Chapter 2

An Intrinsic Characterization of Bonnet Surfaces Based on a Closed Differential Ideal

Paul Bracken

Additional information is available at the end of the chapter

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Abstract

The structure equations for a two-dimensional manifold are introduced and two results based on the Codazzi equations pertinent to the study of isometric surfaces are obtained from them. Important theorems pertaining to isometric surfaces are stated and a theorem due to Bonnet is obtained. A transformation for the connection forms is developed. It is proved that the angle of deformation must be harmonic, and that the differentials of many of the important variables generate a closed differential ideal. This implies that a coordinate system exists in which many of the variables satisfy particular ordinary differential equations, and these results can be used to characterize Bonnet surfaces.

Keywords: manifold, differential form, closed, isometric, differential equation, Bonnet surface

1. Introduction

Bonnet surfaces in three-dimensional Euclidean space have been of great interest for a number of reasons as a type of surface [1, 2] for a long time. Bonnet surfaces are of nonconstant mean curvature that admits infinitely many nontrivial and geometrically distinct isometries, which preserve the mean curvature function. Nontrivial isometries are ones that do not extend to isometries of the whole space E^3. Considerable interest has resulted from the fact that the differential equations that describe the Gauss equations are classified by the type of related Painlevé equations they correspond to and they are integrated in terms of certain hypergeometric transcendents [3–5]. Here the approach first given by Chern [6] to Bonnet surfaces is considered. The development is accessible with many new proofs given. The main intention is to end by deriving an intrinsic characterization of these surfaces which indicates they are analytic. Moreover, it is shown that a type of Lax pair can be given for these surfaces and integrated. Several of the more important functions such as the mean curvature are seen to satisfy nontrivial ordinary differential equations.

Quite a lot is known about these surfaces. With many results the analysis is local and takes place under the assumptions that the surfaces contain no umbilic points and no critical points of the mean curvature function. The approach here allows the elimination of many assumptions and it is found the results are not too different from the known local ones. The statements and proofs have been given in great detail in order to help illustrate and display the interconnectedness of the ideas and results involved.

To establish some information about what is known, consider an oriented, connected, smooth open surface M in E^3 with nonconstant mean curvature function H. Moreover, suppose M admits infinitely many nontrivial and geometrically distinct isometries preserving H. Suppose U is the set of umbilic points of M and V the set of critical points of H. Many global facts are known with regard to U,V and H, and a few will now be mentioned. The set U consists of isolated points, even if there exists only one nontrivial isometry preserving the mean curvature, moreover, $U \subset V$ [7, 8]. Interestingly, there is no point in $V - U$ at which all order derivatives of H are zero, and V cannot contain any curve segment. If the function by which a nontrivial isometry preserving the mean curvature rotates the principal frame is considered, as when there are infinitely many isometries, this function is a global function on M continuously defined [9–11]. As first noted by Chern [6], this function is harmonic. The analysis will begin by formulating the structure equations for two-dimensional manifolds.

2. Structure equations

Over M, there exists a well-defined field of orthonormal frames, which is written as x, e_1, e_2, e_3 such that x$\in M$, e_3 is the unit normal at x, and e_1, e_2 are along principal directions [12]. The fundamental equations for M have the form

$$d\mathrm{x} = \omega_1 e_1 + \omega_2\, e_2, \quad de_1 = \omega_{12} e_2 + \omega_{13} e_3, \quad de_2 = -\omega_{12} e_1 + \omega_{23} e_3, \quad de_3 = -\omega_{13} e_1 - \omega_{23} e_2. \qquad (1)$$

Differentiating each of these equations in turn, results in a large system of equations for the exterior derivatives of the ω_i and ω_{ij}, as well as a final equation which relates some of the forms [13]. This choice of frame and Cartan's lemma allows for the introduction of the two principal curvatures which are denoted by a and c at x by writing

$$\omega_{12} = h\omega_1 + k\omega_2, \qquad \omega_{13} = a\omega_1, \qquad \omega_{23} = c\omega_2. \qquad (2)$$

Suppose that $a > c$ in the following. The mean curvature of M is denoted by H and the Gaussian curvature by K. They are related to a and c as follows

$$H = \frac{1}{2}(a + c), \qquad K = a \cdot c. \qquad (3)$$

The forms which appear in Eq. (1) satisfy the fundamental structure equations which are summarized here [14],

$$d\omega_1 = \omega_{12} \wedge \omega_2, \qquad d\omega_2 = \omega_1 \wedge \omega_{12}$$
$$d\omega_{13} = \omega_{12} \wedge \omega_{23} \qquad d\omega_{23} = \omega_{13} \wedge \omega_{12}, \tag{4}$$
$$d\omega_{12} = ac\ \omega_2 \wedge \omega_1 = -K\ \omega_1 \wedge \omega_2.$$

The second pair of equations of (4) is referred to as the Codazzi equation and the last equation is the Gauss equation.

Exterior differentiation of the two Codazzi equations yields

$$(da-(a-c)h\omega_2) \wedge \omega_1 = 0, \qquad (dc-(a-c)k\omega_1) \wedge \omega_2 = 0. \tag{5}$$

Cartan's lemma can be applied to the equations in (5). Thus, there exist two functions u and v such that

$$\frac{1}{a-c}\ da-h\omega_2 = (u-k)\omega_1, \qquad \frac{1}{a-c}\ dc-k\omega_1 = (v-h)\omega_2. \tag{6}$$

Subtracting the pair of equations in (6) gives an expression for $d\log(a-c)$

$$d\log(a-c) = (u-2k)\ \omega_1-(v-2h)\ \omega_2. \tag{7}$$

Define the variable J to be

$$J = \frac{1}{2}(a-c) > 0. \tag{8}$$

It will appear frequently in what follows. Equation (7) then takes the form

$$d\log J = (u-2k)\omega_1-(v-2h)\omega_2. \tag{9}$$

The ω_i constitute a linearly independent set. Two related coframes called ϑ_i and α_i can be defined in terms of the ω_i and the functions u and v as follows,

$$\vartheta_1 = u\omega_1 + v\omega_2, \quad \vartheta_2 = -v\omega_1 + u\omega_2,$$
$$\alpha_1 = u\omega_1 - v\omega_2, \qquad \alpha_2 = v\omega_1 + u\omega_2. \tag{10}$$

These relations imply that $\vartheta_1 = 0$ is tangent to the level curves specified by H equals constant and $\alpha_1 = 0$ is its symmetry with respect to the principal directions.

