dot{S} &= \frac{d}{dt} S = -\iint \frac{\partial p}{\partial t} (\log p + 1) d\sigma dx = -\iint J \cdot \frac{\nabla p}{p} d\sigma dx, \end{aligned} \tag{36}$$

Eq. (35) is reduced to

$$J = -D(\nabla p + \beta_p \sigma p \nabla \overline{\psi} + p\zeta). \tag{37}$$

From the constraint of Eq. (34), it follows that

$$0 = \int J d\sigma = -\int D(\nabla p + \beta_p \sigma p \nabla \overline{\psi} + p\zeta) d\sigma = -D(\beta_p \overline{\omega} \nabla \overline{\psi} + \zeta) \tag{38}$$

and

$$\begin{aligned} 0 = \iint \sigma J \cdot \nabla \overline{\psi}\, d\sigma dx &= \iint -\sigma D(\nabla p + \beta_p \sigma p \nabla \overline{\psi} + p\zeta) \cdot \nabla \overline{\psi} d\sigma dx \\ &= \iint -\sigma D\left(\nabla p + \beta_p (\sigma p - p\overline{\omega}) \nabla \overline{\psi}\right) \cdot \nabla \overline{\psi} d\sigma dx \\ &= -\int_\Omega D \nabla \overline{\omega} \cdot \nabla \overline{\psi} dx - \beta_p \int_\Omega D\left(\int \sigma^2 p d\sigma - \overline{\omega}^2\right) |\nabla \overline{\psi}|^2 dx \end{aligned} \tag{39}$$

which implies

$$\zeta = -\beta_p \overline{\omega} \nabla \overline{\psi} \tag{40}$$

and

$$\beta_p = -\frac{\int_\Omega D \nabla \overline{\omega} \cdot \nabla \overline{\psi}}{\int_\Omega D \left(\int \sigma^2 p d\sigma - \overline{\omega}^2 \right) |\nabla \overline{\psi}|^2} \tag{41}$$

Thus, we end up with

$$\begin{aligned} &\frac{\partial p}{\partial t} + \nabla \cdot (p\overline{u}) = \nabla \cdot D\left(\nabla p + \beta_p(\sigma - \overline{\omega}) p \nabla \overline{\psi}\right), \quad \beta_p = -\frac{\int_\Omega D \nabla \overline{\omega} \cdot \nabla \overline{\psi}}{\int_\Omega D(\int \sigma^2 p d\sigma - \overline{\omega}^2) |\nabla \overline{\psi}|^2} \\ &D\left(\nabla p + \beta_p(\sigma - \overline{\omega}) p \nabla \overline{\psi}\right) \cdot \nu|_{\partial\Omega} = 0, \quad \overline{\omega} = \int_I \sigma p d\sigma = -\Delta \overline{\psi}, \quad \overline{\psi}|_{\partial\Omega} = 0, \quad \overline{u} = \nabla^\perp \overline{\psi} \end{aligned} \tag{42}$$

by Eqs. (30), (37), (40) and (41), where $D = D(x, t) > 0$.

3. Point vortex model

Point vortex model is regarded as a special case of vorticity patch model, where the patch size shrinks to zero [16]. Here, we give a quantitative description of this limit process, using localization. First, we derive the equilibrium mean field equation of point vortices from that of vorticity patches. Then, we derive relaxation equation for the point vortex model. Fundamental quantities of point vortex model are circulation $\alpha\tilde{\alpha}$, probability $\rho^{\tilde{\alpha}}(x, t)$ and number density $P(d\tilde{\alpha})$. Circulation of each vortex is set to be small to preserve total energy and total circulation in the mean field limit. In the vorticity patch model, on the other hand, vorticity σ and probability $p(x, \sigma, t)$ are the fundamental quantities (**Figure 1**).

