Chapter 5

Symplectic Manifolds: Gromov-Witten Invariants on Symplectic and Almost Contact Metric Manifolds

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Additional information is available at the end of the chapter

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Abstract

In this chapter, we introduce Gromov-Witten invariant, quantum cohomology, Gromov-Witten potential, and Floer cohomology on symplectic manifolds, and in connection with these, we describe Gromov-Witten type invariant, quantum type cohomology, Gromov-Witten type potential and Floer type cohomology on almost contact metric manifolds. On the product of a symplectic manifold and an almost contact metric manifold, we induce some relations between Gromov-Witten type invariant and quantum cohomology and quantum type invariant. We show that the quantum type cohomology is isomorphic to the Floer type cohomology.

Keywords: symplectic manifold, Gromov-Witten invariant, quantum cohomology, Gromov-Witten potential, Floer cohomology, almost contact metric manifold, Gromov-Witten type invariant, quantum type cohomology, Gromov-Witten type potential, Floer type cohomology

1. Introduction

The symplectic structures of symplectic manifolds (M, ω, J) are, by Darboux's theorem 2.1, locally equivalent to the standard symplectic structure on Euclidean space.

In Section 2, we introduce basic definitions on symplectic manifolds [1–5, 10–13] and flux homomorphism. In Section 2.1, we recall J-holomorphic curve, moduli space of J-holomorphic curves, Gromov-Witten invariant and Gromov-Witten potential, quantum product and quantum cohomology, and in Section 2.2, symplectic action functional and its gradient flow line, Maslov type index of critical loop, Floer cochain complex and Floer cohomology, and theorem of Arnold conjecture.

In Section 3, we introduce almost contact metric manifolds $(M, g, \varphi, \eta, \xi, \phi)$ with a closed fundamental 2-form ϕ and their product [4, 7, 8]. In Section 3.1, we study φ-coholomorphic

map, moduli space of φ-coholomorphic maps which represent a homology class of dimension two, Gromov-Witten type cohomology, quantum type product and quantum type cohomology, Gromov-Witten type potentials on the product of a symplectic manifold, and an almost contact metric manifold [5, 6, 13]. In Section 3.2, we investigate the symplectic type action functional on the universal covering space of the contractible loops, its gradient flow line, the moduli space of the connecting flow orbits between critical loops, Floer type cochain complex, and Floer type cohomology with coefficients in a Novikov ring [7, 9, 13].

In Section 4, as conclusions we show that the Floer type cohomology and the quantum type cohomology of an almost contact metric manifold with a closed fundamental 2-form are isomorphic [7, 13], and present some examples of almost contact metric manifolds with a closed fundamental 2-form.

2. Symplectic manifolds

By a symplectic manifold, we mean an even dimensional smooth manifold M^{2n} together with a global 2-form ω which is closed and nondegenerate, that is, the exterior derivative $d\omega = 0$ and the n-fold wedge product ω^n never vanishes.

Examples: (1) The $2n$-dimensional Euclidean space $\mathbb{R}^{2n}$ with coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ admits symplectic form $\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$.

(2) Let M be a smooth manifold. Then its cotangent bundle T^*M has a natural symplectic form as follows. Let $\pi : T^*M \to M$ be the projection map and $x_1, \ldots, x_n$ are local coordinates of M. Then $q_i = x_i \circ \pi, i = 1, 2, \ldots, n$ together with fiber coordinates $p_1, \ldots, p_n$ give local coordinates of T^*M. The natural symplectic form on T^*M is given by

$$\omega = \sum_{i=1}^{n} dq_i \wedge dq_j. \tag{1}$$

(3) Every Kähler manifold is symplectic.

Darboux's Theorem 2.1 ([6]). *Every symplectic form ω on M is locally diffeomorphic to the standard form ω_0 on $\mathbb{R}^{2n}$.*

A symplectomorphism of (M, ω) is a diffeomorphism $\phi \in Diff(M)$ which preserves the symplectic form $\phi^*\omega = \omega$. Denote by $\mathrm{Sym}(M)$ the group of symplectomorphims of M. Since ω is nondegenerate, there is a bijection between the vector fields $X \in \Gamma(TM)$ and 1-forms $\omega(X, \cdot) \in \Omega^1(M)$. A vector field $X \in \Gamma(TM)$ is called symplectic if $\omega(X, \cdot)$ is closed.

Let M be closed, i.e., compact and without boundary. Let $\phi : \mathbb{R} \to \mathrm{Diff}(M)$, $t \mapsto \phi_t$ be a smooth family of diffeomorphisms generated by a family of vector fields $X_t \in \Gamma(TM)$ via,

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$$\frac{d}{dt}\phi_t = X_t \circ \phi_t, \phi_0 = \mathrm{id}. \tag{2}$$

Then $\phi_t \in \mathrm{Symp}(M)$ if and only if $X_t \in \Gamma(TM, \omega)$ the space of symplectic vector fields on M. Moreover, if $X, Y \in \Gamma(TM, \omega)$, then $[X, Y] \in \Gamma(TM, \omega)$ and $\omega([X, Y], \cdot) = dH$, where $H = \omega(X, Y) : M \to R$. Let $H : M \to R$ be a smooth function. Then the vector field X_H on M determined by $\omega(X_H, \cdot) = dH$ is called the Hamiltonian vector field associated with H. If M is closed, then X_H generates a smooth 1-parameter group of diffeomorphisms $\phi_H^t \in \mathrm{Diff}(M)$ such that

$$\frac{d}{dt}\phi_H^t = X_H \circ \phi_H^t, \phi_H^0 = \mathrm{id}. \tag{3}$$

This $\{\phi_H^t\}$ is called the Hamiltonian flow associated with H. The flux homomorphism Flux is defined by

$$\mathrm{Flux}\{\phi_H^t\} = \int_0^1 \omega(X_t, \cdot)dt. \tag{4}$$

Theorem 2.2 ([6]). *$\phi \in Sym(M)$ is a Hamiltonian symplectomorphism if and only if there is a homotopy $[0, 1] \to \mathrm{Sym}(M)$, $t \mapsto \phi_t$ such that $\phi_0 = \mathrm{id}$, $\phi_1 = \phi$, and $\mathrm{Flux}(\{\phi_t\}) = 0$.*

2.1. Quantum cohomology

Let (M, ω) be a compact symplectic manifold. An almost complex structure is an automorphism of TM such that $J^2 = -I$. The form ω is said to tame J if $\omega(v, Jv) > 0$ for every $v \neq 0$. The set $\mathfrak{I}_\tau(M, \omega)$ of almost complex structures tamed by ω is nonempty and contractible. Thus the Chern classes of TM are independent of the choice $J \in \mathfrak{I}_\tau(M, \omega)$. A smooth map $\phi : (M_1, J_1) \to (M_2, J_2)$ from M_1 to M_2 is (J_1, J_2)-holomorphic if and only if

$$d\phi_x \circ J_1 = J_2 \circ d\phi_x \tag{5}$$

Hereafter, we denote by $H_2(M)$ the image of Hurewicz homomorphism $\pi_2 M \to H_2(M, \mathbb{Z})$. A (i, J)-holomorphic map $u : (\Sigma, z_1, \ldots, z_k) \to M$ from a reduced Riemann surface (Σ, j) of genus g with k marked points to (M, J) is said to be stable if every component of Σ of genus 0 (resp. 1), which is contracted by u, has at least 3 (resp. 1) marked or singular points on its component, and the k marked points are distinct and nonsingular on Σ. For a two-dimensional homology class $A \in H_2(M)$ let $\mathcal{M}_{g,k}(M, A; J)$ be the moduli space of (j, J)-holomorphic stable maps which represent A.

