Chapter 13

Design of Dynamic Output Feedback Laws Based on Sums of Squares of Polynomials

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Abstract

We consider the stabilization of nonlinear polynomial systems and the design of dynamic output feedback laws based on the sums of squares (SOSs) decompositions. To design the dynamic output feedback laws, we show the design conditions in terms of the state-dependent linear matrix inequalities (SDLMIs). Because the feasible solutions of the SDLMIs are found by the SOS decomposition, we can obtain the dynamic output feedback laws by using numerical solvers. We show numerical examples of the design of dynamic output feedback laws.

Keywords: sums of squares polynomials, output feedback stabilization, Lyapunov methods, state-dependent LMIs

1. Introduction

In the last few decades, control design methods based on numerical methods have appeared in the control literature. Major progress in the 1980s was the emergence of numerical methods based on linear matrix inequalities (LMIs) [1]. The methods provide the numerical solutions to linear control problems in the formulation of the semidefinite programming. The LMI approach provides the design methods of feedback laws for the asymptotic stabilization, H-infinity control, and robust control. For the nonlinear control problems, the sums of squares (SOS) approach is introduced as a generalization of the LMI approach to nonlinear systems [2–6]. A feature of the sums of squares polynomials is negative semidefiniteness, and this is suitable for the stability analysis of nonlinear systems based on the Lyapunov theory. The studies [2, 3] have shown that the sums of squares decomposition can be solved in the formulation of the semidefinite programming. The result leads to the development of numerical methods for the analysis and synthesis of nonlinear polynomial systems. Applications to control problems are feedback design [7, 8], motion planning [9], modeling, and control of fuzzy systems [10] to mention a few. Applications of the SOS approach to nonpolynomial systems are found in reference [11, 12].

The SOS approach has been the basis of numerical methods for the analysis and the synthesis of nonlinear systems. Although the Lyapunov-based approach offers the methods for the analysis and the synthesis, the construction of Lyapunov functions is often a difficult task. The SOS approach provides a technique to find Lyapunov functions by formulating the Lyapunov inequality conditions into the SOS conditions. The stability of nonlinear systems is analyzed by a direct application of SOS decompositions to the Lyapunov stability analysis. However, applications of the SOS approach to Lyapunov-based feedback design are much complicated because decision variables do not enter the Lyapunov inequalities conditions linearly. So far, two main approaches have been proposed. One is a method in [8], which formulates the design conditions into state-dependent linear matrix inequalities (SDLMIs) conditions. The SDLMIs are solved by the SOS decompositions. The other method is based on an iterative algorithm shown in reference [7], which also considers the enlargement of the regions of attraction of the closed-loop systems.

In the actual control problems, we often cannot measure all the values of the state variables of control systems. This fact leads to the necessity of the design of output feedback laws. The design of output feedback laws is more complicated task than that of state feedback laws because the stability conditions of the closed-loop systems become complex. As far as the authors know, so far, a few output feedback design methods have been proposed, for example, [[7], Section 3.5] and [13–15]. The further developments of design methods for output feedback laws have been desired.

It is well known that we often can design dynamic feedback laws even when the design of static output feedback laws is difficult. This leads to the motivation of developing a design method based on the SOS approach for the design of dynamic output feedback laws. In reference [7], an iterative method for the design of dynamic output feedback laws has been shown. However, we need to give control Lyapunov functions (CLFs) to start the iteration in the method, and this might be a difficult task especially for complex or high-dimensional systems. The state-dependent LMI approach can be an alternative approach because it does not need to give any CLF. However, a concrete method for dynamic output feedback laws has not been shown in this direction yet.

We provide the design methods of dynamic output feedback laws for the stabilization based on the SDLMI approach. This method is based on the design method of state feedback laws based on the SDLMI approach [8]. The proposed method employs a two-step algorithm. We first design a virtual state feedback law for a given system using the method of reference [8]. Then, we design a dynamic output feedback by using an SDLMI again based on the virtual state feedback law. The use of the virtual state feedback inherits the design approach of output feedback laws in reference [16], which indicates the general design approach of output feedback laws not necessarily for the SOS approach. We also show some numerical examples to demonstrate the effectiveness of the proposed method to the actual control problems.

Notation: We denote the set of the real numbers and integers as $\mathbb{R}$ and $\mathbb{Z}$, respectively. The notation $\mathbb{Z}_+$ is the nonnegative integers. The notation $\|x\|$ is the Euclidean norm of a vector x. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$, $|\alpha|$ denotes $|\alpha| = \sum_{i=1}^{n} \alpha_i$. For a matrix $X \in \mathbb{R}^{n\times n}$, He (X) denoes He (X) $= X + X^T$.

2. Preliminary: stability of nonlinear systems

This section provides the stability theory of nonlinear systems. We present the definitions of stability, and then, we introduce the Lyapunov stability theory. The Lyapunov stability theory forms the basis for the analysis and synthesis of the stability of dynamical systems. The theory states that the existence of a kind of functions implies the stability.

This section considers the stability of an autonomous nonlinear system

$$\dot{x} = f(x), \quad x(t_0) = x_0 \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $f{:}\mathbb{R}^n \to \mathbb{R}^n$ is the vector fields, and $x_0 \in \mathbb{R}^n$ is the initial value of the state. In the following, we assume that the origin $x = 0$ is the equilibrium of system (1), that is, $f(0) = 0$, and we consider the stability of the origin.

To begin with, we show the definitions of the stability.