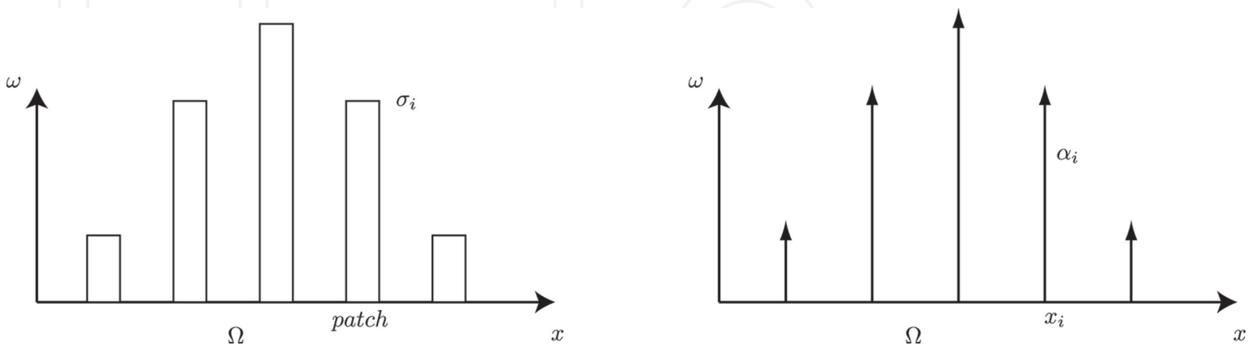

Figure 1. Vorticity distribution: vorticity patch model (left). point vortex model (right).

Here, we use the following localization in order to transform vorticity patch to point vortex (**Figure 2**):

1. Divide each patch into two patches with half area and the same vorticity.
2. Again, divide each patch into two patches with half area: one has doubled vorticity and the other has 0 vorticity.

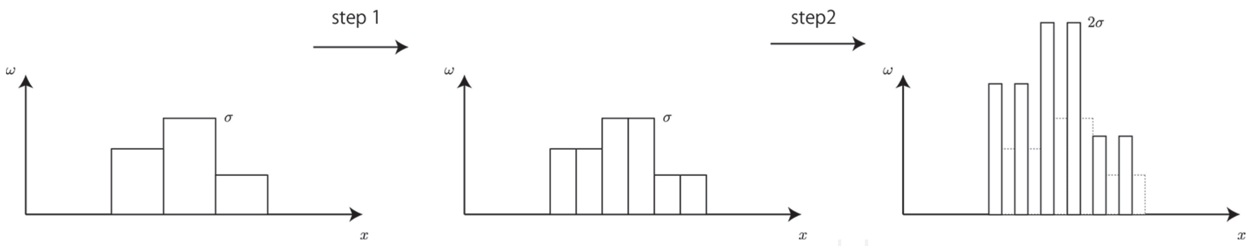

Figure 2. Sketch of localization procedure.

Under this procedure, the number of nonzero patches is doubled and their vorticities are also doubled. At the same time, the area of each patch becomes 1/4 and the number of total patches is quadrupled, while the total circulation is preserved. First, we describe the detailed process for the stationary state of Eq. (7).

Let Ω be divided into many cells with uniform size Δ and let each cell be composed of many patches. Let $N^{(k)}(x,\sigma)dxd\sigma$ be the number of patches in the cell after k-times of the above procedure centered at x of which vorticity was originally σ and let $\sigma^{(k)}$ be the vorticity of these patches after k-times localization. We assume that the number of total vorticity patches in the cell,

$$N_c^{(k)}(\Delta) = \int N^{(k)}(x,\sigma)d\sigma, \tag{43}$$

is independent of x. Then, the number of total patches in Ω, the total area of the patches and the total circulation of the patches after k-times localization procedures, with original vorticity σ, are given by

$$N^{(k)}(\sigma)d\sigma = \int_\Omega N^{(k)}(x,\sigma), \quad M^{(k)}(\sigma)d\sigma = |\Omega| \frac{N^{(k)}(\sigma)d\sigma}{\int N^{(k)}(\sigma)d\sigma}, \tag{44}$$

and

$$\gamma^{(k)}(\sigma)d\sigma = \sigma^{(k)} M^{(k)}(\sigma)d\sigma, \tag{45}$$

respectively.

We obtain

$$N_p = \iint N^{(0)}(x,\sigma)d\sigma dx, \tag{46}$$

recalling Eq. (7). Since

$$\sigma^{(k)} = 2^k\sigma, \tag{47}$$

it holds that

$$N^{(k)}(x,\sigma)dxd\sigma = (4^k{-}2^k)N_c^{(0)}(\Delta)\delta_0(d\sigma) + 2^k N^{(0)}(x,\sigma)dxd\sigma. \tag{48}$$

