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Chapter 3

Topology and Integrability in Lagrangian Mechanics

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Additional information is available at the end of the chapter

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Abstract

This chapter reviews complete integrability in the setting of Lagrangian/Hamiltonian mechanics. It includes the construction of angle-action variables in illustrative examples, along with a proof of the Liouville-Arnol'd theorem. Results on the topology of the configuration space of a mechanical (or Tonelli) Hamiltonian are reviewed and several open problems are high-lighted.

Mathematics Subject Classication (2010): 37J30; 53C17, 53C30, 53D25

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1. Introduction

Lagrangian mechanics employs the least-action principle to derive Newton's equations from a scalar function, the action function L. In classical mechanics, L is the difference of kinetic and potential energies and therefore appears as an artifice. It is somewhat mysterious, then, that the reformulation of Newtonian mechanics in terms of momentum and position, rather than velocity and position as in Lagrangian mechanics, leads immediately to the total energy function H and a plethora of geometric structure that is hidden in the native setting.

Due to the advantages of the Hamiltonian perspective, this chapter studies Lagrangian systems from this dual point of view. The organization of the chapter is this: Section 2 recalls the classic construction of angle-action variables in 1 degree of freedom via several examples, then states and proves the Liouville- Arnol'd theorem; Section 3 discusses the relationship between the topology of the configuration space and the existence of integrable mechanical systems; and it reviews several constructions of integrable systems whose configuration space is the sphere or torus. Section 3 provides a number of open problems that may stimulate interested researchers or students.

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2. Integrability in Hamiltonian mechanics

2.1. Integrability in 1 degree of freedom

One of the central problems in classical mechanics is the *integrability* of the equations of motion. The classical notion of integrability is loosely related to exact solvability, and roughly corresponds to the ability to solve a system of differential equations by means of a finite number of integration steps.

2.1a. *Example: Harmonic oscillator* Let us take the simple harmonic oscillator, or an idealized Hookean spring-mass system, with mass m and spring constant k. If q is the displacement from equilibrium and p the momentum, then the total energy is

$$H = \frac{1}{2m}p^2 + \frac{k}{2}q^2, \quad \text{and equations of motion are} \quad \left\{ \begin{array}{rcl} \dot{q} & = & p/m, \\ \dot{p} & = & -kq \end{array} \right\}. \tag{1}$$

The change of variables $(q, p) = (Q/\lambda, \lambda P)$ transforms the system to, with $\lambda = \sqrt[4]{km}$,

$$H = \frac{\omega}{2}(P^2 + Q^2), \quad \text{and equations of motion are} \quad \left\{ \begin{array}{rcl} \dot{Q} & = & \omega P, \\ \dot{P} & = & -\omega Q \end{array} \right\}, \tag{2}$$

where $\omega = \sqrt{k/m}$. A second change of variables $(Q, P) = (\sqrt{2I}\cos\theta, \sqrt{2I}\sin\theta,)$ transforms the system to

$$H = \omega I, \quad \text{and equations of motion are} \quad \left\{ \begin{array}{rcl} \dot{\theta} & = & \omega, \\ \dot{I} & = & 0 \end{array} \right\}. \tag{3}$$

The differential equations in (3) are trivial to integrate since the right-hand sides are constants. Let us explain the sequence of transformations. The change of coordinates $(q, p) \to (Q, P)$ is an area-preserving linear transformation that transforms the elliptical level sets of H into circles. The transformation $(Q, P) \to (\theta, I)$ is analogous to the introduction of polar coordinates–indeed the transformation $(r,\theta) = (\sqrt{2I},\theta)$ is a transformation to polar coordinates. Because the area form $\mathrm{d}P\ \mathrm{d}Q = r\ \ \mathrm{d}r\ \ \mathrm{d}\theta$, we see that the transformation $\mathrm{d}P\ \mathrm{d}Q = \mathrm{d}I = \mathrm{d}\theta$.

Therefore, the change of coordinates $(q, p) \to (\theta, I)$ not only reveals the exact solutions of the harmonic oscillator equations, it is area preserving.

Suppose that for some reason one did not know to introduce "polar" coordinates. One might still determine the change of coordinates using only that the transformation $(Q, P) \to (\theta, I)$ preserves area. Indeed, since $\mathrm{d}\ (P\ \mathrm{d}Q – I\ \mathrm{d}\theta) = 0$, there is a function $v = v(Q, \theta)$ such that $P\ \mathrm{d}Q – I\ \ \mathrm{d}\theta = dv$ or $P = \frac{\partial v}{\partial Q}$ and $I = -\frac{\partial v}{\partial \theta}$. Then, upon substituting the identity $P = v_Q$ into (2), one obtains

$$v \equiv \int_0^Q \sqrt{2H/\omega – Q^2}\ \mathrm{d}Q = \frac{1}{2}Q\sqrt{2H/\omega – Q^2} – (H/\omega)\ \ \arccos(Q/\sqrt{2H/\omega}), \tag{4}$$

where ≡ indicates that v equals the right-hand side up to the addition of a 2π-periodic function of θ.

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If (Q,P) make a complete circuit around the contour $\{H = c\}$ then one obtains from (4) and the identity that $P = \nu_Q$ that

$$\Delta\nu = \oint_{\{H=c\}} P\, \mathrm{d}Q \; = (H/\omega)\, 2\pi. \tag{5}$$

On the other hand, since $\mathrm{d}^2 = 0$ and I is held constant on the contour, Green's theorem implies that

$$\Delta\nu = \oint_{\{H=c\}} \mathrm{d}\nu + I\, \mathrm{d}\theta = \oint_{\{H=c\}} I\, \mathrm{d}\theta = 2\pi I. \tag{6}$$

Equating (5) and (6) shows that $H = \omega I$.

These calculations show that one may determine H as a function of I without explicit knowledge of the coordinate transformation $(Q,P) \to (\theta,I)$–but one does need to solve the Hamilton-Jacobi equation

$$H(Q, \nu_Q) = c, \tag{7}$$

for ν, as performed in Eq. (4). At this point, if one wants to derive the change of coordinates from ν, Eq. (4) shows that it is easier to write $\nu = \nu(Q,\ I)$, in which case $P\ \mathrm{d}Q + \theta\ \mathrm{d}I = \mathrm{d}\nu$ or

$$\theta = \nu_I = -\arccos\,(Q/\sqrt{2I}), \tag{8}$$

so $Q = \sqrt{2I}\cos(\theta)$ and $P = \nu_Q = \sqrt{2I}\sin(\theta)$.

Let it be observed that if, in Eq. (4), one had chosen the anti-derivative to be arcsin rather than −arccos, then Q would be $\sqrt{2I}\sin(\theta)$ and P would be $\pm\sqrt{2I}\cos(\theta)$. However, because $\mathrm{d}P\ \mathrm{d}Q = \mathrm{d}I\ \mathrm{d}\theta$, one would be obligated to choose the negative square root to define P; otherwise, $\mathrm{d}P\ \mathrm{d}Q = -\mathrm{d}I\ \mathrm{d}\theta$.

2.1b. *Example: the planar pendulum*. Let us take the idealized planar pendulum with a mass-less rigid rod of length l suspended at a fixed end with a bob of mass m at the opposite end (**Figure 1**). The total energy is

$$H = \frac{1}{2m}p^2 + mlg(1-\cos q), \quad \text{and} \quad \left\{ \begin{array}{rcl} \dot{q} & = & p/m, \\ \dot{p} & = & -mlg\sin q \end{array} \right\}. \tag{9}$$

To simplify the exposition, assume that the mass $m = 1$ and let $\omega^2 = 16lg$, where ω is 4 times the frequency of the linearized oscillations at $q = p = 0$. The substitution $q = 2Q$, $p = P/2$ transforms the Hamiltonian to

$$8H = P^2 + \omega^2 \sin^2 Q, \quad \text{and} \quad \left\{ \begin{array}{rcl} \dot{Q} & = & P/4 \\ \dot{P} & = & -\omega^2 \sin(2Q)/8 \end{array} \right\}. \tag{10}$$

If one tries to solve for a generating function $\nu = \nu(Q,I)$ of a coordinate change $(Q,P) \to (\theta,I)$ such that $H = H(I)$, then one obtains from $P = \nu_Q$ that