Let $B := C^\infty(\Sigma, M; A)$ be the space of smooth maps

$$u : \Sigma \to M \tag{6}$$

which represent $A \in H_2(M)$.

Let us consider infinite dimensional vector bundle $E \to B$ whose fiber at u is the space $E_u = \Omega^{0,1}(\Sigma, u^*TM)$ of smooth J-antilinear 1-forms on Σ with values in u^*TM. The map $\overline{\partial}_J : B \to E$ given by

$$\overline{\partial}_J(u) = \frac{1}{2}(du + J \circ du \circ j) \tag{7}$$

is a section of the bundle. The zero set of the section $\overline{\partial}_I$ is the moduli space $\mathcal{M}_{g,k}(M, A; J)$.

For an element $u \in \mathcal{M}_{g,k}(M, A; J)$ we denote by

$$D_u : \Omega^0(\Sigma, u^*TM) = T_uB \to \Omega^{0,1}(\Sigma, u^*TM) \tag{8}$$

the composition of the derivative

$$d(\overline{\partial}_J)_u : T_uB \to T_{(u,0)}E \tag{9}$$

with the projection to fiber $T_{(u,0)}E \to \Omega^{0,1}(\Sigma, u^*TM)$. Then the virtual dimension of $\mathcal{M}_{g,k}(M, A; J)$ is

$$\begin{aligned} \dim \mathcal{M}_{g,k}(M, A; J) &= \operatorname{index} D_u : \Omega^0(\Sigma, u^*TM) \to \Omega^{0,1}(\Sigma, u^*TM) \\ &= 2c_1(TM)A + n(2-2g) + (6g-6) + 2k. \end{aligned} \tag{10}$$

Theorem 2.1.1. *For a generic almost complex structure $J \in \mathfrak{I}_\tau(M, \omega)$ the moduli space $\mathcal{M}_{g,k}(M, A; J)$ is a compact stratified manifold of virtual dimension,*

$$\dim \mathcal{M}_{g,k}(M, A; J) = 2c_1(TM)A + n(2-2g) + (6g-g) + 2k. \tag{11}$$

For some technical reasons, we assume that $c_1(A) \geq 0$ if $\omega(A) > 0$ and A is represented by some J-holomorphic curves. In this case, we call the symplectic manifold M semipositive. We define the evaluation map by

$$\mathrm{ev} : \mathcal{M}_{g,k}(M, A; J) \to M^k,\ \mathrm{ev}([u; z_1, \ldots, z_k]) = (u(z_1), \ldots, u(z_k)). \tag{12}$$

Then the image Im(ev) is well defined, up to cobordism on J, as a $\dim \mathcal{M}_{g,k}(M, A; J) : \equiv m$-dimensional homology class in M^k.

Definition. The Gromov-Witten invariant $\Phi_{g,k}^{M,A}$ is defined by

$$\Phi_{g,k}^{M,A} : H^m(M^k) \to Q, \Phi_{g,k}^{M,A}(a) = \int_{\mathcal{M}_{g,k}(M, A; J)} \mathrm{ev}^* \bullet \alpha \tag{13}$$

where $\alpha = PD(a) \in H_{2nk-m}(M^k)$ and $\bullet$ is the intersection number of ev and α in M^k.

The minimal Chern number N of (M, ω) is the integer $N := \min\ \}7\mathrm{B}\lambda | c_1(A) = \lambda \geq 0, A \in H_2(M)\}$. We define the quantum product $a * b$ of $a \in H^k(M)$ and $b \in H\ \ (M)$ as the formal sum

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$$a * b = \sum_{A \in H_2(M)} (a * b)_A q^{c_1(A)/N} \tag{14}$$

where q is an auxiliary variable of degree $2N$ and $(a * b)_A \in H^{k+l-2c_1(A)}(M)$ is defined by

$$\int_C (a * b)_A = \Phi_{0,3}^{M,A}(a \otimes b \otimes r) \tag{15}$$

for $C \in H_{k+l-2c_1(A)}(M)$, $r = PD(C)$. Hereafter, we use the Gromov-Witten invariants of $g = 0$ and $k = 3$. Then the quantum product $a * b$ is an element of

$$QH^* := H^*(M) \otimes \mathbb{Q}[q] \tag{16}$$

where $\mathbb{Q}[q]$ is the ring of Laureut polynomials of the auxiliary variable q.

Extending $*$ by linearity, we get a product called quantum product

$$* : QH^*(M) \otimes QH^*(M) \to QH^*(M). \tag{17}$$

It turns out that $*$ is distributive over addition, skew-commutative, and associative.

Theorem 2.1.2. *Let (M, ω) be a compact semipositive symplectic manifold. Then the quantum cohomology $(QH^*(M), +, *)$ is a ring.*

Remark. For $A = 0 \in H_2(M)$, the all J-holomorphic maps in the class A are constant. Thus $(a * b)_0 = a \cup b$. The constant term of $a * b$ is the usual cup product $a \cup b$.

We defined the Novikov ring Λ_ω by the set of functions $\lambda : H_2(M) \to Q$ that satisfy the finiteness condition

$$\#\{A \in H_2(M) | \lambda(A) \neq 0, \omega(A) < c\} < \infty \tag{18}$$

for every $c \in R$. The grading is given by $\deg(A) = 2c_1(A)$.

Examples ([5]). (1) Let $p \in H^2(\mathbb{C}P^n)$ and $A \in H_2(\mathbb{C}P^n)$ be the standard generators. There is a unique complex line through two distinct points in $\mathbb{C}P^n$ and so $p * p^n = q$. The quantum cohomology of $\mathbb{C}P^n$ is

$$QH^*(\mathbb{C}P^n; \mathbb{Q}[q]) = \frac{\mathbb{Q}[p, q]}{< p^{n+1} = q >}. \tag{19}$$

(2) Let $G(k, n)$ be the Grassmannian of complex k-planes in $\mathbb{C}^n$. There are two natural complex vector bundles $\mathbb{C}^k \to E \to G(k, n)$ and $\mathbb{C}^{n-k} \to F \to G(k, n)$. Let $x_i = c_i(E^*)$ and $y_i = c_i(F^*)$ be Chern classes of the dual bundles E^* and F^*, respectively. Since $E \oplus F$ is trivial, $\sum_{i=0}^{j} x_i y_{j-i} = 0$, $j = 1, \ldots, n$. By computation $x_k * y_{n-k} = (-1)^{n-k} q$. The quantum cohomology of $G(k, n)$ is

$$QH^*\Big(G(k,n);\mathbb{Q}[q]\Big)=\frac{\mathbb{Q}[x_1,\ldots,x_k,q]}{<y_{n-k+1},\ldots,y_{n-1},y_n+(-1)^{n-k}q>}. \tag{20}$$

Let $\{e_0,\ldots,e_n\}$ be an integral basis of $H^*(M)$ such that $e_0=1\in H^0(M)$ and each e_i has pure degree. We introduce $n+1$ formal variables $t_0,\ldots,t_n$ and the linear polynomial a_t in $t_0,\ldots,t_n$ with coefficients in $H^*(M)$ by $a_t=t_0e_0+\cdots+t_ne_n$. The Gromov-Witten potential of (M,ω) is a formal power series in variables $t_0,\ldots,t_n$ with coefficients in the Novikov ring Λ_ω

$$\begin{aligned}\Phi^M(t)&=\sum_{k\geq 3}\sum_A\frac{1}{k!}\Phi_{0,k}^{M,A}(a_t,\ldots,a_t)q^{\frac{c_1(A)}{N}}\\&=\sum_{k_0+\cdots+k_n\geq 3}\sum_A\frac{\varepsilon(k_0,\ldots,k_n)}{k_0!\ldots k_n!}\Phi_{0,k}^{M,A}(e_0^{k_0}\otimes\cdots\otimes e_n^{k_n})\cdot(t^0)k_0\ldots(t^n)^{k_n}q^{c_1(A)/N}.\end{aligned} \tag{21}$$

Examples ([4]). (1) $\Phi^{\mathbb{C}P^1}(t)=\frac{1}{2}t_0{}^2t_1+\left(e^{t_1}-1-t_1-\frac{t_1{}^2}{2}\right)$.

