Chapter 10

Numerical Random Periodic Shadowing Orbits of a Class of Stochastic Differential Equations

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Additional information is available at the end of the chapter

http://dx.doi.org/10.5772/67010

Abstract

This paper is devoted to the existence of a true random periodic solution near the numerical approximate one for a kind of stochastic differential equations. A general finite-time random periodic shadowing theorem is proposed for the random dynamical systems generated by some stochastic differential equations under appropriate conditions and an estimate of shadowing distance via computable quantities is given. Application is demonstrated in the numerical simulations of random periodic orbits of the stochastic Lorenz system for certain given parameters.

Keywords: random chaotic system, stochastic differential equations, random periodic shadowing, stochastic Lorenz system

1. Introduction

The investigation for the dynamical properties of the random periodic orbits in some specific stochastic differential equations (SDEs) is a difficult problem [1]. In general, numerical computation is still one of the most feasible methods of studying random periodic orbits of SDEs describing many natural phenomena in meteorology, biology and so on [2–4]. As the chaotic systems is sensitive to the initial value and random noise is constantly affected the systems constantly, to estimate a particular solution of a random chaotic system by numerical solutions for a given length of time is even more difficult. Therefore, it is always difficult to infer the existence of a random periodic orbit rigorously from numerical computations. Shadowing property plays important roles in the theory and applications of random dynamical systems (RDS), especially in the numerical simulations of random chaotic systems generated by some SDEs. As we know, numerical experiments can lead to many nice discoveries, a new numerical method is presented to establish the existence of a true random periodic orbit of SDEs which lies near a numerical random periodic orbit. Furthermore, the reliability and feasibility of numerical computations is considered as well.

There are two main motivations for this work. On the one hand, it follows from the classical shadowing lemma that many studies about the periodic dynamics of deterministic chaotic systems have been performed in Ref. [3] and references therein. Many nice works on the numerical analysis of RDS had been completed in Refs. [5] and [6]. On the other hand, our results in this article have been inspired by our earlier work in Refs. [7] and [8], on shadowing orbits of SDEs where we established in a rather general setting. To the best of our knowledge, shadowing is still an interesting method for studying their random periodic dynamic behavior of SDE, and there is no investigations of the random periodic shadowing theorem of SDE exist in the literatures.

In this work, two computational issues should be considered first. One is the definition of (ω, δ)-pseudo random periodic orbit, in which a true random periodic orbit is sufficiently closed. Another issue is that in which conditions the random chaotic systems generated by some SDE possess the so-called pseudo hyperbolicity for certain given parameters. With some additional numerical computations, we can show the existence of a true random periodic orbit near the (ω, δ)-pseudo random periodic orbit under appropriate conditions. Therefore, the main difference between the existing work and the current one is that the random periodic case is concerned, and there is no hyperbolicity assumption on the original systems.

Utilizing the existence of the modified Newton equation's solution, a random periodic shadowing theorem for some kind of SDEs is proposed. The result shows that under some appropriate conditions, there exists a true periodic orbit near the numerical approximative one and the upper bound for the shadowing distance is given.

This paper is organized as follows. In Section 2, background materials on random shadowing for random dynamical system generated by SDEs, including the definitions of (ω, δ)-pseudo random periodic orbit and the pseudo hyperbolic in mean square, are given. The main result on random periodic shadowing is then stated in Section 3. Illustrative numerical experiments for the main theorem are included in Section 4. The numerical implementations in details are presented in the following section. And, the proof for the main result is presented in Section 6. The final section is devoted to summarize the main results in the current work.

2. Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a canonical Wiener space, $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$ be its natural normal filtration, and $W(t)(t\in\mathbb{R})$ is a standard one-dimensional Brownian motion defined on the space $(\Omega, \mathcal{F}, \mathbb{P})$. And, we assume that $\Omega := \{\omega\in C(\mathbb{R},\mathbb{R}) : \omega(0) = 0\}$, which means that the elements of Ω can be identified with paths of a Wiener process $\omega(t) = W_t(\omega)$. We consider a class of Stratonovich SDEs in the form of

$$dx_t = f(x_t)dt + \mu x_t \circ dW_t, \qquad x(0) = x_0(\omega)\in\mathbb{R}^d, \tag{1}$$

where the random variable $x_0(\omega)$ is independent of $\mathcal{F}_0$ and satisfies the inequality $\mathbb{E}|x_0(\omega)|^2 < \infty$, and μ is a nonzero real number.

http://dx.doi.org/10.5772/67010

2.1. Basic assumptions and notations

We define the metric dynamical systems $(\Omega, \mathcal{F}, \mathbb{P}, \theta^t)$ by the mapping $\theta : \mathbb{R} \times \Omega \to \Omega$, such that for $\omega \in \Omega$,

$$\theta^t \omega(s) = \omega(t+s) - \omega(t),$$

where $s, t \in \mathbb{R}$.

Let $O_t(\omega)$ be a one-dimension random stable Ornstein-Uhlenbeck process which satisfies the following linear SDE

$$dO_t = -O_t dt + dW_t.$$

And let

$$z(t, \omega) := \exp(-\mu O_t(\omega)) x_t(\omega) \in \mathbb{R}^d,$$

then SDE (1) can be changed to a random differential equation (RDE) in the form of

$$\frac{dz}{dt} = \exp(-\mu O_t(\omega)) f(\exp(\mu O_t(\omega)) z) + \mu O_t z = f_1(\theta^t \omega, z). \tag{2}$$

It follows from Doss-Sussmann Theorem in Ref. [9] that the solution of RDE (2) is the solution of SDE (1).

In this paper, we make the following assumptions:

- We suppose that $f_1 : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ be a measurable function which is locally bounded, locally Lipschitz continuous with respect to the first variable, and be a C^1 vector field on $\mathbb{R}^d$.

  By Theorem 2.2.2 in Ref. [2], RDE(2) generates a unique RDS φ on the metric dynamical systems $(\Omega, \mathcal{F}, \mathbb{P}, \theta^t)$ as follows

  $$\varphi(s, t, \omega) z = z + \int_s^t f_1(\theta^\tau \omega, \varphi(s, \tau, \omega) z) d\tau \in \mathbb{R}^d, \tag{3}$$

  and which is C^1-class with respect to z in Ref. [8].

  And there exists a diffeomorphism $\varphi : \mathbb{R} \times \mathbb{R} \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$, $\varphi(s, t, \omega, z) := \varphi(s, t, \omega) z \in \mathbb{R}^d$.

  We also make use of the following notations which is similar to the Ref. [8].

- The norm of a random variable $x = (x_1, x_2, \ldots, x_d) \in L^2(\Omega, \mathbb{P})$ is defined in the form of

  $$\|x\|_2 = \left[\int_\Omega [|x_1(\omega)|^2 + |x_2(\omega)|^2 + ,\ldots, + |x_d(\omega)|^2] d\mathbb{P}(\omega) \right]^{\frac{1}{2}} < \infty,$$

  where $L^2(\Omega, \mathbb{P})$ is the space of all square-integrable random variables $x : \Omega \to \mathbb{R}^d$.

