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# **Solving Nonlinear Parabolic Partial Differential Equations Using Multidomain Bivariate Spectral Collocation Method**

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Additional information is available at the end of the chapter

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## **Abstract**

In this study we introduce the multidomain bivariate spectral collocation method for solving nonlinear parabolic partial differential equations (PDEs) that are defined over large time intervals. The main idea is to reduce the size of the computational domain at each subinterval to ensure that very accurate results are obtained within shorter computational time when the spectral collocation method is applied. The proposed method is based on applying the quasi-linearization technique to simplify the nonlinear partial differential equation (PDE) first. The time domain is decomposed into smaller nonoverlapping subintervals. Discretization is then performed on both time and space variables using spectral collocation. The approximate solution of the PDE is obtained by solving the resulting linear matrix system at each subinterval independently. When the solution in the first subinterval has been computed, the continuity condition is used to obtain the initial guess in subsequent subintervals. The solutions at different subintervals are matched together along a common boundary. The examples chosen for numerical experimentation include the Burger's-Fisher equation, the Fitzhugh-Nagumo equation and the Burger's-Huxley equation. To demonstrate the accuracy and the effectiveness of the proposed method, the computational time and the error analysis of the chosen illustrative examples are presented in the tables.

**Keywords:** bivariate interpolation, spectral collocation, quasi-linearisation, multi-domain approach, non-linear evolution PDEs

## 1. Introduction

Most practical problems which model systems in nature lead to nonlinear partial differential equations (PDEs). This is evident in the fields of chemistry, physics, biology, mathematics and engineering. Many assumptions have been made to make some nonlinear PDEs solvable. It has been reported that a vast number of nonlinear PDEs that are encountered in these fields are difficult to solve analytically [1]. The investigation of solutions of such nonlinear PDEs has then been of key interest to many researchers due to their potential applications and more effort has been devoted to search for better and more efficient solution methods for these nonlinear models [2, 3].

The nonlinear PDEs that are solved in this study include the generalized Burger's-Fisher equation, the generalized Burger's-Huxley equation and the Fitzhugh-Nagumo equation. The generalized Burger's-Fisher equation appears in many applications such as shock wave formation, fluid mechanics, turbulence, traffic flows, gas dynamics, heat conduction and sound waves via viscous medium among other fields of applied science [4–6]. The generalized Burger's-Huxley equation models the interaction between reaction mechanisms, diffusion transports and convection effects [7–11]. The Fitzhugh-Nagumo equation arises in genetics, biology, and heat and mass transfer [12, 13].

A number of methods have been applied to solve the nonlinear PDEs such as spectral collocation method [7, 8], Adomian decomposition method [9], homotopy perturbation method [14] and the variational iteration method [4]. The spectral methods have been reported to be strikingly successful if the problem has a smooth solution and falls into various categories, namely Galerkin, Tau and collocation-based methods [15], and therefore, recent advances in the development of numerical methods for solving nonlinear PDEs has focused spectral-based approaches as they require a few grid points to give very accurate results and take less computation time. The spectral collocation-based methods are used often, chiefly because they offer the simplest treatment of boundary conditions. A newly developed spectral collocation method for solving nonlinear PDEs is the bivariate spectral quasi-linearization method (QLM) [16]. This method approximates the solution of the PDE using a bivariate Lagrange interpolation polynomial [17]. It applies quasi-linearization method of Bellman and Kalaba [18] to simplify the nonlinear PDE which is then discretized using spectral collocation on both time and space variables. The method has successfully been used to solve problems defined over shorter time intervals [16]. However, it has been observed that when this method is applied to solve problems defined over large-time intervals, there is no guarantee that the resulting approximate solution will be accurate [16].

In this study, we describe the multidomain bivariate spectral collocation method (MDBSCM) to solutions of nonlinear parabolic PDEs defined over large-time intervals. The MDBSCM is based on decomposing the given domain of approximation in the time variable into smaller subintervals and then solving the PDE independently in each subinterval using the bivariate spectral collocation method. The multidomain approach has been applied to solve nonlinear ordinary differential equations that model chaotic systems described as 1st order systems of equations [19–21]. In this study the same idea is extended to solutions of nonlinear parabolic

PDEs. In the description of the method, the algorithm is kept as simple as possible, while retaining the heart of generality to cover many applications. The extent of the discussion of multidomain approach in this study is limited to nonoverlapping subintervals only.

## 2. Method of solution

In this section, we describe the algorithm to describe how the multidomain bivariate spectral collocation method can be applied to solve nonlinear parabolic PDEs. We shall consider a general second-order nonlinear PDE,

$$\frac{\partial u}{\partial t} = F\left(u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}\right), \quad x \in (a, b), t \in [0, T], \quad (1)$$

subject to boundary conditions

$$u(a, t) = g_a(t), \quad u(b, t) = g_b(t), \quad (2)$$

and initial condition

$$u(x, 0) = f(x), \quad (3)$$

where  $u(x, t)$  is the required solution,  $f(x)$ ,  $g_a(t)$  and  $g_b(t)$  are known functions and  $F$  is a nonlinear operator operating on  $u$  and its first and second spatial derivatives.