(2) $\Phi^{\mathbb{C}P^n}(t)=\frac{1}{6}\sum_{i+j+k=n}t_it_jt_k+\sum_{d>0}\sum_{k_2\ldots k_n}N_d(k_2\ldots k_n)\cdot\frac{t_2^{k_2}\ldots t_n^{k_n}}{k_2!\ldots k_n!}e^{dt_1}q^d$,

where $N_d(k_2\ldots k_n)=\Phi_{0,k}^{\mathbb{C}P^n,dA}(p^2\ldots p^2,\ldots,p^n\ldots p^n)$.

We define a nonsingular matrix (g_{ij}) by $g_{ij}=\int_M e_i\cup e_j$ and denote by (g^{ij}) its inverse matrix.

Theorem 2.1.3 ([4, 5]). *The Gromov-Witten potential Φ^M of (M,ω) satisfies the WDVV-equations:*

$$\sum_{\upsilon,\mu}\partial_{t_i}\partial_{t_j}\partial_{t_\upsilon}\Phi^M(t)g^{\upsilon\mu}\partial_{t_\mu}\partial_{t_k}\partial_t\ \ \Phi^M(t)=\varepsilon_{ijk}\cdot\sum_{\upsilon,\mu}\partial_{t_j}\partial_{t_k}\partial_{t_\upsilon}\Phi^M(t)g^{\upsilon\mu}\partial_{t_\mu}\partial_{t_i}\partial_t\ \ \Phi^M(t), \tag{22}$$

where $\varepsilon_{ijk}=(-1)^{\deg(e_1)(\deg(e_j)+\deg(e_k))}$.

2.2. Floer cohomology

Let a compact symplectic manifold (M,ω) be semipositive. Let $H_{t+1}:M\to R$ be a smooth 1-periodic family of Hamiltonian functions. The Hamiltonian vector field X_t is defined by $\omega(X_t,\cdot)=dH_t$. The solutions of the Hamiltonian differential equation $\dot{x}(t)=X_t(x(t))$ generate a family of Hamiltonian symplectomorphisms $\phi_t:M\to M$ satisfying $\frac{d}{dt}\phi_t=X_t\circ\phi_t$ and $\phi_0=\mathrm{id}$. For every contractible loop $x:\mathbb{R}/\mathbb{Z}\to M$, there is a smooth map $u:D:=\ \}7Bz\in C||z|\leq 1\}\to M$ such that $u(e^{2\pi it})=x(t)$. Two such maps u_1 and u_2 are called equivalent if their boundary sum$(u_1)\#(-u_2)$ is homologus to zero in $H_2(M)$. Denote by $(x,[u_1])=(x,[u_2])$ for equivalent pairs, LM the space of contractible loops and $\tilde{LM}$ the space of equivalence classes. Then $\widetilde{LM}\to LM$ is a covering space whose covering transformation group is $H_2(M)$ via, $A(x,[u])=(x,[A\#u])$ for each $A\in H_2(M)$ and $(x,[u])\in\widetilde{LM}$.

Definition. The symplectic action functional a_H is defined by

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$$a_H : \widetilde{LM} \to R, a_H(x,[u]) = -\int_D u^*\omega - \int_0^1 H_t\Big(x(t)\Big)dt. \tag{23}$$

For each element $\tilde{x} := (x,[u]) \in \widetilde{LM}$ and $\xi \in T_{\tilde{x}}\widetilde{LM}$, we have

$$da_H(x,[u])(\xi) = \int_0^1 \omega\Big(\dot{x}(t) - X_t\Big(x(t)\Big), \xi\Big)dt. \tag{24}$$

Thus the critical points of a_H are one-to-one correspondence with the periodic solutions of $\dot{x}(t) - X_t\Big(x(t)\Big) = 0$. Denote by $\tilde{PH} \subset \tilde{LM}$ the critical points of a_H and by $PH \subset LM$ the set of periodic solutions.

The gradient flow lines of a_H are the solutions $u : \mathbb{R}^2 \to M$ of the partial differential equation

$$\partial_u + J(u)\Big(\partial_t u - X_t(u)\Big) = 0$$

with conditions $u(s,t+1) = u(s,t)$,

$$\lim_{s \to \pm\infty} u(s,t) = x^{\pm}(t) \tag{25}$$

for some $x^- \in PH$.

Let $\mathcal{M}(\tilde{x}^-, \tilde{x}^+)$ be the space of such solutions u with $\tilde{x}^+ = \tilde{x}^- \# u$. This space is invariant under the shift $u(s,t) \mapsto u(s+s_0,t)$ for each $s_0 \in R$. For a generic Hamiltonian function, the space $\mathcal{M}(\tilde{x}^-, \tilde{x}^+)$ is a manifold of dimension

$$\dim \mathcal{M}(\tilde{x}^-, \tilde{x}^+) = \mu(\tilde{x}^-) - \mu(\tilde{x}^+). \tag{26}$$

Here $\mu : \tilde{PH} \to Z$ is a version of Maslov index defined by the path of symplectic matrices generated by the linearized Hamiltonian flow along $x(t)$.

Let $\mu(\tilde{x}) - \mu(\tilde{y}) = 1$. Then $\mathcal{M}(\tilde{x}, \tilde{y})$ is a one-dimensional manifold and the quotient by shift $\mathcal{M}(\tilde{x}, \tilde{y})/\mathbb{R}$ is finite. In this case, we denote by $n(\tilde{x}, \tilde{y}) = \#\Big(\frac{\mathcal{M}(\tilde{x}, \tilde{y})}{\mathbb{R}}\Big)$ the number of connecting orbits from $\tilde{x}$ to $\tilde{y}$ counted with appropriate signs.

We define the Floer cochain group $FC^*(M,H)$ as the set of all functions $\xi : \tilde{PH} \to Q$ that satisfy the finiteness condition,

$$\#\{\tilde{x} \in \widetilde{PH} \,|\, \xi(\tilde{x}) \neq 0, a_H(\tilde{x}) \leq c\} < \infty \tag{27}$$

for every $c \in R$. The complex $FC^*(M,H)$ is a Λ_ω-module with action

$$(\lambda * \xi)(\tilde{x}) := \sum_A \lambda(A)\xi(A\#\tilde{x}). \tag{28}$$

The degree k part $FC^*(M,H)$ consists of all $\xi \in FC^*(M,H)$ that are nonzero only on elements $\tilde{x} \in \widetilde{PH}$ with $\mu(\tilde{x}) = k$. Thus the dimension of $FC^*(M,H)$ as a Λ_ω-module is the number $\#(PH)$.