- The norm of a stochastic process $x(t,\omega)$ with $x_t(\omega) \in L^2(\Omega,\mathbb{P})$ and $t \in \mathbb{R}$ is defined as

$$\|x(t,\omega)\|_2 = \sup_{t\in\mathbb{R}} \|x_t(\omega)\|_2 < \infty.$$

- For a given random matrix A, and the operator norm $|\cdot|$, the norm of A is defined as follows

$$\|A\|_{L^2(\Omega,\mathbb{P})} = [\mathbb{E}(|A|^2)]^{\frac{1}{2}}.$$

- Normally, the norm $\|\cdot\|_2$ and $\|\cdot\|_{L^2(\Omega,\mathbb{P})}$ are denoted as $\|\cdot\|$ for simplicity reason, unless otherwise stated.

2.2. Some extended definitions

Definition 2.1. For a given positive number δ, if there is a sequence of positive times $\{t_k\}_{k=0}^{N+1}, 0 \le t_0 \le t_1 \le, \ldots, \le \tau \le t_{N+1},\ \tau$, and a sequence of random variables

$$\{(y_k(\theta^{t_k}\omega), \mathcal{F}_{t_k})\}_{k=0}^{N},$$

$y_k(\theta^{t_k}\omega)$ is $\mathcal{F}_{t_k}$-adapted, such that

$$f_1(y_k(\theta^{t_k}\omega))y_k(\theta^{t_k}\omega) \neq 0, \qquad \mathbb{P}\text{-almost surely for } k = 0, 1, 2, \ldots, N,$$

and the following inequalities $\mathbb{P}$-almost surely hold

$$\|y_{k+1}(\theta^{t_{k+1}}\omega) - \varphi(t_k, t_{k+1}, \theta^{t_k}\omega) y_k(\theta^{t_k}\omega)]\| \le \delta,\ k = 0, 1, \ldots, N-1,$$

and

$$\|y_N(\theta^{t_N}\omega) - y_0(\theta^{t_0}\omega)\| \le \delta, \tag{4}$$

then the random variables $\{(y_k(\theta^{t_k}\omega), \mathcal{F}_{t_k})\}_{k=0}^{N}$ are said to be a (ω, δ)-pseudo random periodic orbit of RDS (3) generated by SDE (1) in mean-square sense.

Definition 2.2. For a given positive number ε and a (ω, δ)-pseudo random periodic orbit $\{(y_k(\theta^{t_k}\omega), \mathcal{F}_{t_k})\}_{k=0}^{N}$ of RDS (3) generated by SDE (1) with associated times $\{t_k\}_{k=0}^{N+1}$, if there is a sequence of times $\{h_k\}_{k=0}^{N+1}, h_0 \le h_1 \le, \ldots, \le \tau \le h_{N+1},$ such that the following inequalities hold

$$\|y_k(\theta^{t_k}\omega) - x_k(\theta^{h_k}\omega)\| \le \varepsilon, 0 \le t_k - h_k \le \varepsilon, k = 0, 1, \ldots, N,$$

and the random variables $\{(x_k(\theta^{h_k}\omega), \mathcal{F}_{h_k})\}_{k=0}^{N}$ are on the true orbits of RDS (3) generated by SDE (1), that is

$$x_{k+1}(\theta^{h_{k+1}}\omega) = \varphi(h_k, h_{k+1}, \theta^{h_k}\omega) x_k(\theta^{h_k}\omega),\ k = 0, 1, 2, \ldots, N-1,$$

and

$$x_0(\theta^{h_0}\omega) = \varphi(h_N, h_{N+1}, \theta^{h_N}\omega)x_N(\theta^{h_N}\omega), \tag{5}$$

then the (ω, δ)-pseudo random periodic orbit $\{(y_k(\theta^{t_k}\omega), \mathcal{F}_{t_k})\}_{k=0}^{N}$ is said to be (ω, δ)-periodic shadowed by a true orbit of RDS (3) generated by SDE (1) in mean-square sense.

Remark 2.3. As the σ-algebra $\mathcal{F}_t(t{\geq}0)$ is nondecreasing, in order to guarantee the random variables $x_k(\theta^{h_k}\omega)(k = 0, 1, 2,\ldots, N)$ are $\mathcal{F}_{t_k}$-measurable, we need the shadowing condition $0{\leq}t_k{-}h_k{\leq}\varepsilon$ instead of $|t_k{-}h_k|{\leq}\varepsilon$. We refer to the Ref. [2] for the deterministic counterpart. Here, we choose a sequence of times $\{h_k\}_{k=0}^{N+1} = \{t_k\}_{k=0}^{N+1}$ in sequels.

Definition 2.4. The RDS $\varphi : \mathbb{R}\times\mathbb{R}\times\Omega\times\mathbb{R}^d \to \mathbb{R}^d$ is said to be pseudo hyperbolic in mean square if the temple variables $\kappa_1(\omega), \kappa_2(\omega){\geq}1$, $\nu_1(\omega), \nu_2(\omega){\geq}0$ exist, such that the following inequations hold with $\mathbb{R}^d = E^s(\omega){\oplus}E^u(\omega)$,

$$\mathbb{E}\|\varphi(s, t_1, \omega)x\|^2 \leq \kappa_1(\omega)e^{-\nu_1(\omega)(t_1-t_2)}\mathbb{E}\|\varphi(s, t_2, \omega)x\|^2, \forall t_1 \geq t_2 \geq s, x \in E^s(\omega),$$
$$\mathbb{E}\|\varphi(s, t_2, \omega)x\|^2 \leq \kappa_2(\omega)e^{-\nu_2(\omega)(t_1-t_2)}\mathbb{E}\|\varphi(s, t_1, \omega)x\|^2, \forall t_1 \geq t_2{\geq}s, x \in E^u(\omega).$$

This means that there is a splitting into exponentially stable $(E^s(\omega))$ and unstable $(E^u(\omega))$ components. The multiplicative ergodic theorem (MET) of Oseledets in [10] provides the stochastic analogue of the deterministic spectral theory of matrices, and a method to check the pseudo hyperbolicity.

3. Random periodic shadowing for RDS generated by SDEs

3.1. Theoretical foundations

Let $\{(y_k(\theta^{t_k}\omega), \mathcal{F}_{t_k})\}_{k=0}^{N}$ be a (ω, δ)-pseudo random periodic orbit of RDS (3) generated by SDE (1) and $y_k(\theta^{h_k}\omega){\in}L^2(\Omega, \mathbb{P})(k = 0, 1,\ldots, N)$. Assume that we have a sequence of $d \times d$ random matrices $\{(Y_k(\theta^{t_k}\omega), \mathcal{F}_{t_k})\}_{k=0}^{N}$ such that

$$\|Y_{k+1}(\theta^{t_{k+1}}\omega){-}D\varphi(t_k, t_{k+1}, \theta^{t_k}\omega)y_k(\theta^{t_k}\omega)\| \leq \delta, \quad for\ \ k = 0, 1,\ldots, N{-}1,$$

and

$$\|Y_0(\theta^{t_0}\omega){-}D\varphi(t_N, t_{N+1}, \theta^{t_N}\omega)y_N(\theta^{t_N}\omega)\| \leq \delta. \tag{6}$$

A sequence of $d \times (d-1)$ random matrices $(S_k(\theta^{t_k}\omega), \mathcal{F}_{t_k})$ are chosen such that its columns form an approximate orthogonal basis for the subspace orthogonal to $T(x_k)$ and $k = 0, 1, \ldots, N$, where $T(x_k) = f_1(\theta^{t_k}\omega, x_k)$, the approximate orthogonal means that the following inequality holds

$$\|S_k(\theta^{t_k}\omega)S_k^*(\theta^{t_k}\omega){-}I\| \leq \delta_1,$$

for some positive number $\delta_1{\in}(0,\delta)$, where * denotes the transpose of matrix.