From Eq. (48), the related probability

$$p^{(k)}(x,\sigma)dxd\sigma = \frac{N^{(k)}(x,\sigma)dxd\sigma}{N_c^{(k)}(\Delta)} \tag{49}$$

satisfies

$$\begin{aligned} p^{(k)}(x,\sigma)dxd\sigma &= \frac{(4^k{-}2^k)N_c^{(0)}(\Delta)\delta_0(d\sigma) + 2^k N^{(0)}(x,\sigma)dxd\sigma}{(4^k{-}2^k)N_c^{(0)}(\Delta) + 2^k \int N^{(0)}(x,\sigma)d\sigma} \\ &= \frac{(4^k{-}2^k)N_c^{(0)}(\Delta)\delta_0(d\sigma) + 2^k N^{(0)}(x,\sigma)dxd\sigma}{4^k N_c^{(0)}(\Delta)} \end{aligned} \tag{50}$$

and hence,

$$\lim_{k\to\infty} p^{(k)}(x,\sigma)dxd\sigma = \delta_0(d\sigma). \tag{51}$$

We also have

$$M^{(k)}(\sigma)d\sigma = \int_\Omega p^{(k)}(x,\sigma)dx = \lim_{\Delta\to 0}\sum_{i=1}^{|\Omega|/\Delta} \frac{N^{(k)}(x_i,\sigma)dxd\sigma}{N_c^{(k)}(\Delta)}\cdot\Delta \tag{52}$$

which implies

$$\begin{aligned} M^{(k)}(\sigma)d\sigma &= \frac{|\Omega|}{4^k N_p}\lim_{\Delta\to 0}\sum_{i=1}^{|\Omega/\Delta|} N^{(k)}(x_i,\sigma)d\sigma = \frac{|\Omega|}{4^k N_p} N^{(k)}(\sigma)d\sigma \\ &= |\Omega|\left((1{-}2^{-k})\delta_0(d\sigma) + 2^{-k}\frac{N^{(0)}(\sigma)d\sigma}{N_p}\right) \end{aligned} \tag{53}$$

by $\frac{\Delta}{N_c^{(k)}(\Delta)} = \frac{|\Omega|}{4^k N_p}$ and Eq. (48). We have, therefore,

$$\lim_{k\to\infty} M^{(k)}(\sigma)d\sigma = |\Omega|\delta_0(d\sigma). \tag{54}$$

It holds also that

$$\begin{aligned} \gamma^{(k)}(\sigma) &= \int_\Omega \sigma^{(k)} p^{(k)}(x,\sigma)dx = \int_\Omega \sigma p^{(0)}(x,\sigma)dx \\ &= \sigma^{(k)} M^{(k)}(\sigma)d\sigma = \frac{\sigma|\Omega|}{N_p} N^{(0)}(\sigma)d\sigma \end{aligned} \tag{55}$$

and

$$\overline{\omega}^{(k)}(x) = \int \sigma^{(k)} p^{(k)}(x,\sigma)d\sigma = \int \sigma p^{(0)}(x,\sigma)d\sigma. \tag{56}$$

Fundamental quantities constituting of the mean field limit of point vortex model thus arise as $k\to\infty$.

To explore the relationship between the quantities in two models, we take regards to circulation of one patch, total circulation of patches with original vorticity σ and local mean vorticity. Based on

$$\sigma^{(k)} \cdot \frac{|\Omega|}{4^k N_p} = \tilde{\alpha} \ \cdot \alpha, \ k \gg 1, \tag{57}$$

and Eq. (47), we reach the ansatz $\sigma|\Omega| = \tilde{\alpha}$, $\frac{1}{2^k N_p} = \alpha, 2^k N_p = N$. Similarly, we use

$$\frac{\sigma|\Omega|}{N_p} N^{(0)}(\sigma)d\sigma = \tilde{\alpha}P(d\tilde{\alpha}) \tag{58}$$

to put

$$\frac{N^{(0)}(\sigma)d\sigma}{N_p} = \frac{M^{(0)}(\sigma)d\sigma}{|\Omega|} = P(d\tilde{\alpha}) \tag{59}$$

by

$$\frac{\sigma|\Omega|}{N_p} N^{(0)}(\sigma)d\sigma = \sigma|\Omega| \cdot \frac{1}{2^k N_p} \cdot 2^k N_p \cdot \frac{N^{(0)}(\sigma)d\sigma}{N_p} = \tilde{\alpha}\alpha N P(d\tilde{\alpha}) = \tilde{\alpha}P(d\tilde{\alpha}). \tag{60}$$

Finally, we use the identity on local mean vorticity

$$\int \sigma p^{(0)}(x,\sigma)d\sigma = \int \tilde{\alpha}\rho^{\tilde{\alpha}}(x)P(d\tilde{\alpha}) \tag{61}$$

to assign

$$\frac{1}{|\Omega|} p^{(0)}(x,\sigma)d\sigma = \rho^{\tilde{\alpha}}(x)P(d\tilde{\alpha}), \tag{62}$$

regarding

$$\int \sigma p^{(0)}(x,\sigma)d\sigma = \int \sigma|\Omega| \cdot \frac{p^{(0)}(x,\sigma)}{|\Omega|} d\sigma = \int \tilde{\alpha}\rho^{\tilde{\alpha}}(x)P(d\tilde{\alpha}). \tag{63}$$