We define a coboundary operator $\delta : FC^*(M,H) \to FC^{k+1}(M,H)$ by

$$\delta(\xi)(\tilde{x}) = \sum_{\mu(\tilde{y})=k} \boldsymbol{n}(\tilde{x},\tilde{y})\xi(\tilde{y}). \tag{29}$$

The coefficients of $\delta(\delta(\xi)(\tilde{x}))$ are given by counting the numbers of pairs of connecting orbits from $\tilde{x}$ to $\tilde{y}$ where $\mu(\tilde{x})-\mu(\tilde{y}) = 2 = \dim\mathcal{M}(\tilde{x},\tilde{y})$. The quotient $\mathcal{M}(\tilde{x},\tilde{y})/\mathbb{R}$ is a one-dimensional oriented manifold that consists of pairs counted by $\delta(\delta(\xi)(\tilde{x}))$. Thus the numbers cancel out in pairs, so that $\delta\Big(\delta(\xi)\Big) = 0$.

Definition. The cochain complex $(FC^*(M,H),\delta)$ induces its cohomology groups

$$FH^k(M,H) := \frac{\mathrm{Ker}\delta : FC^k(M,H) \to FC^{k+1}(M,H)}{\mathrm{Im}\delta : FC^{k-1}(M,H) \to FC^k(M,H)} \tag{30}$$

which are called the Floer cohomology groups of (M,ω,H,J).

Remark. By the usual cobordism argument, the Floer cohomology groups $FH^*(M,H)$ are independent to the generic choices of H and J. Let $f : M \to R$ be a Morse function such that the negative gradient flow of f with respect to the metric $g(\cdot,\cdot) = \omega(\cdot,J\cdot)$ is Morse-Smale. Let $H = -\varepsilon f : M \to R$ be the time-independent Hamiltonian. If ε is small, then the 1-periodic solutions of $\dot{x}(t)-X_H\Big(x(t)\Big) = 0$ are one-to-one correspondence with the critical points of f. Thus we have $PH = \mathrm{Crit}(f)$ and the Maslov type index can be normalized as

$$\mu(x,[u]) = \mathrm{ind}_f(x)-n \tag{31}$$

where $u : D \to M$ is the constant map $u(D) = x$.

We define a cochain complex $MC^*(M;\Lambda_\omega)$ as the graded Λ_ω-module of all functions

$$\xi : \mathrm{Crit}(f)H_2(M) \to Q \tag{32}$$

that satisfy the finiteness condition

$$\#\{(x,A)|\xi(x,A)\neq 0, \omega(A)\geq c\} < \infty \tag{33}$$

for every $c\in R$. The Λ_ω-module structure is given by $(\lambda * \xi)(x,A) = \sum \lambda(B)\xi(x,A+B)$ and the grading $\deg(x,A) = \mathrm{ind}_f(x)-2c_1(A)$. The gradient flow lines $u : \mathbb{R} \to M$ of f are the solutions of $\dot{u}(s) = -\nabla f(u(s))$. These solutions determine coboundary operator

$$\delta : MC^k(M;\Lambda_\omega) \to MC^{k+1}(M;\Lambda_\omega) \tag{34}$$

$$\delta(\xi)(x,A) = \sum_y n_f(x,y)\xi(y,A) \tag{35}$$

where $n_f(x,y)$ is the number of connecting orbits u from x to y satisfying $\lim_{s\to-\infty} u(s) = x$, $\lim_{s\to+\infty} u(s) = y$, counted with appropriate signs and $\mathrm{ind}_f(x)-\mathrm{ind}_f(y) = 1$.

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Definition–Theorem 2.2.1. (1) *The cochain complex* $(MC^*(M;\Lambda_\omega),\delta)$ *defines a cohomology group*

$$MH^*(M;\Lambda_\omega) := \frac{\mathrm{Ker}\delta : MC^*(M;\Lambda_\omega) \to MC^{*+1}(M;\Lambda_\omega)}{\mathrm{Im}\delta : MC^{*-1}(M;\Lambda_\omega) \to MC^*(M;\Lambda_\omega)} \tag{36}$$

which is called the Morse-Witten cohomology of M.

(2) $MH^*(M;\Lambda_\omega)$ *is naturally isomorphic to the quantum cohomology* $QH^*(M;\Lambda_\omega)$.

Theorem 2.2.2 ([5]). *Let a compact symplectic manifold* (M,ω) *be semipositive. There is an isomorphism*

$$\Phi : FH^*(M,H) \to QH^*(M;\Lambda_\omega) \tag{37}$$

which is linear over the Novikov ring Λ_ω.

Let $H : M \to R$ be a generic Hamiltonian function and $\phi : M \to M$ the Hamiltonian symplectomorphism of H. By Theorems 2.2.1 and 2.2.2

$$FH^*(M,H) \simeq QH^*(M;\Lambda_\omega) \simeq H^*(M) \otimes \Lambda_\omega \tag{38}$$

The rank of $FC^*(M,H)$ as a Λ_ω-module must be at least equal to the dimension of $H^*(M)$. The rank is the number $\#(PH)$ which is the number of the fixed points of ϕ.

Theorem 2.2.3 (Arnold conjecture). *Let a compact symplectic manifold* (M,ω) *be semipositive. If a Hamiltonian symplectomorphism* $\phi : M \to M$ *has only nondegenerate fixed points, then*

$$\#(\mathrm{Fix}(\phi)) \geq \sum_{j=0}^{2n} b_j(M) \tag{39}$$

where $b_j(M)$ *is the jth Betti number of M.*

3. Almost contact metric manifolds

Let be a real $(2n+1)$-dimensional smooth manifold. An almost cocomplex structure on M is defined by a smooth $(1,1)$ type tensor φ, a smooth vector field ξ, and a smooth 1-form η on M such that for each point $x \in M$,

$$\varphi_x^2 = -I + \eta_x \otimes \xi_x, \eta_x(\xi_x) = 1, \tag{40}$$

where $I : T_xM \to T_xM$ is the identity map of the tangent space T_xM.

A Riemannian manifold M with a metric tensor g and with an almost co-complex structure (φ,ξ,η) such that

$$g(X,Y) = g(\varphi X, \varphi Y) + \eta(X)\eta(Y), X, Y \in \Gamma(TM), \tag{41}$$

is called an almost contact metric manifold.

The fundamental 2-form ϕ of an almost contact metric manifold $(M, g, \varphi, \xi, \eta)$ is defined by

$$\phi(X, Y) = g(X, \varphi Y) \tag{42}$$

for all $X, Y \in \Gamma(TM)$. The $(2n + 1)$-form $\phi^n \wedge \eta$ does not vanish on M, and so M is orientable. The Nijehuis tensor [8, 11] of the (1,1) type tensor φ is the (1,2) type tensor field N_φ defined by

$$N_\varphi(X, Y) = [\varphi X, \varphi Y] - [X, Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] \tag{43}$$

for all $X, Y \in \Gamma(TM)$, where $[X, Y]$ is the Lie bracket of X and Y. An almost cocomplex structure (φ, ξ, η) on M is said to be integrable if the tensor field $N_\varphi = 0$, and is normal if $N_\varphi + 2d\eta \otimes \xi = 0$.

Definition. An almost contact metric manifold $(M, g, \varphi, \eta, \xi, \phi)$ is said to be

1. almost cosymplectic (or almost co-Kähler) if $d\phi = 0$ and $d\eta = 0$,
2. contact (or almost Sasakian) if $\phi = d\eta$,
3. an almost C-manifold if $d\phi = 0$, $d\eta \neq 0$, and $d\eta \neq \phi$,
4. cosymplectic (co-Kähler) if M is an integrable almost cosymplectic manifold,
5. Sasakian if M is a normal almost Sasakian manifold,
6. a C-manifold if M is a normal almost C-manifold.

An example of compact Sasakian manifolds is an odd-dimensional unit sphere S^{2n+1}, and the one of the co-Kähler (almost cosymplectic) manifolds is a product MS^1 where M is a compact Kähler (symplectic) manifold, respectively.