Definition 1 (stability). The equilibrium $x = 0$ is said to be Lyapunov stable if for any $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$, such that for any $\|x_0\| < \delta$, the solution $x(t)$ of (1) satisfies that

$$\| x(t) \| < \epsilon, \quad \forall\, t \in [t_0, \infty).$$

Definition 2 (asymptotic stability). The equilibrium $x = 0$ is said to be asymptotically stable if it is stable and there exists $\delta > 0$, such that for any $\|x_0\| < \delta$, the solution of (1) satisfies that

$$x(t) \to 0 \text{ as } t \to \infty.$$

Definition 3 (global asymptotic stability). The equilibrium $x = 0$ is said to be globally asymptotically stable if it is stable and for any $x_0 \in \mathbb{R}^n$, the solution $x(t)$ of (1) satisfies that

$$x(t) \to 0 \text{ as } t \to \infty.$$

To introduce the Lyapunov stability theory, we provide the definitions of the properties of functions.

Definition 4 (positive definiteness). A function $h{:}\ \mathbb{R}^n \to \mathbb{R}$ is said to be positive definite if $h(x) > 0$ for any $x \neq 0$ and $h(0) = 0$.

Definition 5 (positive semidefiniteness). A function $h\colon \mathbb{R}^n \to \mathbb{R}$ is said to be positive semidefinite if $h(x) \geq 0$ for any $x \in \mathbb{R}^n$.

We say that a function $h(x)$ is negative definite (negative semidefinite) if the function $-h(x)$ is positive definite (respectively, positive semidefinite).

Definition 6 (properness). A function $h\colon \mathbb{R}^n \to \mathbb{R}$ is said to be proper if for any $K \in \mathbb{R}$, the sublevel set

$$\{x \in \mathbb{R}^n \mid h(x) \leq K\}$$

is bounded.

The Lyapunov stability theory is stated as follows [17].

Theorem 1. Let U be an open subset of $\mathbb{R}^n$ which contains the origin. Suppose that a function $V\colon U \to \mathbb{R}$ is continuously differentiable, positive definite, and proper. The equilibrium of system (1), $x = 0$, is stable if and only if the function $V(x)$ satisfies that

$$\frac{dV}{dt}(x) = \frac{\partial V}{\partial x}(x) f(x) \leq 0, \ \ \forall\, x \in \mathbb{R}^n.$$

Moreover, the equilibrium of system (1), $x = 0$, is asymptotically stable if and only if the function $V(x)$ satisfies that

$$\frac{dV}{dt}(x) = \frac{\partial V}{\partial x}(x) f(x) < 0, \ \ \forall\, x \neq 0.$$

When $U = \mathbb{R}^n$, the global asymptotic stability holds.

The Lyapunov theory is used to investigate the stability of nonlinear systems. However, to investigate the stability of each system by Lyapunov theory, we need to find a Lyapunov function for it. However, to find the Lyapunov functions is often a difficult task. Further, when we try to design stabilizing feedback laws based on the Lyapunov theory, we also need to find the Lyapunov function candidates for the closed-loop systems. Therefore, we require a method to find Lyapunov functions for each nonlinear system. The SOS approach provides Lyapunov functions as solutions to the SOS conditions.

3. Sums of squares polynomials and state-dependent linear matrix inequalities

This chapter introduces some definitions and results on SOS polynomials. We also introduce that SDLMIs can be solved by the SOS decomposition.

We begin with the definitions of monomials, polynomials, and sums of squares polynomials.

Definition 7 (monomials). Let $z = (z_1, \dots, z_n) \in \mathbb{R}^n$, and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$. A monomial of z, $m_\alpha(z)$, is a function given by

$$m_\alpha(z) = \prod_{i=1}^{n} z_i^{\alpha_i}$$

Definition 8 (polynomials). Consider monomials of z, $m_{\alpha_i}(z)$, where $\alpha_i = (\alpha_{i1}, \alpha_{i2}, \dots, \alpha_{in}) \in \mathbb{Z}_+^n$, and $c_i \in \mathbb{R}$ for $i = 1,\dots,m$. A polynomial of z, $f(z)$, is a function given in the form of

$$f(z) = \sum_{i=1}^{m} c_i\, m_{\alpha_i}(z).$$

The degree of polynomial $f(z)$, d, is given by

$$d = \max_i |\alpha_i|.$$

Let $\mathcal{R}_n$ denote the set of polynomials of n variables. Then, we show the definition of the sums of squares polynomials.

Definition 9 (sums of squares polynomials, SOSs). Let $z = (z_1,\dots,z_n)$. A sum of squares polynomial $\sigma_n(z)$ is a function given in the form of

$$\sigma_n(z) = \sum_{i=1}^{m} f_i\,(z)^2, \quad f_i(z) \in \mathcal{R}_n, \qquad i = 1, \dots, m.$$

The decomposition of given polynomials into SOSs is called as the SOS decomposition. Regarding the SOS decomposition, the following result is shown.

Theorem ([2, 3]). Consider the polynomial of z of degree $2d$, $f(z)$. The polynomial $f(z)$ is an SOS polynomial if and only if there exist a column vector $X(z)$ whose elements are monomials of z of degree no greater than d and a positive semidefinite matrix Q such that

$$f(z) = X(z)^T Q X(z)$$

holds.

We show a simple example of SOSs.

Example 1. Consider a polynomial $f(z)$ given by

$$f(z) = z^2 + 2z + 2,$$

where $z \in \mathbb{R}$. Apparently, this polynomial is expressed as the sum of squares polynomial

$$f(z) = z^2 + 2z + 2 = (z+1)^2 + 1.$$

Regarding Theorem 2, the polynomial is also expressed as

$$f(z) = \begin{bmatrix} z & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix}, \tag{2}$$

and the matrix in the right-hand side of (2) is positive definite.

