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Conditions for Optimality of Singular Controls in Dynamic Systems with Retarded Control

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Abstract

In this chapter, we consider an optimal control problem with retarded control and study a larger class of singular (in the classical sense) controls. For the optimality of singular controls, the various necessary conditions in the recurrent forms are obtained. These conditions contain also the analogs of Kelly, Koppa-Mayer, Gabasov, and equality-type conditions. While proving the main results, the Legendre polynomials are used as variations of control.

Keywords: singular control, optimal control, variation transform method, Legendre polynomial, necessary optimality conditions

1. Introduction

As is known, optimal control problems described by the dynamical systems with retarded control are attracting the attention of many specialists, and the results obtained in this field deal mainly with the first-order necessary optimality conditions [1–8, etc.]. However, theory of singular controls for systems with retarded control has not been studied enough yet [9, 10]. One of the main reasons here is that the methods proposed and developed for ordinary systems (for systems without retardation) in [11–18] are not directly applicable to the singular controls in dynamical systems with aftereffect (see [9, 14–19]). Therefore, to study optimal control problems in the systems with retarded control is of special theoretical interest. Besides, such problems have practical significance as well, because mathematical modelling for some problems of organization of the economic plan and production leads to the problems with retarded control (see, e.g., [20]).

As is known, the concept of singular control was first introduced to the theory of optimal processes by Rozenoer [22] in 1959. First results on the necessary optimality conditions for singular controls have been obtained by Kelley [12] in the case of open set U, and by Gabasov [11] in the case of arbitrary (in particular, closed) set U, where U is a set of values of admissible controls. Afterward, Kelley and Gabasov's conditions as well as the methods for treating singular controls proposed in [11, 13] have been significantly generalized in [10, 14–19, 23–41, etc.] to the cases of (1) controls with higher-order degeneration, (2) multidimensional controls, and (3) various classes of control systems. Considering all these cases, the methods in [11, 13] have been generalized in [17, 37] and for optimality of singular controls, necessary conditions in the form of recurrence sequences are obtained for dynamical systems with delayed in state. Similar results for the problem of dynamic systems with retarded control have been obtained in [10] only for singular controls with full degree of degeneration. Below, by considering a larger class of singular controls, proposing a modified version of the variations transform method [13] and matrix impulse method [11], we generalize all results of [10]. While treating the optimality of singular (in the classical sense) controls, we use the Legendre [[42], p. 413] polynomials as variations of control because such an approach is more convenient.

1. Problem statement. Consider the following optimal control problem with retarded control:

$$S(u) = \varphi\left(x(t_1)\right) \to \min_u \tag{1.1}$$

$$\dot{x}(t) = f\left(x(t), u(t), u(t-h), t\right), \ t \in I := [t_0, t_1], \ x(t_0) = x_0, \tag{1.2}$$

$$u(t) = w(t), \ t \in I_0 := [t_0 - h, t_0), u(t) \in U \subset R^r, \ t \in I. \tag{1.3}$$

Here, U is an open set in r-dimensional Euclidean space R^r, $R^1 = :R: = (-\infty, +\infty)$, $x \in R^n$ is an n-vector with phase coordinates, $u \in U$ is an r-vector of control actions, $h = \text{const} > 0$, x_0, t_0, t_1 are fixed points with $t_1 > t_0 + h$; $\varphi(x): R^n \to R$, $f(x, u, v, t): R^n \times R^r \times R^r \times R \to R^n$, $w(\cdot) \in \tilde{C}^+\left([t_0 - h, t_0], R^r\right)$ are the given functions, where $\tilde{C}^+\left([t_0 - h, t_0], R^r\right)$ is a class of piecewise continuous (continuous from the right at discontinuity points and continuous from the left at the point t_0) vector functions $w(t): [t_0 - h, t_0] \to R^r$.

The function $u(\cdot)$ is said to be an admissible control if it belongs to $\tilde{C}^+\left(I_1, R^r\right)$ and satisfies the condition (1.3), where $I_1 := I_0 \cup I = [t_0 - h, t_1]$.

Note that if the function $f(\cdot)$ and its partial derivative $f_x(\cdot)$ are continuous on $R^n \times R^r \times R^r \times R$, then, by using the method of successive approximations as in [21] it is easy to show that every admissible control $u(\cdot)$ generates a unique absolutely continuous solution

$x(\cdot)$ of the system (1.2), (1.3) where this solution will be assumed as defined everywhere on I.