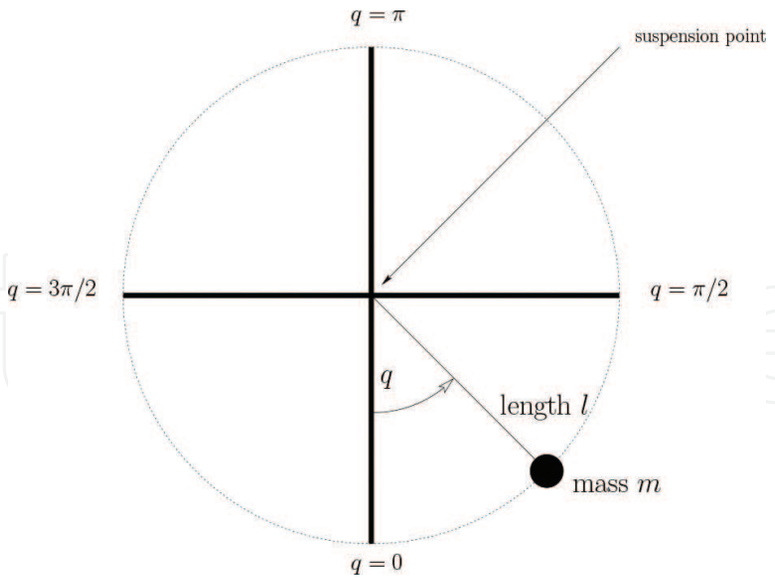


Figure 1. The planar pendulum with potential energy $V = mlg(1-\cos q)$.

$$\nu \equiv \int_0^Q \sqrt{8H - \omega^2 \sin Q}\, dQ = \frac{\omega}{k} E(Q,k) \tag{11}$$

where $\equiv$ indicates equality up to a 2π-periodic function of θ, $H = \omega^2/(8k^2)$ and E is the elliptic integral of the second kind defined by $E(x,\ k) = \int_0^x \sqrt{1-k^2 \sin^2 x}\ dx$.

If (Q,P) make a complete circuit around the contour $\{H = c\}$, then one obtains from Eq. (11) that

$$\Delta\nu = 4\frac{\omega}{k} K(k) \tag{12}$$

where $K(k) = E(Q_+(k),k)$ and $Q_+(k) = \arcsin(1/k)$ if $k > 1$ and $\pi/2$ if $k < 1$ (in which case, K is the complete elliptic integral of the second kind). The area of the shaded region K in **Figure 2** shows the geometric meaning of $K(k)$ for $k > 1$. Along with the identity (6), one obtains

$$I = \frac{2}{\pi}\frac{\omega}{k} K(k), \tag{13}$$

which determines $H = H(I)$ implicitly.

Figure 3 graphs H as a function of I using the definition of I in (13) with $\omega = 1$, along with the graph for the harmonic oscillator. Although H appears to be a smooth function of I on the interval depicted, this is a numerical artifact. Indeed, there are two distinct proofs that H cannot be differentiable in I over the interval [0,1]. Without loss of generality, it is assumed that $\omega = 1$.

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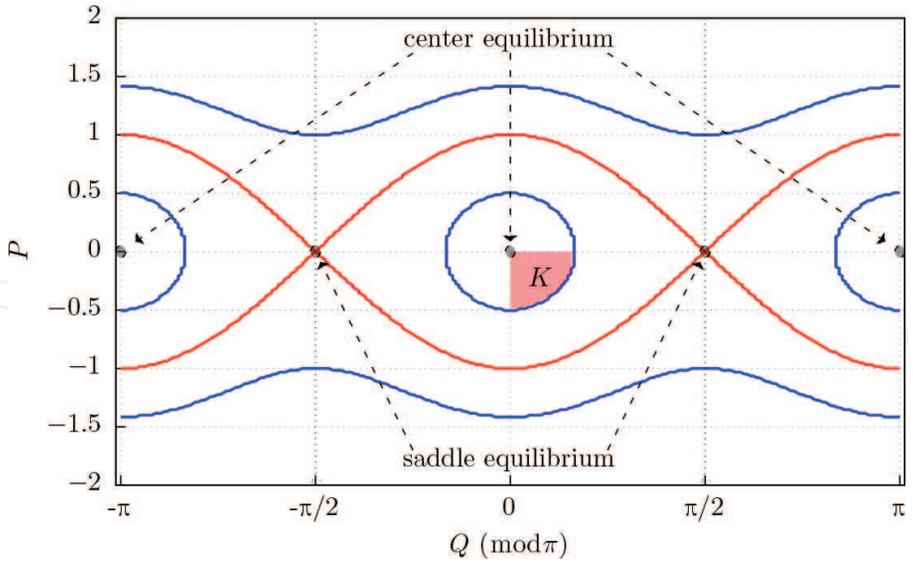


Figure 2. The contours of the pendulum Hamiltonian with $\omega = 1$ (9).

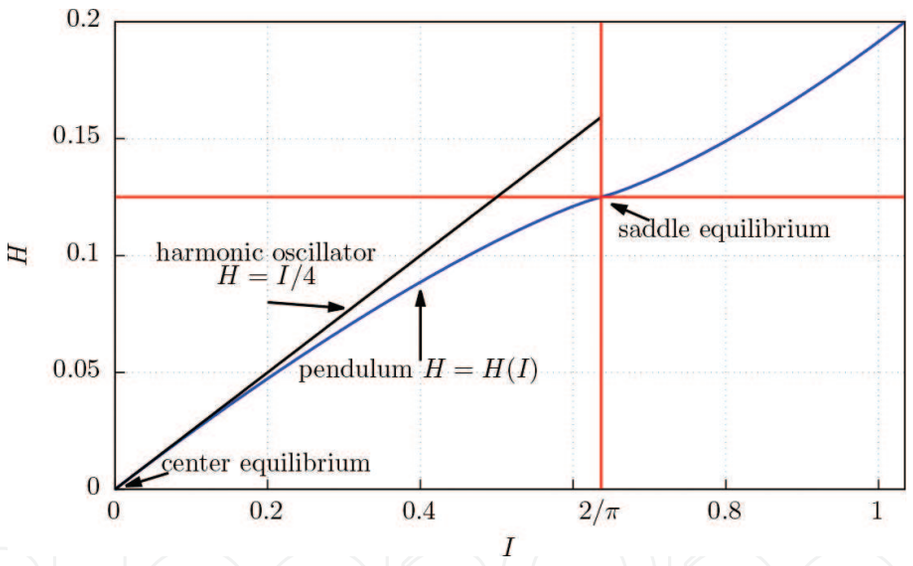


Figure 3. The graph of $H = H(I)$for the pendulum.

The first, calculus-based, proof is this: as $k \to 1^+ (H \to 1/8^-), \partial I/\partial k \to \infty$. If H is a differentiable function of I, then $\partial H/\partial I = 0$ at $I = 2/\pi (H = 1/8)$. But then the entire level set consists of fixed points, which is false.

The second, topological, proof is this: each level set $\{H = c\}, c < 1/8$, is connected; each level set for $c > 1/8$ has exactly two connected components (*c.f.* **Figure 2**). If the generating function v were differentiable in (Q, I) on any rectangle containing $\mathbf{R}/\pi\mathbf{Z} \times \{2/\pi\}$, then Eq. (13) would determine a homeomorphism $H = H(I)$, and so the level sets of H would remain connected on either side of the critical level at height 1/8. Absurd.

To derive the change of coordinates $(Q,P)\to(\theta,I)$ from the generating function v, one uses the identity $\theta = v_I$ and properties of the elliptic integrals to deduce

$$\theta = \frac{\pi}{2}\frac{F(Q,\,k)}{(Q_+(k),k)} \quad\Rightarrow\quad Q = \mathrm{am}_k\left(\frac{2F_+}{\pi}\theta\right) \tag{14}$$

where $F(x,k) = \int_0^x \mathrm{d}x/\sqrt{1-k^2\sin^2 x}$ is the elliptic integral of the first kind, $F_+ = F(Q_+(k),k)$ and $\mathrm{am}_k(u)$ is the Jacobian amplitude function, a local inverse to F ([1], Chapter 2). Along with $P = v_Q$, (14) implies that

$$P = \frac{\omega}{k}\mathrm{dn}_k\left(\frac{2F_+}{\pi}\theta\right), \tag{15}$$

where $\mathrm{dn}_k(u)$is the Jacobian elliptic function.