The SOS decomposition can be solved by some numerical solvers, such as YALMIP [18] and SOSTOOLS [19]. When some coefficients of polynomials are decision variables in an SOS decomposition, by using the numerical solvers, we can find the feasible solutions such that the SOS decomposition holds. Therefore, we can adapt the SOS decomposition to the design of feedback laws in control problems.

With the relation to the stability theory presented in Section 2, the sufficient condition of the stability is given as the SOS conditions.

Theorem 3. [2] Consider system (1). If there exist a positive definite function $\phi: \mathbb{R}^n \to \mathbb{R}$ and an SOS polynomial $\epsilon: \mathbb{R}^n \to \mathbb{R}^n$, such that

$$\begin{aligned} \phi(x) - \epsilon(x) &> 0, \\ \frac{\partial \phi}{\partial x}(x) f(x) + \epsilon(x) &< 0, \qquad \forall\, x \neq 0 \end{aligned}$$

then the equilibrium $x = 0$ is asymptotically stable.

Theorem 3 shows a direct application of the SOSs to the analysis of the stability. This implies that the SOS decomposition can be applied to the synthesis of the stabilizing feedback laws. This chapter develops a method to design dynamic output feedback laws based on the SDLMI approach [8]. The SDLMI is defined as the optimization problem:

$$\begin{aligned} &\text{minimize} \quad \sum_{i=1}^{m} a_i\, c_i \\ &\text{subject to} \;\; F_0(z) + \sum_{i=1}^{m} c_i\, F_i(z) \geq 0, \quad \text{for } \forall\, z \in \mathbb{R}^n, \end{aligned}$$

where $a_i \in \mathbb{R}$ are the fixed coefficients, c_i are the decision variables, the matrix functions F_i: $\mathbb{R}^n \to \mathbb{R}^{q \times q}$ are state-dependent symmetric matrices. The constraint should be satisfied for any $z \in \mathbb{R}^n$. This differs from standard LMIs and is the derivation of the word, state-dependent.

A relation of the SDLMIs and the SOS decompositions is shown as follows.

Theorem 4. ([8]) Let $d > 0$ and $F:\mathbb{R}^n \to \mathbb{R}^{q\times q}$ a symmetric polynomial matrix the elements of which are polynomials of z with degree $2d$. Moreover, consider a vector $v \in \mathbb{R}^q$. If $v^T F(z)v$ is a sum of squares polynomial, then $F(z) \geq 0$ holds for any $z \in \mathbb{R}^q$.

Theorem 4 states that if we find that the polynomial $v^T F(z)v$ is decomposed into an SOS with respect to (z,v), it implies the positive definiteness of $F(z)$ for any $z \in \mathbb{R}^n$. We can derive stability conditions in terms of SDLMIs. This leads to the design of feedback laws for the stabilization based on the combination of the SDLMIs and the SOS decomposition. We develop the synthesis of dynamic output feedback laws based on Theorem 4 in the following sections.

4. Problem setting: stabilization through dynamic output feedback

This chapter considers the stabilization problem via dynamic output feedback laws and the synthesis of the stabilizing feedback laws. This section states the problem setting.

The approach presented here is based on the SDLMI approach, which derives the sufficient conditions of the existence of stabilizing feedback laws as the SDLMI conditions. We can obtain stabilizing feedback control laws and Lyapunov functions by solving the SDLMI conditions using numerical solvers.

Consider a nonlinear system given as

$$\begin{aligned} \dot{x} &= f(x,u), \qquad x(t_0) = x_0 \\ y &= h(x), \end{aligned} \tag{3}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^{n_u}$ is the input, $y \in \mathbb{R}^{n_y}$ is the output, f: $\mathbb{R}^n \times \mathbb{R}^{n_u} \to \mathbb{R}^n$, h: $\mathbb{R}^n \to \mathbb{R}^{n_y}$, and x_0 is the initial state. For the nonlinear systems given by (3), we assume that system (3) is expressed as

$$\begin{aligned} \dot{x} &= A(x)Z(x) + B(x)u, \quad x(t_0) = x_0 \\ y &= C(x)Z(x), \end{aligned} \tag{4}$$

where $Z:\mathbb{R}^n \to \mathbb{R}^N$, A: $\mathbb{R}^n \to \mathbb{R}^{n\times N}$, B: $\mathbb{R}^n \to \mathbb{R}^{n\times n_u}$, C: $\mathbb{R}^n \to \mathbb{R}^{n_y\times N}$. Further, we assume that $Z(x) = 0$, if and only if $x = 0$. We consider the output stabilization of system (4) using a dynamic feedback law in the form of

$$\begin{aligned} \dot{\hat{x}} &= A_c(\hat{x},y)\hat{x} + B_c(\hat{x},y)y, \qquad \hat{x}(t_0) = \hat{x}_0 \\ u &= C_c(\hat{x},y)\hat{x} + D_c(\hat{x},y)y, \end{aligned} \tag{5}$$

where $\hat{x} \in \mathbb{R}^{n_{\hat{x}}}$ is the state of the dynamic feedback law,

$A_c:\mathbb{R}^{n_{\hat{x}}}\times\mathbb{R}^{n_y}\to\mathbb{R}^{n_{\hat{x}}\times n_{\hat{x}}}$, $B_c:\mathbb{R}^{n_{\hat{x}}}\times\mathbb{R}^{n_y}\to\mathbb{R}^{n_{\hat{x}}\times n_y}$, $C_c:\mathbb{R}^{n_{\hat{x}}}\times\mathbb{R}^{n_y}\to\mathbb{R}^{n_u\times n_{\hat{x}}}$, and $D_c:\mathbb{R}^{n_{\hat{x}}}\times\mathbb{R}^{n_y}\to\mathbb{R}^{n_u\times n_y}$, and $\hat{x}_0$ is the initial state.

