Chapter 9

Simultaneous H^∞ Control for a Collection of Nonlinear Systems in Strict-Feedback Form

Jenq-Lang Wu , Chee-Fai Yung and Tsu-Tian Lee

Additional information is available at the end of the chapter

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Abstract

Based on the control storage function approach, a constructive method for designing simultaneous H^∞ controllers for a collection of nonlinear control systems in strict-feedback form is developed. It is shown that under mild assumptions, common control storage functions (CSFs) for nonlinear systems in strict-feedback form can be constructed systematically. Based on the obtained common CSFs, an explicit formula for constructing simultaneous H^∞ controllers is presented. Finally, an illustrative example is provided to verify the obtained theoretical results.

Keywords: nonlinear control systems, simultaneous H^∞ control, state feedback, storage functions, strict-feedback form

1. Introduction

The simultaneous H^∞ control problem concerns with designing a single controller which simultaneously renders a set of systems being internally stable and satisfying an L_2-gain specification. In the last decades, there have been some researchers studying the simultaneous H^∞ control problem in linear case, see references [1–6]. In references [1] and [2], necessary and sufficient conditions for the simultaneous H^∞ control via nonlinear digital output feedback controllers were derived by using the dynamic programming approach. In reference [3], a numerical design method was proposed for designing simultaneous H^∞ controllers. In reference [4], it was shown that the simultaneous H^∞ control problem is equivalent to a strong H^∞ control problem. In reference [5], linear periodically time-varying controllers were employed for simultaneous H^∞ control. In reference [6], a simultaneous H^∞ control problem was solved via the chain scattering framework.

All the results mentioned earlier are derived for linear systems case. Till now, only few results have been reported about simultaneous H^∞ control of nonlinear systems, see references [7, 8]. In reference [7], a control storage function (CSF) method was developed for designing simultaneous H^∞ state feedback controllers for a collection of single-input nonlinear systems. Necessary and sufficient conditions for the existence of simultaneous H^∞ controllers were derived. Moreover, an explicit formula for constructing simultaneous H^∞ feedback controllers was proposed. The CSF approach was first introduced in reference [9]. It is motivated by the control Lyapunov function (CLF) method (please see references [10–18]) for designing stabilizing controllers of nonlinear control systems. One difficulty in applying CSFs/CLFs for solving control problems is that how to derive CSFs/CLFs for nonlinear systems is an open problem unless they are in some particular forms. No systematic methods for constructing CSFs have been proposed in reference [7]. It is important to identify those nonlinear systems whose corresponding CSFs/CLFs exist and can be constructed systematically. In reference [8], the CSF approach was applied to design simultaneous H^∞ controllers for a collection of nonlinear control systems in canonical form. It was shown that under mild assumptions, CSFs can be constructed systematically for nonlinear systems in canonical form; and simultaneous H^∞ control for such systems can be easily achieved. In this chapter, we further study the simultaneous H^∞ control problem for nonlinear systems in strict-feedback form. It is known that the strict-feedback form is more general than the canonical form. Moreover, a restrictive assumption made in reference [8] is relaxed in this chapter. Based on the CSF approach and by using the backstepping technique, we develop a systematic method for constructing simultaneous H^∞ state feedback controllers. The proposed results in reference [8] are special cases of the results presented in this chapter.

2. Problem formulation and preliminaries

In this section, the simultaneous H^∞ control problem to be solved will be formulated and some preliminaries will be presented. For simplifying the expressions, we use the same notations x, u, w, and z to denote the states, control inputs, exogenous inputs, and the controlled outputs of all the considered systems.

2.1. Problem formulation

Consider a collection of nonlinear control systems:

$$\begin{aligned} \dot{x} &= f_i(x) + g_{1i}(x)w + g_{2i}(x)u \\ z &= h_{1i}(x) + k_{11i}(x)w, \quad \mathrm{i} = 1, \ldots, q, \end{aligned} \tag{1}$$

where $x = \left[x_1, x, \cdots, x_n\right]^T \in R^n$ is the state, $w \in R^m$ is the disturbance input, $u \in R$ is the control input, $z \in R^r$ is the controlled output, $f_i: R^n \mapsto R^n, g_{1i}: R^n \mapsto R^{n \times m}, g_{2i}: R^n \mapsto R^n, h_{1i}: R^n \mapsto R^r$, and $k_{11i}: R^n \mapsto R^{r\times m}, i = 1, ..., q,$ are smooth functions. Here we denote the i-th system in Eq. (1) as system S_i. For all i=1,…,q, suppose that $f_i(0) = 0$ and $h_{1i}(0) = 0$. For convenience, define $\bar{x}_j = [x_1, x_2, \cdots, x_j]^T \in R^j, j = 1, ..., n.$ Suppose that $f_i(x)$, $g_{1i}(x)$, and $g_{2i}(x)$, i=1,…,q, have the following forms:

$$f_i(x) = \begin{bmatrix} x_2 + \theta_{i1}(x_1) \\ \vdots \\ x_{j+1} + \theta_{ij}(\bar{x}_j) \\ \vdots \\ x_n + \theta_{i(n-1)}(\bar{x}_{n-1}) \\ \theta_{in}(x) \end{bmatrix}, \; g_{1i}(x) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \rho_i(x) \end{bmatrix}, \; g_{2i}(x) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \eta_i(x) \end{bmatrix}. \tag{2}$$

where $\theta_{ij}: R^j \mapsto R, \rho_i: R^n \mapsto R^{1\times m}$, and $\eta_i: R^n \mapsto R, i = 1, 2, ..., q, j = 1, ..., n,$ are smooth functions with $\theta_{ij}(0) = 0$ and $\eta_i(x) \neq 0$ for each $x \in R^n$. Assume that all functions $\eta_i(x), i = 1, ..., q,$ have the same sign. Without loss of generality, suppose that $\eta_i(x) > 0, i = 1, ..., q$. By Eq. (2), the q possible models can be explicitly expressed as

$$\begin{aligned} \dot{x}_1 &= x_2 + \theta_{i1}(x_1) \\ &\vdots \\ \dot{x}_j &= x_{j+1} + \theta_{ij}(\bar{x}_j) \\ &\vdots \\ \dot{x}_{n-1} &= x_n + \theta_{i(n-1)}(\bar{x}_{n-1}) \\ \dot{x}_n &= \theta_{in}(x) + \rho_i(x)w + \eta_i(x)u, \\ z &= h_{1i}(x) + k_{11i}(x)w, \qquad i = 1, 2, .., q. \end{aligned} \tag{3}$$

Suppose that the following assumption holds.

Assumption 1: $\gamma^2 I - k_{11i}^T(x)k_{11i}(x) > 0.\ \forall x \in R^n$ and $\forall i \in \{1, ..., q\}$.

It is clear that we can always find a positive (semi)definite function $U(x)$ such that, for all $i \in \{1,\ldots q\}$,

$$h_{1i}^T(x)h_{1i}(x) + h_{1i}^T(x)k_{11i}(x)\left(\gamma^2 I - k_{11i}^T(x)k_{11i}(x)\right)^{-1} k_{11i}^T(x)h_{1i}(x) \le U(x),\ \forall x \in R^n.$$

The objective of this chapter is to find a continuous function $p: R^n \mapsto R$ such that the state feedback controller

$$u = p(x) \tag{4}$$

internally stabilizes the systems in Eq. (3) simultaneously; and, for each $T > 0$ and for each $w_i \in L_2[0, T]$, all closed-loop systems, starting from the initial state $x(0) = 0$, satisfy (for a given $\gamma > 0$)

$$\int_0^T z^T(t)z(t)dt \le \hat{\gamma}^2 \int_0^T w^T(t)w(t)dt \quad \text{for some } \hat{\gamma} < \gamma. \tag{5}$$

2.2. Control storage functions

Here we review some important concepts about the CSF method introduced in references [7, 9].

Definition 1 [7, 9]: Consider the system S_i in Eq. (1). A smooth, proper, and positive definite function $V_i : R^n \to R$ is a CSF of S_i if, for each $x \in R^n \backslash \{0\}$ and each $w \in R^m$,

$$\left[\inf_{u \in R} \left\{ \frac{\partial V_i(x)}{\partial x} \left(f_i(x) + g_{1i}(x)w + g_{2i}(x)u \right) + \left(h_{1i}(x) + k_{11i}(x)w \right)^T \left(h_{1i}(x) + k_{11i}(x)w \right) - \gamma^2 w^T w \right\} < 0. \right]$$

For ensuring the continuity of the obtained simultaneous H^∞ controllers, the *L_2-gain small control property* (*L_2-gain SCP*) has been introduced in reference [7].

Definition 2 [7]: A CSF $V_i : R^n \to R$ of S_i satisfies the L_2-gain SCP if for each $\varepsilon > 0$, there is a $\delta_1 > 0$ and a $\delta_2 > 0$ such that, if $x \neq 0$ satisfies $\|x\| < \delta_1$ and w satisfies $\|w\| < \delta_2$, there is some u with $|u| < \varepsilon$ satisfying

$$\frac{\partial V_i(x)}{\partial x} \left(f_i(x) + g_{1i}(x)w + g_{2i}(x)u \right) + \left(h_{1i}(x) + k_{11i}(x)w \right)^T \left(h_{1i}(x) + k_{11i}(x)w \right) - \gamma^2 w^T w < 0.$$

3. Main results

For a single system, it has been shown in reference [7] that the existence of CSFs is a necessary and sufficient condition for the existence of H^∞ controllers. Therefore, for the existence of simultaneous H^∞ controllers for the systems in Eq. (3), the existence of CSFs for these systems is necessary. In references [7] and [9], no systematic methods have been proposed for constructing CSFs. Here, based on the backstepping method, we first derive CSFs explicitly for the systems in Eq. (3).