Let $(M_1^{2n_1+1}, g_1, \varphi_1, \eta_1, \xi_1)$ and $(M_2^{2n_2+1}, g_2, \varphi_2, \eta_2, \xi_2)$ be almost contact metric manifolds. For the product $M := M_1 M_2$, Riemannian metric on M is defined by

$$g\Big((X_1, Y_1), (X_2, Y_2)\Big) = g_1(X_1, X_2) + g_2(Y_1, Y_2). \tag{44}$$

An almost complex structure on M is defined by

$$J(X, Y) = \Big(\varphi_1(X) + \eta_2(Y)\xi_1, \varphi_2(Y) - \eta_1(X)\xi_2\Big). \tag{45}$$

Then $J^2 = -I$ and the fundamental 2-form ϕ on M is $\phi = \phi_1 + \phi_2 + \eta_1 \wedge \eta_2$. If ϕ_1, ϕ_2 and η_1 and η_2 are closed, then ϕ is closed. Thus we have

Theorem 3.1. *Let $(M_1^{2n_1+1}, g_1, \varphi_1, \eta_1, \xi_1)$ be almost contact metric manifolds, $j = 1, 2$, and (M, g, ϕ, J) be the product constructed as above.*

1. *If ϕ_i and η_i, $i = 1,2$, are closed, then ϕ is closed.*
2. *J is an almost complex structure on M.*

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3. *If M_i, $i = 1, 2$, are cosymplectic, then M is Kähler.*

Let $(M_1^{2n_1}, g_1, J_1)$ be a symplectic manifold, and $(M_2^{2n_2+1}, g_2, \varphi_2, \eta_2, \xi_2)$ be an almost contact metric manifold. Then $\xi_1 = \eta_1 = 0$, and $\omega_1 = \phi_1$ on M_1.

Theorem 3.2. *Let $(M, g, \varphi, \eta, \xi)$ be the product constructed as above.*

1. *If M_2 is contact, then M is an almost C-manifold.*

2. *If M_2 is a C-manifold, then M is an almost C-manifold.*

3. *If M_2 is almost cosymplectic, then M is almost cosymplectic.*

3.1. Quantum type cohomology

In [10, 11] we have studied the quantum type cohomology on contact manifolds. In this section, we want to introduce the quantum type cohomologies on almost cosymplectic, contact, and C-manifolds.

Let $(M^{2n+1}, g, \varphi, \eta, \xi)$ be an almost contact metric manifold. Then the distribution $\mathfrak{H} = \{X \in TM | \eta(X) = 0\}$ is an n-dimensional complex vector bundle on M.

Now fix the vector bundle $\mathfrak{H} \to M$. As the symplectic manifolds, a (1,1) type tensor field $\varphi : \mathfrak{H} \to H$ with $\varphi^2 = -I$ is said to be tamed by ϕ if $\phi(X, \varphi X) > 0$ for $X \in \mathfrak{H} \backslash \{0\}$ is said to be compatible if $\phi(\varphi X, \varphi Y) = \phi(X, Y)$.

Assume that the almost contact metric manifold M has a closed fundamental 2-form ϕ, i.e., $d\phi = 0$. An almost contact metric manifold M with the ϕ is called semipositive if for every $A \in \pi_2(M)$, $\phi(A) > 0$, $c_1(\mathfrak{H})(A) \geq 3-n$, then $c_1(\mathfrak{H})(A) > 0$ [13]. A smooth map $u : (\Sigma, j) \to (M, \varphi)$ from a Riemann surface (Σ, j) into (M, φ) is said to be φ-coholomorphic if $du \circ j = \varphi \circ du$.

Let $A \in H_2(M; \mathbb{Z})$ be a two-dimensional integral homology class in M. Let $\mathcal{M}_{0,3}(M; A, \varphi)$ be the moduli space of stable rational φ-coholomorphic maps with three marked points, which represent class A.

Lemma 3.1.1. *For a generic almost complex structure φ on the distribution, $\mathbb{C}^n \to H \to M$, the moduli space $\mathcal{M}_{0,3}(M; A, \varphi)$ is a compact stratified manifold with virtual dimension $2c_1(\mathfrak{H})[A] + 2n$.*

Consider the evaluation map given by

$$\mathrm{ev} : \mathcal{M}_{0,3}(M; A, \varphi) \to M^3, \tag{46}$$

$$\mathrm{ev}(\Sigma; z_1, z_2, z_3, u) = \Big(u(z_1), u(z_2), u(z_3)\Big). \tag{47}$$

We have a Gromov-Witten type invariant given by

$$\Phi_{0,3}^{M,A,\varphi} : H^*(M^3) \to Q \tag{48}$$

$$\Phi_{0,3}^{M,A,\varphi}(\alpha) = \int\limits_{\mathcal{M}_{0,3}(M; A, \varphi)} \mathrm{ev}^*(\alpha) = \mathrm{ev}_*[\mathcal{M}_{0,3}(M; A, \varphi)] \cdot PD(\alpha) \tag{49}$$

which is the number of these intersection points counted with signs according to their orientations.

We define a quantum type product $*$ on $H^*(M)$, for $\alpha \in H^k(M)$ and $\beta \in H \quad (M)$,

$$\alpha * \beta = \sum_{A \in H_2(M)} (\alpha * \beta)_A q^{c_1(\mathfrak{H})[A]/N}, \tag{50}$$

where N is called the minimal Chern number defined by

$$< c_1(\mathfrak{H}), H_2(M) >= N\mathbb{Z} \tag{51}$$

The $(\alpha * \beta)_A \in H^{k+l-2c_1(\mathfrak{H})[A]}(M)$ is defined for each $C \in H_{k+l-2c_1(\mathfrak{H})[A]}(M)$,

$$\int_C (\alpha * \beta)_A = \Phi_{0,3}^{M,A,\varphi}(\alpha \otimes \beta \otimes \gamma), \gamma = PD(C). \tag{52}$$

We denote a quantum type cohomology [11, 13] of M by

$$QH^*(M) := H^*(M) \otimes Q[q] \tag{53}$$

where $\mathbb{Q}[q]$ is the ring of Laurent polynomials in q of degree $2N$ with coefficients in the rational numbers $\mathbb{Q}$. By linearly extending the product $*$ on $QH^*(M)$, we have

Theorem 3.1.2. *The quantum type cohomology $QH^*(M)$ of the manifold M is an associative ring under the product $*$.*

Let $(M_1^{2n_1}, g_1, J_1, \omega_1)$ be a symplectic manifold and $(M_2^{2n_2+1}, g_2, \varphi_2, \eta_2, \xi_2, , \phi_2)$ be an either almost cosymplectic or contact or C-manifold.

Let the product $(M^{2n+1}, g, \varphi, \eta, \xi, \phi)$ be construct as Theorem 3.2 where $n = n_1 + n_2$. Now we will only consider the free parts of the cohomologies. By the Künneth formula, $H^*(M) \simeq H^*(M_1) \otimes H^*(M_2)$ in particular, $H_2(M) \simeq H_2(M_1 \oplus \big(H_1(M_1) \otimes H_1(M_2)\big) \oplus H_2(M_2))$.

Assume that a two-dimensional class$A = A_1 + A_2 \in H_2(M_1) \oplus H_2(2) \subset H_2(M)$.