We have the closed-loop system of (4) with the dynamic output feedback law (5), given by

$$\begin{aligned}\dot{x} &= \{A(x)+B(x)D_c(\hat{x},y)C(x)\}Z(x)+B(x)C_c(\hat{x},y)\hat{x},\\ \dot{\hat{x}} &= A_c(\hat{x},y)\hat{x}+B_c(\hat{x},y)C(x)Z(x).\end{aligned} \tag{6}$$

We consider the stabilization of the closed-loop system (6). To this end, we give a method to design the matrix functions $A_c(\hat{x},y), B_c(\hat{x},y), C_c(\hat{x},y), D_c(\hat{x},y)$ in the next section.

Remark 1. We obtain a system in the form of (4) as an expression of a nonlinear affine system

$$\begin{aligned}\dot{x} &= f(x)+g(x)u,\\ y &= h(x),\end{aligned}$$

by choosing *Z(x)* properly. Note that the choice of *Z(x)* is not unique in general. The systems in the form of (4) can be seen as a generalization of linear systems, given as

$$\begin{aligned}\dot{x} &= Ax+Bu,\\ y &= Cx,\end{aligned}$$

where the matrices *A*, *B*, and *C* are with the appropriate dimensions.

5. Design of dynamic output feedback laws through SOSs

This section provides a design method of dynamic feedback laws (5) for the output stabilization of system (4). We show stability conditions of the closed-loop system of (6) as SDLMI conditions. We can obtain the stabilizing laws by solving the SDLMI conditions via SOS decomposition using numerical solvers.

The main idea of the proposed method is as follows. Instead of the dynamic feedback law (5), assume that there exists a static state feedback law

$$u = k(x), \tag{7}$$

where $k:\mathbb{R}^n\to\mathbb{R}^{n_u}$, such that the feedback law asymptotically stabilizes the origin of system (4). Then, according to the converse Lyapunov theorem, we have a Lyapunov function $U_1(x)$. Then, we consider the design of the dynamic output feedback law (5) so that a function $U(x,\hat{x})$ given by

$$U(x, \hat{x}) = U_1(x) + (k(x) - \hat{x})^T \Sigma (k(x) - \hat{x}) \tag{8}$$

becomes the Lyapunov function of the closed-loop system (6) with some positive definite matrix Σ. When we design the output feedback laws, so that the function $U(x, \hat{x})$ of (8) is a Lyapunov function of the closed-loop system, the value of $\hat{x}$ of the designed output feedback laws in (8) will estimate the value of $k(x)$. A design procedure discussed here can be seen in reference [16], and is called as the direct design. As shown in the following, when we obtain the static feedback law (7) in polynomial forms, we can obtain the SDLMI conditions where the stability of the closed-loop system (6) is guaranteed by function (8).

In the following, if the matrix $B(x)$ of (4) has rows all the elements of which are zero, we denote the corresponding row indices as $J = \{j_1, \dots, j_p\}$. We also employ the notation $\tilde{x} = (x_{j_1}, x_{j_2}, \dots, x_{j_p})$.

As discussed above, we design a stabilizing state feedback law as the first step. The state feedback law also can be designed by using SDLMIs. We introduce the following result shown in reference [8].

Theorem 5. ([8]) Suppose that there exist a symmetric polynomial matrix P: $\mathbb{R}^n \to \mathbb{R}^{N \times N}$, a polynomial matrix $K: \mathbb{R}^n \to \mathbb{R}^{n_u \times N}$, a parameter $\epsilon_1 > 0$, and an SOS polynomial $\epsilon_2: \mathbb{R}^n \to \mathbb{R}$, such that

$$\begin{aligned}
&v^T (P(\tilde{x}) - \epsilon_1 I) v, \\
&-v \left\{ \begin{aligned} &P(\tilde{x}) A(x)^T M(x)^T + M(x) A(x) P(\tilde{x}) + K(x)^T B(x)^T M(x)^T \\ &\qquad + M(x) B(x) K(x) - \sum_{j \in J} \frac{\partial P}{\partial x_j}(\tilde{x}) \big(A_j(x) Z(x)\big) + \epsilon_2(x) I \end{aligned} \right\} v
\end{aligned} \tag{9}$$

are SOS polynomials, where $v \in \mathbb{R}^N$, $A_j(x)$ is the j-th row of $A(x)$, and

$$M(x) = \frac{\partial Z}{\partial x}(x). \tag{10}$$

Then, the origin of (4) is asymptotically stabilized by a state feedback given by

$$u = k(x) = K(x) P(\tilde{x})^{-1} Z(x). \tag{11}$$

For the design of the output feedback laws, we show the following theorem as the main result, which gives a design condition of the feedback law (5) in terms of state-dependent matrix inequalities.