Let

$$s_1(x_1) = x_1,$$

$$\hat{V}_1(x_1) = \frac{1}{2} s_1^2(x_1).$$

It is easy to show that we can find a function $\varphi_1 : R \to R$ and a positive definite function $\mu_1 : R \to R$ such that

$$x_1 \cdot (\varphi_1(x_1) + \theta_{i1}(x_1)) \le -\mu_1(x_1) \text{ , for all } i = 1, \ldots, q.$$

For j=2,…,n, let

$$s_j(\bar{x}_j) = x_j - \varphi_{j-1}(\bar{x}_{j-1}),$$

$$\hat{V}_j(\bar{x}_j) = \hat{V}_{j-1}(\bar{x}_{j-1}) + \frac{1}{2} s_j^2(\bar{x}_j).$$

Similarly, we can find functions $\varphi_j : R^j \to R, j = 2, ..., n-1,$ and positive definite function $\mu_j : R^j \to R, j = 2, ..., n-1,$ such that

$$\sum_{l=1}^{j-1} \frac{\partial \hat{V}_j(\bar{x}_j)}{\partial x_l} \left(x_{l+1} + \theta_{il}(\bar{x}_l) \right) + \frac{\partial \hat{V}_j(\bar{x}_j)}{\partial x_j} \left(\varphi_j(\bar{x}_j) + \theta_{ij}(\bar{x}_j) \right) \le -\sum_{l=1}^{j} \mu_l(\bar{x}_l), \quad \text{for all } i = 1, \ldots, p.$$

Then, it is clear that the function

$$\hat{V}(x) \equiv \frac{1}{2} \sum_{j=1}^{n} s_j^2(\bar{x}_j)$$

is positive definite, and radially unbounded.

Now, we discuss the existence of common CSFs for the systems in Eq. (3). For convenience, we say that a continuous function $\nu(\bar{x}_j)$ is dominated by a continuous function $\upsilon(\bar{x}_j)$ if there exists a constant $c > 0$ such that $\nu(\bar{x}_j) < c\upsilon(\bar{x}_j)$ for all $\bar{x}_j \ne 0$.

***Theorem 1*:** Consider the systems in Eq. (3). Suppose that *Assumption 1* holds. If the functions $\varphi_j : R^j \to R$, j=1,…,n-1, are such that $U(x)\Big|_{s_n(x)=0}$ is dominated by $\sum_{l=1}^{n-1}\mu_l(\bar{x}_l)$, then there exists a common *CSF* that satisfies the L_2-*gain SCP* for all the systems in Eq. (3).

Proof: Let $V(x) = K \cdot \hat{V}(x)$, where $K > 0$ will be specified later. For system S_i, define the corresponding Hamiltonian function as

$$H_i(x,w,u) \equiv \dot{V}(x) + \left(h_{1i}(x) + k_{11i}(x)w\right)^T \left(h_{1i}(x) + k_{11i}(x)w\right) - \gamma^2 w^T w.$$

By the backstepping method, we can show that

$$\begin{aligned} &H_i(x,w,u) \\ &= K\sum_{j=1}^{n} s_j(\bar{x}_j)\cdot \dot{s}_j(\bar{x}_j) + \left(h_{1i}(x) + k_{11i}(x)w\right)^T \left(h_{1i}(x) + k_{11i}(x)w\right) - \gamma^2 w^T w \\ &\le -K\sum_{i=1}^{n-1}\mu_j(\bar{x}_j) \ + Ks_n(x)\left(s_{n-1}(\bar{x}_{n-1}) + \theta_{in}(x) + \rho_i(x)w + \eta_i(x)u - \sum_{l=1}^{n-1}\frac{\partial\phi_{n-1}(\bar{x}_{n-1})}{\partial x_l}\cdot\left(x_{l+1} + \theta_{il}(\bar{x}_l)\right)\right) \\ &+ h_{1i}^T(x)h_{1i}(x) + h_{1i}^T(x)k_{11i}(x)w + w^T k_{11i}^T(x)h_{1i}(x) - w^T\left(\gamma^2 I - k_{11i}^T(x)k_{11i}(x)\right)w. \end{aligned}$$