2.1c. *Example: a mechanical system*. Let $V = V(Q)$ be a smooth potential function of a 1-degree-of-freedom Hamiltonian system with

$$H = \frac{1}{2}P^2 + V(Q). \tag{16}$$

If one attempts to find the generating function $v = v(Q,I)$ of an area-preserving transformation $(Q,P)\to(\theta,I)$ that transforms $H = H(I)$, then one deduces that

$$v \equiv \int_{Q_0}^{Q} \sqrt{2(H-V(Q))}\,\mathrm{d}Q, \tag{17}$$

up to a function depending only on I. Then, in a complete circuit around the connected contour $\{H = c\}$, one has $2\pi = \Delta\theta = \Delta v_I$ identically, so

$$2\pi I = \oint_{\{H=c\}} P\,\mathrm{d}Q. \tag{18}$$

and, upon solving (18) for $H = H(I)$, one inverts

$$\theta = \frac{1}{\sqrt{2}}\int_{Q_0}^{Q}\frac{H_I}{\sqrt{H-V(Q)}}\mathrm{d}Q, \tag{19}$$

to obtain $Q = Q(\theta,I)$, and finally $P = v_Q$ yields $P = P(\theta,I)$. Since the change of coordinates is area-preserving, the Hamiltonian form of the equations of motion are preserved, so the resulting equations are

$$H = H(I) \quad \text{and} \quad \left\{\begin{array}{ccc}\dot\theta & = & \partial H/\partial I\\ \dot I & = & 0\end{array}\right\}. \tag{20}$$

2.2. The generating function

The above three examples use a generating function $v = v(Q,I)$ of a mixed system of coordinates in order to create an area-preserving change of coordinates to angle-action variables $(\theta,\,I)$.

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2.2a. *Question: why do the angle-action variables exist?* In order to understand the generating function, it is necessary to clarify the existence of the coordinates (θ, I), which are commonly called angle-action variables. Let $H : X \to \mathbf{R}$ be a smooth function from an oriented surface X to the reals. If it is assumed that $A \subset X$ is an open, connected, saturated $(H^{-1}(H(A)) \cap A = A)$ subset of the domain of $H, H|A$has no critical points and $H|A$ is proper, then $H|A$ is a submersion onto the interval $B = H(A) \subset \mathbf{R}$. Since $H|A$ is proper, for each $b \in B$, the level set $(H|A)^{-1}(b)$ is a compact one-manifold and hence its components are circles. Since A is connected and $H|A$ is critical-point free, the level set must be connected, so it is a circle. Therefore, the submersion theorem implies that A is diffeomorphic to $A' = S^1 \times B$.[1] To make this system of coordinates concrete, note that there is a complete vector field U on A such that $\mathrm{d}H(U) \equiv 1$. Let $\gamma \subset A$ be a segment of an integral curve of U which is maximal (i.e. an integral curve that strictly contains γ intersects $X - A$). For each $a \in A$, let $t = t(a)$ be the time along the flow line of the Hamiltonian vector field X_H beginning at the initial condition $\gamma \cap H^{-1}(H(a))$. The function t is multi-valued, since the flow line is closed, so it should be considered as a function on the universal cover of A.

Since the tangent space at $a \in A$ is spanned by X_H and U, Ω is determined by $\Omega(X_H, U)$. But $Q(X_H, U) = -\mathrm{d}H(U) = -1$, so $\Omega = \mathrm{d}H\mathrm{d}t$.

Let T be such that $2\pi T$ is the least period of the function t (i.e. the first return time to γ). Then $T = T(H)$ is a function of H alone. Define θ by

$$\theta = t/T(H) \pmod{2\pi} \quad \text{and } I \text{ by} \quad \mathrm{d}I = T(H)\,\mathrm{d}H. \tag{21}$$

The function θ is the normalized time along the flow lines of Hamiltonian vector field X_H, while $\mathrm{d}H/\mathrm{d}I = 1/T(I)$ is the frequency. One computes that the oriented area form $\Omega = \mathrm{d}H\mathrm{d}t = \mathrm{d}I\mathrm{d}\theta$. Moreover, in the coordinates (θ, I), the Hamiltonian vector field

$$X_H = \left\{ \begin{array}{ccccc} \dot{\theta} & = & 1/T(H) & = & \dfrac{\mathrm{d}H}{\mathrm{d}I}, \\ \dot{I} & = & 0 & & \end{array} \right\}. \tag{22}$$

This proves the existence of an area-preserving diffeomorphism $\phi : D \times \mathbf{S}^1 \to A$, where $D \subset \mathbf{R}$ is an open interval, such that the Hamiltonian H is transformed to a function of I alone; and ϕ is as smooth as H and the area form Ω are (e.g. if both are real-analytic, then ϕ is real-analytic).

2.2b. Question: what kind of "function" is v? In the first instance, v is not single-valued. Indeed, one postulates the area-preserving change of coordinates $\phi : (Q, P) \to (\theta, I)$ to deduce that

$$\mathrm{d}(P\mathrm{d}Q + \theta\mathrm{d}I) = 0, \tag{23}$$

so that *locally* there is a function v such that

$$P\,\mathrm{d}Q + \theta\mathrm{d}I = \mathrm{d}v. \tag{24}$$

[1] If one prefers a purely "elementary" proof, one might apply the inverse function theorem at this point.

But since θ is an angle variable, this equation can only hold globally modulo $2\pi\mathbf{Z}\ \mathrm{d}I$. So, in this formulation of the generating function, ν can only be defined globally modulo $2\pi\mathbf{Z}\ I$. Or, equivalently, ν_I is a function with values in the circle $\mathbf{R}/2\pi\mathbf{Z}$.

The way to resolve these ambiguities or difficulties is simple: the domain of the change of coordinates ϕ must be non-simply connected (a disjoint union of open annuli, in fact, as can be deduced from the discussion above) and so one should view (24) as holding globally on the universal cover of this annulus where θ is a single-valued real function (*c.f.* 21). In this case, the lift of a closed contour $\{H = c\}$ is a path that projects to the contour and whose endpoints differ by a deck transformation–which in the angle-action coordinates is $(\theta, I) \to (\theta + 2\pi, I)$. Since I is constant along this path, the path integral of $P\mathrm{d}Q$ equals the path integral of $\mathrm{d}\nu$, i.e. $\Delta\nu$, the change in ν from one preimage to its translate. With this understanding, Eq. (18) is correct. And, indeed, one sees that the integral in Eq. (17) is defined not on the domain of the coordinate change ϕ but on its universal cover; the same is true for the integral in Eq. (19), but the marvellous fact about that integral is that it is 2π-periodic: this follows from the observation that $\Delta\theta = 2\pi$ identically around a closed connected contour in $\{H = c\}$.

So to answer the question that started the section, the generating function ν is a function defined on the universal cover of the union of regular compact levels of H which implicitly defines a 2π-periodic change of coordinates to "angle-action" variables (θ, I).

2.3. Integrability in 2 or more degrees of freedom and Tonelli Hamiltonians

Integrability in 2 or more degrees of freedom is substantially more involved than the case of 1 degrees of freedom. Of course, a sum of n distinct, non-interacting 1-degree-of-freedom Hamiltonians is a simple case; and upon reflection, a not-so- simple case, because this condition is not coordinate independent. Indeed, a necessary and sufficient condition is that the Hamiltonian vector field be Hamiltonian with respect to two distinct non-degenerate Poisson brackets $\{,\}_i$ that are compatible in the sense that the linear space spanned by the brackets is a space of Poisson brackets, and maximal in the sense that a "recursion" operator naturally defined from the two brackets has a maximal number of functionally independent eigenvalue fields [2].

Let us turn now to a definition which generalizes mechanical Hamiltonians.

Definition 2.1 *(Tonelli Hamiltonian). Let Σ be a smooth n-manifold and $T^*\Sigma$ its cotangent bundle. A smooth function $H : T^*\Sigma \to \mathbf{R}$ which satisfies $(\mathrm{T}_1) H|T^*_x\Sigma$ is strictly convex for each $x \in \Sigma$; and $(T_2)\ H(x, tp)/t \to \infty$ uniformly as $t \to \infty$, is called a Tonelli Hamiltonian.*

As noted, Tonelli Hamiltonians are natural generalizations of mechanical systems. For this reason, Σ will be referred as the configuration space of the Hamiltonian H.