Theorem 6. Suppose that there exist a symmetric matrix $P_1 \in \mathbb{R}^{N \times N}$, a polynomial matrix $K: \mathbb{R}^n \to \mathbb{R}^{n_u \times N}$, a parameter $\epsilon_1 > 0$, and an SOS polynomial $\epsilon_2: \mathbb{R}^n \to \mathbb{R}$, such that

$$v^T(P_1 - \epsilon_1 I)v,$$
$$-v^T\{P_1 A(x)^T M(x)^T + M(x)A(x)P_1 + K(x)^T B(x)^T M(x)^T + M(x)B(x)K(x) + \epsilon_2(x)I\}v \tag{12}$$

are SOS polynomials, where $v \in \mathbb{R}^N$ and $M(x)$ is given as 10. Further, suppose that there exist a symmetric matrix $P_2 : \mathbb{R}^{(N+n_{\hat{x}}) \times (N+n_{\hat{x}})}$, and an SOS polynomial $\epsilon_3 : \mathbb{R}^n \times \mathbb{R}^{n_{\hat{x}}} \to \mathbb{R}$, such that

$$-w^T\left(\begin{pmatrix} \Lambda_{11}(x,\hat{x}) & \Lambda_{12}(x,\hat{x}) \\ \Lambda_{12}^T(x,\hat{x}) & \Lambda_{22}(x,\hat{x}) \end{pmatrix} + \epsilon_3(x,\hat{x})I\right)w \tag{13}$$

is an SOS polynomial where $w \in \mathbb{R}^{N+n_{\hat{x}}}$, and

$$\begin{aligned}
\Lambda_{11}(x,\hat{x}) &= \mathrm{He}(P_1^{-1}M(x)\big(A(x) + B(x)D_c(\hat{x},y)C(x)\big) \\
&\quad + P_1^{-1}K(x)^T P_2\left(\left\{\frac{\partial k}{\partial x}(x)\big(A(x) + B(x)D_c(\hat{x},y)C(x)\big) - B_c(\hat{x},y)C(x)\right\}\right), \\
\Lambda_{12}(x,\hat{x}) &= P_1^{-1}M(x)B(x)C_c(\hat{x},y) + P_1^{-1}K(x)^T P_2\left(\frac{\partial k}{\partial x}(x)B(x)C_c(\hat{x},y) - A_c(x)\right) \\
&\quad + \left\{B_c(\hat{x},y)C(x) - \frac{\partial k}{\partial x}(x)\big(A(x) + B(x)D_c(\hat{x},y)C(x)\big)\right\}^T P_2, \\
\Lambda_{22}(x,\hat{x}) &= \mathrm{He}\left(P_2\left(A_c(\hat{x},y) - \frac{\partial k}{\partial x}(x)B(x)C_c(\hat{x},y)\right)\right)
\end{aligned}$$

where the matrices $A_c(\hat{x},y)$, $B_c(\hat{x},y)$, $C_c(\hat{x},y)$, *and* $D_c(\hat{x},y)$, are given in (5). Then, the dynamic output feedback law (5) globally asymptotically stabilizes the origin of the system (4).

Proof. According to Theorem 6, the function

$$U_1(x) = Z(x)^T P_1^{-1} Z(x)$$

is the Lyapunov function of the closed-loop system of (4) with the state feedback law

$$u = k(x) = K(x)P_1^{-1}Z(x).$$

Then, to consider a dynamic output feedback law in the form of (5), we consider a function given by

$$V(x,\hat{x}) = U_1(x) + U_2(x,\hat{x}), \tag{14}$$

where the function $U_2(x,\hat{x})$ is given by

$$U_2(x,\hat{x}) = (k(x)-\hat{x})^T P_2(k(x)-\hat{x}).$$

Then, the time derivative of function (14) along the trajectory of the closed-loop system (6) is given as

$$V(x,\hat{x}) = U_1(x) + U_2(x,\hat{x}),$$

where

$$\begin{aligned} U_1(x) &= Z(x)^T P_1^{-1} Z(x) + Z(x)^T P_1^{-1} Z(x) \\ &= Z(x)^T \mathrm{He}\left(P_1^{-1} M(x)\{A(x) + B(x) D_c(\hat{x},y) C(x)\}\right) Z(x) \\ &+ \hat{x}^T C_c(\hat{x},y)^T B(x)^T M(x)^T P_1^{-1} Z(x) \\ &+ Z(x)^T P_1^{-1} M(x) B(x) C_c(\hat{x},y)\hat{x}, \end{aligned} \tag{15}$$

and

$$\begin{aligned} \dot{U}_2(x,\hat{x}) &= \left(\dot{k}(x)-\dot{\hat{x}}\right)^T P_2(k(x)-\hat{x}) + (k(x)-\hat{x})^T P_2\left(\dot{k}(x)-\dot{\hat{x}}\right) \\ &= Z(x)^T He\left(P_1^{-1}K(x)^T P_2\left\{\frac{\partial k}{\partial x}(x)\big(A(x)+B(x)D_c(\hat{x},y)C(x)\big) - B_c(\hat{x},y)C(x)\right\}\right)Z(x) \\ &+Z(x)^T\left[\left\{-\frac{\partial k}{\partial x}(x)\big(A(x)+B(x)D_c(\hat{x},y)C(x)\big)+B_c(\hat{x},y)C(x)\right\}^T P_2\right. \\ &\left.+P_1^{-1}K(x)^T P_2\left(\frac{\partial k}{\partial x}(x)B(x)C_c(\hat{x},y)-A_c(\hat{x},y)\right)\right]\hat{x}^T \\ &+\hat{x}^T\left[\left\{\frac{\partial k}{\partial x}(x)B(x)C_c(\hat{x},y)-A_c(\hat{x},y)\right\}^T P_2K(x)P_1^{-1}\right. \\ &\left.+P_2\left\{-\frac{\partial k}{\partial x}(x)\big(A(x)+B(x)D_c(\hat{x},y)C(x)\big)+B_c(\hat{x},y)C(x)\right\}\right]Z(x) \\ &+\hat{x}^T \mathrm{He}\left(P_2\left(A_c(\hat{x},y)-\frac{\partial k}{\partial x}(x)B(x)C_c(\hat{x},y)\right)\right)\hat{x}. \end{aligned} \tag{16}$$