Now a sequence of $(d-1) \times (d-1)$ random matrices $A_k(\theta^{t_k}\omega)$ is chosen which satisfy

$$\|A_k(\theta^{t_k}\omega) - S^*_{k+1}(\theta^{t_{k+1}}\omega)Y_k(\theta^{t_k}\omega)S_k(\theta^{t_k}\omega)\| \le \delta, \ \ for \ \ k = 0, 1, \ldots N-1,$$

and

$$\|A_N(\theta^{t_N}\omega) - S^*_0(\theta^{t_0}\omega)Y_N(\theta^{t_N}\omega)S_N(\theta^{t_N}\omega)\| \le \delta.$$

Next, a linear operator L is defined as follows. If random variables $\xi = \{\xi_k(\theta^{t_k}\omega)\}_{k=0}^N$ are in the space $(\mathbb{R}^{d-1})^{N+1}$, then we let $L\xi = \{[L\xi]_k\}_{k=0}^N$ to be

$$[L\xi]_k = \xi_{k+1}(\theta^{t_{k+1}}\omega) - A_k(\theta^{t_k}\omega)\xi_k(\theta^{t_k}\omega), \ \ \ for \ \ k = 0, 1, \ldots, N-1.$$

and

$$[L\xi]_N = \xi_0(\theta^{t_0}\omega) - A_N(\theta^{t_N}\omega)\xi_N(\theta^{t_k}\omega).$$

It follows from Section 4.2 that the operator L has right inverses and we choose one such right inverse L^{-1}.

At last, we define some constants. Let U be a convex subset of $\mathbb{R}^d$ containing the value of the (ω, δ)-pseudo orbit $\{(y_k(\theta^{t_k}\omega), \mathcal{F}_{t_k})\}_{k=0}^N$. Therefore, we define

$$\Delta t = \inf_{0 \le k \le N} \Delta t_{k+1} = \inf_{0 \le k \le N} (t_{k+1} - t_k).$$

Next, we choose a positive number $0 < \varepsilon_0 \le \Delta t$ such that $\|x - y_k(\theta^{t_k}\omega)\| \le \varepsilon_0$, then the solution $\varphi(s, t, \omega)x(s \le t)$ is defined and remains in U for $0 < t \le t_k + \varepsilon_0$ $\mathbb{P}$-almost surely.

Finally, we define

$$\begin{aligned} M_0 &= \sup_{x \in U} \|f_1(\theta^t\omega, x(t))\|, \\ M_1 &= \sup_{x \in U} \|Df_1(\theta^t\omega, x(t))\|, \\ M_2 &= \sup_{x \in U} \|D^2 f_1(\theta^t\omega, x(t))\| \end{aligned}$$

and

$$\Theta = \sup_{0 \le k \le N-1} \|Y_k(\theta^{t_k}\omega)\|,$$

where

$$Df_1 = \left[\frac{\partial f_1(\theta^t\omega, x(t))}{\partial x_i}\right],$$

We first introduce the following lemma which has been proved in the Ref. [8] and will be applied to the main theorem [11].

Lemma 3.1 Let $\mathcal{X}$ and $\mathcal{Y}$ be finite-dimensional random vector spaces of the same dimension, and $\mathbb{B}$ be an open subset of $\mathcal{X}$. Let v_0 be a given element of $\mathbb{B}$. Suppose that $G : \mathbb{B} \to \mathcal{Y}$ be a C^2 function and satisfy:

i. the derivative $DG(v_0)$ of function G at $v_0 \in \mathbb{B}$ is right inverse with $\mathcal{K}$;

ii. $\mathbb{B}$ contains a closed ball whose center is v_0 and radius is $\bar{\varepsilon}$, where $\bar{\varepsilon} = 2\|\mathcal{K}\|\|G(v_0)\|$;

iii. the inequality $2M\|\mathcal{K}\|^2\|G(v_0)\| \leq 1$ holds, where

$$M = \sup\{\|D^2G(v)\| : v \in \mathbb{B}, \|v - v_0\| \leq \bar{\varepsilon}\};$$

Then, there is a solution $\bar{v}$ of the equation $G(\bar{v}) = 0$ satisfying $\|\bar{v} - v_0\| \leq \bar{\varepsilon}$.

3.2. Main results

Now, we state the main theorem and postponed its proof in the latter section.

Theorem 3.2. For a given bounded (ω, δ)-pseudo random periodic orbit of RDS (3) generated by SDE (1) $\{(y_k(\theta^{t_k}\omega), \mathcal{F}_{t_k})\}_{k=0}^{N}$, assume that

$$C := \max\{M_0^{-1}(1 + \Theta\|L^{-1}\|), \|L^{-1}\|\}. \tag{7}$$

If the quantities shown in Section 3.1 together with δ and ε_0 satisfy:

i. $C_1 = C\delta < \frac{1}{2}$;

ii. $C_2 = 4C\delta < \varepsilon_0$;

iii. $C_3 = 8C^2\delta(M_0M_1 + 2M_1\exp(M_1\Delta t) + M_2\Delta t \cdot \exp(2M_1\Delta t)) \leq 1$;

Then there exists a sequence of times $\{h_k\}_{k=0}^{N+1}(h_0 \leq h_1 \leq, \ldots, \leq h_{N+1} \leq t_{N+1})$ such that the (ω, δ)-pseudo random periodic orbit $\{(y_k(\theta^{t_k}\omega), \mathcal{F}_{t_k})\}_{k=0}^{N}$ is (ω, δ)-periodic shadowed by a true random periodic orbit of SDE (1) containing points $\{(x_k(\theta^{h_k}\omega), \mathcal{F}_{h_k})\}_{k=0}^{N}$ in mean-square. Moreover, shadowing distance satisfies $\varepsilon \leq 4C\delta$.

4. Numerical experiments

Here, we apply the random periodic shadowing theorem to rigorously establish the existence of random periodic orbits of the stochastic Lorenz equation. And, this section will provide numerical experiments to compute the shadowing distance.

4.1. Experimental preparation

Consider the following Stratonovich stochastic Lorenz equation (SSLE) in $\mathbb{R}^3$,

$$dX_t = f(X_t)dt + \mu X_t \circ dW_t(\omega), \quad X(0) = x_0 \in \mathbb{R}^3 \tag{8}$$

where $X_t = (x, y, z)^T \in \mathbb{R}^3$, x, y and z are the state variables, σ, ρ and β are positive constant parameters, and

$$f(X_t) = \begin{pmatrix} -\sigma x + \sigma y \\ \rho x - y - xz \\ -\beta z + xy \end{pmatrix}, \mu X_t = \begin{pmatrix} \mu x \\ \mu y \\ \mu z \end{pmatrix}.$$

Make the transformation as follows:

$$\begin{cases} \overline{x}(t,\omega) = \exp(-\mu O_t(\omega))x \\ \overline{y}(t,\omega) = \exp(-\mu O_t(\omega))y \\ \overline{z}(t,\omega) = \exp(-\mu O_t(\omega))z, \end{cases}$$

It follows from the transformation that the above SSLE (8) can be transformed to the random differential equation (RDE) in the following form

$$\begin{cases} \dfrac{d\overline{x}}{dt} = \sigma(-\overline{x} + \overline{y}) + \mu O_t(\omega)\overline{x} \\ \dfrac{d\overline{y}}{dt} = -\overline{x}\,\overline{z} + \rho\overline{x} - \overline{y} + \mu O_t(\omega)\overline{y} \\ \dfrac{d\overline{z}}{dt} = \overline{x}\,\overline{y} - \beta\overline{z} + \mu O_t(\omega)\overline{z}. \end{cases} \tag{9}$$

The existence and uniqueness of solution of RDE (9) can be proved by the same approaches as proposed in the Refs. [2] and [12] though a normally required linear growth condition does not be satisfied. Hence, a RDS φ can be generated by the solution operator of RDE (9).