Lemma 3.1.3. *Let $(M, g, \varphi, \eta, \xi, \phi)$ be the product $M = M_1 M_2$ constructed as above. For a generic almost cocomplex structure φ on M*

(1) *the moduli space $\mathcal{M}_{0,3}(M; A, \varphi)$ is homeomorphic to the product*

$$\mathcal{M}_{0,3}(M_1, A_1, J_1)\mathcal{M}_{0,3}(M_2, A_2, \varphi), \tag{54}$$

$$\dim \mathcal{M}_{0,3}(M, A, \varphi) = 2[c_1(TM_1)(A_1) + c_1(\mathfrak{H}_2)(A_2)] + 2(n_1 + n_2). \tag{55}$$

Theorem 3.1.4. *For the product $(M, g, \varphi, \eta, \xi, \phi) = (M_1, g_1, J_1, \omega_1)(M_2, g_2, \varphi_2, \eta_2, \xi_2, \phi_2)$, if $A = A_1 + A_2 \in H_2(M_1) \oplus H_2(M_2) \subset H_2(M)$, then the Gromov-Witten type invariants satisfy the following equality*

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$$\Phi_{0,3}^{M,A,\varphi} = \Phi_{0,3}^{M_1,A_1,J_1} \cdot \Phi_{0,3}^{M_2,A_2,\varphi_2}. \tag{56}$$

The complex $(n_1 + n_2)$-dimensional vector bundle

$$TM_1 \oplus \mathfrak{H}_2 \to M = M_1 M_2 \tag{57}$$

has the first Chern class $c_1(TM_1 \oplus \mathfrak{H}_2) = c_1(TM_1) + c_1(\mathfrak{H}_2)$.
The minimal Chern numbers N_1 and N_2 are given by $N_1\mathbb{Z} =< c_1(TM_1), H_2(M_1) >$ and

$$N_2\mathbb{Z} =< c_1(\mathfrak{H}_2), H_2(M_2) > . \tag{58}$$

For cohomology classes

$$\alpha = \alpha_1 \otimes \alpha_2 \in H^{k_1}(M_1) \otimes H^{k_2}(M_2) \subset H^k(M), \tag{59}$$

$$\beta = \beta_1 \otimes \beta_2 \in H^{l_1}(M_1) \otimes H^{l_2}(M_2) \subset H^l(M), \tag{60}$$

$k_1 + k_2 = k$, the quantum type product $\alpha * \beta$ is defined by

$$\alpha * \beta = \sum_{\substack{A_1 \in H_2(M_1) \\ A_2 \in H_2(M_2)}} (\alpha_1 * \beta_1)_{A_1} q^{c_1(A_1)/N_1} \otimes (\alpha_2 * \beta_2)_{A_2} q^{c_1(A_2)/N_2} \tag{61}$$

where q_i is a degree $2N_i$ auxiliary variable, $i = 1, 2$, and the cohomology class $(\alpha_i * \beta_i)_{A_i} \in H_{k_i + l_i - 2c_1(A_i)}(M_i)$ is defined by the Gromov-Witten type invariants as follows:

$$\int_{C_i} (\alpha_i * \beta_i)_{A_i} = \Phi_{0,3}^{M_i,A_i,\varphi_i}(\alpha_i \otimes \beta_i \otimes \gamma_i) \tag{62}$$

where $C_i \in H_{k_i + l_i - 2c_1(A_i)}(M_i)$, $\gamma_i = PD(C_i)$ and $\varphi_1 := J_1, i = 1, 2$, respectively.
The quantum type cohomology of M is defined by the tensor product

$$QH^*(M) = H^*(M) \otimes Q[q_1, q_2], \tag{63}$$

where $\mathbb{Q}[q_1, q_2]$ is the ring of Laurent polynomials of variables q_1 and q_2 with coefficients in $\mathbb{Q}$. Extend the product $*$ linearly on the quantum cohomology $QH^*(M)$; similarly, we define the quantum cohomology rings

$$\begin{cases} QH^*(M_1) = H^*(M_1) \otimes Q[q_1], \\ QH^*(M_2) = H^*(M_2) \otimes Q[q_2]. \end{cases} \tag{64}$$

Theorem 3.1.5. *There is a natural ring isomorphism between quantum type cohomology rings constructed as above,*

$$QH^*(M) = QH^*(M_1) \otimes QH^*(M_2). \tag{65}$$

Let (M, g, φ, ϕ) be the product of a compact symplectic manifold $(M_1^{2n_1}, g_1, J_1, \omega_1)$ and an either almost cosymplectic or contact or C-manifold $(M_2^{2n_2+1}, g_2, \varphi_2, \eta_2, \xi_2, \phi_2)$. We choose integral

bases, $e_0, e_1, \ldots, e_{k_1}$ of $H^*(M_1)$ and $f_0, f_1, \ldots, f_{k_2}$ of $H^*(M_2)$ such that $e_0 = 1 \in H^0(M_1)$, $f_0 = 1 \in H^0(M_2)$ and each basis element has a pure degree. We introduce a linear polynomial of $k_1 + 1$ variables $t_0, t_1, \ldots, t_{k_1}$, with coefficients in $H^*(M_1)$

$$a_t := t_0 e_0 + t_1 e_1 + \cdots + t_{k_1} e_{k_1}, \tag{66}$$

and a linear polynomial of $k_2 + 1$ variables $s_0, s_1, \cdots, s_{k_2}$ with coefficients in $H^*(M_2)$

$$a_s := s_0 f_0 + s_1 f_1 + \cdots + s_{k_2} f_{k_2}. \tag{67}$$

By choosing the coefficients in $\mathbb{Q}$, the cohomology of M is

$$H^*(M) \cong H^*(M_1) \otimes H^*(M_2). \tag{68}$$

Then, $H^*(M)$ has an integral basis $\{e_i \otimes f_i | i = 0, \ldots, k_1, j = 0, \ldots, k_2\}$. The rational Gromov-Witten type potential of the product (M, ω) is a formal power series in the variables $\{t_i, s_j | i = 0, \ldots, k_1, j = 0, \ldots, k_2\}$ with coefficients in the Novikov ring Λ_ω as follows:

$$\begin{aligned} \Psi_0^M(t,s) &= \sum_A \sum_m \frac{1}{m!} \Phi_{0,m}^{M,A,\varphi}(a_t \otimes a_s, \ldots, a_t \otimes a_s) e^{-\int_A \phi} \\ &= \sum_{A_1} \sum_{m_1} \frac{1}{m_1!} \Phi_{0,m_1}^{M_1,A_1,J_1}(a_t, \ldots, a_t) e^{-\int_{A_1} \omega_1} \cdot \sum_{A_2} \sum_{m_2} \frac{1}{m_2!} \Phi_{0,m_2}^{M_2,A_2,J_2}(a_s, \ldots, a_s) e^{-\int_{A_2} \phi_2} \\ &= \Psi_0^{M_1}(t) \cdot \Psi_0^{M_2}(s). \end{aligned} \tag{69}$$

Theorem 3.1.6. *The rational Gromov-Witten type potential of (M, φ) is the product of the rational Gromov-Witten potentials of M_1 and M_2, that is,*

$$\Psi_0^M(t,s) = \Psi_0^{M_1}(t) \cdot \Psi_0^{M_2}(s). \tag{70}$$

3.2. Floer type cohomology

In this subsection, we assume that our manifold $(M^{2n+1}, g, \varphi, \eta, \xi, \phi)$ is either a almost cosymplectic, contact, or C-manifold.

Let $H_t = H_{t+1} : M \to R$ be a smooth 1-periodic family of Hamiltonian functions. Denoted by $X_t : M \to TM$ the Hamiltonian vector field of H_t.

The vector fields X_t generate a family of Hamiltonian contactomorphisms $\psi_t : M \to M$ satisfying $\frac{d}{dt}\psi_t = X_t \circ \psi_t$ and $\psi_0 = \mathrm{id}$.