Therefore, the time derivative of the function $V(x,\hat{x})$ along the solution of system (6) is given as

$$\begin{aligned} \dot{V}(x,\hat{x}) &= \dot{U}_1(x) + \dot{U}_2(x,\hat{x}) \\ &= Z(x)^T \Lambda_{11}(x,\hat{x}) Z(x) + Z(x)^T \Lambda_{12}(x,\hat{x})\hat{x} + \hat{x}^T \Lambda_{12}(x,\hat{x})^T Z(x) \\ &+ \hat{x}^T \Lambda_{22}(x,\hat{x})\hat{x} \\ &= \begin{bmatrix} Z(x)^T & \hat{x}^T \end{bmatrix} \begin{bmatrix} \Lambda_{11}(x,\hat{x}) & \Lambda_{12}(x,\hat{x}) \\ \Lambda_{12}(x,\hat{x})^T & \Lambda_{22}(x,\hat{x}) \end{bmatrix} \begin{bmatrix} Z(x) \\ \hat{x} \end{bmatrix}. \end{aligned} \tag{17}$$

Then, condition (13) of the theorem and Theorem 4 imply that

$$\begin{bmatrix} \Lambda_{11}(x,\hat{x}) & \Lambda_{12}(x,\hat{x}) \\ \Lambda_{12}(x,\hat{x})^T & \Lambda_{22}(x,\hat{x}) \end{bmatrix} < 0, \quad \forall (x,\hat{x}) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_{\hat{x}}}. \tag{18}$$

From (17) and (18), we can conclude that $\dot{V}(x,\hat{x})$ is negative definite. Therefore, according to Theorem 1, we can conclude that the origin of the closed-loop system is globally asymptotically stable. This completes the proof.

When we design the dynamic output feedback law (5) according to Theorem 6, we first solve the SOS decomposition of condition (12) to find the matrix P_1. Then, if we can obtain the feasible solutions of the matrix P_1 and the function $K(x)$ satisfying condition (12), we try to find the matrix functions $A_c(\hat{x}, y)$, $B_c(\hat{x}, y)$, $C_c(\hat{x}, y)$, $D_c(\hat{x}, y)$, the matrix P_2, and the SOS polynomial $\epsilon_3 > 0$ satisfying condition (13). At this time, because the decision variables do not enter in (13) linearly, we set $P_2 = I$ in general. Then, we can consider the SOS decomposition for (13). If we can find the feasible solution of condition (13), we will obtain the stabilizing feedback laws in the form of (5).

Remark 2. The condition of (12) in Theorem 6 corresponds to the condition of (9) in Theorem 5. Note that the matrix P_1 in Theorem 6 is a constant matrix, although the matrix $P(x)$ in Theorem 5 is the function of x. This is due to the fact that the inverse of the matrix P_1 appears in (16). If the matrix P_1 is the polynomial matrix in Theorem 6, we cannot employ the SOS decomposition. Therefore, we limit ourselves to the case of the constant matrices in Theorem 6.

6. Numerical examples of dynamic output feedback stabilization

6.1. Numerical example 1

This section shows some numerical examples of the dynamic output feedback stabilization by the proposed method shown in Section 5.

We show the first example of the stabilization. Consider a system given by

$$\begin{aligned} \dot{x}_1 &= 0.5\, x_1 - 0.1 x_1^3 + u, \\ \dot{x}_2 &= x_1^2 - x_2, \\ y &= x_1, \end{aligned} \tag{19}$$

where $x = (x_1, x_2)^T$ is the state, $y \in \mathbb{R}$ is the output, and $u \in \mathbb{R}$ is the input. In order to design a dynamic output feedback law for the stabilization of system (19) based on the result presented in the previous section, we choose $Z(x)=(x_1, x_2)^T$. Then, we have the expression of system (19) in the form of (4), where

$$A(x) = \begin{pmatrix} 0.5 - 0.1x_1^2 & 0 \\ x_1 & -1 \end{pmatrix}, \quad B(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C(x) = (1 \quad 0).$$

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We consider the output feedback stabilization of system (19) using the dynamic feedback law (5). We consider a low-dimensional dynamic feedback, and we assume that $n_{\hat{x}} = 1$. According to Theorem 6, by choosing $P_2 = I$, we obtained the matrix P_1 and the function $K(x)$ by solving the SOS decomposition of (12) using YALMIP. We consider the function $K(x)$ with zero degree. The obtained matrix P_1 and the function $K(x)$ are given as

$$P_1 = \begin{bmatrix} 1.2306\times10^{-2} & -9.8824\times10^{-11} \\ -9.8824\times10^{-11} & 5.2061\times10^{-2} \end{bmatrix},$$
$$K(x) = \begin{bmatrix} -6.9660\times10^{-3} & -4.9775\times10^{-6} \end{bmatrix}.$$

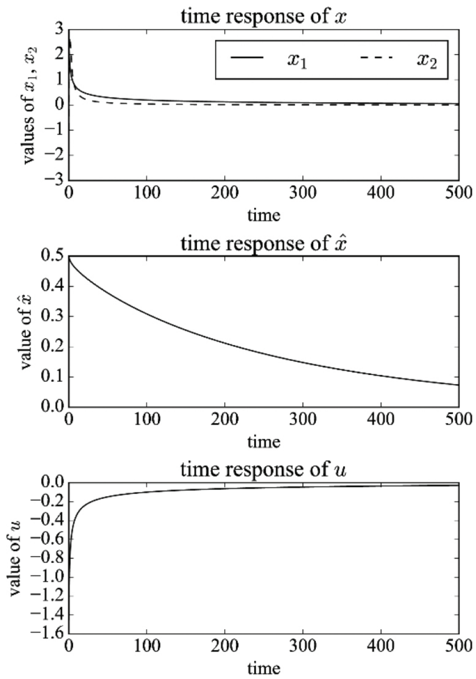

Figure 1. Time responses of x, $\hat{x}$, and u of (19) with dynamic output feedback law (5) with degree zero one.