After some manipulations, we have

$$\begin{aligned} H_i(x,w,u) &\le a_i(x) + b_i(x)\cdot u - (w - w_{i*}(x))^T\left(\gamma^2 I - k_{11i}^T(x)k_{11i}(x)\right)(w - w_{i*}(x)) \\ &\le a_i(x) + b_i(x)\cdot u, \end{aligned} \qquad (6)$$

where

$$\begin{aligned} a_i(x) &= -K\sum_{j=1}^{n-1}\mu_j(\bar{x}_j) + Ks_n(x)\left(s_{n-1}(\bar{x}_{n-1}) + \theta_{in}(x) - \sum_{l=1}^{n-1}\frac{\partial\varphi_{n-1}(\bar{x}_{n-1})}{\partial x_l}\cdot\left(x_{l+1} + \theta_{il}(\bar{x}_l)\right)\right) + h_{1i}^T(x)h_{1i}(x) \\ &\quad + \left(\frac{K}{2}s_n(x)\rho_i^T(x) + k_{11i}^T(x)h_{1i}(x)\right)^T\left(\gamma^2 I - k_{11i}^T(x)k_{11i}(x)\right)^{-1}\left(\frac{K}{2}s_n(x)\rho_i^T(x) + k_{11i}^T(x)h_{1i}(x)\right) \\ b_i(x) &= K\eta_i(x)s_n(x) \\ w_{i*}(x) &= \left(\gamma^2 I - k_{11i}^T(x)k_{11i}(x)\right)^{-1}\left(\frac{K}{2}s_n(x)\rho_i^T(x) + k_{11i}^T(x)h_{1i}(x)\right) \end{aligned}$$

Therefore, $V(x) = K \cdot \hat{V}(x)$ is a CSF of S_i if

$$\forall x \neq 0 \text{ such that } b_i(x) = 0 \Rightarrow a_i(x) < 0.$$

As $U(x)\big|_{s_n(x)=0}$ is dominated by $\sum_{l=1}^{n-1} \mu_l(\bar{x}_l)$, we can choose a $K > 0$ such that

$$U(x)\big|_{s_n(x)=0 \text{ and } x\neq 0} - K\sum_{j=1}^{n-1} \mu_j(\bar{x}_j) < 0.$$

Notice that $b_i(x) = 0$ if and only if $s_n(x) = 0$. Therefore,

$$a_i(x)\,|_{b_i(x)=0 \text{ and } x\neq 0} \leq -K\sum_{j=1}^{n-1}\mu_j(\bar{x}_j) + U(x)\,|_{s_n(x)=0 \text{ and } x\neq 0} < 0. \tag{7}$$

This shows that $V(x)$ is a CSF for the i-th system in Eq. (3). Since Eq. (7) holds for all $i \in \{1,\ldots,q\}$, $V(x)$ is a common CSF for all the systems in Eq. (3).

Now we prove that $V(x)$ satisfies the L_2-gain SCP. Note that if we can find a continuous stabilizing feedback law $d_i(x)$ with $d_i(0) = 0$ such that $H_i(x, w, d_i(x)) < 0$ for each $x \in R^n\backslash\{0\}$ and each $w \in R^m$, then $V(x)$ satisfies the L_2-gain SCP. Let

$$\begin{aligned} d_i(x) = -\frac{1}{\eta_i(x)}\Bigg(& s_{n-1}(\bar{x}_{n-1}) + \theta_{in}(x) - \sum_{l=1}^{n-1}\frac{\partial \varphi_{n-1}(\bar{x}_{n-1})}{\partial x_l}\cdot\left(x_{l+1} + \theta_{il}(\bar{x}_l)\right) \\ & + \rho_i(x)\left(\gamma^2 I - k_{11i}^T(x)k_{11i}(x)\right)^{-1}\left(\frac{K}{4}s_n(x)\rho_i^T(x) + k_{11i}^T(x)h_{1i}(x)\right)\Bigg) - \hat{\mu}_n(x) \end{aligned}$$

where the continuous function $\hat{\mu}_n(x)$ with $\hat{\mu}_n(0) = 0$ is such that $s_n(x)\hat{\mu}_n(x) > 0$ if $s_n(x) \neq 0$, and

$$-K\sum_{j=1}^{n-1}\mu_j(\bar{x}_j) - K\eta_i(x)s_n(x)\hat{\mu}_n(x) + U(x) < 0 \quad \forall x \neq 0.$$

Note that such $\hat{\mu}_n(x)$ always exists since $U(x)\big|_{s_n(x)=0}$ is dominated by $\sum_{l=1}^{n-1}\mu_l(\bar{x}_l)$. Clearly, $d_i(x)$ is continuous in R^n and $d_i(0) = 0$. By Eq. (6), we have

$$\begin{aligned}H_i(x,w,d_i(x)) &\le a_i(x)+b_i(x)d_i(x)\\ &= -K\sum_{j=1}^{n-1}\mu_j(\bar{x}_j)-K\eta_i(x)s_n(x)\hat{\mu}_n(x)\\ &\quad + h_{1i}^T(x)h_{1i}(x)+h_{1i}^T(x)k_{11i}(x)\left(\gamma_i^2 I-k_{11i}^T(x)k_{11i}(x)\right)^{-1}k_{11i}^T(x)h_{1i}(x)\\ &\le -K\sum_{j=1}^{n-1}\mu_j(\bar{x}_j)-K\eta_i(x)s_n(x)\hat{\mu}_n(x)+U(x)<0,\ \forall x\neq 0, \forall w.\end{aligned}$$

This implies that $V(x)$ satisfies the L_2-gain SCP and completes the proof.