These relations are summarized in the following **Table 1**:

Vorticity patch model	Point vortex model
$\sigma\|\Omega\|$	$\tilde{\alpha}$
$\frac{1}{2^k N_p}$	α
$2^k N_p$	N
$\frac{N^{(0)}(\sigma)d\sigma}{N_p}$	$P(d\tilde{\alpha})$
$\frac{1}{\|\Omega\|} p^{(0)}(x,\sigma)d\sigma$	$\rho^{\tilde{\alpha}}(x)P(d\tilde{\alpha})$

Table 1. Relation between vorticity patch model and point vortex model for $\tilde{\alpha}$.

After k-times localization, the first equation in Eq. (29) takes the form

$$-\Delta\overline{\psi} = \int \sigma^{(k)} M^{(k)}(\sigma) \frac{p^{(k)}(x,0)e^{-\beta_p \sigma^{(k)}\overline{\psi}}}{\int_\Omega p^{(k)}(x,0)e^{-\beta_p\sigma^{(k)}\overline{\psi}}} d\sigma$$
$$= \int \frac{\sigma|\Omega|}{N_p} N^{(0)}(\sigma) \frac{p^{(k)}(x,0)e^{-\beta_p 2^k \sigma\overline{\psi}}}{\int_\Omega p^{(k)}(x,0)e^{-\beta_p 2^k\sigma\overline{\psi}}} d\sigma \tag{64}$$
$$= \int \sigma|\Omega| \frac{p^{(k)}(x,0)e^{-\beta_p \frac{2^k}{|\Omega|}\cdot\sigma|\Omega|\cdot\overline{\psi}}}{\int_\Omega p^{(k)}(x,0)e^{-\beta_p\frac{2^k}{|\Omega|}\cdot\sigma|\Omega|\cdot\overline{\psi}}} \frac{N^{(0)}(\sigma)}{N_p} d\sigma.$$

From **Table 1**, the right-hand side on Eq. (64) is replaced by

$$\int \tilde{\alpha} \frac{p^{(k)}(x,0)e^{-\frac{\beta_N}{N}\tilde{\alpha}\overline{\psi}}}{\int_\Omega p^{(k)}(x,0)e^{-\frac{\beta_N}{N}\tilde{\alpha}\overline{\psi}}} P(d\tilde{\alpha}) \tag{65}$$

for $\beta_N = 4^k \frac{N_p}{|\Omega|}\beta_p = N \cdot \frac{2^k\beta_p}{|\Omega|}$. Sending $k \to \infty$, we obtain the first equation of (6) with $\beta = \frac{\beta_N}{N}$ by Eq. (51). This means that the vorticity patch model is transformed to the point vortex model applied to the mean field limit by taking the localization procedure.

We can derive also relaxation equation of point vortex model from that of vorticity patch model. By Eq. (37), the value of the diffusion flux J for $\sigma = 0$ is

$$J(x,0,t) = -D(x,t)\Big(\nabla p(x,0,t) + p(x,0,t)\zeta(x,t)\Big) \tag{66}$$

and hence

$$\zeta(x,t) = \frac{-D(x,t)^{-1}J(x,0,t) + \nabla p(x,0,t)}{p(x,0,t)}. \tag{67}$$

Flux is thus given by

$$J(x,\sigma,t) = -D(x,t)\left(\nabla p(x,\sigma,t) + \beta_p(t)\sigma p(x,\sigma,t)\nabla\overline{\psi}(x,t) - p(x,\sigma,t)\frac{D(x,t)^{-1}J(x,0,t) + \nabla p(x,0,t)}{p(x,0,t)}\right). \tag{68}$$

We reach

$$\frac{\partial p}{\partial t} + \nabla\cdot(p\overline{u}) = \nabla\cdot D\left(\nabla p + \beta_p \sigma p \nabla\overline{\psi} - p\left[\frac{D^{-1}J + \nabla p}{p}\right]_{\sigma=0}\right) \tag{69}$$

with

$$\beta_p = \beta_p(t) = -\frac{\int_\Omega D\nabla\overline{\omega}\cdot\nabla\overline{\psi} - \int_\Omega D\overline{\omega}\left[\frac{D^{-1}J+\nabla p}{p}\right]_{\sigma=0}\cdot\nabla\overline{\psi}}{\iint D\sigma^2 p|\nabla\overline{\psi}|^2 d\sigma dx} \tag{70}$$