Then, by using P_1 and $K(x)$, we found the feasible solution $A_c(\hat{x}, y), B_c(\hat{x}, y), C_c(\hat{x}, y), D_c(\hat{x}, y)$, which are two degree, to the SOS decomposition of condition (13). Therefore, we obtain the dynamic output feedback laws that stabilizes system (19), given by

$$\begin{aligned} A_c(\hat{x}, y) &= -0.003412109272 + 1.3668\times 10^{-5}\, y - 0.0034187\, y^2 + 1.4670\times 10^{-5}\, \hat{x} \\ &\quad - 2.3942\times 10^{-5}\, y\,\hat{x} - 0.0034045\, \hat{x}^2, \\ B_c(\hat{x}, y) &= -6.91051768\times 10^{-5} + 1.4074\times 10^{-6}\, y + 1.6941\times 10^{-5}\, y^2 - 5.1525\times 10^{-6}\, \hat{x} \\ &\quad + 7.2076\times 10^{-6}\, y\,\hat{x} - 1.1067\times 10^{-5}\, \hat{x}^2, \\ C_c(\hat{x}, y) &= 9.38813958\times 10^{-6} + 3.7173\times 10^{-5}\, y + 1.9926\times 10^{-4}\, y^2 - 5.5249\times 10^{-6}\hat{x} \\ &\quad + 1.3925\times 10^{-4}\, y\,\hat{x} + 3.5555\times 10^{-6}\, \hat{x}^2, \\ D_c(\hat{x}, y) &= -0.5020898379 - 3.4848\times 10^{-4}\, y + 0.0023489\, y^2 - 2.2606\times 10^{-5}\, \hat{x} \\ &\quad + 2.1822\times 10^{-4}\, y\,\hat{x} - 0.0068477\, \hat{x}^2. \end{aligned} \tag{20}$$

Figure 1 shows the time responses of the state variables $x(t)$, $\hat{x}(t)$ and $u(t)$ of the closed-loop system (19) with the designed dynamic output feedback (20). The initial values are chosen as $x(0) = (3,-1)$,and $\hat{x}(0) = 0.5$. In **Figure 1**, the states $x(t)$ and $\hat{x}(t)$ converge to the origin.

Then, we also obtain a dynamic output feedback control law in the case where the elements of $K(x)$ are degree zero, and the elements of $A_c(\hat{x},y), B_c(\hat{x},y), C_c(\hat{x},y)$, and $D_c(\hat{x},y)$ are degree three with respect to $\hat{x}$ and y. Again, by solving the SOS decomposition following Theorem 6, we obtain the value of the matrix P_1 and the function $K(x)$ as same as above.

We also obtain the values of $A_c(\hat{x},y), B_c(\hat{x},y), C_c(\hat{x},y)$, and $D_c(\hat{x},y)$ as

$$
\begin{aligned}
A_c(\hat{x},y) = {} & -0.04345107574 - 1.9588\times 10^{-7}\, y - 0.044197\, y^2 - 8.8208\times 10^{-8}\,\hat{x} \\
& -4.5391\times 10^{-7}\, y\,\hat{x} - 0.043558\,\hat{x}^2 - 2.1533\times 10^{-5}\, y^3 + 3.9213\times 10^{-9}\, y^2\hat{x} \\
& -1.5638\times 10^{-5}\, y\,\hat{x}^2 - 2.1580\times 10^{-9}\,\hat{x}^3, \\
B_c(\hat{x},y) = {} & -9.486666512\times 10^{-5} + 2.8266\times 10^{-7}\, y + 1.2691\times 10^{-4}\, y^2 - 4.6792\times 10^{-7}\,\hat{x} \\
& -8.6168\times 10^{-7}\, y\,\hat{x} + 2.8204\times 10^{-5}\,\hat{x}^2 - 0.0031033\, y^3 + 5.6498\times 10^{-7}\, y^2\,\hat{x} \\
& -0.0022537\, y\,\hat{x}^2 - 3.1099\times 10^{-7}\hat{x}^3, \\
C_c(\hat{x},y) = {} & 0.0001750991329 - 3.9137\times 10^{-7}\, y + 9.3325\times 10^{-4} y^2 \\
& +1.5899\times 10^{-8}\,\hat{x} + 4.6876\times 10^{-8}\, y\,\hat{x} + 3.3911\times 10^{-4}\,\hat{x}^2 + 3.8040\times 10^{-5}\, y^3 \\
& -6.9264\times 10^{-9}\, y^2\,\hat{x} + 2.7626\times 10^{-5}\, y\,\hat{x}^2 + 3.8121\times 10^{-9}\,\hat{x}^3, \\
D_c(\hat{x},y) = {} & -0.5184067865 - 2.2545\times 10^{-7}\, y - 0.023064\, y^2 - 1.8349\times 10^{-8}\,\hat{x} \\
& +3.8378\times 10^{-9}\, y\,\hat{x} - 0.034449\,\hat{x}^2 + 2.1533\times 10^{-5}\, y^3 - 3.9199\times 10^{-9}\, y^2\,\hat{x} \\
& +1.5638\times 10^{-5}\, y\,\hat{x}^2 + 2.1580\times 10^{-9}\,\hat{x}^3.
\end{aligned}
$$