Squaring both sides of the relation $2H = a + c$ and subtracting the relation $4K = 4ac$ yields $4(H^2-K) = (a-c)^2$. The Hodge operator, denoted by $*$, will play an important role throughout. It produces the following result on the basis forms ω_i,

$$*\omega_1 = \omega_2, \qquad *\omega_2 = -\omega_1, \qquad *^2 = -1. \tag{11}$$

Moreover, adding the expressions for da and dc given in Eq. (6), there results

$$\frac{1}{a-c}(da + dc) = (u-k)\omega_1 + h\omega_2 + (v-k)\omega_2 + k\omega_1 = u\omega_1 + v\omega_2 = \vartheta_1. \tag{12}$$

Finally, note that

$$\alpha_1 + 2 \ *\omega_{12} = u\omega_1 - v\omega_2 + 2 \ *(h\omega_1 + k\omega_2) = (u-2k)\omega_1 - (v-2h)\omega_2 = d\log J. \tag{13}$$

Therefore, the Codazzi equations (12) and (13) can be summarized using the definitions of H and J as

$$dH = J\vartheta_1, \qquad d\log J = \alpha_1 + 2 \ *\omega_{12.} \tag{14}$$

3. A theorem of Bonnet

Suppose that M^* is a surface which is isometric to M such that the principal curvatures are preserved [10–12]. Denote all quantities which pertain to M^* with the same symbols but with asterisks, as for example

$$a^* = a, \qquad c^* = c.$$

The same notation will be applied to the variables and forms which pertain to M and M^*. When M and M^* are isometric, the forms ω_i are related to the ω_i^* by the following transformation

$$\omega_1^* = \cos\tau\,\omega_1 - \sin\tau\,\omega_2, \qquad \omega_2^* = \sin\tau\,\omega_1 + \cos\tau\,\omega_2. \tag{15}$$

Theorem 3.1 Under the transformation of coframe given by Eq. (15), the associated connection forms are related by

$$\omega_{12}^* = \omega_{12} - d\tau. \tag{16}$$

Proof: Exterior differentiation of ω_1^* produces

$$\begin{aligned} d\omega_1^* &= -\sin\tau\,d\tau\wedge\omega_1 + \cos\tau\,d\omega_1 - \cos\tau\,d\tau\wedge\omega_2 - \sin\tau\,d\omega_2 \\ &= d\tau\wedge(-\sin\tau\omega_1 - \cos\tau\omega_2) + \cos\tau\,\omega_{12}\wedge\omega_2 - \sin\tau\omega_1\wedge\omega_{12} = (-d\tau + \omega_{12})\wedge\omega_2^*. \end{aligned}$$

Similarly, differentiating ω_2^* gives

$$\begin{aligned} d\omega_2^* &= \cos\tau\wedge\omega_1 + \sin\tau\,d\omega_1 - \sin\tau\,d\tau\wedge\omega_2 + \cos\tau\,d\omega_2 \\ &= d\tau\wedge(\cos\tau\omega_1 - \sin\tau\omega_2) + \sin\tau\omega_{12}\wedge\omega_2 + \cos\tau\omega_1\wedge\omega_{12} = \omega_1^*\wedge(-d\tau + \omega_{12}). \end{aligned}$$

There is a very important result that can be developed at this point. In the case that $a = a^*$ and $c = c^*$, the Codazzi equations imply that

$$\alpha_1 + 2 * \omega_{12} = d\log(a - c) = d\log(a^* - c^*) = \alpha_1^* + 2 * \ \omega_{12}^*.$$

Apply the operator $*$ to both sides of this equation, we obtain

$$\alpha_2 - 2\omega_{12} = \alpha_2^* - 2\omega_{12}^*.$$

Substituting for ω_{12}^* from Theorem 3.1, this is

$$2d\tau = \alpha_2 - \alpha_2^*. \tag{17}$$

Lemma 3.1

$$\vartheta_1 = \vartheta_1^*.$$

Proof: This can be shown in two ways. First from Eq. (15), express the ω_i in terms of the ω_i^*

$$\omega_1 = \cos\tau\ \omega_1^* + \sin\tau\ \omega_2^*, \qquad \omega_2 = -\sin\tau\ \omega_1^* + \cos\tau\ \omega_2^*. \tag{18}$$

Therefore,

$$\vartheta_1 = u\omega_1 + v\omega_2 = u(\cos\tau\ \omega_1^* + \sin\tau\omega_2^*) + v(-\sin\tau\omega_1^* + \cos\tau\omega_2^*) = u^*\omega_1^* + v^*\omega_2^* = \vartheta_1^*,$$

where $u^* = u\cos\tau - v\sin\tau$ and $v^* = u\sin\tau + v\cos\tau$. □

Lemma 3.1 also follows from the fact that $dH = dH^*$ and Eq. (8).

Lemma 3.2

$$\alpha_2^* = \sin(2\tau)\ \alpha_1 + \cos(2\tau)\ \alpha_2.$$

Proof:

$$\begin{aligned}\alpha_2^* &= (u\sin\tau + v\cos\tau)(\cos\tau\omega_1 - \sin\tau\omega_2) + (u\cos\tau - v\sin\tau)(\sin\tau\omega_1 + \cos\tau\omega_2)\\ &= (u\sin(2\tau) + v\cos(2\tau))\omega_1 + (-v\sin(2\tau) + u\cos(2\tau))\omega_2\\ &= \sin(2\tau)\alpha_1 + \cos(2\tau)\alpha_2.\end{aligned}$$

Substituting α_2^* from Lemma 3.2 into Eq. (13), $d\tau$ can be written as

$$d\tau = \frac{1}{2}(\alpha_2 - \sin(2\tau)\alpha_1 - \cos(2\tau)\alpha_2) = \frac{1}{2}((1 - \cos(2\tau))\alpha_2 - \sin(2\tau)\alpha_1). \tag{19}$$

Introduce the new variable $t = \cot(\tau)$ so $dt = -\csc^2(\tau)\ d\tau$ and $\sin\tau = \frac{1}{\sqrt{1+t^2}}$, $\cos\tau = \frac{1}{\sqrt{1+t^2}}$, hence the following lemma.

Lemma 3.3

$$dt = t\alpha_1 - \alpha_2.$$

This is the total differential equation which must be satisfied by the angle τ of rotation of the principal directions during the deformation. If the deformation is to be nontrivial, it must be that this equation is completely integrable.

Theorem 3.2 A surface M admits a nontrivial isometric deformation that keeps the principal curvatures fixed if and only if

$$d\alpha_1 = 0, \qquad d\alpha_2 = \alpha_1 \wedge \alpha_2. \tag{20}$$

Proof: Differentiating both sides of Lemma 3.3 gives

$$dt \wedge \alpha_1 + t d\alpha_1 - d\alpha_2 = (t\alpha_1 - \alpha_2) \wedge \alpha_1 + t d\alpha_1 - d\alpha_2 = 0.$$

Equating the coefficients of t to zero gives the result (20).

This theorem seems to originate with Chern [6] and is very useful because it gives the exterior derivatives of the α_i. When the mean curvature is constant, $dH = 0$, hence it follows from Eq. (14) that $\vartheta_1 = 0$. This implies that $u = v = 0$, and so α_1 and α_2 must vanish. Hence, $dt = 0$ which implies that, since the α_i is linearly independent, t equals a constant. Thus, we arrive at a theorem originally due to Bonnet.