### 2.1. The quasi-linearization method

The quasi-linearization method (QLM) of Bellman and Kalaba [18] is a technique that is used to simplify nonlinear ordinary and partial differential equations. The technique has been adopted and generalized in further studies presented in [22, 23]. The QLM is based on the Newton-Raphson method and is constructed from the linear terms of Taylor series expansion about an initial approximation to solution. The QLM assumes that the difference between solutions at two successive iterations denoted by  $u_s$  and  $u_{s+1}$  is very small. Applying the QLM on Eq. (1) yields

$$\begin{aligned} F(u, u', u'') \approx & F(u_s, u'_s, u''_s) + \frac{\partial F}{\partial u}(u_s, u'_s, u''_s)(u - u_s) + \frac{\partial F}{\partial u'}(u_s, u'_s, u''_s)(u' - u'_s) \\ & + \frac{\partial F}{\partial u''}(u_s, u'_s, u''_s)(u'' - u''_s), \end{aligned} \quad (4)$$

where prime denotes differentiation with respect to  $x$  and  $s$  denotes the iteration level. Eq. (4) can be written in compact form as

$$F(u, u', u'') \approx F(u_s, u'_s, u''_s) + \sum_{\gamma=0}^2 \frac{\partial F}{\partial u^{(\gamma)}}(u_s, u'_s, u''_s) (u^{(\gamma)} - u_s^{(\gamma)}), \quad (5)$$

where  $u^{(0)} = u$ . Using the expanded form of Eq. (5) in Eq. (1), we obtain the QLM scheme for approximating the solution  $u_{s+1}(x, t)$  at the  $(s+1)$ th iteration level as

$$\alpha_{2,s}(x, t)u''_{s+1} + \alpha_{1,s}(x, t)u'_{s+1} + \alpha_{0,s}(x, t)u_{s+1} - \dot{u}_{s+1} = R_s(x, t), \quad (6)$$

where

$$\begin{aligned} \alpha_{\gamma,s}(x, t) &= \frac{\partial F}{\partial u^{(\gamma)}}(u_s, u'_s, u''_s), \quad \gamma = 0, 1, 2, \\ R_s(x, t) &= \sum_{\gamma=0}^2 \alpha_{\gamma,s}(x, t)u_s^{(\gamma)} - F(u_s, u'_s, u''_s). \end{aligned} \quad (7)$$

The dot here denotes differentiation with respect to the time  $t$ . Starting with an initial approximation  $u_0$ , the QLM scheme is solved iteratively until a solution with desired accuracy requirements is obtained. The multidomain approach is implemented on the linearized scheme (6) as illustrated below. For the purpose of this study, we shall apply the multidomain approach on the time ( $t$ ) variable only.

Let  $t \in \Gamma$  where  $\Gamma \in [0, T]$ . The domain  $\Gamma$  is decomposed into  $p$  nonoverlapping intervals as

$$\Gamma_k = [t_{k-1}, t_k], \quad t_{k-1} < t_k, \quad t_0 = 0, \quad t_p = T, \quad k = 1, 2, \dots, p. \quad (8)$$

The domain  $t \in [t_{k-1}, t_k]$  in each of the  $k$ th subdomain is first transformed to  $\tau \in [-1, 1]$  using the linear transformation

$$t = \frac{1}{2}(t_k - t_{k-1})\tau + \frac{1}{2}(t_k + t_{k-1}), \quad (9)$$

before the spectral collocation is applied. Similarly, the spatial domain  $x \in [a, b]$  is transformed to  $\eta \in [-1, 1]$  using the linear transformation

$$x = \frac{1}{2}(b - a)\eta + \frac{1}{2}(b + a). \quad (10)$$

The collocation nodes are the symmetrically distributed Gauss-Lobatto grid points defined on the interval  $[-1, 1]$  by,

$$\{\tau_j\}_{j=0}^M = \cos\left(\frac{j\pi}{M}\right), \quad \{\eta_i\}_{i=0}^N = \cos\left(\frac{i\pi}{N}\right). \quad (11)$$

To distinguish between the solutions at different subdomains we shall use,  $u^{(k)}$ ,  $k = 1, 2, \dots, p$ , to denote solution at the  $k$ th subinterval. The PDE is solved independently in each subinterval. In the first subinterval we must solve,

$$\alpha_{2,s}(x,t) \frac{\partial^2 u_{s+1}^{(1)}}{\partial x^2} + \alpha_{1,s}(x,t) \frac{\partial u_{s+1}^{(1)}}{\partial x} + \alpha_{0,s}(x,t) u_{s+1}^{(1)} - \frac{\partial u_{s+1}^{(1)}}{\partial t} = R_s^{(k)}(x,t), \quad x \in [a,b], t \in [0, t_1], \quad (12)$$

subject to boundary and initial conditions

$$u^{(1)}(a,t) = g_a(t), \quad u^{(1)}(b,t) = g_b(t), \quad u^{(1)}(x,0) = f(x). \quad (13)$$

After the solution in the first interval  $\Gamma_1$  has been computed, the solutions at the subsequent  $k$ th subinterval are computed by using the solution at the right hand boundary of the  $(k-1)$ th interval as an initial solution. Thus in the next subintervals,  $k = 2, 3, \dots, p$ , we must solve

$$\alpha_{2,s}(x,t) \frac{\partial^2 u_{s+1}^{(k)}}{\partial x^2} + \alpha_{1,s}(x,t) \frac{\partial u_{s+1}^{(k)}}{\partial x} + \alpha_{0,s}(x,t) u_{s+1}^{(k)} - \frac{\partial u_{s+1}^{(k)}}{\partial t} = R_s^{(k)}(x,t), \quad (14)$$

$$x \in [a,b], \quad t \in [t_{k-1}, t_k],$$

subject to boundary and initial conditions

$$u^{(k)}(a,t) = g_a(t), \quad u^{(k)}(b,t) = g_b(t), \quad u^{(k)}(x, t_{k-1}) = u^{(k-1)}(x, t_{k-1}). \quad (15)$$