If Q_i are coordinates on Σ and $\Theta = \sum_i P_i\, \mathrm{d}Q_i$ are the coordinates of the 1-form Θ, then the canonical symplectic structure $\Omega = \mathrm{d}\Theta = \sum \mathrm{d}P_i \wedge \mathrm{d}Q_i$ on $T^*\Sigma$. The symplectic form Ω equips the space of smooth functions on $T^*\Sigma$ with a Poisson bracket denoted $\{,\}$ that satisfies

$$\{P_i, Q_j\} = -\{Q_j, P_i\} = \delta_{ij} \quad \{Q_i, Q_j\} = \{P_i, P_i\} = 0 \tag{25}$$

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for all i,j. The Poisson bracket is fundamental to Hamiltonian mechanics. For each smooth function H, one has a smooth vector field $X_H = \{H,\}$, and the skew symmetry of the bracket implies that H is preserved by the flow. One says that H_1 and H_2 Poisson commute if $\{H_1, H_2\}\equiv 0$.

A fundamental result in Hamiltonian mechanics is the Liouville-Arnol'd theorem, which provides a semi-local description of a completely integrable Hamiltonian and the Poisson bracket.

Theorem 2.1 *(Liouville-Arnol'd). Let $H : T^*\Sigma \to \mathbf{R}$ be a smooth Hamiltonian. Assume there exists n functionally independent, Poisson commuting conserved quantities $F = (F_1 = H, \ldots, F_n) : T^*\Sigma \to \mathbf{R}^n$. If $L \subset F^{-1}(c)$ is a compact component of a regular level set, then there is a neighbourhood W of L and a diffeomorphism $\phi = (\theta, I) : \mathbf{T}^n \times \mathbf{B}^n \to W$ such that*

$$F = F(I) \quad \{I_i, \theta_j\} = \delta_{ij}, \quad \{I_i,\ I_j\} = \{\theta_i, \theta_j\} = 0,$$
$$X_{F_i} = \sum \frac{\partial F_i(I)}{\partial I_j} \frac{\partial}{\partial \theta_j},$$

that maps L to $\mathbf{T}^n \times \{0\}$.

In such a situation, it is said that $H$is Liouville, or completely, integrable. The torus $\mathbf{T}^n \times \{I_0\}$ is a *Liouville torus*, the neighbourhood $\mathbf{T}^n \times \mathbf{B}^n$ is a *toroidal ball* and the conserved quantities are *first integrals*. Systems with k first integrals, of which $l < k$ Poisson commute with all k first integrals, where $k + l = 2n$ are called *non-commutatively integrable*; when $k = 2n-1$, the system is also called *super-integrable c.f.* [3, 4].

There are several proofs of the Liouville-Arnol'd theorem in the literature. The basic ideas are already captured in the one-dimensional case discussed in Section 2.2.

It can be assumed, without loss, that $L = F^{-1}(c)$. Since $c \in \mathbf{R}^n$ is a regular value of F, the submersion theorem implies that there is an open neighbourhood C of c consisting of regular values of F and the open set $F^{-1}(C)$ is diffeomorphic to $L \times C$. Therefore, there is a smooth n-dimensional submanifold $M \subset F^{-1}(C)$ such that M transversely intersects each level set $L_f = F^{-1}(f), f \in C$. Possibly by shrinking the open set C, it can be assumed that M is Lagrangian: $\Omega|M \equiv 0$.[2]

Because the functions $F_i, \ldots, F_n$ Poisson commute and are functionally independent, the Hamiltonian vector fields $X_{F_1}, \ldots, X_{F_n}$ span the tangent space $T_x L_f$, for each $x \in L_f$, $f \in C$. Because L_f is compact, each vector field is complete, so there is a well-defined flow map $\phi_{F_i} : \mathbf{R} \times F^{-1}(C) \to F^{-1}(C)$. Because $F_i, \ldots, F_n$ Poisson commute, the respective flow maps commute, so there is an action of $\mathbf{R}^n$ on $F^{-1}(C)$ defined by

$$\phi^t = \phi_{F_n}^{t_n} \circ \cdots \circ \phi_{F_1}^{t_1} \tag{26}$$

for all $t \in \mathbf{R}^n$. Define a map

[2]The existence of M is a consequence of Darboux's theorem. Of course, a less elementary proof would appeal to Weinstein's theorem and Moser's isotopy lemma.

$$\Phi(t,\ m) = \phi^t(m), \quad t \in \mathbf{R}^n, m \in M. \tag{27}$$

This is a smooth map which is a local diffeomorphism of $\mathbf{R}^n \times M$ with $F^{-1}(C)$. Indeed, ϕ^t carries each level L_f into itself and carries M into a submanifold $\phi^t(M)$ transverse to L_f at $\phi^t(m)$; on the other hand, the derivative of ϕ^t with respect to t is a surjective linear map onto $T_{\phi^t(m)}L_f$. Therefore, $\mathrm{d}\Phi$ is surjective, so injective, hence Φ is a local diffeomorphism onto its image. Compactness and connectedness of the levels L_f imply that the image of Φ is $F^{-1}(C)$.

For each $m \in M$, let $P(m) \subset \mathbf{R}^n$ be the set of t such that $\Phi(t, m) = m$. Since each level set is compact, $P(m)$ is a discrete subgroup of $\mathbf{R}^n$ isomorphic to $\mathbf{Z}^n$. This is the "period lattice" of the action ϕ. If one selects a basis of $P(m)$, one obtains a map $M \to \mathrm{GL}(n; \mathbf{R}), m \to 2\pi T(m)$. The implicit function theorem implies that there is a smooth map amongst these maps. Moreover, since $F|M$ is a bijection onto its image, one can take the components of F as coordinates on M, or in other words, $T = T(F)$.

Define functions $\theta = (\theta_1, \ldots, \theta_n)$ by

$$\theta = T(F)^{-1} \cdot t \pmod{2\pi}, \quad \theta : \mathbf{R}^n \times C \to \mathbf{R}^n / 2\pi\mathbf{Z}^n. \tag{28}$$

The flow map Φ therefore induces a diffeomorphism $F^{-1}(C) \to \mathbf{T}^n \times C : x \to (\theta(x), F(x))$.

To complete the proof, one might show that each vector field $\partial/\partial\theta_i$ is Hamiltonian with Hamiltonian function I_i and that F is functionally dependent on I so that (θ, I) is a canonical system of coordinates on $F^{-1}(C)$. This is performed indirectly. Define the functions $I_i = I_i(F)$ by

$$2\pi I_i = \oint_{\Gamma_i(F)} \xi, \tag{29}$$

where $\xi = P \cdot \mathrm{d}Q$ is the primitive of the symplectic form Ω and $\Gamma_i(F)$ is the cycle on L_F on which θ_i increases from 0 to 2π and the other angle variables are held equal to 0. To show that (θ, I) is a system of coordinates on $F^{-1}(C)$, one computes the Jacobian $[\partial I_i/\partial F_j]$:

$$2\pi \frac{\partial I_i}{\partial F_j} = \lim_{s \to 0} \frac{1}{s} \int\int_{C_j(F,s)} \Omega, \tag{30}$$

where, in the (t, F) coordinate system,

$$C_j(F,\ s) = \{(uT(F + ve_j)_i, F + ve_j) | u \in [0, 2\pi],\ v \in [0, s]\}$$

is the "cylinder" obtained by sweeping out the cycles $\Gamma_j(F + ve_j)$ as the j-th component of Fincreases from F_j to $F_j + s$, and T_i is the i-th column of the period matrix T. Since

$$\Omega\left(\frac{\partial}{\partial F_j},\ \frac{\partial}{\partial t_k}\right) = \frac{\partial F_k}{\partial F_j} = \delta_{jk}, \tag{31}$$

which implies

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$$\frac{\partial I_i}{\partial F_j} = T_{ji}. \tag{32}$$

Since the period matrix T is non-singular, the transformation $(\theta, F) \to (\theta, I)$ is a diffeomorphism.

Finally, the functions $I_1, \ldots, I_n$ Poisson commute and since M is Lagrangian, the functions $t_i, \ldots, t_n$ Poisson commute, which implies $\theta_1, \ldots, \theta_n$ Poisson commute. And, since $\{F_i, t_j\} = \delta_{ij}$, this implies that $\{I_i, \theta_j\} = \delta_{ij}$.