Theorem 3.3 A surface of constant mean curvature can be isometrically deformed preserving the principal curvatures. During the deformation, the principal directions rotate by a fixed angle.

4. Connection form associated to a coframe and transformation properties

Given the linearly independent one forms ω_1,ω_2, the first two of the structure equations uniquely determine the form ω_{12}. The ω_1,ω_2 is called the orthonormal coframe of the metric

$$ds^2 = \omega_1^2 + \omega_2^2,$$

and ω_{12} is the connection form associated with it.

Theorem 4.1 Suppose that $A > 0$ is a function on M. Under the change of coframe

$$\omega_1^* = A\omega_1, \qquad \omega_2^* = A\omega_2, \tag{21}$$

the associated connection forms are related by

$$\omega_{12}^* = \omega_{12} + * \, d\log A. \tag{22}$$

Proof: The structure equations for the transformed system are given as

$$d\omega_1^* = \omega_{12}^* \wedge \omega_2^*, \qquad d\omega_2^* = \omega_1^* \wedge \omega_{12}^*.$$

Using Eq. (21) to replace the ω_i^* in these, we obtain

$$d\log A \wedge \omega_1 + d\omega_1 = \omega_{12}^* \wedge \omega_2, \qquad d\log A \wedge \omega_2 + d\omega_2 = \omega_1 \wedge \omega_{12}^*.$$

The ω_i satisfy a similar system of structure equations, so replacing $d\omega_i$ here yields

$$(\omega_{12}^* - \omega_{12}) \wedge \omega_2 = d\log A \wedge \omega_1, \qquad (\omega_{12}^* - \omega_{12}) \wedge \omega_1 = -d\log A \wedge \omega_2.$$

Since the form ω_i satisfies the equations $*\omega_1 = \omega_2$ and $*\omega_2 = -\omega_1$, substituting these relations into the above equations and using $\Omega_k \wedge (*\Theta_k) = \Theta_k \wedge (*\Omega_k)$, we obtain that in the form

$$\omega_1 \wedge *(\omega_{12}^* - \omega_{12}) = -\omega_1 \wedge d \log A, \qquad \omega_2 \wedge *(\omega_{12}^* - \omega_{12}) = -\omega_2 \wedge d \log A.$$

Cartan's lemma can be used to conclude from these that there exist functions f and g such that

$$*(\omega_{12}^* - \omega_{12}) = -d\log A - f\omega_1, \qquad *(\omega_{12}^* - \omega_{12}) = -d\log A + g\omega_2.$$

Finally, apply $*$ to both sides and use $*^2 = -1$ to obtain

$$\omega_{12}^* - \omega_{12} = *d\log A + f\omega_2, \qquad \omega_{12}^* - \omega_{12} = *d\log A + g\omega_1.$$

The forms ω_i are linearly independent, so for these two equations to be compatible, it suffices to put $f = g = 0$, and the result follows. □

For the necessity in the Chern criterion, Theorem 3.2, no mention of the set V of critical points of H is needed. In fact, when H is constant, this criterion is met and the sufficiency also holds with τ constant. However, when H is not identically constant, we need to take the set V of critical points into account for the sufficiency. In this case, $M - V$ is also an open, dense, and connected subset of M. On this subset $J > 0$ and the function A can be defined in terms of the functions u and v as

$$A = +\sqrt{u^2 + v^2} > 0. \tag{23}$$

To define more general transformations of the ω_i, define the angle ψ as

$$u = A\cos(\psi), \qquad v = A\sin(\psi). \tag{24}$$

This angle, which is defined modulo 2π, is continuous only locally and could be discontinuous in a nonsimply connected region of $M - V$. With A and ψ related to u and v by Eq. (24), the forms ϑ_i and α_i can be written in terms of A and ψ as

$$\vartheta_1 = A(\cos(\psi)\,\omega_1 + \sin(\psi)\omega_2), \quad \vartheta_2 = A(-\sin(\psi)\,\omega_1 + \cos(\psi)\,\omega_2),$$
$$\alpha_1 = A(\cos(\psi)\,\omega_1 - \sin(\psi)\,\omega_2), \quad \alpha_2 = A(\sin(\psi)\,\omega_1 + \cos(\psi)\,\omega_2). \tag{25}$$

The forms ω_i, ϑ_i, α_i define the same structure on M and we let ω_{12}, ϑ_{12}, α_{12} be the connection forms associated to the coframes ω_1,ω_2; ϑ_1,ϑ_2; α_1,α_2. The next theorem is crucial for what follows.

Theorem 4.2

$$\vartheta_{12} = d\psi + \omega_{12} + *d\log A = 2d\psi + \alpha_{12}. \tag{26}$$

Proof: Each of the transformations which yield the ϑ_i and α_i in the form (25) can be thought of as a composition of the two transformations which occur in the Theorems 3.1 and 4.1. First apply the transformation $\omega_i \to A\omega_i$ and $\tau \to -\psi$ with $\omega_i^* \to \vartheta_i$ in Eq. (15), we get the ϑ_i equations in Eq. (25). Invoking Theorems 3.1 and 4.1 in turn, the first result is obtained

$$\vartheta_{12} = d\psi + \omega_{12} + *\,d\log A.$$

The transformation to the α_i is exactly similar except that $\tau \to \psi$, hence

$$\alpha_{12} = -d\psi + \omega_{12} + \; * \; d\log A.$$

This implies $*d\log A = \alpha_{12} + d\psi - \omega_{12}$. When replaced in the first equation of (26), the second equation appears. Note that from Theorem 3.2, $\alpha_{12} = \alpha_2$, so the second equation can be given as $\vartheta_{12} = 2d\psi + \alpha_2$.

Differentiating the second equation in Eq. (14) and using $d\alpha_1 = 0$, it follows that

$$d * \omega_{12} = 0. \tag{27}$$

Lemma 4.1 The angle ψ is a harmonic function $d * d\psi = 0$ and moreover, $d * \vartheta_{12} = 0$.

Proof: From Theorem 4.2, it follows by applying $*$ through Eq. (26) that

$$*\vartheta_{12} = *\omega_{12} + *d\psi - d\log A = 2 * d\psi - \alpha_1. \tag{28}$$

Exterior differentiation of this equation using $d * \omega_{12} = 0$ immediately gives

$$d * d\psi = 0.$$

This states that ψ is a harmonic function. Equation (28) also implies that $d * \vartheta_{12} = 0$.