In the solution process, the approximate solution that is searched for takes a form of a bivariate Lagrange interpolation polynomial. The solution at each subinterval is approximated as

$$u^{(k)}(x,t) \approx U^{(k)}(\eta, \tau) = \sum_{p=0}^N \sum_{q=0}^M U^{(k)}(\eta_p, \tau_q) L_p(\eta) L_q(\tau). \quad (16)$$

The first and second spatial derivatives are evaluated at the collocation nodes  $(\eta_i, \tau_j)$  for  $j = 0, 1, 2, \dots, M$  as follows

$$\frac{\partial u^{(k)}}{\partial x}(\eta_i, \tau_j) = \mathbf{D} \mathbf{U}_j^{(k)} = \left( \frac{2}{b-a} \right) \hat{\mathbf{D}} \mathbf{U}_j^{(k)}, \quad \frac{\partial^2 u^{(k)}}{\partial x^2}(\eta_i, \tau_j) = \mathbf{D}^2 \mathbf{U}_j^{(k)} \quad (17)$$

where  $\hat{\mathbf{D}} = \left( \frac{b-a}{2} \right) \mathbf{D}$  of size  $(N+1) \times (N+1)$  is the standard first-order Chebyshev differentiation matrix as defined in [15]. The time derivative is evaluated at the collocation nodes  $(\eta_i, \tau_j)$  for  $i = 0, 1, 2, \dots, N$  as

$$\frac{\partial u^{(k)}}{\partial t}(\eta_i, \tau_j) = \sum_{q=0}^M d_{j,q} \mathbf{U}_q^{(k)} = \sum_{q=0}^M \left( \frac{2}{t_k - t_{k-1}} \right) \hat{d}_{j,q} \mathbf{U}_q^{(k)}, \quad (18)$$

where  $\hat{d}_{j,q} = \left( \frac{t_k - t_{k-1}}{2} \right) d_{j,q}$ ,  $j, q = 0, 1, 2, \dots, M$  of size  $(M+1) \times (M+1)$  is the standard first-order Chebyshev differentiation matrix,

$$\mathbf{U}_j^{(k)} = [u^{(k)}(x_0, t_j), u^{(k)}(x_1, t_j), \dots, u^{(k)}(x_N, t_j)]^T \quad (19)$$

and  $T$  denotes matrix transpose. Using the definitions (17)–(18), we express Eq. (6) in matrix form as

$$\left[ \alpha_{2,s}(\mathbf{x}, t_j) \mathbf{D}^2 + \alpha_{1,s}(\mathbf{x}, t_j) \mathbf{D} + \alpha_{0,s}(\mathbf{x}, t_j) \right] \mathbf{U}_j^{(k)} - \sum_{q=0}^M d_{j,q} \mathbf{U}_q^{(k)} = \mathbf{R}_s^{(k)}(\mathbf{x}, t_j). \quad (20)$$

By changing the indices, Eq. (20) can be written as

$$\left[ \alpha_{2,s}(\mathbf{x}, t_i) \mathbf{D}^2 + \alpha_{2,s}(\mathbf{x}, t_i) \mathbf{D} + \alpha_{0,s}(\mathbf{x}, t_i) \right] \mathbf{U}_i^{(k)} - \sum_{j=0}^{M-1} d_{i,j} \mathbf{U}_j^{(k)} = \mathbf{R}_s^{(k)}(\mathbf{x}, t_i) + d_{i,M} \mathbf{U}_{t_M}^{(k)}, \quad (21)$$

where

$$\mathbf{U}_{t_M}^{(1)} = f(\mathbf{x}), \quad \text{for } k=1 \quad \text{and} \quad \mathbf{U}_{t_M}^{(k)} = \mathbf{U}_{t_0}^{(k-1)}, \quad \text{for } k=2, 3, \dots, P. \quad (22)$$

Eq. (21) constitutes an  $M(N+1) \times M(N+1)$  matrix system given by

$$\begin{bmatrix} \mathbf{A}_{0,0} & \mathbf{A}_{0,1} & \mathbf{A}_{0,2} & \dots & \mathbf{A}_{0,M-1} \\ \mathbf{A}_{1,0} & \mathbf{A}_{1,1} & \mathbf{A}_{1,2} & \dots & \mathbf{A}_{1,M-1} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \mathbf{A}_{M-1,0} & \mathbf{A}_{M-1,1} & \mathbf{A}_{M-1,2} & \dots & \mathbf{A}_{M-1,M-1} \end{bmatrix} \begin{bmatrix} \mathbf{U}_0^{(k)} \\ \mathbf{U}_1^{(k)} \\ \vdots \\ \mathbf{U}_{M-1}^{(k)} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_0^{(k)} \\ \mathbf{R}_1^{(k)} \\ \vdots \\ \mathbf{R}_{M-1}^{(k)} \end{bmatrix}, \quad (23)$$