In this experiment, it appears numerically that the stochastic Lorenz equations have asymptotically stable random periodic orbit for the parameter values $\sigma = 10, \rho = 100.5, \beta = \frac{8}{3}$.

Firstly, we generate Brownian trajectories in the following way

$$W_0 = 0, W_{(i+1)\Delta t} = W_{i\Delta t} + \psi_{i+1}$$

where

$$\psi_i = N(0, \sqrt{\Delta t}), i = 1, 2, \ldots, N$$

Secondly, it follows from the reference [13] that a global attractor, i.e., a forward invariant random compact set $\mathcal{U}$ of RDS φ generated by RDE (9) is the closed ball $\mathbb{B}_1$ with center zero and radius $\mathcal{R}(\omega)$, that is, $\mathbb{B}_1 = \{X_t \in \mathbb{R}^3 : \|X_t\| \le \mathcal{R}(\omega)\}$, where

$$\mathcal{R}(\omega) = c_2 \int_{-t_N}^{0} \exp(c_1 s - 2\sigma W_s(\omega)) ds$$

and

$$c_1 = \min(1, \beta, \sigma), c_2 > 0, 2\langle BX_t, X_t\rangle < -c_1|X_t|^2 + c_2,$$
$$B = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -\beta \end{pmatrix}.$$

It has been proved in Ref. [13] that the RDS φ generated by Eq. (8) lies in the forward invariant random compact set $\mathcal{U}$ for $\mathbb{P}$-almost surely $\omega \in \Omega$ on the finite interval.

4.2. Numerical results

We first present the results of our computations of the (ω, δ)-pseudo random periodic orbits for the stochastic Lorenz equation. To generate a good (ω, δ)-pseudo random periodic orbit, we numerically computed the orbit for some time with a rough guess of initial value. In this experiment, we take the initial value $(x_0, y_0, z_0) = (1.76, -4.48, 80.99)$, time step size $\Delta t = 0.00007$ and iterative step $N = 100000$. The (ω, δ)-pseudo random periodic orbits of Eq. (9) in **Figure 1** are generated by the Euler-Maruyama scheme in Ref. [14] and the refined initial data. This also shows that there exists a forward invariant random compact set.

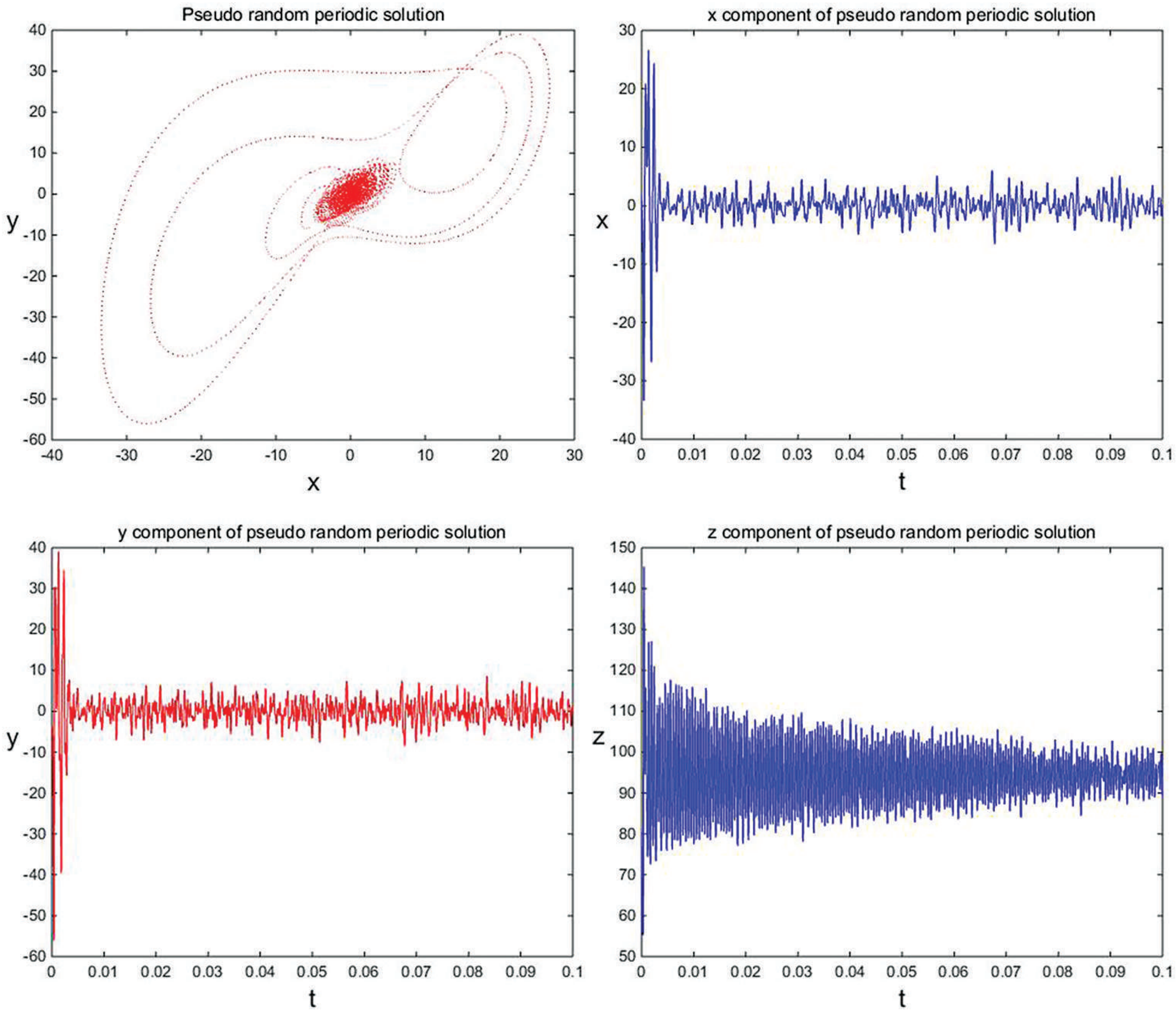

Figure 1. (ω, δ)-pseudo random periodic orbits of SLS.

Secondly, we briefly describe the details of the computation of the key quantities listed in **Table 2**. It follows from the methods shown in Section 3, and we can determine the parameters of Theorem 3.2. **Tables 1** and **2** present the important quantities and the necessary inequalities pertaining to this (ω, δ)-pseudo random periodic orbit.

Parameters	Value	Parameters	Value
Δt	0.00007	ε_0	2.01
X_0	(1.76, −4.48, 80.99)	M_0	≤ 9.8037
N	10^5	M_1	≤ 0.0185
Approx. period	$\tau = 0.1837$	M_2	0.0014
X_{2623}	(−0.6911, −7.7293, 81.6553)	Θ	≤ 1.0013
$\Vert X_{2623} - X_0 \Vert$	4.1241	δ	≤ 4.1265
		$\Vert L^{-1} \Vert$	$\leq 4.8218e - 03$

Table 1. Value of the parameters.