5. Construction of the closed differential ideal associated with M

Exterior differentiation of the first equation in (14) and using the second equation produces

$$d\vartheta_1 + (\alpha_1 + 2 * \omega_{12})\wedge\vartheta_1 = 0. \tag{29}$$

The structure equation for the ϑ_i will be needed,

$$d\vartheta_1 = \vartheta_{12}\wedge\vartheta_2 = - * \vartheta_{12}\wedge\vartheta_1. \tag{30}$$

From the second equation in Eq. (26), we have $*\omega_{12} - d\log A + \alpha_1 = *d\psi$, and putting this in the first equation of Eq. (26), we find

$$- * \vartheta_{12} + \alpha_1 + 2 \ * \omega_{12} = 2\, d \log A. \tag{31}$$

Using Eq. (31) in Eq. (30),

$$d\vartheta_1 + (\alpha_1 + 2 * \omega_{12})\wedge\vartheta_1 = 2\, d \log A\wedge\vartheta_1. \tag{32}$$

Replacing $d\vartheta_1$ by means of Eq. (29) implies the following important result

$$d \log A\wedge\vartheta_1 = 0. \tag{33}$$

Equation (33) and Cartan's lemma imply that there exists a function B such that

$$d \log A = B\vartheta_1. \tag{34}$$

This is the first in a series of results which relates many of the variables in question such as J, B, and ϑ_{12} directly to the one-form ϑ_1. To show this requires considerable work. The way to proceed is to use the forms α_i in Theorem 3.2 because their exterior derivatives are known. For an arbitrary function on M, define

$$df = f_1\alpha_1 + f_2\alpha_2. \tag{35}$$

Differentiating Eq. (35) and extracting the coefficient of $\alpha_1\wedge\alpha_2$, we obtain

$$f_{21} - f_{12} + f_2 = 0. \tag{36}$$

In terms of the α_i, $*d\psi = \psi_1\alpha_2 - \psi_2\alpha_1$, Lemma 4.1 yields

$$\psi_{11} + \psi_{22} + \psi_1 = 0. \tag{37}$$

Finally, since $*\vartheta_{12} = 2 * d\psi - \alpha_1$, substituting for $*d\psi$, we obtain that

$$*\vartheta_{12} = -(2\psi_2 + 1)\alpha_1 + 2\psi_1\, \alpha_2. \tag{38}$$

Differentiating structure equation (30) and using Lemma 4.1,

$$*\vartheta_{12} \wedge d\vartheta_1 = 0,$$

so,

$$*\vartheta_{12} \wedge \vartheta_{12} \wedge \vartheta_2 = 0$$

This equation implies that either ϑ_{12} or $*\vartheta_{12}$ is a multiple by a function of the form ϑ_2. Hence, for some function p,

$$\begin{aligned} \vartheta_{12} &= -p\vartheta_2, \quad *\vartheta_{12} = p\vartheta_1, \\ \vartheta_{12} &= p\vartheta_1, \quad *\vartheta_{12} = p\vartheta_2, \end{aligned} \tag{39}$$

Substituting the first line of Eq. (39) back into the structure equation, we have

$$d\vartheta_1 = 0. \tag{40}$$

The second line yields simply $d\vartheta_1 = p\vartheta_1 \wedge \vartheta_2$. Only the first case is examined now. Substituting Eq. (40) into Eq. (29), the following important constraint is obtained

$$(\alpha_1 + 2 * \omega_{12}) \wedge \vartheta_1 = 0. \tag{41}$$

Theorem 5.1 The function ψ satisfies the equation

$$2\psi_1 \cos(2\psi) + (2\psi_2 + 1)\sin(2\psi) = 0. \tag{42}$$

Proof: By substituting $*d\psi$ into Eq. (28) we have

$$*\vartheta_{12} = 2 * (\psi_1\alpha_1 + \psi_2\alpha_2) - \alpha_1 = -(2\psi_2 + 1)\alpha_1 + 2\psi_1\alpha_2. \tag{43}$$

Substituting Eq. (43) into Eq. (26) and solving for $*\omega_{12}$, we obtain that

$$*\omega_{12} = *\vartheta_{12} - * d\psi + d\log A = *\vartheta_{12} - * d\psi + B\vartheta_1 = *d\psi - \alpha_1 + B\vartheta_1.$$

This can be put in the equivalent form

$$2 * \omega_{12} + \alpha_1 = 2 * d\psi - \alpha_1 + 2B\vartheta_1. \tag{44}$$

Taking the exterior product with ϑ_1 and using $d\psi_1$, we get

$$(\alpha_1 + 2 * \omega_{12})\wedge\vartheta_1 = (2 * d\psi - \alpha_1)\wedge\vartheta_1 = (2\psi_1 * \alpha_1 + 2\psi_2 * \alpha_2 - \alpha_1)\wedge\vartheta_1$$
$$= (2\psi_1 \cos(2\psi) + (2\psi_2 + 1)\sin(2\psi))\vartheta_2\wedge\vartheta_1.$$

Imposing the constraint (41), the coefficient of $\vartheta_1\wedge\vartheta_2$ can be equated to zero. This produces the result (42).

As a consequence of Theorem 5.1, a new function C can be introduced such that

$$2\psi_1 = C\sin(2\psi), \qquad 2\psi_2 + 1 = -C\cos(2\psi). \tag{45}$$

Differentiation of each of these with respect to the α_i basis, we get for $i = 1, 2$ that

$$2\psi_{1i} = C_i \sin(2\psi) + 2\psi_i\, C\cos(2\psi), \qquad 2\psi_{2i} = -C_i\cos(2\psi) + 2\psi_i\, C\sin(2\psi).$$

Substituting $f = \psi$ into Eq. (36) and using the fact that ψ satisfies Eq. (37) gives the pair of equations

$$-C_1\cos(2\psi) - C_2\sin(2\psi) + 2\psi_1 C\sin(2\psi) - (2\psi_2 + 1)C\cos(2\psi) - 1 = 0,$$
$$C_1\sin(2\psi) - C_2\cos(2\psi) + 2\psi_1 C\cos(2\psi) + (2\psi_2 + 1)C\sin(2\psi) = 0.$$

This linear system can be solved for C_1 and C_2 to get

$$C_1 + C(2\psi_2 + 1) + \cos(2\psi) = 0, \quad C_2 - 2C\psi_1 + \sin(2\psi) = 0. \tag{46}$$

By differentiating each of the equations in (46), it is easy to verify that C satisfies Eq. (36), namely, $C_{12} - C_{21} - C_2 = 0$. Hence, there exist harmonic functions which satisfy Eq. (42). The solution depends on two arbitrary constants, the values of ψ and C at an initial point.

Lemma 5.1

$$dC = (C^2 - 1)\vartheta_1, \qquad * \vartheta_{12} = C\vartheta_1. \tag{47}$$

Proof: It is easy to express the ϑ_i in terms of the α_i,

$$\vartheta_1 = \cos(2\psi)\alpha_1 + \sin(2\psi)\alpha_2, \qquad \vartheta_2 = -\sin(2\psi)\alpha_1 + \cos(2\psi)\alpha_2. \tag{48}$$

Therefore, using Eqs. (45) and (46), it is easy to see that

$$dC = C_1\alpha_1 + C_2\alpha_2 = (C^2 - 1)(\cos(2\psi)\alpha_1 + \sin(2\psi)\alpha_2) = (C^2 - 1)\vartheta_1.$$

Using Eq. (45), it follows that

$$*\vartheta_{12} = -(2\psi_2 + 1)\alpha_1 + 2\psi_1\alpha_2 = C\cos(2\psi)\alpha_1 + C\sin(2\psi)\alpha_2$$
$$= C(\cos(2\psi)\alpha_1 + \sin(2\psi)\alpha_2) = C\vartheta_1.$$

This implies that $\vartheta_{12} = -C\vartheta_2$.