where

$$\begin{aligned} A_{i,i} &= \alpha_{2,s}(\mathbf{x}, t_i) \mathbf{D}^2 + \alpha_{2,s}(\mathbf{x}, t_i) \mathbf{D} + \alpha_{0,s}(\mathbf{x}, t_i) - d_{i,i} \mathbf{I}, \\ A_{i,j} &= -d_{i,j} \mathbf{I}, \quad i \neq j, \\ \mathbf{R}_i^{(k)} &= \mathbf{R}_s^{(k)}(\mathbf{x}, t_i) + d_{i,M} \mathbf{U}_M^{(k)}, \\ \alpha_{\mu,s}(\mathbf{x}, t_i) &= \begin{bmatrix} \alpha(\mathbf{x}_0, t_i) & & & \\ & \alpha(x_1, t_i) & & \\ & & \ddots & \\ & & & \alpha(x_N, t_i) \end{bmatrix}, \quad \mu = 0, 1, 2, \end{aligned} \quad (24)$$

and  $\mathbf{I}$  is an identity matrix of size  $(N+1) \times (N+1)$ . The boundary conditions at the collocation points are

$$U^{(k)}(x_N, t_i) = g_a(t_i), \quad U^{(k)}(x_0, t_i) = g_b(t_i). \quad (25)$$

These boundary conditions are imposed on the main diagonal submatrices of the matrix system (23) to obtain a new system which takes the form

$$\begin{bmatrix} \hat{\mathbf{A}}_{0,0} & \hat{\mathbf{A}}_{0,1} & \hat{\mathbf{A}}_{0,2} & \dots & \hat{\mathbf{A}}_{0,M-1} \\ \hat{\mathbf{A}}_{1,0} & \hat{\mathbf{A}}_{1,1} & \hat{\mathbf{A}}_{1,2} & \dots & \hat{\mathbf{A}}_{1,M-1} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \hat{\mathbf{A}}_{M-1,0} & \hat{\mathbf{A}}_{M-1,1} & \hat{\mathbf{A}}_{M-1,2} & \dots & \hat{\mathbf{A}}_{M-1,M-1} \end{bmatrix} \begin{bmatrix} \mathbf{U}_0^{(k)} \\ \mathbf{U}_1^{(k)} \\ \vdots \\ \mathbf{U}_{M-1}^{(k)} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{R}}_0^{(k)} \\ \hat{\mathbf{R}}_1^{(k)} \\ \vdots \\ \hat{\mathbf{R}}_{M-1}^{(k)} \end{bmatrix}, \quad (26)$$

where



$$\hat{A}_{i,i} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ \hline & A_{i,i} & & & & \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}, \quad \hat{A}_{i,j} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ \hline & A_{i,i} & & & & \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \quad i \neq j, \quad (27)$$

$$\hat{\mathbf{R}}_i = \begin{bmatrix} \frac{g_b(t_i)}{\mathbf{R}_i} \\ \frac{g_a(t_i)}{\mathbf{R}_i} \end{bmatrix}.$$

The matrix system (26) is solved for  $U^{(k)}$ ,  $k = 1, 2, \dots, p$ . The solutions at different subdomains are matched together along common boundaries to give the desired approximate solution. The patching condition is given by

$$u^{(k)}(x, t_{k-1}) = u^{(k-1)}(x, t_{k-1}), \quad (28)$$

which denotes the solution at the boundaries of the subintervals.

### 3. Numerical experimentation

In this section, we illustrate the practical applicability of the multidomain approach in solving nonlinear parabolic PDEs by considering the solutions of well-known nonlinear PDEs that have been reported in the literature.

**Example 1.** We consider the modified Burger's-Fisher equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} + u(1-u), \quad x \in (0, 5), \quad t \in [0, 10], \quad (29)$$

subject to boundary conditions

$$u(0, t) = \frac{1}{2} + \frac{1}{2} \tanh\left(\frac{5t}{8}\right), \quad u(5, t) = \frac{1}{2} + \frac{1}{2} \tanh\left(\frac{5t}{8} - \frac{5}{4}\right), \quad (30)$$

and initial condition

$$u(x, 0) = \frac{1}{2} - \frac{1}{2} \tanh\left(\frac{x}{4}\right). \quad (31)$$

The exact solution is given in [24] as

$$u(x, t) = \frac{1}{2} + \frac{1}{2} \tanh\left(\frac{5t}{8} - \frac{x}{4}\right). \quad (32)$$

Eq. (29) is an example of a generalized Burger's-Fisher equation that was solved in [4] using variational iteration method. Applying the QLM, we obtain the linearized system

$$\alpha_{2,s}(x, t)u''_{s+1} + \alpha_{1,s}(x, t)u'_{s+1} + \alpha_{0,s}(x, t)u_{s+1} - \dot{u}_{s+1} = R_s(x, t), \quad (33)$$

where

$$\alpha_{2,s}(x, t) = 1, \quad \alpha_{1,s}(x, t) = -u_s, \quad \alpha_{0,s}(x, t) = -u'_s + 1 - 2u_s, \quad R_s(x, t) = -u_s u'_s - u_s^2. \quad (34)$$