To derive simultaneous H^∞ controllers, define (for i=1,…,q)

$$p_i(x)\equiv\begin{cases}-\dfrac{a_i(x)+\sqrt{a_i^2(x)+\beta_i b_i^4(x)}}{b_i(x)}, & \text{if } s_n(x)\neq 0\\ 0, & \text{if } s_n(x)=0\end{cases}$$

where $\beta_i > 0$, i=1,…,q, are given constants. Since $V(x)$ satisfies the L_2-gain SCP, the functions $p_i(x)$, i=1,…,q, are continuous in R^n [16]. We have the following results.

Theorem 2: Consider the collection of systems in Eq. (3). Suppose that *Assumption 1* holds. If the functions $\varphi_j: R^j \to R, j = 1, ..., n-1$, are such that $U(x)\big|_{s_n(x)=0}$ is dominated by $\sum_{l=1}^{n-1}\mu_l(\bar{x}_l)$, then a continuous function $p: R^n \to R$ exists such that the feedback law defined in Eq. (4) internally stabilizes the collection of systems in Eq. (3) simultaneously; and moreover, all the closed-loop systems satisfy the L_2-gain requirement specified in Eq. (5). In this case,

$$u=p(x)\equiv\begin{cases}\min\limits_{i\in\{1,2,\ldots,q\}}\{p_i(x)\}, & \text{if } s_n(x)>0\\ 0, & \text{if } s_n(x)=0\\ \max\limits_{i\in\{1,2,\ldots,q\}}\{p_i(x)\}, & \text{if } s_n(x)<0\end{cases} \tag{8}$$

is a simultaneous H^∞ controller for all the systems in Eq. (3).

Proof: Since the functions $p_i(x)$, i=1,2,…,q, are continuous in R^n, from the definition of $p(x)$, its continuity is obvious. In the following, we first prove the achievement of L_2-gain requirement [Eq. (5)], and then the internal stability of all the closed-loop systems.

A. L_2-gain requirement

Since $H_i(x, w, u) \le a_i(x) + b_i(x)u$, if we can show that

$$a_i(x)+b_i(x)p(x)<0 \quad \forall x\neq 0, i=1,\ldots,q, \tag{9}$$

Then, with the controller defined in Eq. (8), all the closed-loop systems satisfy the L_2-gain requirement specified in Eq. (5).

1. $s_n(x) = 0$ and $x \neq 0$.

In this case, $u = p(x) = 0$ and $b_i(x) = 0$. Then, by Eq. (7),

$$a_i(x)+b_i(x)\cdot p(x)=a_i(x)<0, \quad i=1,\ldots,q.$$

2. $s_n(x) > 0$.

In this case, since $b_i(x) > 0$, we have

$$\begin{aligned}a_i(x)+b_i(x)p(x) &= a_i(x)+b_i(x)\cdot \min_{j\in\{1,2,\ldots,q\}}\{p_j(x)\}\\ &\leq a_i(x)+b_i(x)\cdot p_i(x)\\ &= -\sqrt{a_i^2(x)+\beta_i b_i^4(x)}<0, \quad i=1,\ldots,q.\end{aligned}$$

3. $s_n(x) < 0$

Similarly, in this case we can show that

$$a_i(x)+b_i(x)p(x)<0, \quad i=1,\ldots,q.$$

These discussions imply that Eq. (9) holds. That is, all the possible closed-loop systems satisfy the L_2-gain requirement specified in Eq. (5).

B. Internal stability

To prove internal stability, notice that Eq. (6) implies that, along the trajectories of system S_i under $w = 0$,

$$\begin{aligned}H_i(x,0,p(x)) &= \frac{\partial V(x)}{\partial x}\big(f_i(x)+g_{2i}(x)p(x)\big)+h_{1i}(x)^T h_{1i}(x)\\ &\leq a_i(x)+b_i(x)p(x)<0 \ \ \forall x\neq 0.\end{aligned}$$

That is, for each $i \in \{1,\ldots,q\}$, along the trajectories of system S_i, we have

$$\dot{V}(x) = \frac{\partial V(x)}{\partial x}\left(f_i(x) + g_{2i}(x)p(x)\right) < 0 \quad \forall x \neq 0.$$

This shows that all the closed-loop systems are internally stable.

***Remark 1*:** The systems considered in reference [8] are special cases of the systems considered in this chapter. If we let $\theta_{ij}(\bar{x}_j) = 0, i = 1, 2, ..., q$, and j=1,2,…,n-1, the systems in Eq. (3) will reduce to the systems considered in reference [8]. On the other hand, in reference [17], it is assumed that $U(s)$ is in quadratic form. In this chapter, we relax this restrictive assumption.