It is possible to obtain formulas for B_1, B_2. Using Eq. (48) in Eq. (34), the derivatives of $\log A$ can be written down

$$(\log A)_1 = B\cos(2\psi), \qquad (\log A)_2 = B\sin(2\psi). \tag{49}$$

Differentiating each of these in turn, we obtain for $i = 1, 2$,

$$(\log A)_{1i} = B_i \cos(2\psi) - 2B\psi_i \sin(2\psi), \qquad (\log A)_{2i} = B_i \sin(2\psi) + 2B\psi_i \cos(2\psi). \tag{50}$$

Taking $f = \log A$ in Eq. (36) produces a first equation for the B_i,

$$B_1 \sin(2\psi) + 2B\psi_1 \cos(2\psi) - B_2 \cos(2\psi) + 2B\psi_2 \sin(2\psi) + B\sin(2\psi) = 0. \tag{51}$$

If another equation in terms of B_1 and B_2 can be found, it can be solved simultaneously with Eq. (51). There exists such an equation and it can be obtained from the Gauss equation in (4) which we put in the form

$$d\omega_{12} = -ac\, \omega_1 \wedge \omega_2 = -ac\, A^{-2}\, \alpha_1 \wedge \alpha_2.$$

Solving Eq. (26) for ω_{12}, we have

$$\omega_{12} = d\psi + \alpha_2 + (\log A)_2 \alpha_1 - (\log A)_1 \alpha_2.$$

The exterior derivative of this takes the form,

$$d\omega_{12} = [1 - (\log A)_{11} - (\log A)_{22} - (\log A)_1] \alpha_1 \wedge \alpha_2.$$

Putting this in the Gauss equation,

$$-(\log A)_{11} - (\log A)_{22} + \{-(\log A)_1 + 1\} + acA^{-2} = 0.$$

Replacing the second derivatives from Eq. (50), we have the required second equation

$$-B_1 \cos(2\psi) - B_2 \sin(2\psi) + B\{2\psi_1 \sin(2\psi) - (2\psi_2 + 1)\cos(2\psi)\} + 1 + acA^{-2} = 0. \tag{52}$$

Solving Eqs. (51) and (52) together, the following expressions for B_1 and B_2 are obtained

$$B_1 + B(2\psi_2 + 1) - (1 + acA^{-2})\cos(2\psi) = 0, \qquad B_2 - 2B\psi_1 - (1 + acA^{-2})\sin(2\psi) = 0. \tag{53}$$

Given these results for B_1 and B_2, it is easy to produce the following two Lemmas.

Lemma 5.2

$$dB = (BC + 1 + acA^{-2})\vartheta_1, \qquad d\log J = (C + 2B)\vartheta_1. \tag{54}$$

Proof: Substituting Eq. (53) into dB, we get

$$dB = B_1\alpha_1 + B_2\alpha_2 = (BC + 1 + acA^{-2})(\cos(2\psi)\alpha_1 + \sin(2\psi)\alpha_2) = (BC + 1 + acA^{-2})\,\vartheta_1.$$

Moreover,

$$d\log J = \alpha_1 + 2 * \omega_{12} = \alpha_1 + 2(*\vartheta_{12} - * d\psi + d\log A) = \alpha_1 + 2 * \vartheta_{12} - 2 * d\psi + 2d\log A$$
$$= *\vartheta_{12} + 2d\log A = C\vartheta_1 + 2B\vartheta_1.$$

Lemma 5.3

$$d\psi = -\frac{1}{2}\sin(2\psi)\vartheta_1 - \frac{1}{2}(C + \cos(2\psi))\vartheta_2. \tag{55}$$

Proof:

$$2d\psi = 2\psi_1\alpha_1 + 2\psi_2\alpha_2 = C\sin(2\psi)\alpha_1 - (C\cos(2\psi) + 1)\alpha_2$$
$$= C\sin(2\psi)(\cos(2\psi)\vartheta_1 - \sin(2\psi)\vartheta_2) - (C\cos(2\psi) + 1)(\sin(2\psi)\vartheta_1 + \cos(2\psi)\vartheta_2)$$
$$= -\sin(2\psi)\vartheta_1 - (C + \cos(2\psi))\vartheta_2.$$

In the interests of completeness, it is important to verify the following theorem.

Theorem 5.2 The function B satisfies Eq. (36) provided ψ satisfies both Eqs. (37) and (41).

Proof: Differentiating B_1 and B_2 given by Eq. (53), the left side of Eq. (36) is found to be

$$B_{21} - B_{12} + B_2 = 2B_1\psi_1 + B_2(2\psi_2 + 1) + 2B(\psi_{11} + \psi_{22} + \psi_1) + A^{-2}((ac)_1\sin(2\psi) - (ac)_2\sin(2\psi))$$
$$-2acBA^{-2}(\cos(2\psi)\sin(2\psi) - \sin(2\psi)\cos(2\psi)) + (1 + acA^{-2})(2\psi_1\cos(2\psi) + (2\psi_2 + 1)\sin(2\psi))$$
$$= 2(1 + acA^{-2})(2\psi_1\cos(2\psi) + (2\psi_2 + 1)\sin(2\psi)) + A^{-2}((ac)_1\sin(2\psi) - (ac)_2\cos(2\psi)).$$

To simplify this, Eq. (37) has been substituted. Using Eq. (48) and $*d(ac) = (ac)_1\alpha_2 - (ac)_2\alpha_1$, it follows that

$$*d(ac)\wedge\vartheta_2 = ((ac)_1\sin(2\psi) - (ac)_2\cos(2\psi))\alpha_1\wedge\alpha_2.$$

Note that the coefficient of $\alpha_1\wedge\alpha_2$ in this appears in the compatibility condition. To express it in another way, begin by finding the exterior derivative of $4ac = (a + c)^2 - (a - c)^2$,

$$4d(ac) = 2(a + c)(a - c)\vartheta_1 - 2(a - c)^2(\alpha_1 + 2 * \omega_{12}).$$

Applying the Hodge operator to both sides of this, gives upon rearranging terms

$$2 * \frac{d(ac)}{a - c} = (a + c)\vartheta_2 - (a - c)(\alpha_2 - 2\omega_{12}).$$

Consequently, we can write

$$-\frac{2}{(a - c)^2} * d(ac)\wedge\vartheta_2 = (\alpha_2 - 2\omega_{12})\wedge\vartheta_2 = -(2\psi_1\cos(2\psi) + (2\psi_2 + 1)\sin(2\psi))\alpha_1\wedge\alpha_2.$$

Therefore, it must be that

$$-(ac)_1 \sin(2\psi) + (ac)_2 \cos(2\psi) = -\frac{1}{2}(a-c)^2(2\psi_1 \cos(2\psi) + (2\psi_2 + 1)\sin(2\psi)).$$

It follows that when $f = B$, Eq. (36) finally reduces to the form

$$(1 + H^2 A^{-2})[2\psi_1 \cos(2\psi) + (2\psi_2 + 1)\sin(2\psi)] = 0.$$

The first factor is clearly nonzero, so the second factor must vanish. This of course is equivalent to the constraint (41).