Therefore, after k-times localization procedure, it holds that

$$\begin{aligned}&\frac{\partial\sigma^{(k)}p^{(k)}}{\partial t}+\nabla\cdot\left(\sigma^{(k)}p^{(k)}\overline{u}\right)\\&=\nabla\cdot D\left(\nabla\sigma^{(k)}p^{(k)}+\beta_p(\sigma^{(k)})^2p^{(k)}\nabla\overline{\psi}-\sigma^{(k)}p^{(k)}\left[\frac{D^{-1}J^{(k)}+\nabla p^{(k)}}{p^{(k)}}\right]_{\sigma=0}\right).\end{aligned} \tag{71}$$

Putting $\beta_N=4^k\frac{N_p}{|\Omega|}\beta_p$, similarly, we obtain

$$\frac{\partial}{\partial t}\left(\tilde{\alpha}\rho^{\tilde{\alpha}}P(d\tilde{\alpha})\right)+\nabla\cdot\left(\tilde{\alpha}\rho^{\tilde{\alpha}}P(d\tilde{\alpha})\overline{u}\right)=\nabla\cdot\left(D\left(\nabla\left(\tilde{\alpha}\rho^{\tilde{\alpha}}P(d\tilde{\alpha})\right)+\beta\tilde{\alpha}^2\rho^{\tilde{\alpha}}P(d\tilde{\alpha})\nabla\overline{\psi}\right)\right), \tag{72}$$

from

$$\begin{aligned}&\lim_{k\to\infty}p^{(k)}(x,\sigma,t)=\delta_0(d\sigma),\quad \lim_{k\to\infty}J^{(k)}(x,0,t)=0\\&\sigma^{(k)}p^{(k)}(x,\sigma,t)=\sigma p^{(0)}(x,\sigma,t)=\sigma|\Omega|\cdot\frac{p^{(0)}(x,\sigma,t)}{|\Omega|}\approx\tilde{\alpha}\rho^{\tilde{\alpha}}(x,t)P(d\tilde{\alpha})\\&(\sigma^{(k)})^2p^{(k)}(x,\sigma,t)=2^k\sigma\cdot\sigma p^{(0)}(x,\sigma,t)=\frac{2^k}{|\Omega|}\cdot(\sigma|\Omega|)^2\cdot\frac{p^{(0)}(x,\sigma,t)}{|\Omega|}\approx\frac{2^k}{|\Omega|}\tilde{\alpha}^2\rho^{\tilde{\alpha}}(x,t)P(d\tilde{\alpha})\end{aligned} \tag{73}$$

Here, we assume $\lim_{k\to\infty}J^{(k)}(x,0,t)=0$, because $\int J^{(k)}(x,\sigma,t)d\sigma=0$ and the 0-vorticity patch becomes dominant in the system. Then, we obtain Eq. (9) by Eq. (72).

4. Relaxation dynamics

If $P(d\tilde{\alpha})=\delta_1(d\tilde{\alpha})$, it holds that $\overline{\omega}=\overline{\omega}_2$ in Eq. (11). Then, we obtain

$$\omega_t+\nabla\cdot\omega\nabla^{\perp}\psi=\nabla\cdot(\nabla\omega+\beta\omega\nabla\psi),\quad \frac{\partial\omega}{\partial\nu}+\beta\omega\frac{\partial\psi}{\partial\nu}|_{\partial\Omega}=0,\quad \omega|_{t=0}=\omega_0(x)\geq 0 \tag{74}$$

$$-\Delta\psi=\omega,\quad \psi|_{\partial\Omega}=0,\quad \beta=-\frac{\int_\Omega\nabla\omega\cdot\nabla\psi}{\int_\Omega\omega|\nabla\psi|^2} \tag{75}$$

assuming $D=1$. Conservations of total mass and energy

$$\|\omega(\cdot,t)\|_1=\lambda,\quad \left(\psi(\cdot,t),\omega(\cdot,t)\right)=e, \tag{76}$$

are derived from Eq. (13), while increase in entropy of Eq. (16) is reduced to

$$\frac{d}{dt}\int_\Omega\Phi(\omega)=-\int_\Omega\omega|\nabla(\log\omega-\beta\psi)|^2\leq 0, \tag{77}$$

where $\Phi(s)=s(\log s-1)+1$.