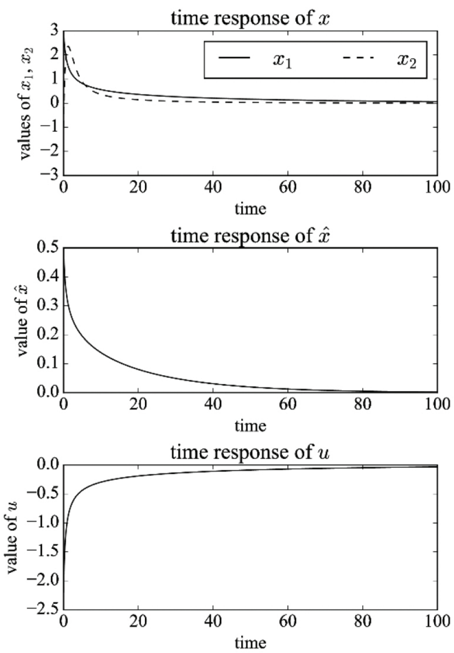

Figure 2. Time responses of x, $\hat{x}$, and u of (19) with dynamic output feedback law (5) with degree zero one.

The obtained feedback control law also stabilizes system (19). **Figure 2** shows the time responses of the state $x(t)$, $\hat{x}(t)$ and the input $u(t)$ of the closed-loop systems with the initial values $x(0) = (3,-1)$, and $\hat{x}(0) = 0.5$. The state converges to the origin, and the value of $u(t)$ also converges to zero.

6.2. Numerical example 2

We consider the following example, which models a circuit with negative-resistance oscillator, taken from reference [17] and modified. Consider a system given by

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1 + x_2 - \tfrac{1}{3}x_2^3 + u, \\ y &= x_2, \end{aligned} \tag{21}$$

where $x = (x_1, x_2)^T$ is the state, $u \in \mathbb{R}$ is the input, and y is the output. To design the dynamic output feedback laws, we express system (21) of form (4) as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -x_1 - \dfrac{1}{3}x_2^3 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u,$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -x_1 - \dfrac{1}{3}x_2^3 \\ x_2 \end{bmatrix}.$$

Following the design procedure in the previous section, we design the dynamic feedback control law with $n_{\hat{x}} = 1$.First, we obtain the constant matrix P_1 and the polynomial matrix $K(x)$ with degree zero. The matrix P_1 and $K(x)$ with zero degree are obtained as

$$\begin{aligned} P_1 &= \begin{bmatrix} 0.0155674 & 0.0012554 \\ 0.0012554 & 0.0155125 \end{bmatrix} \\ K(x) &= \begin{bmatrix} -0.0013102 & -0.0167686 \end{bmatrix} \end{aligned}$$

Then, we solve the SOS decomposition (13) to find the matrices $A_c(\hat{x}, y), B_c(\hat{x}, y), C_c(\hat{x}, y)$, and $D_c(\hat{x}, y)$ with degree one. By choosing $P_2 = I$, the feasible solutions are obtained as

$$\begin{aligned} A_c(\hat{x}, y) &= -0.1675653629 - 9.5745 \times 10^{-9} y + 8.1304 \times 10^{-11} \hat{x}, \\ B_c(\hat{x}, y) &= -0.1582768369 + 5.2303 \times 10^{-8} y - 3.9643 \times 10^{-10} \hat{x}, \\ C_c(\hat{x}, y) &= 6.655333476 \times 10^{-5} + 3.0370 \times 10^{-11} y + 3.9639 \times 10^{-12} \hat{x}, \\ D_c(\hat{x}, y) &= -1.081399695 - 4.5067 \times 10^{-11} y + 4.1856 \times 10^{-12} \hat{x}. \end{aligned}$$

Figure 3 shows the time responses of the states x, $\widehat{x}$ and the input u of the closed-loop systems. The figure shows that the states x and $\widehat{x}$converge to the origin. Also, the figure shows that the input values converge to zero as the states converge to the origin.

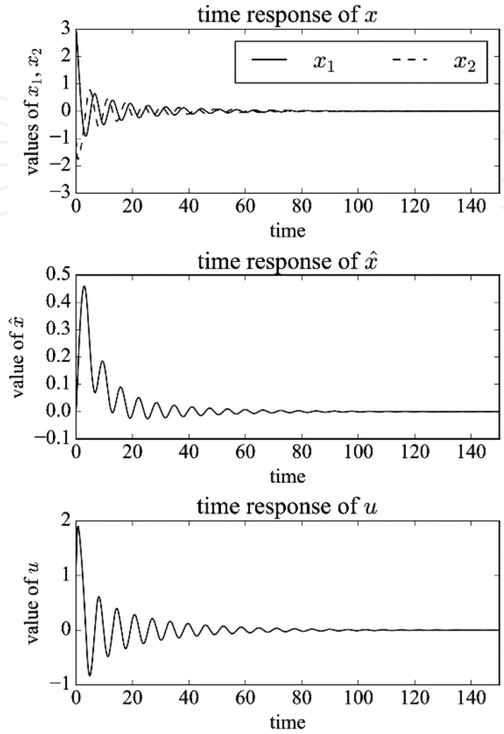

Figure 3. Time responses of x, $\widehat{x}$, and u of (21) with dynamic output feedback law (5) with degree zero one.

7. Conclusion

We considered the design of dynamic output feedback laws via the SOS decomposition. For the design of the feedback laws, we derived the design conditions as the state-dependent matrix inequalities. According to the derived conditions, we can design the stabilizing feedback laws as the feasible solutions to the SDLMIs by using the numerical solvers. Future works include to derive less conservative conditions and to develop design methods of dynamic output feedback laws for advanced control, such as H-infinity control.

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