The remainder of the theorem follows from the fact that the angle-action coordinates (θ, I) are canonical and $F = F(I)$.

3. Topology of configuration spaces

The central problem in the theory of completely integrable Tonelli Hamiltonian systems is to

Problem 3.1. *Determine necessary conditions on the configuration space Σ for the existence of a completely integrable Tonelli Hamiltonian H.*

This is a broad, overarching problem which has motivated research by many authors over an almost 40-year period, including many of the author's publications. It is helpful to pose several sub-problems which address aspects of this problem and that appear to be amenable to solution. The remainder of this section is devoted to an elaboration of this problem, along with known results. We start with two-dimensional configuration spaces.

3.1. Surfaces of genus more than one

As a rule, completely integrable Tonelli Hamiltonians are quite rare, as are the configuration spaces Σ which support such Hamiltonians. Indeed, in two dimensions, the compact surfaces that are *known* to support a completely integrable Tonelli Hamiltonian are the 2-sphere, $\mathbf{S}^2$, the 2-torus $\mathbf{T}^2$ and their non-orientable counterparts. With some quite mild restrictions on the singular set–called condition א–, and assuming that the Hamiltonian is Riemannian, Bialy has proven these are the only compact examples [5]. This extended an earlier result of V. V. Kozlov [6]; the author has obtained a similar result for super-integrable Tonelli Hamiltonians [7].

V. Bangert has suggested to the author that Bialy's argument should extend to prove the non-existence of a C^2 integral that is independent of the Hamiltonian when Σ is a compact surface of negative Euler characteristic (*c.f.* [8]). The idea of such a proof would be the following (assuming that H is Riemannian): Suppose that H enjoys a C^2 integral F that is independent on a dense set, hence that the union of Liouville tori is dense. Let $\Gamma \subset H^{-1}(\frac{1}{2})$ be the union of orbits which project to minimizing geodesics. It is known, due to results of Manning and Katok [9, 10], that $\overline{\Gamma}$ contains a hyperbolic invariant set Λ on which the flow is conjugate to a horseshoe. Let $\lambda \subset \Lambda$ be a closed orbit of the geodesic flow of period T. Since the union of Liouville tori is dense, for each $\epsilon > 0$, there is a Liouville torus $L_{\lambda,\epsilon}$ that contains an orbit of

the geodesic flow that remains within a distance ϵ of λ over the interval $[0, T]$. Hence, $\pi_1(L_{\lambda,\epsilon})$ has a homotopy class mapping onto λ. Since λ is minimizing, it has no conjugate points and so for ϵ sufficiently small, the same is true for the orbit on $L_{\lambda,\epsilon}$ over the time interval $[0, T]$. This implies that the image of $\pi_1(L_{\lambda,\epsilon})$ is (free) cyclic and the kernel is generated by a cycle that bounds a disc–in classical terminology, this means that $L_{\lambda,\epsilon}$ is *compressible*. It follows that $L_{\lambda,\epsilon}$ bounds a solid torus $T_\lambda \cong \mathbf{T}^1 \times \mathbf{B}^2$ that is invariant for the geodesic flow. The integral $F|T_\lambda$ induces a singular fibration of the solid torus by invariant 2-tori.

Thus, for each closed orbit λ in the hyperbolic invariant set Λ, we have produced an invariant solid torus T_λ that shadows λ–at least in some rough, homotopic sense. This fact alone should suffice to achieve a contradiction.

Problem 3.2. *Let Σ be a compact surface of negative Euler characteristic. Extend the above argument to prove the non-existence of a smooth Tonelli Hamiltonian $H : T^*\Sigma \to \mathbf{R}$ with a second C^2 integral F that is independent on a dense set; or give an example of a completely integrable Tonelli Hamiltonian $H : T^*\Sigma \to \mathbf{R}$.*

V. Bangert proposes similar problems in his contribution in ([8], Problems 1.1, 1.2).

There is a similar, but possibly more accessible, problem for twist maps. Recall that if we discretize time, the notion of a Tonelli Hamiltonian is replaced by that of a *twist map $f : T^*\Sigma \to T^*\Sigma$ which is a symplectomorphism that satisfies a condition analogous to* T_1. If f enjoys n independent, Poisson commuting first integrals, then the Liouville-Arnol'd theorem implies that some power of f acts a translation on the Liouville tori. We noted above that the Hamiltonian flow of a Tonelli Hamiltonian has a horseshoe on an energy level.

Problem 3.3. *Let $f : T^*\mathbf{T}^1 \to T^*\mathbf{T}^1$ be a twist map. If f has a horseshoe and a C^1 first integral F, is F necessarily constant on an open set?*

3.2. The 2-torus

Let us turn now to the torus. The 2-torus $\mathbf{T}^2$ admits a family of completely integrable Riemannian Hamiltonians which are called *Liouville*. These are of the form

$$H = \frac{p_x^2 + p_y^2}{2[f(x) + g(y)]} \quad F = \frac{g(y)p_x^2 - f(x)p_y^2}{f(x) + g(y)} \tag{33}$$

where $f, g : \mathbf{T}^1 \to \mathbf{R}$ are smooth positive functions and (x, y, p_x, p_y) is a canonical system of coordinates on $T^*\mathbf{T}^2$. The degenerations of the Liouville family include the rotationally symmetric ($f \equiv const.$) and flat ($f, g \equiv const.$).

The Liouville family is obtained from two uncoupled mechanical oscillators with periodic potentials,

$$G = \frac{1}{2}(p_x^2 + p_y^2) + a(x) + b(y), \tag{34}$$

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on an energy level $E = \alpha + \beta > \max a + \max b$ such that $f = \alpha$–$a, g = \beta$–b. The Maupertuis principle states that orbits of the Hamiltonian flow of G on the energy level $\{G = E\}$ are orbits of the Hamiltonian flow of H up to a change in time along the orbit. The complete integrability of G is explained in Sections 2.1c and 2.3.

It is a remarkable fact that the Liouville family exhausts the list of known completely integrable Riemannian Hamiltonians whose configuration space is $\mathbf{T}^2$. Indeed, in 1989, Fomenko conjectured that these are the *only* examples possible when the second integral in polynomial-in-momenta [11]. Most recently, in 2012, Kozlov, Denisova and Treschëv reiterate Fomenko's conjecture ([12], p. 908).

Let us note that it is a well-known fact that, if the first integral F is real- analytic, then $F = \sum_{N \geq 0} F_N$ where each term F_N is polynomial-in-momenta with real-analytic coefficients, homogeneous and of degree N and since $\{H, F_N\}$ is polynomial-in-momenta, homogeneous and of degree $N + 1$, each graded piece of F is a first integral. So, there is no loss in generality in restricting attention to polynomial-in-momenta first integrals–and, indeed, a slight increase in generality because the coefficients of the polynomial-in-momenta first integral are not assumed to be real-analytic.

In [13, 14], Kozlov and Denisova prove that if, when (x, y) are isothermal coordinates, and

$$H = \frac{1}{2\Lambda}(p_x^2 + p_y^2), \tag{35}$$

with the conformal factor Λ a trigonometric polynomial, then the existence of a second independent first integral that is polynomial-in-momenta implies that H is Liouville.

In [12], Denisova, Kozlov and Treschëv prove that, if one only assumes Λ is smooth, then H has no irreducible polynomial-in-momenta first integral F that is of degree 3 or 4 that is independent of H. Mironov separately proves the non-existence of F of degree 5, but as noted in ([12], p. 909), Λ satisfies an extra unstated hypothesis [15]. The line of attack used in these papers is pioneered in [16], where Kozlov and Treschëv introduce the notion of the *spectrum* $S \subset 2n\mathbf{Z}^2$ *of the function* Λ as the support of the Fourier transform of Λ. This spectrum is finite iff Λ is a trigonometric polynomial; Denisova and Kozlov prove that, in this case, any first integral of H is dependent on H unless the spectrum S is contained in a pair of orthogonal lines through (0, 0), in which case H is Liouville and has a second independent first integral that is quadratic-in-momenta. Without the hypothesis that S is finite, the problem becomes significantly more delicate. The bulk of [12], for example, is devoted to a study of solutions to a PDE that characterizes the first integral F by means of Fourier analysis.