***Remark 2*:** In this chapter, we consider the case that the controlled output z is independent of the control input u. In this situation, a much simpler formula (not a special case of the formula in reference [7]) is proposed for constructing simultaneous H^∞ controllers. In the case that the controlled output z depends on u, necessary and sufficient conditions for the existence of simultaneous H^∞ controllers and a formula for constructing simultaneous H^∞ controllers can be derived by the results in reference [7].

4. An illustrative example

Consider the following nonlinear systems:

$$S_i : \begin{cases} \dot{x}_1 = x_2 + \theta_{i1}(x) \\ \dot{x}_2 = \theta_{i2}(x) + \rho_i(x)w + \eta_i(x)u \\ z = h_{1i}(x) + k_{11i}(x)w, \qquad\qquad i = 1, 2, \text{and } 3 \end{cases} \tag{10}$$

where

$$\begin{aligned} \theta_{11}(x) &= x_1, \theta_{21}(x) = x_1\sin(x_1),\ \theta_{31}(x) = -x_1\cos(x_1), \\ \theta_{12}(x) &= x_1^2 + x_2^3, \theta_{22}(x) = x_1(1-2x_2),\ \theta_{32}(x) = x_1\cos(x_2) + 2x_2\sin(5x_1), \\ \rho_1(x) &= -1 + x_1,\ \rho_2(x) = x_1x_2,\ \rho_3(x) = x_1 - x_2^2, \\ \eta_1(x) &= 1 + (x_1 + x_2)^2,\ \eta_2(x) = 2 - \cos(x_1),\ \eta_3(x) = 2 + x_2^2, \\ h_{11}(x) &= x_1\cos(x_2^2),\ h_{12}(x) = -x_1\sin(x_1),\ h_{13}(x) = x_2, \\ k_{111}(x) &= -1 + \cos(x_1),\ k_{112}(x) = 1,\ k_{113}(x) = 1 + \sin(5x_2). \end{aligned}$$

It can be shown that

$$h_{1i}^T(x)h_{1i}(x)+h_{1i}^T(x)k_{11i}(x)\left(\gamma^2-k_{11i}^T(x)k_{11i}(x)\right)^{-1}k_{11i}^T(x)h_{1i}(x)\le U(x),\quad i=1,2,\text{ and }3$$

with

$$U(x)=\frac{9}{5}x_1^2+\frac{9}{5}x_2^2.$$

Let $\gamma = 3$. It can be verified that *Assumption 1* holds. Let

$$\begin{aligned}
s_1(\bar{x}_1) &= x_1\\
\varphi_1(x_1) &= -2x_1\\
\mu_1(x_1) &= x_1^2\\
s_2(\bar{x}_2) &= x_2-\varphi_1(\bar{x}_1)=2x_1+x_2.
\end{aligned}$$

Then,

$$\hat{V}(x)=\frac{1}{2}\left(s_1^2(x_1)+s_2^2(\bar{x}_2)\right)$$

is positive, definite, and radially unbounded. By choosing $K = 10$, it can be shown that

$$-K\mu_1(x_1)+U(x)\Big|_{s_2(\bar{x}_2)=0}<0,\quad \forall x_1\ne 0.$$

Therefore,

$$V(x)=K\hat{V}(x)=5\left(s_1^2(x_1)+s_2^2(\bar{x}_2)\right)$$

is a common CSF for the three systems in Eq. (10). For $i = 1$, 2, and 3, define

$$\begin{aligned}
a_i(x) =& -K\mu_1(x)+Ks_2(x)\left(s_1(x_1)+\theta_{i2}(x)-\frac{\partial\varphi_1(x_1)}{\partial x_1}(x_2+\theta_{i1}(x))\right)+h_{1i}^T(x)h_{1i}(x)\\
&+\left(\frac{K}{2}s_n(x)\rho_i^T(x)+k_{11i}^T(x)h_{1i}(x)\right)^T\left(\gamma^2 I-k_{11i}^T(x)k_{11i}(x)\right)^{-1}\left(\frac{K}{2}s_n(x)\rho_i^T(x)+k_{11i}^T(x)h_{1i}(x)\right)\\
b_i(x) =& K\eta_i(x)s_2(x)
\end{aligned}$$

and (with $\beta_1 = \beta_2 = \beta_3 = 0.1$)

$$p_i(x) \equiv \begin{cases} -\dfrac{a_i(x) + \sqrt{a_i^2(x) + \beta_i b_i^4(x)}}{b_i(x)}, & \text{if } s_2(x) \neq 0 \\ 0, & \text{if } s_2(x) = 0. \end{cases}$$

From *Theorem 2*, the following controller

$$u = p(x) \equiv \begin{cases} \min\{p_1(x), p_2(x), p_3(x)\}, & \text{if } 2x_1 + x_2 > 0 \\ 0, & \text{if } 2x_1 + x_2 = 0 \\ \max\{p_1(x), p_2(x), p_3(x)\}, & \text{if } 2x_1 + x_2 < 0 \end{cases} \tag{11}$$

is a simultaneous H^∞ controller for the three systems in Eq. (10). With arbitrarily chosen disturbance inputs, **Figures 1–3** show the states, control inputs, disturbance inputs, and controlled outputs of these three systems starting at different initial states with the same controller defined in Eq. (11). It can be seen that all the three closed-loop systems are internally stable and satisfy the required L_2-gain specification. That is, the controller defined in Eq. (11) is indeed a simultaneous H^∞ controller for the three systems in Eq. (10).