In each subinterval  $k = 1, 2, \dots, p$ , we must solve

$$\alpha_{2,s}(x, t) \frac{\partial^2 u_{s+1}^{(k)}}{\partial x^2} + \alpha_{1,s}(x, t) \frac{\partial u_{s+1}^{(k)}}{\partial x} + \alpha_{0,s}(x, t) u_{s+1}^{(k)} - \frac{\partial u_{s+1}^{(k)}}{\partial t} = R_s^{(k)}(x, t), \quad (35)$$

$$x \in (0, 5), \quad t \in [t_{k-1}, t_k],$$

$$u^{(k)}(0, t) = \frac{1}{2} + \frac{1}{2} \tanh\left(\frac{5t}{8}\right), \quad u^{(k)}(5, t) = \frac{1}{2} + \frac{1}{2} \tanh\left(\frac{5t}{8} - \frac{5}{4}\right), \quad (36)$$

$$u^{(1)}(x, 0) = \frac{1}{2} - \frac{1}{2} \tanh\left(\frac{x}{4}\right), \quad k = 0, \text{ and } u^{(k)}(x, t_{k-1}) = u^{(k-1)}(x, t_{k-1}), \quad k = 2, 3, \dots, p. \quad (37)$$

The matrices resulting from application of the spectral collocation in (33) are

$$\begin{aligned} \mathbf{A}_{i,i} &= \mathbf{D}^2 + \alpha_{1,s}(\mathbf{x}, t_i) \mathbf{D} + \alpha_{0,s}(\mathbf{x}, t_i) - d_{i,i} I, \\ \mathbf{A}_{i,j} &= -d_{i,j} I, \quad \text{when } i \neq j, \\ \mathbf{B}_{i,s}^{(k)} &= \mathbf{R}_{i,s}^{(k)}(\mathbf{x}, t_i) + d_{i,M} \mathbf{U}_M^{(k)}, \end{aligned} \quad (38)$$

The initial condition at different subintervals is given by

$$\begin{aligned} \mathbf{U}_{t_M}^{(1)} &= f(x) = \frac{1}{2} - \frac{1}{2} \tanh\left(\frac{x}{4}\right), \quad \text{for } k = 1 \quad \text{and} \\ \mathbf{U}_{t_M}^{(k)} &= \mathbf{U}_{t_0}^{(k-1)}, \quad \text{for } k = 2, 3, \dots, p. \end{aligned} \quad (39)$$

The boundary conditions at the collocation points are

$$U^{(k)}(x_N, t_i) = g_a(t_i) = \frac{1}{2} + \frac{1}{2} \tanh\left(\frac{5t_i}{8}\right), \quad U^{(k)}(x_0, t_i) = g_b(t_i) = \frac{1}{2} + \frac{1}{2} \tanh\left(\frac{5t_i}{8} - \frac{5}{4}\right). \quad (40)$$

Making the relevant substitution, a matrix system similar to (26) is solved to obtain the approximate solution.

**Example 2.** We consider the modified Fitzhugh-Nagumo equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(u-1)(1-u), \quad x \in (1, 5), \quad t \in (0, 1], \quad (41)$$

subject to boundary conditions

$$u(1, t) = \frac{1}{2} \left[ 1 - \coth\left(\frac{-1}{2\sqrt{2}} + \frac{t}{4}\right) \right], \quad u(5, t) = \frac{1}{2} \left[ 1 - \coth\left(\frac{-5}{2\sqrt{2}} + \frac{t}{4}\right) \right], \quad (42)$$

and initial condition

$$u(x, 0) = \frac{1}{2} \left[ 1 - \coth\left(\frac{-x}{2\sqrt{2}}\right) \right]. \quad (43)$$

The exact solution is given in [12]

$$u(x, t) = \frac{1}{2} \left[ 1 - \coth\left(\frac{-x}{2\sqrt{2}} + \frac{t}{4}\right) \right]. \quad (44)$$

Eq. (41) is an example of a generalized Fitzhugh-Nagumo equation [12, 13]. Applying the QLM, we obtain a linearized system similar to that given in Eq. (33). The coefficients in this example are given by:

$$\alpha_{2,s}(x, t) = 1, \quad \alpha_{1,s}(x, t) = 0, \quad \alpha_{0,s}(x, t) = -1 + 4u_s - 3u_s^2, \quad R_s(x, t) = 2u_s^2 - 2u_s^3. \quad (45)$$

In each subinterval  $k = 1, 2, \dots, p$ , we must solve:

$$\alpha_{2,s}(x,t)\frac{\partial^2 u_{s+1}^{(k)}}{\partial x^2} + \alpha_{1,s}(x,t)\frac{\partial u_{s+1}^{(k)}}{\partial x} + \alpha_{0,s}(x,t)u_{s+1}^{(k)} - \frac{\partial u_{s+1}^{(k)}}{\partial t} = R_s^{(k)}(x,t),$$

$$x \in (1,5), \quad t \in (t_{k-1}, t_k],$$
(46)

$$u^{(k)}(1,t) = \frac{1}{2} \left[ 1 - \coth \left( \frac{-1}{2\sqrt{2}} + \frac{t}{4} \right) \right], \quad u^{(k)}(5,t) = \frac{1}{2} \left[ 1 - \coth \left( \frac{-5}{2\sqrt{2}} + \frac{t}{4} \right) \right],$$
(47)

$$u^{(1)}(x,0) = \frac{1}{2} \left[ 1 - \coth \left( \frac{-x}{2\sqrt{2}} \right) \right], \quad k=0, \quad \text{and} \quad u^{(k)}(x, t_{k-1}) = u^{(k-1)}(x, t_{k-1}),$$

$$k=2,3,\dots,p.$$
(48)