In the stationary state, we obtain $\log\omega + \beta\psi = \text{constant}$ by Eq. (77). Hence, it follows that

$$-\Delta\psi = \omega, \quad \psi|_{\partial\Omega} = 0, \;\; \omega = \frac{\lambda e^{-\beta\psi}}{\int_\Omega e^{-\beta\psi}}, \;\; -\beta = \frac{\int_\Omega \nabla\omega \cdot \nabla\psi}{\int_\Omega \omega|\nabla\psi|^2}, \quad e = \int_\Omega \omega\psi \tag{78}$$

from Eq. (76). Here, the third equation implies the fourth equation as

$$(\nabla\omega, \nabla\psi) = -\beta\int_\Omega \omega|\nabla\psi|^2. \tag{79}$$

Using

$$v = \beta\psi, \quad \mu = \frac{\beta\lambda}{\int_\Omega e^{-\beta\psi}}, \tag{80}$$

therefore, Eq. (78) is reduced to

$$-\Delta v = \mu e^{-v}, \quad v|_{\partial\Omega} = 0, \quad \frac{e}{\lambda^2} = \frac{\int_\Omega |\nabla v|^2}{\left(\int_{\partial\Omega} -\frac{\partial v}{\partial \nu}\right)^2}\;. \tag{81}$$

In fact, to see the third equality of (81), we note

$$e = (\omega, \psi) = \beta^{-1}\frac{\lambda\int_\Omega e^{-v}v}{\int_\Omega e^{-v}} \tag{82}$$

which implies

$$\mu = \frac{\lambda}{\int_\Omega e^{-v}} \cdot \frac{\lambda}{e}\frac{\int_\Omega e^{-v}v}{\int_\Omega e^{-v}} = \frac{\lambda^2}{e} \cdot \frac{\int_\Omega e^{-v}v}{\left(\int_\Omega e^{-v}\right)^2} \tag{83}$$

and hence

$$\frac{e}{\lambda^2} = \frac{1}{\mu} \cdot \frac{\int_\Omega e^{-v}v}{\left(\int_\Omega e^{-v}\right)^2} = \frac{\|\nabla v\|_2^2}{\left(\int_{\partial\Omega} -\frac{\partial v}{\partial \nu}\right)^2}\;. \tag{84}$$

If $\mu < 0$, system of Eq. (81) except for the third equation is equivalent to the Gel'fand problem

$$-\Delta w = \sigma e^w, \quad w|_{\partial\Omega} = 0 \tag{85}$$

with $\sigma = -\mu$. If Ω is simply connected, there is a non-compact family of solutions as $\mu \uparrow 0$, which are uniformly bounded near the boundary [8, 9]. Hence, there arises

$$\lim_{\mu\uparrow 0}\frac{e}{\lambda^2} = +\infty \tag{86}$$

for this family. For $\mu \geq 0$, on the contrary, system of Eq. (81) except for the third equation admits a unique solution $v = v_\mu(x)$. Regarding Eq. (76), therefore, it is necessary that

$$\lim_{\mu\uparrow+\infty} \frac{\| \nabla v_\mu \|_2^2}{\left(\int_{\partial\Omega} -\frac{\partial v_\mu}{\partial \nu}\right)^2} = 0 \tag{87}$$

for any orbit to Eqs. (74), (75) to be global-in-time and compact, for any $\lambda, e > 0$ in Eq. (76).

If $\Omega = B \equiv \{x \in \mathbb{R}^2 |\ |x| < 1\}$, it actually holds that Eq. (87). In this case, we have $v = v(r)$, $r = |x|$, and the result follows from an elementary calculation. More precisely, putting $u = v - \log \mu$, $s = \log r$, we obtain

$$u_{ss} + e^{-u+2s} = 0, \quad s < 0, \quad u(0) = -\log\mu, \quad \lim_{s\downarrow-\infty} u_s e^{-s} = 0, \quad \frac{\| \nabla v \|_2^2}{\left(\int_{\partial\Omega} -\frac{\partial v}{\partial \nu}\right)^2} = \frac{I}{2\pi}, \tag{88}$$

where $I = \frac{\int_{-\infty}^{0} u_s^2 ds}{u_s(0)^2}$. Using $w = u - 2s$, $p = \frac{1}{\sqrt{2}}(e^{-w} + 2)^{1/2}$, we have