6. Intrinsic characterization of *M*

During the prolongation of the exterior differential system, the additional variables ψ, A, B, and C have been introduced. The significance of the appearance of the function C, is that the process terminates and the differentials of all these functions can be computed without the need to introduce more functions. This means that the exterior differential system has finally closed.

The results of the previous section, in particular, the lemmas, can be collected such that they justify the following.

Proposition 6.1 The differential system generated in terms of the differentials of the variables ψ, A, B, and C is closed. The variables H, J, A, B, C remain constant along the ϑ_2-curves so $\vartheta_1 = 0$. Hence, an isometry that preserves H must map the ϑ_1, ϑ_2 curves onto the corresponding ϑ_1^*, ϑ_2^* curves of the associated surface M^* which is isometric to M.

Along the ϑ_1, ϑ_2 curves, consider the normalized frame,

$$\zeta_1 = \cos(\psi)e_1 + \sin(\psi)e_2, \qquad \zeta_2 = -\sin(\psi)e_1 + \cos(\psi)e_2. \tag{56}$$

The corresponding coframe and connection form are

$$\xi_1 = \cos(\psi)\omega_1 + \sin(\psi)\omega_2, \quad \xi_2 = -\sin(\psi)\omega_1 + \cos(\psi)\omega_2, \quad \xi_{12} = d\psi + \omega_{12}. \tag{57}$$

Then ϑ_1 can be expressed as a multiple of ξ_1 and ϑ_2,ϑ_{12} in terms of ξ_2, and the differential system can be summarized here:

$$\begin{gathered} \vartheta_1 = A\xi_1, \qquad \vartheta_2 = A\xi_2, \quad \vartheta_{12} = \xi_{12} + *d\log A = -CA\xi_2, \\ d\log A = AB\xi_1, \qquad dB = A(BC + 1 + acA^{-2})\xi_1, \qquad dC = A(C^2 - 1)\xi_1, \\ dH = AJ\xi_1, \quad dJ = AJ(2B + C)\xi_1. \end{gathered} \tag{58}$$

The condition $d\vartheta_1 = 0$ is equivalent to

$$dA \wedge \xi_1 + Ad\xi_1 = 0.$$

This implies that $d\xi_1 = 0$ since dA is proportional to ξ_1. Also, $d * \vartheta_{12} = 0$ is equivalent to $d * \xi_{12} = 0$.

Moreover, $d * \xi_{12} = 0$ is equivalent to the fact that the ξ_1,ξ_2 curves can be regarded as coordinate curves parameterized by isothermal parameters. Therefore, along the ξ_1,ξ_2 curves, orthogonal isothermal coordinates denoted (s,t) can be introduced. The first fundamental form of M then takes the form,

$$I = \xi_1^2 + \xi_2^2 = E(s)(ds^2 + dt^2). \tag{59}$$

Now suppose we set $e(s) = \sqrt{E(s)}$, then

$$\xi_1 = e(s)\, ds, \qquad \xi_2 = e(s)\, dt, \qquad \xi_{12} = \frac{e'(s)}{e^2(s)}\xi_2 = \frac{e'(s)}{e(s)}\, dt. \tag{60}$$

This means such a surface is isometric to a surface of revolution. Since ψ, $d^* \xi_{12} = 0$, Eq. (57) implies that $d^* \omega_{12} = 0$. This can be stated otherwise as the principal coordinates are isothermal and so M is an isothermic surface.

Since A, B, C, H, and J are functions of only the variable s, this implies that H and J, or H and K, are constant along the t curves where s is constant. This leads to the following proposition.

Proposition 6.2

$$dH \wedge dK = 0, \qquad \xi_{12} = -(C + B)A\xi_2. \tag{61}$$

This is equivalent to the statement M is a Weingarten surface.

Proof: The first result follows from the statement about the coordinate system above. Since $\vartheta_{12} = \xi_{12} + * d\log A = -CA\xi_2$ and $dA = A^2 B\xi_1$,

$$\xi_{12} = -CA\xi_2 - * \, d\log A = -CA\xi_2 - * \, A^{-1}\, dA = -CA\xi_2 - AB * \xi_1 = -(C + B)A\xi_2$$

Consequently, the geodesic curvature of each ξ_2 curve, s constant, is

$$\frac{e'(s)}{e^2(s)} = -A(B + C),$$

which is constant.

To express the ω_i in terms of ds and dt, start by writing ω_i in terms of the ξ_i and then substituting Eq. (60),

$$\omega_1 = \cos(\psi) e\, ds - \sin(\psi) e\, dt, \qquad \omega_2 = \sin(\psi) e\, ds + \cos(\psi) e\, dt. \tag{62}$$

Subscripts (s,t) denote differentiation and $H_s = H'$ is used interchangeably. Beginning with $dH = H'\, ds$ and using Eq. (62), we have

$$dH = H_1\omega_1 + H_2\omega_2 = (H_1 \cos(\psi) + H_2 \sin(\psi))\, e\, ds + (-H_1 \sin(\psi) + H_2 \cos(\psi))\, e\, dt = H'\, ds.$$

Equating coefficients of differentials, this implies that

$$H_1 e \cos(\psi) + H_2 e \sin(\psi) = H', \qquad -H_1 \sin(\psi) + H_2 \cos(\psi) = 0.$$

Solving this as a linear system we obtain H_1, H_2,

$$H_1 = \frac{H'}{e} \cos(\psi), \qquad H_2 = \frac{H'}{e} \sin(\psi). \tag{63}$$

Noting that $u = H_1/J$ and $v = H_2/J$, using Eq. (57) the forms α_i can be expressed in terms of ds,dt

$$\alpha_1 = \frac{H'}{J}(\cos(2\psi)\, ds - \sin(2\psi)\, dt), \qquad \alpha_2 = \frac{H'}{J}(\sin(2\psi)\, ds + \cos(2\psi)\, dt). \tag{64}$$

Substituting ξ_1 from Eq. (60) into $dH = AJ\xi_1$,

$$dH = H' ds = AJ\xi_1 = AJ\, e(s)\, ds.$$

Therefore, $H' = AJe > 0$ and so $H(s)$ is an increasing function of s. Now define the function $Q(s)$ to be

$$Q = \frac{H'}{J} = A \cdot e > 0. \tag{65}$$

Substituting Eq. (65) into Eq. (64), α_i is expressed in terms of Q as well. Equations (20) in Theorem 3.2 can easily be expressed in terms of ψ and Q.