Inequalities	Value
C	≤ 0.1025
C_1	≤ 0.4229
C_2	≤ 1.6918
C_3	≤ 0.0757
Shadowing distance ε	1.2688
Shadowing time t	70

Table 2. Comparison of the inequalities.

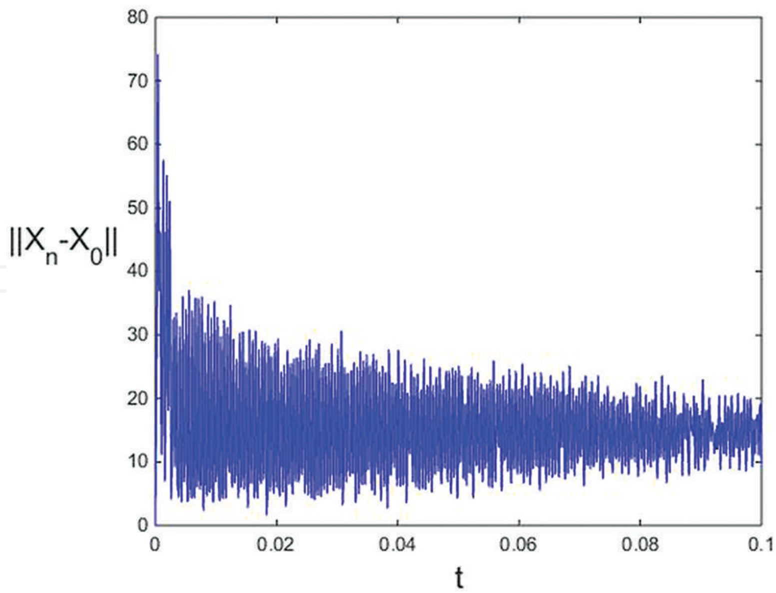

Figure 2. The distance $\Vert X_n - X_0 \Vert$.

In conclusion, there is explicit dependent relationship between the shadowing distance and the pseudo orbit error, and there exists the true periodic orbit in the appropriate neighborhood of the (ω, δ)-pseudo random periodic orbit of SLS (**Figure 2**). **Figures 3a** and **3b** demonstrate the relation

http://dx.doi.org/10.5772/67010

between (ω, δ)-pseudo random periodic orbits and true periodic orbits of Eq. (8). The blue lines denote (ω, δ)-pseudo random periodic orbit for the random dynamical system, and the domain between two blue lines has at least a true orbit for the corresponding random dynamical system.

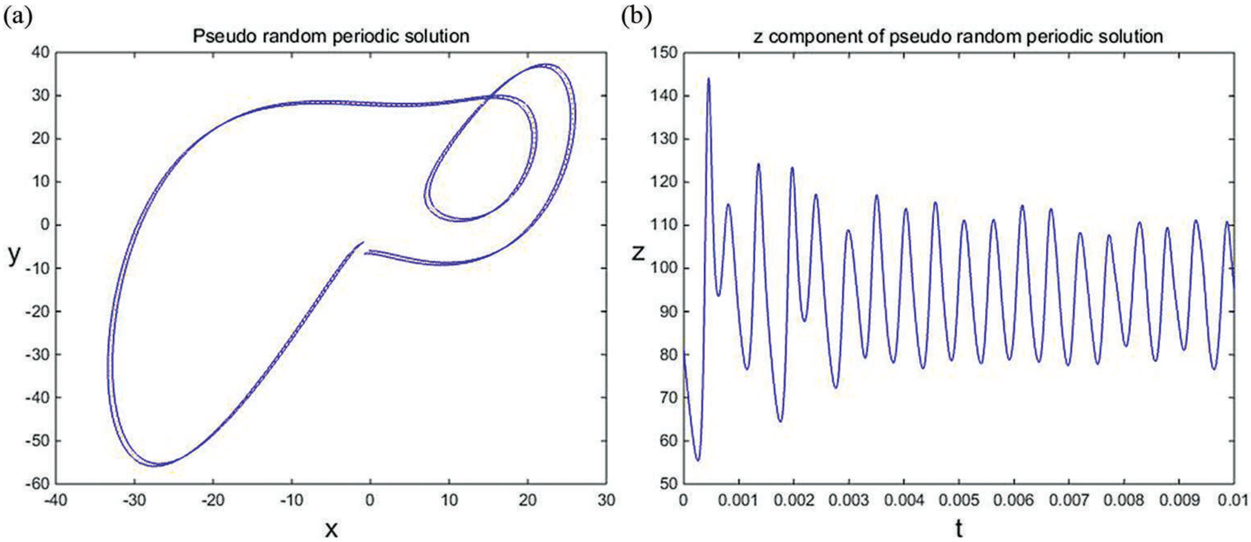

Figure 3. (a) The symbolic drawing of the relation between true orbit and pseudo orbit plane. (b) The approximative structure of pseudo random periodic solution projected on the z plane.

5. Choice of the operator L^{-1}

We are going to verify that the linear operator L along the obtained (ω, δ)-pseudo random periodic orbit $\{(y_k(\theta^{t_k}\omega), \mathcal{F}_{t_k})\}_{k=0}^{N}$ is invertible for $\mathbb{P}$-almost surely $\omega \in \Omega$.

Let $g = \{g_k(\theta^{t_k}\omega)\}_{k=0}^{N}$ be in $\mathcal{Y}$. To find $\xi = L^{-1}g$, we have to solve the random difference equation

$$\xi_{k+1}(\theta^{t_{k+1}}\omega) = A_k(\theta^{t_k}\omega)\xi_k(\theta^{t_k}\omega) + g_k(\theta^{t_k}\omega), \;\; for \;\; k = 0, \ldots N-1,$$
$$\xi_0(\theta^{t_0}\omega) = A_N(\theta^{t_N}\omega)\xi_N(\theta^{t_N}\omega) + g_N(\theta^{t_N}\omega).$$

With the same choice of the parameters as Section 3, it can be shown that random matrix $A_k(\theta^{t_k}\omega)$ is upper triangular with positive diagonal entries. Therefore, there is an integer l such that for most k, the first l diagonal entries of $A_k(\theta^{t_k}\omega)$ exceed 1 and the rest are less than 1 in mean square for $\mathbb{P}$-almost surely $\omega \in \Omega$ [15]. We can partition the random matrix $A_k(\theta^{t_k}\omega)$ in the form

$$A_k(\theta^{t_k}\omega) = \begin{bmatrix} P_k(\theta^{t_k}\omega) & Q_k(\theta^{t_k}\omega) \\ 0 & R_k(\theta^{t_k}\omega) \end{bmatrix}, k = 0, 1, \ldots, N,$$

where $P_k(\theta^{t_k}\omega)$ is $l \times l$ random matrix,$Q_k(\theta^{t_k}\omega)$ is $l \times (d-l-1)$ random matrix, and $R_k(\theta^{t_k}\omega)$ is $(d-l-1) \times (d-l-1)$ random matrix.