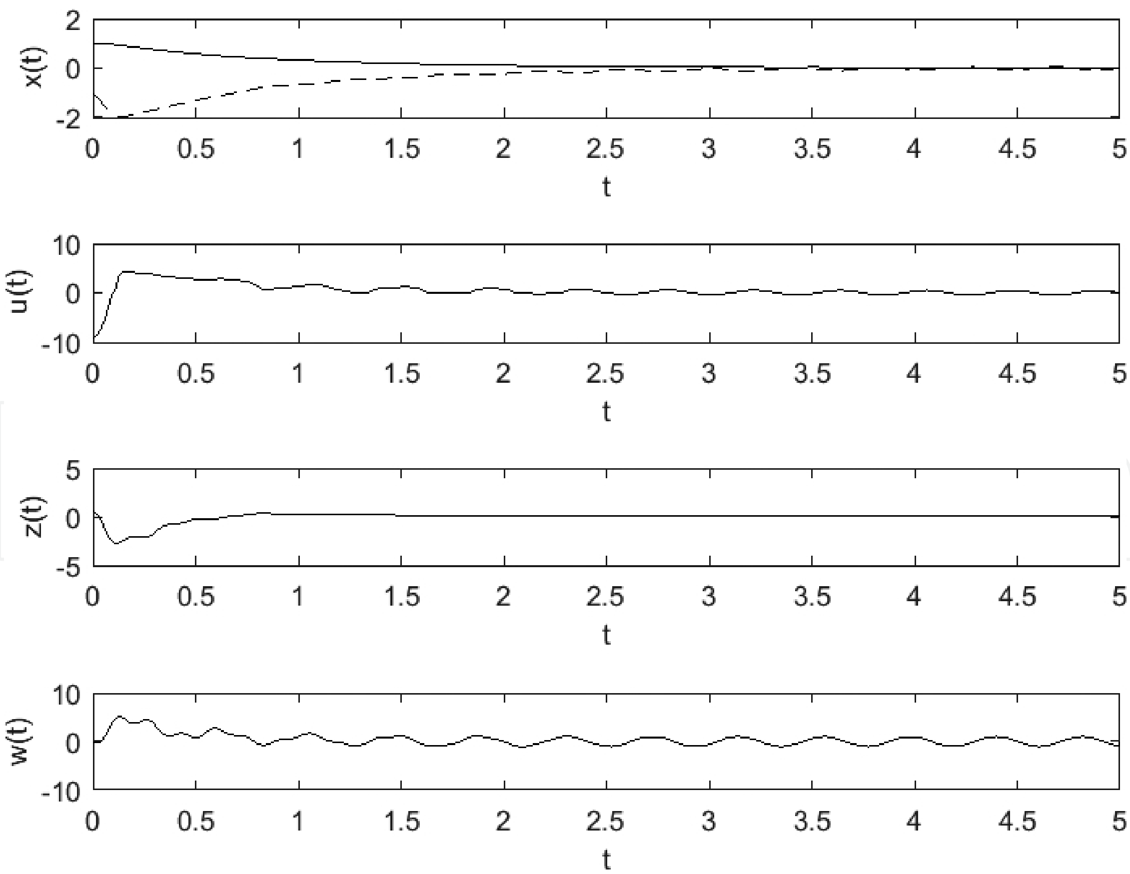

Figure 1. Responses of the system S_1 controlled by the controller defined in Eq. (11).