The application of the spectral collocation in (33) results into the following set of coefficient matrices:

$$\begin{aligned} \mathbf{A}_{i,i} &= \mathbf{D}^2 + \alpha_{0,s}(\mathbf{x}, t_i) - d_{i,i} I, \\ \mathbf{A}_{i,j} &= -d_{i,j} I, \quad \text{when } i \neq j, \\ \mathbf{B}_{i,s}^{(k)} &= \mathbf{R}_{i,s}^{(k)}(x, t_i) + d_{i,M} \mathbf{U}_M^{(k)}, \end{aligned}$$
(49)

The initial condition at different subintervals is given by:

$$\mathbf{U}_{t_M}^{(1)} = f(x) = \frac{1}{2} - \frac{1}{2} \tanh \left( \frac{x}{4} \right), \quad \text{for } k=1 \quad \text{and} \quad \mathbf{U}_{t_M}^{(k)} = \mathbf{U}_{t_0}^{(k-1)}, \quad \text{for } k=2,3,\dots,p.$$
(50)

The boundary conditions at the collocation points are given by:

$$\begin{aligned} U^{(k)}(x_N, t_i) &= g_a(t_i) = \frac{1}{2} \left[ 1 - \coth \left( \frac{-1}{2\sqrt{2}} + \frac{t_i}{4} \right) \right], \\ U^{(k)}(x_0, t_i) &= g_b(t_i) = \frac{1}{2} \left[ 1 - \coth \left( \frac{-5}{2\sqrt{2}} + \frac{t_i}{4} \right) \right]. \end{aligned}$$
(51)

**Example 3.** We consider the modified Burger's-Huxley equation:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} + u(1-u)(u-0.1), \quad x \in (0,1), \quad t \in [0,10],$$
(52)

subject to boundary conditions

$$u(0,t) = \frac{1}{2} - \frac{1}{2} \tanh\left[\frac{1}{2}(-0.9t)\right], \quad u(1,t) = \frac{1}{2} - \frac{1}{2} \tanh\left[\frac{1}{2}(1-0.9t)\right], \quad (53)$$

and initial condition

$$u(x,0) = \frac{1}{2} - \frac{1}{2} \tanh\left(\frac{x}{2}\right). \quad (54)$$

The exact solution is given in [25] as

$$u(x,t) = \frac{1}{2} - \frac{1}{2} \tanh\left[\frac{1}{2}(x-0.9t)\right]. \quad (55)$$

Eq. (52) is an example of a generalized Burger's-Huxley equation [25]. Applying the QLM, we obtain a linearized system similar to that given in Eq. (33). The coefficients in this example are given by

$$\begin{aligned} \alpha_{2,s}(x,t) &= 1, \quad \alpha_{1,s}(x,t) = -u_s, \quad \alpha_{0,s}(x,t) = -0.1 - u'_s + 2.2u_s - 3u_s^2, \\ R_s(x,t) &= -u_s u'_s + 1.1u_s^2 - 2u_s^3. \end{aligned} \quad (56)$$

In each subinterval  $k = 1, 2, \dots, p$ , we must solve

$$\begin{aligned} \alpha_{2,s}(x,t) \frac{\partial^2 u_{s+1}^{(k)}}{\partial x^2} + \alpha_{1,s}(x,t) \frac{\partial u_{s+1}^{(k)}}{\partial x} + \alpha_{0,s}(x,t) u_{s+1}^{(k)} - \frac{\partial u_{s+1}^{(k)}}{\partial t} &= R_s^{(k)}(x,t), \\ x \in (0,1), \quad t \in (t_{k-1}, t_k], \end{aligned} \quad (57)$$

$$u^{(k)}(0,t) = \frac{1}{2} - \frac{1}{2} \tanh\left[\frac{1}{2}(-0.9t)\right], \quad u^{(k)}(1,t) = \frac{1}{2} - \frac{1}{2} \tanh\left[\frac{1}{2}(1-0.9t)\right], \quad (58)$$

$$u^{(1)}(x,0) = \frac{1}{2} - \frac{1}{2} \tanh\left(\frac{x}{2}\right), \quad k=0, \quad \text{and} \quad u^{(k)}(x, t_{k-1}) = u^{(k-1)}(x, t_{k-1}), \quad k=2,3,\dots,p. \quad (59)$$

The application of the spectral collocation in (33) results into the following set of coefficient matrices

$$\begin{aligned} \mathbf{A}_{i,i} &= \mathbf{D}^2 + \alpha_{1,s}(\mathbf{x}, t_i) \mathbf{D} + \alpha_{0,s}(\mathbf{x}, t_i) - d_{i,i} I, \\ \mathbf{A}_{i,j} &= -d_{i,j} I, \quad \text{when } i \neq j, \\ \mathbf{B}_{i,s}^{(k)} &= \mathbf{R}_{i,s}^{(k)}(\mathbf{x}, t_i) + d_{i,M} \mathbf{U}_M^{(k)}, \end{aligned} \quad (60)$$

The initial condition at different subintervals is given by

$$\mathbf{U}_{t_M}^{(1)} = f(x) = \frac{1}{2} - \frac{1}{2} \tanh\left(\frac{x}{2}\right), \text{ for } k=1 \text{ and } \mathbf{U}_{t_M}^{(k)} = \mathbf{U}_{t_0}^{(k-1)}, \text{ for } k=2, 3, \dots, p. \quad (61)$$