It follows from multiplicative ergodic theorem that the Lyapunov exponents of $A_k(\theta^{t_k}\omega)$ are nonzero. Then it suggests that the RDS φ generated by SDE (1) along the obtained (ω, δ)-

pseudo orbit $\{(y_k(\theta^{t_k}\omega), \mathcal{F}_{t_k})\}_{k=0}^N$ is pseudo hyperbolicity in mean square for $\mathbb{P}$-almost surely $\omega \in \Omega$. It can be written as

$$\begin{cases} \xi_{k+1}^{(1)}(\theta^{t_{k+1}}\omega) = P_k(\theta^{t_k}\omega)\xi_k^{(1)}(\theta^{t_k}\omega) + Q_k(\theta^{t_k}\omega)\xi_k^{(2)}(\theta^{t_k}\omega) + g_k^{(1)}(\theta^{t_k}\omega) \\ \xi_{k+1}^{(2)}(\theta^{t_{k+1}}\omega) = R_k(\theta^{t_k}\omega)\xi_k^{(2)}(\theta^{t_k}\omega) + g_k^{(2)}(\theta^{t_k}\omega) \end{cases}$$

for $k = 0, 1, \ldots, N-1$, and

$$\begin{cases} \xi_0^{(1)}(\theta^{t_0}\omega) = P_N(\theta^{t_N}\omega)\xi_N^{(1)} + Q_N(\theta^{t_N}\omega)\xi_N^{(2)}(\theta^{t_N}\omega) + g_N^{(1)}(\theta^{t_N}\omega) \\ \xi_0^{(2)}(\theta^{t_0}\omega) = R_N(\theta^{t_N}\omega)\xi_N^{(2)}(\theta^{t_N}\omega) + g_N^{(2)}(\theta^{t_N}\omega) \end{cases}$$

Let $\xi_0^{(2)}(\theta^{t_0}\omega) = 0$ solve forwards the second equation of the first equations above. The substitute it into the first equation with $\xi_k^{(2)}(\theta^{t_k}\omega)$, and let $\xi_N^{(2)}(\theta^{t_N}\omega) = 0$, then solve it backwards. Finally, the solutions $\xi_k^{(1)}(\theta^{t_k}\omega)$ are obtained. Therefore, the right inverse L^{-1} is obtained as

$$[L^{-1}g]_k = [\xi_k^{(1)}(\theta^{t_k}\omega), \xi_k^{(2)}(\theta^{t_k}\omega)]^T, k = 0, 1, \ldots, N.$$

Hence, invertibility of the operator L is proved, which is an important for the application of the random shadowing lemma.

6. Proof of the main theorem

Proof. For a given (ω, δ)-pseudo random periodic orbit $\{(y_k(\theta^{t_k}\omega), \mathcal{F}_{t_k})\}_{k=0}^N$ of RDS φ (3) generated by SDE (1), and an associated sequence of $d \times d$ random matrices $\{Y_k(\theta^{t_k}\omega)\}_{k=0}^N$ satisfying Eq. (6). Our aim is to show that $\{(y_k(\theta^{t_k}\omega), \mathcal{F}_{t_k})\}_{k=0}^N$ is (ω, δ)-periodic shadowed by a true random periodic orbit containing $\{(x_k(\theta^{h_k}\omega), \mathcal{F}_{h_k})\}_{k=0}^N$, where $x_k(\theta^{h_k}\omega)$ lies in the random hyperplane $\mathcal{H}_k(\theta^{t_k}\omega)$ through $y_k(\theta^{t_k}\omega)$.

Suppose that the random hyperplane $\mathcal{H}_k(\theta^{t_k}\omega)$ is approximately normal to $T(y_k) = f_1(\theta^{t_k}\omega, y_k)$ at the point $y_k(\theta^{t_k}\omega)$. Therefore, we only need to find a sequence of times $\{h_k\}_{k=0}^{N+1} = \{t_k\}_{k=0}^{N+1}$, $h_0 \le h_1 \le, \ldots, \le h_{N+1} \le t_{N+1}$ and a sequence of points $\{(x_k(\theta^{h_k}\omega), \mathcal{F}_{t_N})\}_{k=0}^N$ with $x_k(\theta^{h_k}\omega) \in \mathcal{H}_k(\theta^{t_k}\omega)$ being contained in the ε-neighborhood of $y_k(\theta^{t_k}\omega)$ such that

$$x_{k+1}(\theta^{h_{k+1}}\omega) = \varphi(h_k, h_{k+1}, \theta^{h_k}\omega)x_k(\theta^{h_k}\omega), \ \ for \ \ k = 0, 1, \ldots, N-1,$$

and

$$x_0(\theta^{h_0}\omega) = \varphi(h_N, h_{N+1}, \theta^{h_N}\omega)x_N(\theta^{h_N}\omega).$$

By the assumption, we obtain that $S_k(\theta^{t_k}\omega)$ is a $d \times (d-1)$ random matrix whose columns form an approximative orthogonal basis for $\mathcal{H}_k(\theta^{t_k}\omega)$. We first define the random hyperplane

$\mathcal{H}_k(\theta^{t_k}\omega)$ as the image of $\mathbb{R}^{d-1}$ through the map $\mathbf{z}\mapsto y_k(\theta^{t_k}\omega) + S_k(\theta^{t_k}\omega)\mathbf{z}$, which can be viewed as a subspace of the tangent space at $y_k(\theta^{t_k}\omega)$.

Therefore, the problem of finding appropriate sequences of h_k and x_k becomes that of finding a sequence of times $\{h_k\}_{k=0}^{N+1} := \{t_k\}_{k=0}^{N+1}$ and a sequence of points $\{(z_k(\theta^{h_k}\omega), \mathcal{F}_{t_N})\}_{k=0}^{N}$ in $\mathbb{R}^{d-1}$ such that

$$\begin{aligned} &y_{k+1}(\theta^{t_{k+1}}\omega) + S_{k+1}(\theta^{t_{k+1}}\omega)\mathbf{z}_{k+1}(\theta^{h_{k+1}}\omega) \\ &= \varphi(h_k, h_{k+1}, \theta^{h_k}\omega)(y_k(\theta^{t_k}\omega) + S_k(\theta^{t_k}\omega)\mathbf{z}_k(\theta^{h_k}\omega)), \ \ k = 0, 1,\ldots, N-1, \end{aligned}$$

and

$$y_0(\theta^{t_0}\omega) + S_0(\theta^{t_0}\omega)\mathbf{z}_0(\theta^{h_0}\omega) = \varphi(h_N, h_{N+1}, \theta^{h_N}\omega)(y_N(\theta^{t_N}\omega) + S_N(\theta^{t_N}\omega)\mathbf{z}_N(\theta^{h_N}\omega)).$$

We next introduce the space $\mathcal{X} = \mathbb{R}^{N+2}\times(\mathbb{R}^{d-1})^{N+1}$ with norm

$$\|(\{s_k\}_{k=0}^{N+1}, \{\zeta_k\}_{k=0}^{N})\| = \max\left\{ \sup_{0\leq k\leq N+1} |s_k|, \sup_{0\leq k\leq N} \|\zeta_k\| \right\},$$

and the space $\mathcal{Y} = (\mathbb{R}^d)^{N+1}$ with norm

$$\|\{g_k\}_{k=0}^{N}\| = \max_{0\leq k\leq N} \|g_k\|,$$

where $s_k \in \mathbb{R}$, $\zeta_k \in \mathbb{R}^{d-1}$ and $g_k \in \mathbb{R}^d$.