An alternative approach, due to Bialy and Mironov, is to observe that the equation $\{H, F\} = 0$ coupled with the hypothesis that F is polynomial-in-momenta of degree N implies that when we write F as

$$F = \sum_{j=0}^{N} a_j(x, y) p_x^{N-j} p_y^j \tag{36}$$

then the coefficients $a_0, \ldots, a_n$ satisfy a *semi-linear* PDE [17, 18]. Indeed, there is a system of coordinates (τ, v) on $\mathbf{T}^2$ such that, when F is written in the adapted canonical coordinates as $F = \sum_{j=0}^{N} u_j(\tau, v)(p_v/g)^j p_r^{(N-j)}$ then this equation is of the form

$$u_v + T(u)u_r = 0 \tag{37}$$

where $u_0 = 1, u_1 = g, u = (u_1, u_2, \ldots, u_N)$ and

$$T(u)ij = \begin{cases} u_{i+1} & \text{if } j = i+1, \\ (i+1)u_{i+1} - (N-1-i)u_{i-1} & \text{if } j = 1, \\ 0 & \text{otherwise,} \end{cases} \tag{38}$$

where we adopt the convention that $u_{-1} = u_{N+1} \equiv 0$.

A standard technique to solve a quasi-linear PDE like (37) is to diagonalize it, that is, to find Riemann invariants, so that it is equivalent to

$$r_v + \Delta(r)r_r = 0 \quad \text{where} \quad \Delta(r) = \operatorname{diag}(\delta_1(r), \ldots, \delta_N(r)), \quad r = (r_1, \ldots, r_N). \tag{39}$$

To find Riemann invariants, Bialy and Mironov employ the following trick: let $p_v = g\cos(\theta)$, $p_r = \sin(\theta)$ parameterize cotangent fibres of $H^{-1}(\frac{1}{2})$. The invariance condition $\{H, F\} = 0$ translates to $F_v g^{-1}\cos(\theta) + F_\tau \sin(\theta) = 0$ along the locus where $F_\theta = 0$, i.e. where dF and dH are co-linear. If one supposes that $\theta_i = \theta_i(\tau, v), i = 1, \ldots, N$, is a smooth parameterization of the critical-point set, then the critical values $r_i = F(r, v, \theta_i(r, v))$ are Riemann invariants with $\delta_i = g(\tau, v) \times \tan(\theta_i)$. Of course, the main problem is to determine the relationship between the Liouville foliation–the singular foliation of $T^*\mathbf{T}^2$ by the Liouville tori and their degenerations–and the system 39.

In ([18], Theorems 1 and 2), Bialy and Mironov prove that if $N \leq 4$, then in any region where a multiplier δ_i is non-real, the metric is Liouville. One can view the result of Bialy and Mironov as a partial confirmation of Fomenko's conjecture and an important step toward resolving that conjecture.

The key step in Bialy and Mironov's proof is to show that, in any region where δ_i is non-real, the imaginary part of the Riemann invariant r_i satisfies an elliptic PDE. It appears that the properties of this PDE are key to proving stronger results.

Problem 3.4. *Extend Bialy and Mironov's work to show that there are no regions where any multiplier δ_i is non-real on $\mathbf{T}^2$, i.e. show that (39) is a hyperbolic system.*

There is good reason to believe that the multipliers δ_i are always real. When $\varnothing \subsetneq \delta_i^{-1}(\mathbf{C} \backslash \mathbf{R}) \subsetneq \mathbf{T}^2$, Bialy and Mironov prove that the Riemann invariant r_i is real and constant, say $r_i = s_i$. This implies that the common level set $F^{-1}(s_i) \cap H^{-1}(\frac{1}{2})$, a subset of the complexified cotangent bundle $T^*_{\mathbf{C}}\mathbf{T}^2$, has a tangent with the fibres of $T^*_{\mathbf{C}}\mathbf{T}^2$ on an open set. That picture is dramatically at odds with the real picture, where the tangency can occur along a one-cycle at most. Because of this, it seems likely that there is a geometric proof of Problem 3.4.

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Hyperbolicity of Eq. (39) has additional meaning. As the previous paragraph alluded to, the points where $F_\theta = 0$ are the critical points of the canonical projection map $\pi : T^*\mathbf{T}^2 \to \mathbf{T}^2$ restricted to a common level $F^{-1}(r) \cap H^{-1}\left(\frac{1}{2}\right)$. Such tori necessarily bound a solid torus in $H^{-1}\left(\frac{1}{2}\right)$ and are not minimizing. Based on Fomenko's conjecture, it is expected that these solid tori must be quite rigid in a well-defined sense: in homology, they should generate at most two transverse subgroups of $H_1(T^*\mathbf{T}^2)$.

There is an alternative approach to Fomenko's conjecture that is based on topological entropy. In a series of papers based on Glasmachers dissertation results, Glasmachers and Knieper study Riemannian Hamiltonians on $T^*\mathbf{T}^2$ with zero topological entropy [19, 20]. They prove the closure of one of the above-mentioned solid tori is a union of one or two closed, minimizing geodesic orbits and their stable and unstable manifolds ([20], Theorem 3.7c).[3] The picture that emerges from their work is that there is a family of minimizing closed geodesics of the same homology class, and their stable and unstable manifolds, which bound a family of invariant solid tori. Bialy [5] describes the boundary of this set as a separatrix chain. The projection of the separatrix chain covers $\mathbf{T}^2$. A neighbourhood of the separatrix chain in the complement is fibred by invariant Lagrangian tori that are graphs, i. e. that are a union of minimizing orbits. The multipliers δ_i, or rather the angles θ_i mentioned above, define sections of the unit cotangent bundle trapped within a separatrix chain.

Let us reformulate this as:

Problem 3.5. *Prove the vanishing of the topological entropy of the geodesic flow of a Riemannian Hamiltonian on $T^*\mathbf{T}^2$ that is completely integrable with a polynomial-in-momenta first integral F.*

In various special cases, such as when F is real-analytic or Morse-Bott, it is known that the topological entropy vanishes [21].

Finally, since topological entropy is an important invariant in the study of these systems, let us state a number of problems that are directly relevant to the preceding discussion. If one assumes Fomenko's conjecture is true and that the Liouville family of Riemannian Hamiltonians equals the set of completely integrable Riemannian Hamiltonians on $\mathbf{T}^2$, then it should be true that

Problem 3.6. *The topological entropy of a non-Liouville Riemannian Hamiltonian on $T^*\mathbf{T}^2$ is positive.*

Glasmachers and Knieper [20, 19] have studied the structure of geodesic flows with zero topological entropy on $T^*\mathbf{T}^2$. The picture that emerges is the phase portrait looks remarkably like that of an integrable system. It seems likely that their results admit a strengthening: in particular, they are unable to determine the number of primitive homology classes represented by non-minimizing geodesics (for Liouville metrics, this is at most 4).

On the other hand, it is known, from results of Contreras, Contreras and Paternain and Knieper and Weiss that an open and dense set of Riemannian Hamiltonians have positive

[3] Although the minimizing orbits have stable and unstable manifolds, it is not suggested that they are hyperbolic.

topological entropy [22–24]. In the case of this particular problem, the natural point of departure is to look at Riemannian Hamiltonians that are close to Liouville, i.e. where the conformal factor in (35) is of the form

$$\Lambda_\epsilon = \Lambda_0 + \epsilon\Lambda_1 + O(\epsilon^2) \tag{40}$$

where Λ_0 is Liouville-and has no $\mathbf{T}^1$ symmetry–and Λ_ϵ is not Liouville for all $\epsilon \neq 0$. Based on the study in [25, 26] of the phase portrait of such systems, it should be possible to prove that the perturbed flow develops transverse homoclinic points.

3.3. The 2-sphere

The unit two-dimensional sphere $\mathbf{S}^2 \subset \mathbf{R}^3$ admits a completely integrable geodesic flow. Indeed, the geodesic flow of an ellipsoid is also completely integrable with the second integral of motion that is, in general, a quadratic form in the momenta.