The boundary conditions at the collocation points are given by

$$\begin{aligned} U^{(k)}(x_N, t_i) &= g_a(t_i) = \frac{1}{2} - \frac{1}{2} \tanh\left[\frac{1}{2}(-0.9t_i)\right], \\ U^{(k)}(x_0, t_i) &= g_b(t_i) = \frac{1}{2} - \frac{1}{2} \tanh\left[\frac{1}{2}(1 - 0.9t_i)\right]. \end{aligned} \quad (62)$$

## 4. Results and discussion

In this section, we present the results for the absolute error values at selected values of  $x$  and  $t$  and the computational time that is obtained when Examples 1–3 are solved using both the bivariate spectral collocation method (single-domain approach) and the multidomain bivariate spectral collocation method. The absolute error is evaluated as

$$Abs_{x_i, t_j} = |u_e(x_i, t_j) - u_a(x_i, t_j)|, \quad 0 \leq i \leq N, 0 \leq j \leq M, \quad (63)$$

where  $u_e(x_i, t_j)$  is the exact solution and  $u_a(x_i, t_j)$  is the approximate solution at the collocation points  $(x_i, t_j)$ . In each example, two tables have been presented to compare the performance of the two approaches that are used in solving each example. The results indicate that multidomain bivariate spectral collocation method is very accurate and the results are generated faster when compared to solving the same problem over single domain. The results obtained from approximating the solution of (29) are given below. **Table 1** shows the results generated when the bivariate spectral collocation method (single domain) is used whereas **Table 2** presents the results obtained when using the multidomain bivariate spectral collocation method. The bivariate spectral collocation method gives on average absolute errors of  $10^{-6}$  whereas those obtained when the multidomain bivariate spectral collocation method is used are  $10^{-12}$ , an indication that the multidomain approach is more accurate. The computational time in the case of the multidomain approach is lesser (0.019602 sec) than (0.103606 sec) that is obtained when solving the problem over single domain.

| $x$            | $t$          |              |              |              |
|----------------|--------------|--------------|--------------|--------------|
|                | 2.0          | 4.0          | 6.0          | 8.0          |
| 0.4775         | 4.91994e-007 | 3.31285e-007 | 1.99894e-008 | 6.67958e-008 |
| 1.3650         | 4.36608e-007 | 7.17567e-007 | 1.76673e-008 | 1.70637e-007 |
| 2.5000         | 3.26724e-006 | 1.43695e-006 | 1.42849e-007 | 2.19591e-007 |
| 3.6350         | 2.35521e-007 | 2.54426e-006 | 1.48557e-007 | 6.72760e-007 |
| 4.5225         | 7.14695e-006 | 6.81188e-006 | 2.08164e-006 | 2.15409e-007 |
| CPU time (sec) | 0.103606     |              |              |              |

**Table 1.** MatLab solution: Absolute error values obtained when solving Example 1 using  $N = M = 20$ , single domain, with Lagrange basis and Gauss-Lobatto nodes, iterations=10.

| $x$            | $t$          |              |              |              |
|----------------|--------------|--------------|--------------|--------------|
|                | 2.0          | 4.0          | 6.0          | 8.0          |
| 0.4775         | 4.77396e-014 | 7.99361e-015 | 1.16573e-014 | 2.66454e-014 |
| 1.3650         | 7.41074e-013 | 1.29896e-014 | 1.13243e-014 | 2.81997e-014 |
| 2.5000         | 3.31513e-013 | 3.83027e-014 | 1.25455e-014 | 2.44249e-015 |
| 3.6350         | 3.37175e-012 | 7.40519e-014 | 8.88178e-015 | 4.21885e-015 |
| 4.5225         | 2.55729e-012 | 1.52323e-013 | 2.68674e-014 | 2.10942e-014 |
| CPU time (sec) | 0.019602     |              |              |              |

**Table 2.** MatLab solution: Absolute error values obtained when Example 1 is solved using  $N = 20, M = 5, p = 10$ , with Lagrange basis and Gauss-Lobatto nodes, iterations=10.

The results obtained from approximating the solution of (41) are given in **Tables 3 and 4**. **Table 3** shows the results generated when the bivariate spectral collocation method (single domain) is used whereas **Table 4** presents the results obtained when using the multidomain bivariate spectral collocation method. The results are similar to those of Example 1, thus the multidomain approach is more efficient than single-domain approach when it is applied in solving nonlinear parabolic PDEs defined over large-time domain.

| $x$            | $t$          |              |              |              |
|----------------|--------------|--------------|--------------|--------------|
|                | 0.2          | 0.4          | 0.6          | 0.8          |
| 1.0039         | 1.35033e-006 | 1.14349e-005 | 1.42724e-005 | 2.50691e-006 |
| 1.9283         | 8.19001e-008 | 1.97335e-008 | 4.17650e-008 | 4.41645e-009 |
| 3.0000         | 6.03945e-009 | 1.22066e-008 | 9.50467e-009 | 8.55427e-009 |
| 4.0717         | 8.95142e-011 | 1.11732e-009 | 2.53453e-009 | 3.16956e-009 |
| 4.9372         | 2.23332e-012 | 2.90505e-011 | 1.05300e-010 | 1.94311e-010 |
| CPU time (sec) | 0.135906     |              |              |              |

**Table 3.** MatLab solution: Absolute error values obtained when solving Example 2 using  $N = 50, M = 10$ , single domain, with Lagrange basis and Gauss-Lobatto nodes, iterations=10.