Theorem 6.1 Equation (20) is equivalent to the following system of coupled equations in ψ and Q:

$$\begin{aligned} \sin(2\psi)(\log(Q))_s + 2\cos(2\psi)\psi_s - 2\sin(2\psi)\psi_t = 0, \\ \cos(2\psi)(\log(Q))_s - 2\sin(2\psi)\psi_s - 2\cos(2\psi)\psi_t = Q. \end{aligned} \tag{66}$$

Moreover, Eq. (66) is equivalent to the following first-order system

$$\psi_s = -\frac{1}{2} Q \sin(2\psi), \qquad \psi_t = \frac{1}{2}(\log(Q))_s - \frac{1}{2} Q \cos(2\psi). \tag{67}$$

System (67) can be thought of as a type of Lax pair. Moreover, Eq. (67) implies that ψ is harmonic as well. Differentiating ψ_s with respect to s and ψ_t with respect to t, it is clear that ψ satisfies Laplace's equation in the (s,t) variables $\psi_{ss} + \psi_{tt} = 0$. This is another proof that ψ is harmonic.

Theorem 6.2 The function $Q(s)$ satisfies the following second-order nonlinear differential equation

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$$Q''(s)Q(s)-(Q'(s))^2 = Q^4(s). \tag{68}$$

There exists a first integral for this equation of the following form

$$Q'(s)^2 = Q(s)^4 + \kappa Q(s)^2, \quad \kappa \in \mathbb{R}. \tag{69}$$

Proof: Equation (68) is just the compatibility condition for the first-order system (67). The required derivatives are

$$\psi_{st} = -\frac{Q}{2}\cos(2\psi)((\log Q)_s - Q\cos(2\psi)), \quad \psi_{ts} = \frac{1}{2}(\log Q)_{ss} - \frac{1}{2}Q_s\cos(2\psi) + Q\sin(2\psi)\psi_s.$$

Equating derivatives $\psi_{st} = \psi_{ts}$, the required (68) follows.

Differentiating both sides of Eq. (69) we get

$$Q''(s) = 2Q(s)^3 + \kappa Q(s). \tag{70}$$

Isolating $\kappa Q(s)$ from Eq. (69) and substituting it into Eq. (70), Eq. (68) appears.

It is important to note that the function C which appears when the differential ideal closes can be related to the function Q.

Corollary 6.1

$$C = \left(\frac{1}{Q}\right)'. \tag{71}$$

Proof: Using ϑ_i from Eq. (58) in Lemma 5.3, in the s,t coordinates

$$2d\psi = -\sin(2\psi)\, Ae\, ds - (C + \cos(2\psi))\, Ae\, dt = \psi_s ds + \psi_t\, dt$$

Hence using Eq. (67), this implies that $2\psi_s = -\sin(2\psi)\, Ae = -Q\sin(2\psi)$, hence $Q = Ae$. The second equation in Eq. (67) for ψ_t implies that $(C + \cos(2\psi))\, Ae = Q\cos(2\psi) - (\log Q)'$. Replacing $Ae = Q$, this simplifies to the form (71).

7. Integrating the Lax pair system

It is clear that the first-order equation in (67) for $Q(s)$ is separable and can be integrated. The integral depends on whether K is zero or nonzero:

$$Q(s) = \frac{1}{\varepsilon s + \gamma}, \quad K = 0; \qquad \log\left(\frac{2(K + \sqrt{K}\sqrt{Q^2 + K})}{Q}\right) = \varepsilon\sqrt{K}s + \gamma, \quad K \neq 0. \tag{72}$$

Here $\varepsilon = \pm 1$ and γ is the last constant of integration. Taking specific choices for the constants, for example, $e^{\gamma} = 2\sqrt{K}$ when $K \neq 0$ and $a = \sqrt{K}$, the set of solutions (72) for $Q(s)$ can be summarized below.

$Dom(s)$	$Q(s)$	$Dom(s)$	$Q(s)$
$s > 0$	$\frac{1}{s}$	$s < 0$	$-\frac{1}{s}$
$0 < s < \frac{\pi}{a}$	$\frac{a}{\sin(as)}$	$-\frac{\pi}{a} < s < 0$	$-\frac{a}{\sin(as)}$
$s > 0$	$\frac{a}{\sinh(as)}$	$s < 0$	$-\frac{a}{\sinh(as)}$

(73)

It is presumed that other choices of the constants can be geometrically eliminated in favor of Eq. (73). The solutions (73) are then substituted back into linear system (67). The first equation in (67) implies that either

$$\psi \equiv 0, \quad \mathrm{mod}\,\frac{\pi}{2}; \qquad \frac{2\psi_s}{\sin(2\psi)} = -Q. \tag{74}$$

Substitute $\psi \equiv 0$ into the second equation in (67). It implies that $(\log Q)_s = Q$ and $\psi = \pi/2$ gives $(\log Q)_s = -Q$. In both cases $Q(s)$ is a solution which already appears in Eq. (73).

For the second case in Eq. (74), the equation can be put in the form

$$(\log|\tan(\psi)|)_s = -Q.$$

Integrating we have for some function $y(t)$ to be determined,

$$\tan(\psi) = e^{-\int Q(s)ds} \cdot y(t). \tag{75}$$

Therefore, $\tan(\psi)$ can be obtained by substituting for $Q(s)$ for each of the three cases in Eq. (73). The upper sign holds for $s > 0$ and the lower sign holds if $s < 0$.

i. $Q(s) = \pm s^{-1}$, $-\int Q(s)ds = \log|s|^{\mp}$ and

$$\tan(\psi) = s^{\mp} \cdot y(t). \tag{76}$$

ii. $Q(s) = \pm \frac{a}{\sin(as)}$, $-\int Q(s)ds = \log|\csc(as) - \cot(as)|^{\mp}$ and

$$\tan(\psi) = \left(\tan\left(\frac{as}{2}\right)\right)^{\mp} \cdot y(t). \tag{77}$$

iii. $Q(s) = \pm \frac{a}{\sinh(as)}$, $-\int Q(s)ds = \mp \mathrm{arctanh}(e^{as})$, and

$$\tan(\psi) = (\tanh(\frac{as}{2}))^{\mp} \cdot y(t). \tag{78}$$

In case (ii), if $s > 0$ and $y(t) = \pm 1$ then $\psi = \pm\frac{1}{2}(as + \pi)$, mod$\pi$, and if $s < 0$ and $y(t) = \pm 1$, then $\psi = \pm\frac{1}{2}as$, modπ.

It remains to integrate the second equation of the Lax pair (67) using solutions for both $Q(s)$ and $\tan(\psi)$. The first case (i) is not hard and will be shown explicitly here. The others can be done, and more complicated cases are considered in the Appendix.