$$p = -1 + 2(1 - ce^{2s})^{-1} \tag{89}$$

with $c \uparrow 1$ as $\mu \uparrow +\infty$. It follows that

$$I = (1-c)^2 \int_{-\infty}^{0} \frac{e^{4s}}{(1-ce^{2s})^2} ds \tag{90}$$

with

$$\int_{-\infty}^{0} \frac{e^{4s}}{(1-ce^{2s})^2} ds = \frac{1}{2c(1-c)} + \frac{1}{2c^2} \log(1-c) \tag{91}$$

and hence

$$\lim_{c\uparrow 1} I = 0. \tag{92}$$

If β is constant in Eq. (9), it is the mean field limit of Brownian vortices [15]. It is nothing but the Smoluchowski-Poisson equation [9, 17] and obeys the feature of canonical ensemble, provided with total mass conservation and decrease of free energy:

$$\frac{d\mathcal{F}}{dt} = -\int_\Omega \omega |\nabla(\log \omega + \beta\psi)|^2, \quad \mathcal{F}(\omega) = \int_\Omega \Phi(\omega) - \frac{1}{2}\left((-\Delta)^{-1}\omega, \omega\right). \tag{93}$$

Then, there arises the blowup threshold $\beta = -8\pi/\lambda$ [18]. Here, we show the following theorem, where $G = G(x, x')$ denotes the Green's function for the Poisson part,

$$-\Delta G(\cdot, x') = \delta_{x'}, \quad G(\cdot, x')|_{\partial\Omega} = 0, \quad x' \in \Omega \tag{94}$$

and

$$\rho_\varphi(x, x') = \nabla\varphi(x) \cdot \nabla_x G(x, x') + \nabla\varphi(x') \cdot \nabla_{x'} G(x, x'), \quad \varphi \in X, \tag{95}$$

where $X = \left\{\varphi \in C^2(\overline{\Omega}) | \frac{\partial\varphi}{\partial\nu}\Big|_{\partial\Omega} = 0\right\}$. It holds that $\rho_\varphi \in L^\infty(\Omega \times \Omega)$. The proof is similar as in Lemma 5.2 of [17] for the case of Neumann boundary condition.

Theorem 1: Let $\Omega = B$ and ω_0 be a smooth function in the form of $\omega_0 = \omega_0(r) > 0$ with $\omega_{0r} < 0$, $0 < r \le 1$. Let $T \in (0, +\infty]$ be the maximal existence time of the classical solution to Eqs. (74), (75) and λ be the total mass defined by Eq. (76). Then, it follows that

$$\lim\sup_{t\uparrow T}\beta(t) < -\frac{8\pi}{\lambda} \ \Rightarrow \ T < +\infty \tag{96}$$

and

$$T < +\infty \ \Rightarrow \ \lim\inf_{t\uparrow T}\beta(t) = -\infty. \tag{97}$$

In particular, we have

$$\lim\inf_{t\uparrow T}\beta(t) > -\infty \Rightarrow \ T = +\infty, \ \lim\sup_{t\uparrow T}\beta(t) \ge -\frac{8\pi}{\lambda} \ . \tag{98}$$

Proof: From the assumption, it follows that $(\omega, \psi) = \big(\omega(r,t), \psi(r,t)\big)$ and

$$\omega_r, \ \psi_r < 0, \quad 0 < r \le 1.$$

Then, we obtain

$$M \equiv \frac{\lambda}{2\pi} \ge \int_0^r r\omega dr \ge \omega(r,t) \int_0^r rdr = \frac{r^2}{2}\omega \tag{99}$$

and hence

$$\omega(r,t) \le \frac{2M}{r^2}, \quad 0 < r \le 1. \tag{100}$$

It holds also that

$$-\beta = \frac{\int_0^1 \omega_r \psi_r r dr}{\int_0^1 \omega \psi_r^2 r dr} > 0 \tag{101}$$

which implies

$$\omega_t = \Delta\omega + \beta\nabla\psi \cdot \nabla\omega + \beta\omega\Delta\psi = \Delta\omega + \beta\nabla\psi \cdot \nabla\omega - \beta\omega^2 \geq \Delta\omega + \beta\nabla\psi \cdot \nabla\omega \tag{102}$$

with

$$-\frac{\partial\omega}{\partial\nu} = \beta\omega\frac{\partial\psi}{\partial\nu} > 0 \quad \text{on} \quad \partial\Omega \times (0, T). \tag{103}$$

The comparison theorem now guarantees $\omega \geq \delta \equiv \min_{\overline{\Omega}} \omega_0 > 0$ and hence

$$\int_\Omega \omega|\nabla\psi|^2 \geq \delta\int_\Omega |\nabla\psi|^2 = \delta e. \tag{104}$$