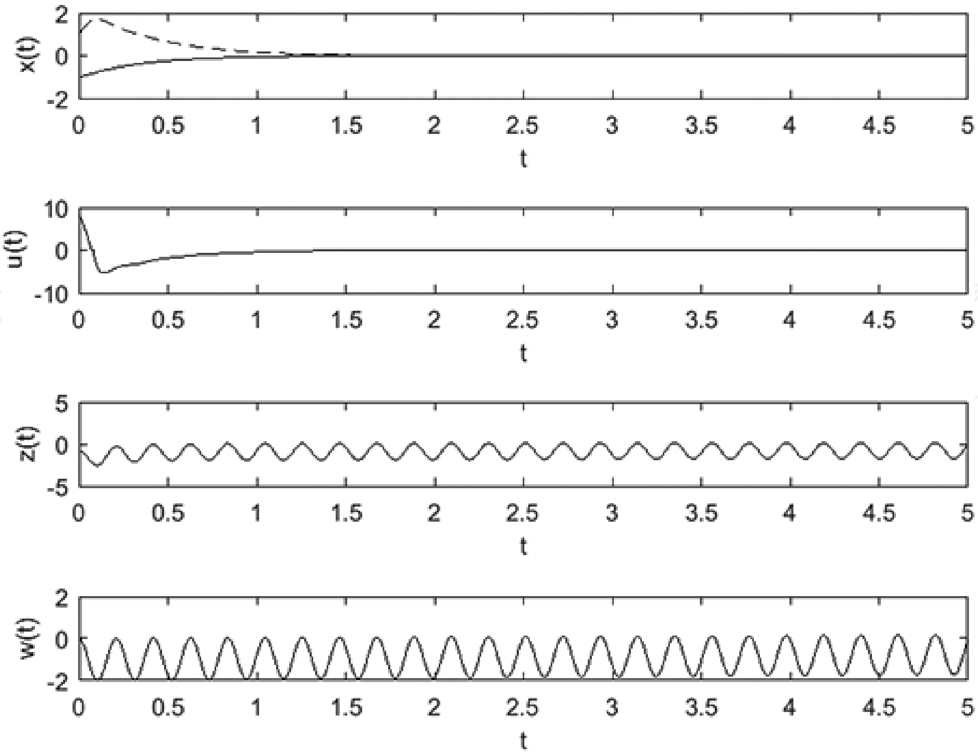

Figure 2. Responses of the system S_2 controlled by the controller defined in Eq. (11).

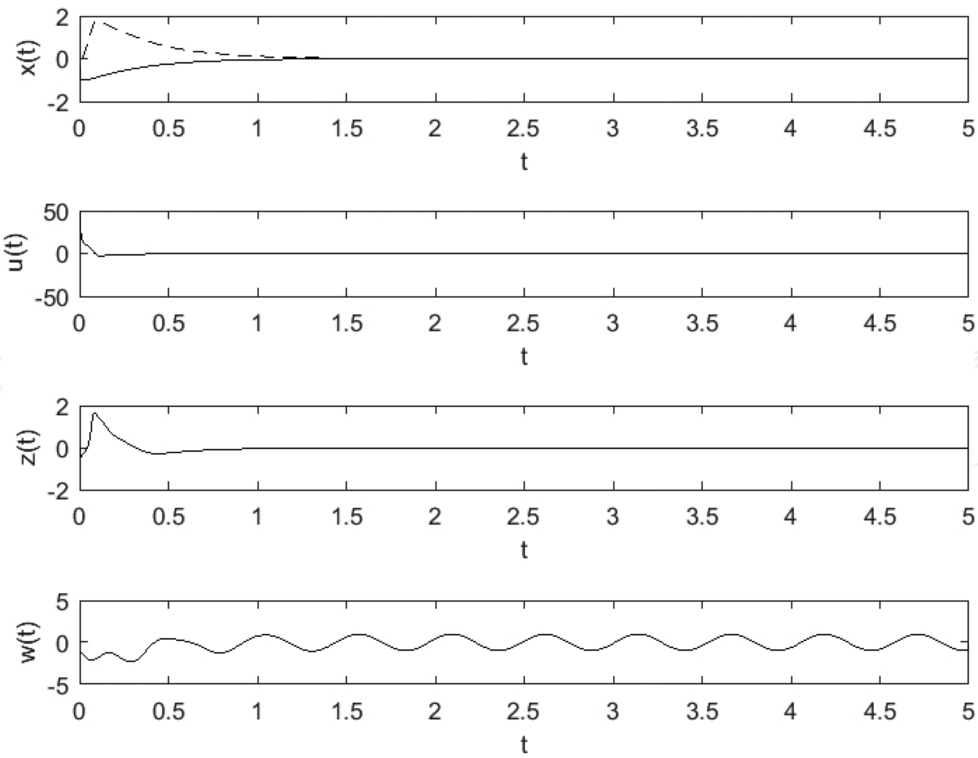

Figure 3. Responses of the system S_3 controlled by the controller defined in Eq. (11).

5. Conclusions

In this chapter, a systematic way for constructing simultaneous H^∞ state feedback controllers of nonlinear control systems in strict-feedback form is proposed. It is shown that the existence of common CSFs guarantees the existence of simultaneous H^∞ controllers. An explicit formula for constructing simultaneous H^∞ controllers is derived. The simulation example is given for verifying the theoretical results. The simulation results show, as expected, that the designed controller can simultaneously stabilize the considered systems and such that all closed-loop systems satisfy the specified disturbance attenuation requirement. Possible further works include considering nonlinear control systems in more general forms, applying the approach to time-varying case, and considering the output feedback case.

Author details

Jenq-Lang Wu[1*], Chee-Fai Yung[1] and Tsu-Tian Lee[2]

*Address all correspondence to: wujl@mail.ntou.edu.tw

1 Department of Electrical Engineering, National Taiwan Ocean University, Keelung, Taiwan, Republic of China

2 Department of Electrical Engineering, Tamkang University, New Taipei City, Taiwan, Republic of China

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