(i) Consider $Q(s) = s^{-1}$ and $\tan(\psi) = s^{-1} \cdot y(t)$. The second equation in (67) simplifies considerably to $y_t = -1$, therefore,

$$y(t) = -(t + \sigma), \qquad \tan(\psi) = -\frac{(t + \sigma)}{s}. \tag{79}$$

For $Q(s) = -s^{-1}$ and $\tan(\psi) = s \cdot y(t)$, the second equation of (67) becomes $y_t = -y^2$, therefore,

$$y(t) = \frac{1}{t + \sigma}, \qquad \tan(\psi) = \frac{s}{t + \sigma.} \tag{80}$$

8. A third-order equation for H and fundamental forms

Since $\xi_{12} = (\log e(s))' dt$, using Eq. (60) ω_{12} can be written as

$$\omega_{12} = \xi_{12} - d\psi = (\log e(s))' \, dt - d\psi. \tag{81}$$

Using Eqs. (14) and (64) for α_1, it follows that

$$d\log(J) = Q(\cos(2\psi)\, ds - \sin(2\psi)\, dt) - 2 * (\psi_t\, dt + \psi_s\, ds) + 2 * (\log(e(s)))' \, dt.$$

when ω_i are put in the s,t coordinates, using $*\omega_1 = \omega_2$, it can be stated that $*ds = dt$ and $*dt = -ds$. Consequently, $d\log(J)$ simplifies to

$$d\log(J) = (Q\cos(2\psi) + 2\psi_t - 2(\log(e(s)))')\, ds + (-Q\sin(2\psi) - 2\psi_s)\, dt. \tag{82}$$

First-order system (67) permits this to be written using $e(s) = \sqrt{E(s)}$ as

$$(\log(J))' + (\log(E))' = (\log(Q))'. \tag{83}$$

Hence, there exists a constant τ independent of s such that $E \cdot J = \tau Q$ or

$$E = \tau \frac{Q}{J} = \tau \frac{Q^2}{H'}. \tag{84}$$

This result (84) for E is substituted into the Gauss equation $-((log(E))_{ss}+(log(E))_{tt})=2E(H^2-J^2)$ giving

$$(\log(E))'' = 2(\log(Q))'' - (\log(H_s))'' = 2Q^2 - \left(\frac{H''}{H'}\right)'. \tag{85}$$

Therefore, the Gauss equation transforms into a third-order differential equation in the s variable,

$$\left(\frac{H''}{H'}\right)' + 2\tau H = 2Q^2\left(1 + \tau \frac{H^2}{H'}\right). \tag{86}$$

Thus, a characterization of Bonnet surfaces is reached by means of the solutions to these equations. This equation determines the function $H(s)$ and after that the functions $J(s)$ and $E(s)$. Therefore, Bonnet surfaces have as first fundamental form the expression

$$I = E(s)(ds^2 + dt^2), \qquad E(s) = \tau \frac{Q^2(s)}{H'(s)}. \tag{87}$$

Since ψ is the angle from the principal axis e_1 to the s curve with t equals constant, the second fundamental form is given by

$$II = L\, ds^2 + 2M\, ds\, dt + N\, dt^2. \tag{88}$$

where the coefficients L, M, N are given by

$$\begin{aligned} L &= E(H + J\cos(2\psi)) = EH + \tau Q \cos(2\psi), \\ M &= -EJ\sin(2\psi) = -\tau Q\ \sin(2\psi), \\ N &= E(H - J\cos(2\psi)). \end{aligned} \tag{89}$$

Appendix

It is worth seeing how the second equation in (67) can be integrated for cases (ii) and (iii). Only the case $s > 0$ will be done with $Q(s)$ taken from Eq. (73).

(a) Differentiating $\tan(\psi)$ given in Eq. (77), we obtain that

$$\psi_t = \frac{\tan\left(\frac{as}{2}\right)}{\tan^2\left(\frac{as}{2}\right) + y^2} y_t(t).$$

The following identities are required to simplify the result,

$$\tan(as) = \frac{2\tan(\frac{as}{2})}{1-\tan^2(\frac{as}{2})}, \qquad \cos(2\psi) = \frac{\tan^2(\frac{as}{2})-y^2}{\tan^2(\frac{as}{2})+y^2}.$$

Substituting ψ_t into Eq. (67), we obtain

$$\frac{2\tan(\frac{as}{2})}{\tan^2(\frac{as}{2})+y^2} y_t = -a\cot(as) - \frac{a}{\sin(as)} \frac{\tan^2(\frac{as}{2})-y^2}{\tan^2(\frac{as}{2})+y^2}.$$

Simplifying this, we get

$$\frac{4}{a} y_t = -\frac{1}{2}\left(1-\tan^2\left(\frac{as}{2}\right)\right) - \frac{1}{2}\left(\cot^2\left(\frac{as}{2}\right)-1\right)y^2 - \sec^2\left(\frac{as}{2}\right) + \csc^2\left(\frac{as}{2}\right)y^2.$$

This simplifies to the elementary equation,

$$y_t = \frac{a}{2}(y^2-1), \qquad y(t) = -\tanh\left(\frac{at}{2}+\eta\right).$$

Here η is an integration constant. To summarize then,

$$\tan(\psi) = \tanh\left(\frac{at}{2}+\eta\right) \cdot \tan\left(\frac{as+\pi}{2}\right).$$

(b) Consider now $s > 0$ and take $Q(s)$ from the last line of Eq. (73). Differentiating $\tan(\psi)$ from (78), we get

$$\psi_t = \frac{\coth(\frac{as}{2})}{1+\coth^2(\frac{as}{2})y^2} y_t(t).$$

In this case, the following identities are needed,

$$\tanh(as) = \frac{2\tanh(\frac{as}{2})}{1+\tanh^2(\frac{as}{2})}, \qquad \cos(2\psi) = \frac{1-\coth^2(\frac{as}{2})y^2}{1+\coth^2(\frac{as}{2})y^2}.$$

Therefore, Eq. (67) becomes

$$2\frac{\coth(\frac{as}{2})}{1+\coth^2(\frac{as}{2})y^2} y_t = -a\coth(as) - \frac{a}{\sinh(as)} \frac{\tanh^2(\frac{as}{2})-y^2}{\tanh^2(\frac{as}{2})+y^2}.$$

This reduces to

$$-\frac{4}{a} y_t = \left(1+\tanh^2\left(\frac{as}{2}\right)+\operatorname{sech}^2\left(\frac{as}{2}\right)\right) + \left(\coth^2\left(\frac{as}{2}\right)+1-\operatorname{csch}^2\left(\frac{as}{2}\right)\right)y^2.$$

Simplifying and integrating, it has been found that

$$y_t = -\frac{a}{2}(1 + y^2), \qquad y(t) = -\tan\left(\frac{at}{2} + \eta\right).$$

To summarize then, it has been shown that,

$$\tan(\psi) = \cot\left(\frac{at}{2} + \eta\right) \cdot \coth\left(\frac{as}{2}\right).$$

These results apply to the case $s > 0$ and similar results can be found for the case $s < 0$ as well.

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Author details

Paul Bracken

Address all correspondence to: paul.bracken@utrgv.edu

Department of Mathematics, University of Texas, Edinburg, TX, USA

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