For Eq. (96) to prove, we use the second moment. First, the Poisson part of Eq. (75) is reduced to

$$-r\psi_r = \int_0^r r\omega dr \equiv A(r). \tag{105}$$

Second, it follows that

$$\begin{aligned} \frac{d}{dt}\int_0^1 \omega r^3 dr &= -\int_0^1 (\omega_r + \beta\omega\psi_r) 2r \cdot r dr \\ &= -2r^2\omega|_{r=0}^{r=1} + \int_0^1 4r\omega - 2\beta\omega\psi_r r^2 dr \\ &= -2\omega|_{r=1} + 4M + 2\beta\int_0^1 AA_r dr \\ &= -2\omega|_{r=1} + 4M + \beta M^2 \leq 4M + \beta M^2 \end{aligned} \tag{106}$$

from $A(1) = M$. Under the hypothesis of Eq. (96), we have $\delta > 0$ such that

$$4M + \beta M^2 \leq -\delta, \quad t \uparrow T. \tag{107}$$

Then, $T = +\infty$ gives a contradiction.

Now, we assume $T < +\infty$. First, equality in (106) implies

$$\int_0^T -\beta(t) dt \leq C \tag{108}$$

by Eq. (100). Second, we have

$$\frac{d}{dt}\int_{\Omega}\omega\varphi = \int_{\Omega}\omega\Delta\varphi + \frac{\beta}{2}\iint_{\Omega\times\Omega}\rho_{\varphi}\omega\otimes\omega \tag{109}$$

and hence

$$\int_0^T \left|\frac{d}{dt}\int_{\Omega}\omega\varphi\right| dt \le C_{\varphi}, \varphi \in X. \tag{110}$$

Inequality (110) takes place of the monotonicity formula used for the Smoluchowski-Poisson equation, which guarantees the continuation of $\omega(x,t)dx$ up to $t = T$ as a measure on $\overline{\Omega}$ [9, 17]. Thus, there is $\mu = \mu(dx,t) \in C_*\left([0,T], \mathcal{M}(\overline{\Omega})\right)$ such that $\mu(dx,t) = \omega(x,t)dx$ for $0 \le t < T$. By Eq. (100), therefore, it holds that

$$\omega(x,t)dx \rightharpoonup c\delta_0(dx) + f(x)dx \quad \text{in} \quad \mathcal{M}(\overline{\Omega}), \quad t\uparrow T, \tag{111}$$

with $c \ge 0$ and $0 \le f = f(x) \in L^1(\Omega)$. From the elliptic regularity, we obtain

$$\liminf_{t\uparrow T}\ \psi(x,t) \ge \frac{c}{2\pi}\log\frac{1}{|x|} \quad \text{loc. unif. in } \ \overline{\Omega}\setminus\{0\}. \tag{112}$$

Then, $e = \left(\omega(\cdot,t), \psi(\cdot,t)\right) \ge \left(\omega(\cdot,t), min\{k, \psi(\cdot,t)\}\right)$ implies $e \ge \frac{c}{2\pi}\min\left\{k, \log\frac{1}{|x|}\right\}$ for k = 1,2,. Hence, it holds that $c = 0$ in Eq. (111).

If the conclusion in Eq. (97) is false, we have the ε regularity in Eqs. (74), (75) [9, 17]. Thus, there is $\varepsilon_0 = \varepsilon_0^k > 0$, such that

$$\limsup_{t\uparrow T}\|\omega(\cdot,t)\|_{L^1(\Omega\cap B(x_0,R))} < \varepsilon_0 \ \Rightarrow\ \limsup_{t\uparrow T}\|\omega(\cdot,t)\|_{L^\infty(\Omega\cap B(x_0,R/2))} < +\infty \tag{113}$$

for $0 < R \ll 1$. The hypothesis in Eq. (113) is valid for $x_0 = 0$ by Eq. (111), c = 0, which contradicts to $T < +\infty$.

5. Conclusion

We study the relaxation dynamics of the point vortices in the incompressible Euler fluid, using the vorticity patch which varies with uniform vorticity and constant area. The mean field limit equation is derived, which has the same form as the one derived for the Brownian point vortex model. This equation governs the last stage of self-organization, not only in the point vortices but also in the two-dimensional center guiding plasma and the rotating superfluid helium, from quasi-equilibrium to equilibrium. Mathematical analysis assures this property for radially symmetric case, provided that the inverse temperature is bounded below.

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