Let $a : \mathbb{R}/\mathbb{Z} \to M$ be a contractible loop, then there is a smooth map $u : D \to M$, defined on the unit disk $D = \{z \in C | |z| \leq 1\}$, which satisfies $u(e^{2\pi i t}) = a(t)$. Two such maps $u_1, u_2 : D \to M$ are called equivalent if their boundary sum $u_1 \# (-u_2) : S^2 \to M$ is homologus to zero in $H_2(M)$.

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Let $\tilde{a} := (a, [u])$ be an equivalence class and denoted by $\widetilde{LM}$ the space of equivalence classes. The space $\widetilde{LM}$ is the universal covering space of the space LM of contractible loops in M whose group of deck transformation is $H_2(M)$.

The symplectic type action functional $a_H : \widetilde{LM} \to R$ is defined by

$$a_H(a, [u]) = -\int_D u^*\phi - \int_0^1 H_t\Big(a(t)\Big)dt, \tag{71}$$

then satisfies $a_H(A\#\tilde{a}) = a_H(\tilde{a}) - \phi(A)$.

Lemma 3.2.1. *Let (M, ϕ) the manifold with a closed fundamental 2-form ϕ and fix a Hamiltonian function $H \in C^\infty(\mathbb{R}/\mathbb{Z}M)$. Let $(a, [u]) \in LM$ and $V \in T_aLM = C^\infty(\mathbb{R}/\mathbb{Z}, a^*TM)$. Then*

$$(da_H)_{(a,[u])}(V) = \int_0^1 \phi\Big(\dot{a} - X_{H_t}(a), V\Big)dt. \tag{72}$$

We denote by $P(\tilde{H}) \subset \widetilde{LM}$ the set of critical points of a_H and by $P(H) \subset LM$ the corresponding set of periodic solutions.

Consider the downward gradient flow lines of a_H with respect to an L^2-norm on LM. The solutions are

$$u : \mathbb{R}^2 \to M, (s, t) \mapsto u(s, t) \tag{73}$$

of the partial differential equation

$$\partial_s(u) + \varphi(u)\Big(\partial_t u - X_t(u)\Big) = 0 \tag{74}$$

with periodicity condition

$$u(s, t + 1) = u(s, t) \tag{75}$$

and limit condition

$$\lim_{s \to -\infty} u(s, t) = a(t), \lim_{s \to +\infty} u(s, t) = b(t), \tag{76}$$

where $a, b \in P(H)$.

Let $\mathcal{M}(\tilde{a}, \tilde{b}) := M(\tilde{a}, \tilde{b}, H, \varphi)$ be the space of all solutions $u(s, t)$ satisfying (74)–(76) with

$$\tilde{a}\#u = \tilde{b}. \tag{77}$$

The solutions are invariant under the action $u(s, t) \mapsto u(s + r, t)$ of the time shift $r \in R$. Equivalent classes of solutions are called Floer connecting orbits.

For a generic Hamiltonian function H, the space $\mathcal{M}(\tilde{a},\tilde{b})$ is a finite dimensional manifold of dimension

$$\dim\mathcal{M}(\tilde{a},\tilde{b}) = \mu(\tilde{a}) - \mu(\tilde{b}), \tag{78}$$

where the function $\mu : P(\tilde{H}) \to Z$ is a version of the Maslov index defined by the path of unitary matrices generated by the linealized Hamiltonian flow along $a(t)$ on D .

If $H_t \equiv H$ is a C^2-small Morse function, then a critical point $(a,[u])$ of H_t is a constant map $u(D) = a$ with index $\mathrm{ind}_H(a)$.

If $\mu(\tilde{a}) - \mu(\tilde{b}) = 1$, then the space $\mathcal{M}(\tilde{a},\tilde{b})$ is a one-dimensional manifold with $\mathbb{R}$ action by time shift and the quotient $\mathcal{M}(\tilde{a},\tilde{b})/\mathbb{R}$ is a finite set. In fact, $\mu(\tilde{a}) \in \pi_1(U(n)) \simeq Z$.

If $\mu(\tilde{a}) - \mu(\tilde{b}) = 1$, $\tilde{a},\tilde{b} \in P(\tilde{H})$, then we denote

$$\eta(\tilde{a},\tilde{b}) := \#\left(\frac{\mathcal{M}(\tilde{a},\tilde{b})}{\mathbb{R}}\right), \tag{79}$$

where the connection orbits are to be counted with signs determined by a system of coherent orientation s of the moduli space $\mathcal{M}(\tilde{a},\tilde{b})$. These numbers give us a Floer type cochain complex.

Let $FC^*(M,H)$ be the set of functions

$$\xi : P(\tilde{H}) \to R \tag{80}$$

that satisfy the finiteness condition

$$\#\{\tilde{x} \in P(\tilde{H}) \,|\, \xi(\tilde{x}) \neq 0, a_H(\tilde{x}) \leq c\} < \infty \tag{81}$$

for all $c \in \mathbb{R}$.

Now we define a coboundary operator

$$\delta^k : FC^k(M,H) \to FC^{k+1}(M,H), \tag{82}$$

$$(\delta^k \xi)(\tilde{a}) = \sum_{\mu(\tilde{a}) = \mu(\tilde{b})+1} \eta(\tilde{a},\tilde{b})\xi(\tilde{b}) \tag{83}$$

where $\xi \in FC^k(M,H)$, $\mu(\tilde{a}) = k+1$ and $\mu(\tilde{b}) = k$.

Lemma 3.2.2. *Let (M,φ) be a semipositive almost contact metric manifold with a closed functional 2-forms. The coboundary operators satisfy $\delta^{k+1} \circ \delta^k = 0$, for all k.*

Definition - Theorem 3.2.3. (1) *For a generic pair (H,φ) on M, the cochain complex (FC^*,δ) defines cohomology groups*

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$$FH^*(M,\phi,H,\varphi) := \frac{\mathrm{Ker}\delta}{\mathrm{Im}\delta} \tag{84}$$

which are called the Floer type cohomology groups of the (M,ϕ,H,φ).

(2) *The Floer type cohomology group* $FH^*(M,\phi,H,\varphi)$ *is a module over Novikov ring* Λ_ϕ *and is independent of the generic choices of H and* φ.

4. Quantum and Floer type cohomologies

In this section, we assume that our manifold M is a compact either almost cosymplectic or contact or C-manifold. In Section 3.1, we study quantum type cohomology of M and in Section 3.2 Floer type cohomology of M. Consequently, we have:

Theorem 4.1. *Let* $(M,g,\varphi,\eta,\xi,\phi)$ *be a compact semipositive almost contact metric manifold with a closed fundamental 2-form* ϕ. *Then, for every regular pair* (H,φ), *there is an isomorphism between Floer type cohomology and quantum type cohomology*

$$\Phi : FH^*(M,\phi,H,\varphi)\tilde{\to}QH^*(M,\Lambda_\phi). \tag{85}$$

Proof. Let $h : M \to R$ be a Morse function such that the negative gradient flow of h with respect to the metric $\phi\Big(\cdot\,,\varphi(\cdot)\Big) + \eta\otimes\eta$ is Morse-Smale and consider the time-independent Hamiltonian

$$H_t := -\varepsilon h, t\in R. \tag{86}$$

If ε is sufficiently small, then the 1-periodic solutions of

$$\dot{a}(t) = X_t(a(t)) \tag{87}$$

are precisely the critical point of h. The index is

$$\mu(a,u_a) = n\text{–ind}_h(a) = \text{ind}_{-h}(a)\text{–}n \tag{88}$$

where $u_a : D \to M$ is the constant map $u_a(z) = a$.