The fundamental problem is to describe the moduli space of completely integrable Hamiltonians on $T^*\mathbf{S}^2$. The sub-problem of describing the integrable Riemannian (resp. natural or mechanical) Hamiltonians H has received wide-spread attention. When H is Riemannian, the most common approach is to assume the second integral F is polynomial-in-momenta, and without loss of generality, homogeneous. If the degree of F is fixed, then the problem of determining $H\&F$ is reducible to a non-linear PDE in the coefficients of F. When the degree is 1, the first integral F is a momentum map of a $\mathbf{T}^1$ isometry group (see below). When the degree is 2, then the Hamiltonian is *Liouville*, a classical result due to Darboux *c.f.* [27]. In degree 3, there is the well-known case due to Goryachev-Chaplygin, and more recent cases due to Selivanova, Dullin and Matveev and Dullin, Matveev and Topalov and Valent [28–33]. In degree 4, Selivanova and Hadeler & Selivanova have produced a family of examples using the results of Kolokol'tsov [34, 27]. Beyond degree 4, Kiyohara has provided a construction of a smooth Riemannian metric H with an independent first integral F of degree k for any $k \geq 1$. In this construction, the metric H depends on a functional modulus, and so for each k, the set is infinite dimensional [35].

3.4. Super-integrable systems with a linear-in-momenta first integral

Let us review the work of Matveev and Shevchishin in more detail [36]. These authors impose an additional formal constraint that the metric possess one first integral that is linear-in-momenta. In conformal coordinates (x, y) where $H = \frac{1}{2}c(x)(p_x^2 + p_y^2)$, the existence of a cubic integral is reduced to a second-order ODE involving c.

From a geometric perspective, it is more natural to introduce coordinates adapted to the isometry group. That is, the existence of a linear-in-momenta first integral is equivalent to the existence of an isometry group containing $\mathbf{T}^1$. The action of $\mathbf{T}^1$ on $\mathbf{S}^1$ induces a cohomogeneity-1 structure. The fixed set of the $\mathbf{T}^1$ action is a set of points $\{p_-, p_+\}$ which are equidistant along any minimal geodesic; and the principal $\mathbf{T}^1$-orbits are orthogonal to these geodesics. If $\gamma : [-T, T] \to \mathbf{S}^2$ is a minimal geodesic such that $\gamma(\pm T) = p_\pm$, then we can let (r, θ) be 'polar'

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coordinates adapted to this structure. The Hamiltonian H and polynomial-in-momenta integral F can be written in the adapted coordinates as

$$H = \frac{1}{2}\left(p_r^2 + s(r)p_\theta^2\right), F = e^{iv\theta} \times \sum_{j=0} a_j(r) p_r^j p_\theta^{N-j}, \tag{41}$$

where $v \in \mathbf{Z}$, $3 \leq N$ is a positive integer and the coefficients a_j are to be determined. The equation $\{H, F\} \equiv 0$ is equivalent to a differential system that couples the coefficients $a_0, \ldots, a_N$, s and an anti-derivative S of vs:

$$dS = vs\, \mathrm{d}r, \tag{42a}$$

$$\mathrm{d}a_j = \frac{1}{2}(N + 2 - j)a_{j-2}\, \mathrm{d}s - a_{j-1}\, \mathrm{d}S; (j = 0, \ldots, N), \tag{42b}$$

$$\mathrm{d}s = 2va_N/a_{N-1}\, \mathrm{d}r \tag{42c}$$

where $a_{-2} = a_{-1} = 0$. It is clear that the general solution of (42b), without the compatibility condition (42c), is obtained via repeated quadratures of products of s and S. The compatibility condition distinguishes those solutions which may arise from (41). The behaviour of s at $r = \pm T$ ultimately determines whether the solution obtained arises from a $\mathbf{T}^1$-invariant Riemannian Hamiltonian H and an independent first integral F on $T^*\mathrm{S}^2$.

In case $N = 3$, the differential system reduces to a third-order nonlinear ODE similar to that studied by Chazy, in his generalization of the Painlêvé classification ([37], Eq. (6)). Based on the work of Matveev and Shevchishin [36], we know the solutions to this equation are real-analytic and define a parameterized family of super-integrable Riemannian metrics with cubic-in-momenta first integral. The latter authors do not solve the ODE explicitly.

Problem 3.7. *Solve the $N = 3$ case of the differential system (42).*

It appears to the author that this differential system may be soluble via hypergeometric functions. A successful resolution to the $N = 3$ case will naturally lead to the higher degree cases, which appear to be somewhat more involved.

Problem 3.8. *Solve the higher degree cases of the differential system (42).*

3.5. Super-integrable systems with a higher degree first integral

The author believes that the differential system 42 provides the key to understanding the subspace of super-integrable Riemannian Hamiltonians which admit a cohomogeneity-1 structure. Super-integrability alone does not imply the existence of such a cohomogeneity-1 structure. Without this additional hypothesis, there is very little known. Indeed, the extremely valuable construction of Kiyohara is the only construction that provides a smooth Riemannian Hamiltonian with a polynomial-in-momenta first integral of degree $N > 3$–super-integrable or not [35, 38].

Let us explain Kiyohara's construction in some detail. Let H_0 be the Riemannian Hamiltonian of the standard unit sphere in $\mathbf{R}^3$. Let F_0, F_1 be linear-in-momenta first integrals of H_0 that are

linearly independent and let $l \geq k \geq 1$ be integers such that $N = k + l \geq 3$. Define a polynomial-in-momenta first integral $G_0 = F_0^k F_1^l$. For almost all $q \in \mathbf{S}^2$, the functions G_0, $H_0 | T_q^* \mathbf{S}^2$ are dependent along two distinct lines through 0; this defines a pair of mutually transverse line bundles $L_\pm^*$ over $\mathbf{S}^2 \backslash \{p_0^\pm, p_1^\pm\}$. The excluded, singular set consists of the anti-podal points $p_j^\pm$ where F_j vanishes identically on the fibre (equivalently, the corresponding Killing field vanishes). This pair of line bundles provides a branched double covering

$$\Phi : \mathbf{T}^2 = \mathbf{R}^2 / 2\pi\mathbf{Z}^2 \to \mathbf{S}^2 \tag{43}$$

with simple branch points at $\{p_0^\pm, p_1^\pm\} = \Phi(\pi\mathbf{Z}^2)$. The line bundles $L_\pm^*$ pullback to the line bundles R $\mathrm{d}x_j$ on $\mathbf{T}^2 = \{(x_1, x_2) \bmod 2\pi\mathbf{Z}\}$. Kiyohara shows that in these coordinates, the pull-back of the function r which measures the time along the unique geodesic γ through $\{p_0^\pm, p_1^\pm\}$ (see **Figure 4**) satisfies the second-order PDE

$$\frac{\partial^2 r}{\partial x_1 \partial x_2} + \frac{1}{B_1 + B_2} \frac{\partial B_1}{\partial x_2} \frac{\partial r}{\partial x_1} + \frac{1}{B_1 + B_2} \frac{\partial B_2}{\partial x_1} \frac{\partial r}{\partial x_2} = 0 \tag{44}$$

where B_1 and B_2 are functions that describe the line bundles $L_\pm$ in terms of the basis $\{\mathrm{d}r, \sin(r)\, \mathrm{d}\theta\}$.

Kiyohara writes a function $R = r_0 + r$ where r_0 is the solution to (44) given by $\Phi^* r$ and r is a solution of (44) with C^2 small boundary conditions satisfying

$$r(s, 0) = u_1(s), r(0, s) = u_2(s), \tag{45a}$$

$$\text{where} \quad u_i(s) = u_i(-s) = u_i(\pi - s), \quad \text{forall } i, s, \text{ and} \tag{45b}$$

$$u_i([-\epsilon, \epsilon]) = 0. \tag{45c}$$

Then, by means of this perturbed function R, Kiyohara writes down an explicit formula for the perturbed Riemannian Hamiltonian H and polynomial-in-momenta first integral F. The condition for the Poisson bracket $\{H, F\}$ to vanish is shown to reduce to the satisfaction of Eq. (44) by R for the *given* values of B_1 and B_2 (this legerdemain is the real trick that makes the construction work).

Condition (45b) ensures that R factors through Φ to a function on $\mathbf{S}^2$, while the condition (45c) ensures that R is C^∞ on $\mathbf{S}^2$ and coincides with r on a neighbourhood of the branch set $\{p_0^\pm, p_1^\pm\}$ (hence that H and F coincide with H_0 and F_0, respectively, on a neighbourhood of the cotangent fibres of the branch set).

Let us now state several problems related to Kiyohara's construction. First, Kiyohara's vanishing condition on the boundary values (45c) is used to deduce the Riemannian Hamiltonians are not real-analytic. Since all the remaining constructions involve real-analytic data, this serves to show his examples are genuinely different.