| $x$            | $t$          |              |              |              |
|----------------|--------------|--------------|--------------|--------------|
|                | 0.2          | 0.4          | 0.6          | 0.8          |
| 1.0039         | 7.57705e-012 | 3.20854e-012 | 6.38556e-012 | 2.23954e-011 |
| 1.9283         | 8.88178e-015 | 2.10942e-014 | 3.37508e-014 | 2.22045e-015 |
| 3.0000         | 1.55431e-015 | 7.32747e-015 | 1.77636e-014 | 1.95399e-014 |
| 4.0717         | 6.83897e-014 | 5.66214e-014 | 8.79297e-014 | 3.50830e-014 |
| 4.9372         | 1.11511e-012 | 2.31637e-012 | 7.83817e-014 | 4.10783e-014 |
| CPU time (sec) | 0.026619     |              |              |              |

**Table 4.** MatLab solution: Absolute error values obtained when Example 2 is solved using  $N = 50$ ,  $M = 5$ ,  $p = 100$ , with Lagrange basis and Gauss-Lobatto nodes, iterations=10.

The results obtained from approximating the solution of (52) are given below. **Table 5** shows the results generated when the bivariate spectral collocation method (single domain) is used whereas **Table 6** presents the results obtained when using the multidomain bivariate spectral collocation method. The results indicate that the multidomain approach is very accurate and computationally faster when it is applied to solve nonlinear PDEs defined over large-time intervals.

| $x$            | $t$          |              |              |              |
|----------------|--------------|--------------|--------------|--------------|
|                | 2.0          | 4.0          | 6.0          | 8.0          |
| 0.0010         | 1.61398e-005 | 9.61285e-005 | 7.30576e-005 | 7.70011e-006 |
| 0.2321         | 2.99891e-005 | 1.09535e-004 | 9.97674e-005 | 2.21837e-006 |
| 0.5000         | 2.15105e-00  | 9.64207e-005 | 1.07763e-004 | 7.47171e-006 |
| 0.7679         | 1.07315e-005 | 1.90612e-005 | 6.33297e-005 | 5.44788e-006 |
| 0.9843         | 3.75286e-005 | 1.10078e-004 | 2.79722e-005 | 6.65481e-007 |
| CPU time (sec) | 0.026619     |              |              |              |

**Table 5.** MatLab solution: Absolute error values obtained when solving Example 3 using  $N = 50$ ,  $M = 10$ , single domain, with Lagrange basis and Gauss-Lobatto nodes, iterations=10.

| $x$            | $t$          |              |              |              |
|----------------|--------------|--------------|--------------|--------------|
|                | 2.0          | 4.0          | 6.0          | 8.0          |
| 0.0010         | 5.21284e-010 | 4.77699e-010 | 3.01078e-010 | 3.88833e-010 |
| 0.2321         | 2.05589e-011 | 4.15302e-011 | 4.59769e-011 | 2.75941e-011 |
| 0.5000         | 2.27406e-011 | 1.40028e-011 | 1.33067e-011 | 1.88674e-011 |
| 0.7679         | 5.37154e-011 | 3.15495e-011 | 1.85438e-011 | 2.47591e-012 |
| 0.9843         | 2.02505e-010 | 6.77300e-011 | 2.10308e-010 | 1.70352e-010 |
| CPU time (sec) | 0.028104     |              |              |              |

**Table 6.** MatLab solution: Absolute error values obtained when Example 3 is solved using  $N = 50$ ,  $M = 5$ ,  $p = 100$ , with Lagrange basis and Gauss-Lobatto nodes, iterations=10.

The lesser computational time that is evident in the case when the multidomain approach is applied to solve the nonlinear PDE is attributed to the fact that the multidomain approach uses very few number of collocation points in each subinterval for the time variable than in the single-domain approach. This reduction in the number of collocation points significantly reduces the size of the resulting coefficient matrices. The small -sized coefficient matrices are



less dense and take less CPU time to produce results. The high accuracy and less computational time substantiate our claim that the multidomain bivariate spectral collocation method is a powerful numerical method for solving nonlinear parabolic PDEs that are defined over large-time intervals. The QLM is a powerful technique for simplifying nonlinear PDEs as very accurate results are obtained after 10 iterations only. The spectral collocation-based methods yield very accurate results with a few number of grid points as the approximate solution that is searched for is a higher degree polynomial. In the numerical experimentation, the symmetrically distributed Gauss-Lobatto (G-L) collocation points have been used instead of equispaced grid points as the G-L nodes have a feature that tends to uniformly distribute the approximation errors across the entire interval of approximation [26]. The equispaced nodes, on the other hand, produce oscillations near the end of interval of approximation, a behavior referred to as Runge phenomena [27].

## 5. Conclusion

The multidomain bivariate spectral collocation method has been used successfully to solve nonlinear parabolic PDEs that arise in a wide range of applications like genetics, biology, heat and mass transfer and wave processes. The approximate results confirm that the multidomain bivariate spectral collocation method is very accurate and computationally faster when it is used to solve nonlinear parabolic PDEs that are defined over large-time domains. This approach is an alternative to other numerical methods that can be used to solve nonlinear parabolic partial differential equations. The multidomain bivariate spectral collocation method being more accurate and computationally faster can therefore be adopted and extended to solve similar problems that model real-life phenomenon.

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