The downward gradient flow lines $u : \mathbb{R} \to M$ of h are solutions of the ordinary differential equation

$$\dot{u}(s) = J(u)X_t(u). \tag{89}$$

These solutions determine a coboundary operator

$$\delta : C^*(M,h,\Lambda_\phi) \to C^*(M,h,\Lambda_\phi). \tag{90}$$

This coboundary operator is defined on the same cochain complex as the Floer coboundary δ, and the cochain complex has the same grading for both complex $C^*(M,h,\Lambda_\phi)$, which can be identified with the graded Λ_ϕ module of all functions

$$\xi : \mathrm{Crit}(h)H_2(M) \to R \tag{91}$$

that satisfy the finiteness condition

$$\#\{(a,A)|\xi(a,A)\neq 0, \phi(A)\geq c\} < \infty \tag{92}$$

for all $c\in R$. The Λ_ϕ-module structure is given by

$$(\nu * \xi)(a,A) = \sum_B \nu(B)\xi(a,A+B), \tag{93}$$

the grading is $\deg(a,A) = \mathrm{ind}_h(a) - 2c_1(A)$, and the coboundary operator δ is defined by

$$(\delta\xi)(a,A) = \sum_b n_h(a,b)\xi(b,A), (a,A)\in \mathrm{Crit}(h)H_2(M), \tag{94}$$

where $n_h(a,b)$ is the number of connecting orbits from a to b of shift equivalence classes of solutions of

$$\{\ \dot{u}(s) + \nabla u(s) = 0, \lim_{s\to-\infty} u(s) = a, \lim_{s\to+\infty} u(s) = b, \tag{95}$$

counted with appropriate signs.

Here we assume that the gradient flow of h is Morse-Smale and so the number of connecting orbits is finite when $\mathrm{ind}_h(a) - \mathrm{ind}_h(b) = 1$. Then the coboundary operator δ is a Λ_ϕ-module homomorphism of degree one and satisfies $\delta\circ\delta = 0$. Its cohomology is canonically isomorphic to the quantum type cohomology of M with coefficients in Λ_ϕ.

For each element $\tilde{a}\in P(\tilde{H})$ we denote $\mathcal{M}(\tilde{a},H,\varphi)$ by the space of perturbed φ-cohomomorphic maps $u : \mathbb{C} \to M$ such that $u(re^{2\pi it})$ converges to a periodic solution $a(t)$ of the Hamiltonian system H_t as $r \to \infty$. The space $\mathcal{M}(\tilde{a},H,\varphi)$ has dimension $n - \mu(\tilde{a})$. Now fix a Morse function $h : M \to R$ such that the downward gradient flow $u : \mathbb{R} \to M$ satisfying (95) is Morse-Smale. For a critical point $b\in\mathrm{Crit}(h)$ the unstable manifold $W^u(b,h)$ of b has dimension $\mathrm{ind}_h(b)$ and codimension $2n - \mathrm{ind}_h(b)$ in the distribution D.

The submanifold $\mathcal{M}(b,\tilde{a})$ of all $u\in M(\tilde{a},H,\varphi)$ with $u(0)\in W^u(b)$ has dimension

$$\dim\mathcal{M}(b,\tilde{a}) = \mathrm{ind}_h(b) - \mu(\tilde{a}) - n. \tag{96}$$

If $\mathrm{ind}_h(b) = \mu(\tilde{a}) + n$, then $\mathcal{M}(b,\tilde{a})$ is 0zero-dimensional and hence the numbers $n(b,\tilde{a})$ of its elements can be used to construct the chain map defined by

$$\Phi : FC^*(M,H) \to C^*(M,h,\Lambda_\phi) \tag{97}$$

$$(\Phi\xi)(b,A)\Lambda \sum_{\mathrm{ind}_h(b)=\mu(\tilde{a})+n} n(b,\tilde{a})\xi(A\#\tilde{a}) \tag{98}$$

which is a Λ_ϕ-module homomorphism and raises the degree by n. The chain map Φ induces a homomorphism on cohomology

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$$\Phi : FH^*(M, \Lambda_\phi) \to H^*(M, h, \Lambda_\phi) = \frac{\mathrm{Ker}\delta}{\mathrm{Im}\delta} \simeq QH^*(M, \Lambda_\phi). \tag{99}$$

Similarly, we can construct a chain map,

$$\Psi : C^*(M, h, \Lambda_\phi) \to FC^*(M, H) \tag{100}$$

$$(\Psi\xi)(\tilde{a}) := \sum_{\mu(\tilde{a})+n=\mathrm{ind}_h(b)-2c_1(A)} n((-A)\#\tilde{a}, b)\xi(b, A). \tag{101}$$

Then $\Phi\circ\Psi$ and $\Psi\circ\Phi$ are chain homotopic to the identity. Thus we have an isomorphism Φ.

We have studied the Gromov-Witten invariants on symplectic manifolds (M, ω, J) using the theory of J-holomorphic curves, and the Gromov-Witten type invariants on almost contact metric manifolds $(N, g, \varphi, \eta, \xi, \phi)$ with a closed fundamental 2-form ϕ using the theory of φ-coholomorphic curves. We also have some relations between them. We can apply the theories to many cases.

Examples 4.2.

1. The product of a symplectic manifold and a unit circle.
2. The circle bundles over symplectic manifolds.
3. The almost cosymplectic fibrations over symplectic manifolds.
4. The preimage of a regular value of a Morse function on a Kähler manifold.
5. The product of two cosymplectic manifolds is Kähler.
6. The symplectic fibrations over almost cosymplectic manifolds.
7. The number of a contactomorphism is greater than or equal to the sum of the Betti numbers of an almost contact metric manifold with a closed fundamental 2-form.

Examples 4.3. Let N be a quintic hypersurface in $\mathbb{C}P^4$ which is called a Calabi-Yau threefold. Then N is symply connected, $c_1(TN) = 0$ and its Betti numbers $b_0 = b_6 = 1$, $b_1 = b_5 = 0$, $b_2 = b_4 = 1$ and $b_3 = 204$.

Let A be the standard generator in $H_2(N)$ and $h \in H^2(N)$ such that $h(A) = 1$. The moduli space $\mathcal{M}_{0,3}(N, A)$ has the dimension zero. The Gromov-Witten invariant $\Phi_{0,3}^{N,A}(a_1, a_2, a_3)$ is nonzero only when $\deg(a_i) = 2$, $i = 1, 2, 3$. In fact, $\Phi_{0,3}^{N,A}(h, h, h) = 5$ [4, 5]. The quantum cohomology of N is $QH^*(N) = H^*(N) \otimes \Lambda$ where Λ is the universal Novikov ring [5].

Let (N, g_1, ω_1, J_1) be the standard Kähler structure on N and $\left(S^1, g_2, \varphi_2 = 0, \eta_2 = d\theta, \xi_2 = \frac{d}{d\theta}, \phi_2 = 0\right)$ the standard contact structure on S^1. Then the product $M = NS^1$ has a canonical cosymplectic structure $(M, g, \varphi, \eta, \xi, \phi)$ as in Section 3. The quantum type cohomology of M is

$$QH^*(M) = QH^*(N) \otimes QH^*(S^1) \tag{102}$$

Let $\psi_1 : N \to N$ be a Hamiltonian symplectomorphism with nondegenerate critical points. Then $\#Fix(\psi_1) \geq \sum_{i=0}^{6} b_i(N) = 208$.

Let $\psi_2 : M \to M$ be a Hamiltonian contactomorphism with nondegenerate critical points. Then $\#Fix(\psi_2) \geq \sum_{i=0}^{7} b_i(M) = 416$.

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