Problem 3.9. *Does Kiyohara's construction extend to real-analytic boundary conditions u_1, u_2 that satisfy (45b) ? Do these real-analytic metrics include other known cases?*

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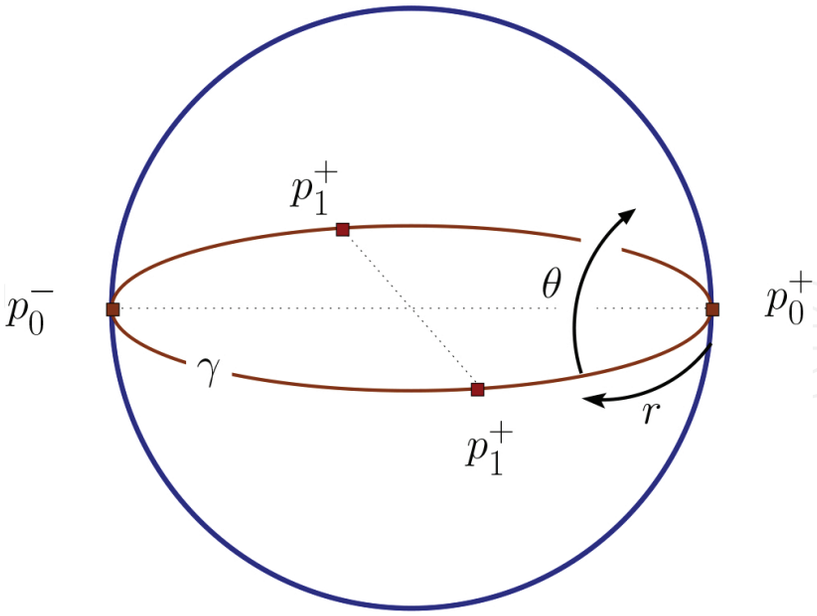


Figure 4. Kiyohara's construction. The zero set of the pair of Killing fields determines the equatorial geodesic γ. A choice of zeros $\{p_0^+, p_1^+\}$ determines the polar coordinate system (r, θ).

In particular, the obtained metrics are unlikely to have a $\mathbf{T}^1$ isometry group, so the question is really whether the known examples in degree 3 and 4 are obtainable via this construction [12–14, 28–30, 34, 39, 40].

Second, Kiyohara's construction produces a polynomial-in-momenta first integral F factors as $A_0^l A_1^m$ where A_i are linear-in-momenta functions. It is clear that the reducibility of the first integral F is forced by the desire to use a very simple branched covering.

Problem 3.10. *Is reducibility of the first integral F necessary?*

It ought to be fruitful to ask three related questions. The reducibility of F is very special, with just two distinct factors.

Problem 3.11. *Is it possible to extend Kiyohara's construction so that the polynomial-in-momenta first integral* F *has more than 2 distinct linear factors?*

It would be natural to try to extend the construction to the case where the zeros all lie on the same geodesic γ. More generally, one might attempt to mirror Kiyohara's construction but in a more abstract way: start with a simple ramified covering $\Phi : \sum \to \mathbf{S}^2$ with a branch set $Y \subset \mathbf{S}^2$. Let F_0 be a product of linear first integrals of H_0 that vanishes identically on $T^*_Y\mathbf{S}^2$ and not elsewhere. The stumbling block is that we need to clarify the intrinsic geometric meaning of the PDE that governs the perturbed systems (44).

Problem 3.12. *Describe in explicit terms the third, independent first integral of H that is of least degree.*

Kiyohara proves in his paper that H is super-integrable (he proves the geodesic flow is 2π-periodic, in fact), but that proof does not proceed by finding this third first integral.

3.6. Three-dimensional configuration spaces

In comparison to the wealth of results and examples for surfaces that were surveyed above, comparatively little is known about the three-dimensional analogues. Tăĭmanov tells us that if the Tonelli Hamiltonian is completely integrable with real-analytic first integrals, then the three-dimensional configuration space $\sum$ has a finite covering $\hat{p} : \hat{\sum} \to \sum$ such that the fundamental group $\pi_1(\hat{\sum})$ is abelian and of rank at most 3 [41–43]. Based on the resolution of the Poincaré conjecture, this result implies that, up to finite covering the only such configuration spaces are

$$\mathbf{S}^3, \qquad \mathbf{S}^2 \times \mathbf{T}^1 \qquad \text{or} \qquad \mathbf{T}^3. \tag{46}$$

The author generalized Kozlov's result on surfaces to three-manifolds. In this result, if the Tonelli Hamiltonian is completely integrable and the singular set is topologically tame, then Tăĭamanov's list extends to include those three-manifolds $\sum$ such that $\pi_1(\sum)$ is almost solvable (equivalently, due to the resolution of the geometrization conjecture, $\sum$ admits either a *Nil* or *Sol* geometry) [44]. Both results are sharp, like Kozlov's, in the sense that all such admissible configuration spaces admit a geometric structure and the Riemannian Hamiltonian of such a structure is completely integrable with first integrals of the requisite type [45, 46].

There are a large number of questions that this strand of research has opened. Let us sketch a few.

3.7. The 3-sphere

The case of $\mathbf{S}^3$ is perhaps best understood. It has been known since Jacobi proved the complete integrability of the geodesic flow of an ellipsoid via separation of variables, that the Liouville family of metrics on $\mathbf{S}^3$ is completely integrable. These systems possess three independent quadratic-in-momenta first integrals.

Based on the analogous problem for the two-sphere,

Problem 3.13. *Describe the structure of the super-integrable Riemannian Hamiltonians on* $\mathbf{S}^3$.

Researchers who specialize in super-integrable classical and quantum systems have developed tools for constructing and classifying super-integrable systems *c.f.* [47–49]. Unfortunately, some key ingredients in these constructions lead to systems with singularities.

The first method is based on the cohomogeneity-1 structure of $\mathbf{S}^3$ with the group $G = \mathrm{SO}(3)$ acting as the linear isometry group of $\mathbf{R}^3 \subset \mathbf{R}^4$. If one represents

$$\mathbf{S}^3 = \{(x, r) | x \in \mathbf{R}^3, r \in \mathbf{R}, |x|^2 + |r|^2 = 1\}, \tag{47}$$

then we see that G acts freely on $T^*\mathbf{S}^3 \backslash T^*_F\mathbf{S}^3$ where $F = \{(0, \pm 1)\}$ is the fixed-point set of the G-action on $\mathbf{S}^3$. This is enough to see that any G-invariant Hamiltonian on $T^*\mathbf{S}^3$ is non-commutatively integrable (analogous to the same fact for $\mathbf{S}^2$). If $K : \mathrm{so}(3)^* \to \mathrm{R}$ is a positive-

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definite quadratic form, and $\Psi : T^*\mathbf{S}^3 \to \mathrm{so}(3)^*$ is the momentum map of the SO(3)-action, then an invariant Riemannian Hamiltonian can be written as

$$H = \frac{1}{2}p_r^2 + \frac{1}{2}s(r)\Psi^* K, \tag{48}$$

for some function $s > 0$ such that $s \times (1 \pm r)^2 \to const.\neq 0$ as $r \to \mp 1$.

If one employs the *ansatz* of Matveev & Shevchishin (*c.f.* Section 3.3), one would like to find first integrals that are polynomial-in-momenta of the form

$$F = \sum_{j=0}^{N} b_j(x,r)\, p_r^j\, \Psi^* \eta_{N-j} \tag{49}$$

where $\eta_{N-j} : \mathrm{so}(3)^* \to \mathbf{R}$ is a homogeneous polynomial of degree $N-j$. In (41), the pre-factor $\exp(iv\theta)$ appears to ensure that the coefficients of the first integral F are common eigenfunctions of the Casimir $\Delta_{\mathbf{S}^1} = \frac{\partial^2}{\partial\theta^2}$ parameterized by r. In the current case, the *ansatz* suggests that the coefficients b_j should factor as $\phi_\lambda(\theta)a_j(r)$ where ϕ_λ is an eigenfunction of the Casimir $\Delta_{\mathbf{S}^2}$ with eigenvalue λ and $\theta = x/|x|$.

Problem 3.14. *Extend the construction sketched, above to higher dimensional spheres.*

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