Now, we let $\mathbb{B}$ be a properly chosen ε-open neighborhood of $v_0 = (\{t_k\}_{k=0}^{N+1}, 0)$ in $\mathcal{X}$ which contain the point $v = (\{s_k\}_{k=0}^{N+1}, \{\zeta_k\}_{k=0}^{N})$. And, we introduce the function $G : \mathbb{B} \to \mathcal{Y}$ given by

$$\begin{aligned} [G(v)]_k &= y_{k+1}(\theta^{t_{k+1}}\omega) + S_{k+1}(\theta^{t_{k+1}}\omega)\zeta_{k+1}(\theta^{s_{k+1}}\omega) \\ &\quad -\varphi(s_k, s_{k+1}, \theta^{s_k}\omega)(y_k(\theta^{t_k}\omega) + S_k(\theta^{t_k}\omega)\zeta_k(\theta^{s_k}\omega)), \ \ for \ \ k = 0, 1,\ldots, N-1, \end{aligned}$$

and

$$\begin{aligned} [G(v)]_N = \ \ & y_0(\theta^{t_0}\omega) + S_0(\theta^{t_0}\omega)\zeta_0(\theta^{s_0}\omega) \\ & -\varphi(s_N, s_{N+1}, \theta^{s_N}\omega)(y_k(\theta^{t_N}\omega) + S_N(\theta^{t_N}\omega)\zeta_N(\theta^{s_N}\omega)). \end{aligned} \tag{10}$$

It is the fact that Theorem 3.2 will be proved if we can find a solution $\overline{v} = (\{h_k\}_{k=0}^{N+1}, \{\mathbf{z}_k(\theta^{h_k}\omega)\}_{k=0}^{N})$ of the equation

$$G(\overline{v}) = 0, \ \ a.s.$$

in the closed ball of radius ε about $v_0 = (\{t_k\}_{k=0}^{N+1}, 0)$.

In order to apply Lemma 3.1, those hypotheses (i) – (iii) for the map G as Eq. (10) should be verified.

Step I:

First and foremost, it follows from the construction of pseudo orbits that $\|G(v_0)\| \leq \delta$. Secondly, the Gateaux derivative of the map G at v_0 with $u = \left(\{\tau_k\}_{k=0}^{N+1}, \{\xi_k(\theta^{t_k}\omega)\}_{k=0}^{N}\right) \in \mathcal{X}$ is given by

$$\begin{aligned}
[DG(v_0)u]_k &= \lim_{\varepsilon \to 0} \frac{[G(v_0 + \varepsilon u) - G(v_0)]_k}{\varepsilon} \\
&= -\tau_k T(y_{k+1}) + S_{k+1}(\theta^{t_{k+1}}\omega) \cdot \xi_{k+1}(\theta^{t_{k+1}}\omega) \\
&\quad - D\varphi(t_k, t_{k+1}, \theta^{t_k}\omega) y_k(\theta^{t_k}\omega) \cdot S_k(\theta^{t_k}\omega) \cdot \xi_k(\theta^{t_k}\omega),
\end{aligned}$$

for $k = 0, 1, \ldots, N-1$, and

$$\begin{aligned}
[DG(v_0)u]_N = & -\tau_N T(y_N) + S_0(\theta^{t_0}\omega) \cdot \xi_0(\theta^{t_0}\omega) \\
& - D\varphi(t_N, t_{N+1}, \theta^{t_N}\omega) y_N(\theta^{t_N}\omega) \cdot S_N(\theta^{t_N}\omega) \cdot \xi_N(\theta^{t_N}\omega).
\end{aligned} \tag{11}$$

We will approximate $DG(v_0)$ by another operator. Now, we define the operator $\mathcal{T} : \mathcal{X} \to \mathcal{Y}$ for $u \in \mathcal{X}$. Let $\mathcal{T}_k u$ be the approximation of $[DG(v_0)u]_k$ in Ref. [16], we have

$$\begin{aligned}
\mathcal{T}_k u = & -\tau_k T(y_{k+1}) + S_{k+1}(\theta^{t_{k+1}}\omega) \cdot \xi_{k+1}(\theta^{t_{k+1}}\omega) \\
& - Y_k(\theta^{t_k}\omega) \cdot S_k(\theta^{t_k}\omega) \cdot \xi_k(\theta^{t_k}\omega), \quad k = 0, 1, \ldots, N-1,
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{T}_N u = & -\tau_N T(y_N) + S_0(\theta^{t_0}\omega) \cdot \xi_0(\theta^{t_0}\omega) \\
& - Y_N(\theta^{t_N}\omega) \cdot S_N(\theta^{t_N}\omega) \cdot \xi_N(\theta^{t_N}\omega).
\end{aligned} \tag{12}$$

Now, we need to prove that $\mathcal{T}$ is invertible. Therefore, we must show that for all $g = \{g_k\}_{k=0}^{N} \in \mathcal{Y}$, there is a solution of the following equation

$$\mathcal{T}_k u = g_k,$$

that is, for $k = 0, 1, \ldots, N-1$,

$$-\tau_k T(y_{k+1}) + S_{k+1}(\theta^{t_{k+1}}\omega)\xi_{k+1}(\theta^{t_{k+1}}\omega) - Y_k(\theta^{t_k}\omega) S_k(\theta^{t_k}\omega)\xi_k(\theta^{t_k}\omega) = g_k(\theta^{t_k}\omega),$$

and

$$\begin{aligned}
& -\tau_N T(y_N) + S_0(\theta^{t_0}\omega) \cdot \xi_0(\theta^{t_0}\omega) - Y_N(\theta^{t_N}\omega) \cdot S_N(\theta^{t_N}\omega) \cdot \xi_N(\theta^{t_N}\omega) \\
= \ & g_N(\theta^{t_N}\omega).
\end{aligned} \tag{13}$$

As we know, the matrix

$$\left[\frac{T(y_k)}{\|T(y_k)\|} \middle| S_k(\theta^{t_k}\omega) \right]$$

is orthogonal for each k. Then this set of equations is equivalent to the following two sets of equations, one set obtained by premultiplying the kth member in Eq. (13) by $T^*(y_{k+1})$ and

$T^*(y_0)$, respectively, the other set obtained by premultiplying the kth member in Eq. (13) by $S^*_{k+1}(\theta^{t_{k+1}}\omega)$ and $S^*_0(\theta^{t_0}\omega)$, respectively. Therefore, we obtain for $k = 0, 1, \ldots, N-1$,

$$-\tau_k\|T(y_{k+1})\|^2 - T(y_{k+1})^* Y_k(\theta^{t_k}\omega)S_k(\theta^{t_k}\omega)\xi_k(\theta^{t_k}\omega) = T(y_{k+1})^* g_k(\theta^{t_k}\omega),$$

and

$$-\tau_N\|T(y_0)\|^2 - T(y_0)^* Y_N(\theta^{t_N}\omega)S_N(\theta^{t_N}\omega)\xi_N(\theta^{t_N}\omega) = T(y_0)^* g_N(\theta^{t_N}\omega), \tag{14}$$

$$\xi_{k+1}(\theta^{t_{k+1}}\omega) - A_k(\theta^{t_k}\omega)\xi_k(\theta^{t_k}\omega) = S^*_k(\theta^{t_{k+1}}\omega)g_k(\theta^{t_k}\omega), \quad k = 0, 1, \ldots, N-1,$$

and

$$\xi_0(\theta^{t_0}\omega) - A_N(\theta^{t_N}\omega)\xi_N(\theta^{t_N}\omega) = S^*_N(\theta^{t_0}\omega)g_N(\theta^{t_N}\omega). \tag{15}$$

If we write $\overline{g} = \{S^*_k(\theta^{t_k}\omega)\mathbf{g}_k(\theta^{t_k}\omega)\}_{k=0}^N$, it follows from the condition (7) that the solution of Eq. (15) is

$$\xi_k = (L^{-1}\overline{g})_k. \tag{16}$$

If Eq. (16) is substituted into Eq. (14), we obtain for $k = 0, 1, \ldots, N-1$,

$$\tau_k = -\frac{T(y_{k+1})^*}{\|T(y_{k+1})\|^2} \cdot [Y_k(\theta^{t_k}\omega)S_k(\theta^{t_k}\omega)L^{-1}S_{k+1}(\theta^{t_{k+1}}\omega) + 1]\mathbf{g}_k(\theta^{t_k}\omega),$$

and

$$\tau_N = -\frac{T(y_0)^*}{\|T(y_0)\|^2} \cdot [Y_N(\theta^{t_N}\omega)S_N(\theta^{t_N}\omega)L^{-1}S_0(\theta^{t_0}\omega) + 1]g_N(\theta^{t_N}\omega). \tag{17}$$

Taking into account Eqs. (16) and (17), we define the right inverse of $\mathcal{T}_k$ in the form of

$$\mathcal{T}_k^{-1}\mathbf{g} = [\{\tau_k\}_{k=0}^{N+1}, \{\xi_k(\theta^{t_k}\omega)\}_{k=0}^N].$$

It follows from Eq. (17) that $\mathcal{T}$ is invertible and the following inequality holds

$$\|\mathcal{T}^{-1}\| \le C. \tag{18}$$

Therefore, we can construct the invertibility of $DG(v_0)$. By the operator theory, we obtain

$$\mathcal{K} = [\mathbb{I} + \mathcal{T}^{-1}(DG(v_0) - \mathcal{T})]^{-1}\mathcal{T}^{-1}. \tag{19}$$

By Eqs. (11) and (12) and the assumption (i) of Theorem 3.2, we obtain that

$$\begin{aligned} & \|\mathcal{T}^{-1}(DG(v_0) - \mathcal{T})\| \le \|\mathcal{T}^{-1}\|\|DG(v_0) - \mathcal{T}\| \\ \le\ & \|\mathcal{T}^{-1}\| \cdot [\sup \|(D\varphi(t_k, t_{k+1}, \theta^{t_w}\omega)y_k(\theta^{t_k}\omega) - Y_k(\theta^{t_k}\omega)S_k(\theta^{t_k}\omega)\xi_k(\theta^{t_k}\omega)\|] \\ \le\ & C\delta < \frac{1}{2}. \end{aligned}$$

Then the inverse $[\mathbb{I} + \mathcal{T}^{-1}(DG(v_0) - \mathcal{T})]^{-1}$ exits and $\mathcal{K}$ is a right inverse of $DG(v_0)$. Furthermore,

$$\|[\mathbb{I} + \mathcal{T}^{-1}(DG(v_0) - \mathcal{T})]^{-1}\| \leq 2.$$

Therefore, we have verified hypothesis (*i*) of Lemma 3.1.

Step II:

It follows from Eqs. (18)–(20) that we have

$$\|\mathcal{K}\| \leq 2C.$$

and

$$\|G(v_0)\| \ = \sup_k \|y_{k+1}(\theta^{t_{k+1}}\omega) - \varphi(t_k, t_{k+1}, \theta^{t_k}\omega) y_k(\theta^{t_k}\omega)\| \leq \delta.$$

By the assumption (*ii*) of Theorem 3.2, we obtain that

$$\varepsilon = 2\|\mathcal{K}\|\|G(v_0)\| \leq 4C\delta < \varepsilon_0.$$

That is, the closed ball of radius ε around v_0 is contained in the open set B. Therefore, we have verified hypothesis (*ii*) of Lemma 3.1.

Step III:

We only need to estimate $\|D^2G(v)\|$. Then we choose $\overline{u} = (\{r_k\}_{k=0}^{N+1}, \{\eta_k\}_{k=0}^{N})$ and calculate the second order Gateaux differential of $G(v)$ for $k = 0, 1, \ldots, N$ as follows

$$\begin{aligned}
&[DG(v)u\overline{u}]_k := \lim_{t \to 0} \frac{[DG(v + t\overline{u})u - DG(v)u]_k}{|t|} \\
= \ & -\tau_k r_k DT[y_k(\theta^{t_k}\omega) + S_k(\theta^{t_k}\omega)\zeta_k(\theta^{t_k}\omega)] \cdot T[y_k(\theta^{t_k}\omega) + S_k(\theta^{t_k}\omega)\zeta_k(\theta^{t_k}\omega)] \\
& -\tau_k DT[y_k(\theta^{t_k}\omega) + S_k(\theta^{t_k}\omega)\zeta_k(\theta^{t_k}\omega)] \cdot \\
& \quad D\varphi(t_k, t_{k+1}, \theta^{t_k}\omega)(y_k(\theta^{t_k}\omega) + S_k(\theta^{t_k}\omega)\zeta_k(\theta^{t_k}\omega)) \cdot S_k(\theta^{t_k}\omega)\eta_k(\theta^{t_k}\omega) \\
& -r_k DT[y_k(\theta^{t_k}\omega) + S_k(\theta^{t_k}\omega)\zeta_k(\theta^{t_k}\omega)] \cdot \\
& \quad D\varphi(t_k, t_{k+1}, \theta^{t_k}\omega)(y_k(\theta^{t_k}\omega) + S_k(\theta^{t_k}\omega)\zeta_k(\theta^{t_k}\omega)) \cdot S_k(\theta^{t_k}\omega)\xi_k(\theta^{t_k}\omega) \\
& -D^2\varphi(t_k, t_{k+1}, \theta^{t_k}\omega)(y_k(\theta^{t_k}\omega) \\
& +S_k(\theta^{t_k}\omega)\zeta_k(\theta^{t_k}\omega)) \cdot [S_k(\theta^{t_k}\omega)\xi_k(\theta^{t_k}\omega)] \cdot [S_k(\theta^{t_k}\omega)\eta_k(\theta^{t_k}\omega)].
\end{aligned}$$

By the norm property, i.e., subadditivity, we obtain

$$M = \sup_k \|D^2G(v)\| \leq M_0M_1 + 2M_1 \exp(M_1\Delta t) + M_2\Delta t \exp(2M_1\Delta t).$$

It follows from the assumption (*iii*) of Theorem 3.2 and

$$\|G(v_0)\| \leq \delta, \|\mathcal{K}\|^2 \leq 4C^2,$$

that

$$2M\|\mathcal{K}\|^2\|G(v_0)\| \leq 1.$$

Then we have verified hypothesis (*iii*) of Lemma 3.1. Therefore, the conclusion follows from Lemma 3.1. This finishes the proof.

Remark 6.1 *The proof is similar to the paper [8], and we extend it to the random periodic case.*

7. Conclusion

The main result presented here is the random periodic shadowing theorem of the RDS generated by some SDEs. To conduct the study, we have extended the random shadowing theorem to the random periodic scenario by taking advantage of mean square and stochastic calculus. We show that the existence of the random periodic shadowing orbits of the SSLE so that the numerical experiments are performed and match the results of theoretical analysis. Although some progresses are made, more accurate numerical methods of estimating the shadowing distance are needed in practice, which will be presented in our further work.

Acknowledgements

The authors like to express their gratitude to Prof. Jialin Hong for his helpful discussion. This work is partially supported by NSFC (No. 11021101, 11290142, 91130003, 91530118) and the Fundamental Research Funds for the Central Universities, HUST, No. 2016YXMS226 and Young Teacher's Education and Science Research Project in the Education Department of Fujian Province, No.JAT160182.

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