If the admissible control $u^0(t)$, $t \in I_1$ is a solution of the problem (1.1)–(1.3), we will call it an optimal control, while the corresponding trajectory $x^0(t)$, $t \in I$ of the system (1.2)–(1.3) will be called an optimal trajectory. The pair $\left(u^0(\cdot), x^0(\cdot)\right)$ will be called an optimal process.

While studying the problem (1.1)–(1.3), we will also use the following assumptions:

(A1) let the functional $\varphi(x)$: $R^n \to R$ be twice continuously differentiable in the space R^n;

(A2) let the function $f(\cdot)$ and its partial derivatives $f_z(\cdot)$, $f_{zz}(\cdot)$ be continuous in the space $R^n \times R^r \times R^r \times R$, where $z = (x, u, v)$;

(A3) let the function $f(\cdot)$ be three times continuously differentiable in the totality of its arguments in the space $R^n \times R^r \times R^r \times R$;

(A4) let the inclusions $\dot{w}(\cdot) \in \tilde{C}\left(\left[t_0 - h, t_0\right], R^r\right)$ and $\dot{u}^0(\cdot) \in \tilde{C}\left(I_1, R^r\right)$ hold for the derivatives $\dot{w}(\cdot)$ and $\dot{u}^0(\cdot)$, where $\tilde{C}\left([a, b], R^r\right)$ is a class of piecewise continuous (continuous from the right and left at the points a and b, respectively) vector functions $c(t): [a, b] \to R^r$;

(A5) let the function $f(\cdot)$ be sufficiently smooth in the totality of its arguments in the space $R^n \times R^r \times R^r \times R$;

(A6) let the initial function $w(\cdot) \in \tilde{C}^+\left(\left[t_0 - h, t_0\right], R^r\right)$ and admissible control $u^0(\cdot)$ be sufficiently piecewise smooth, that is, $\dfrac{d^m}{dt^m} w(t) \in \tilde{C}\left(\left[t_0 - h, t_0\right], R^r\right)$ and $\dfrac{d^m}{dt^m} u^0(t) \in \tilde{C}\left(I_1, R^r\right)$, $m = 1, 2, \ldots$.

Especially note that more precise assumptions on the analytic properties of $\varphi(\cdot), f(\cdot), u(\cdot), w(\cdot)$ will directly follow from the representation of optimality criteria obtained below.

2. The second variation of the objective functional and the definition of a singular (in the classical sense) control

Let assumptions (A1) and (A2) be fulfilled, and $\left(u^0(\cdot), x^0(\cdot)\right)$ be some admissible process. If the process $\left(u^0(\cdot), x^0(\cdot)\right)$ is optimal, then, by using the known technique (see, e.g., [27, p. 51]), it is easy to get

$$\delta^1 S\left(u^0; \delta u(\cdot)\right) = 0, \delta^2 S\left(u^0; \delta u(\cdot)\right) \geq 0, \ \forall \delta u(\cdot) \in \tilde{C}^+\left(I_1, R^r\right), \delta u(t) = 0, t \in I_0. \tag{2.1}$$

Here

$$\delta^1 S\left(u^0; \delta u(\cdot)\right) := -\int_{t_0}^{t_1}\left[H_u^T(t)\delta u(t) + H_v^T(t)\delta u(t-h)\right]dt,$$
$$\delta u(\cdot) \in \tilde{C}^+\left(I_1, R^r\right), \delta u(t) = 0, t \in I_0, \tag{2.2}$$

$$\delta^2 S\left(u^0; \delta u(\cdot)\right) := \delta x^T(t_1)\varphi_{xx}\left(x^0(t_1)\right)\delta x(t_1) - \int_{t_0}^{t_1}\Big\{\delta x^T(t)H_{xx}(t)\delta x(t)$$
$$+\delta u^T(t)H_{uu}(t)\delta u(t) + \delta u^T(t-h)H_{vv}(t)\delta u(t-h) + 2\Big[\delta x^T(t)H_{xu}(t)\delta u(t) \tag{2.3}$$
$$+\delta x^T(t)H_{xv}(t)\delta u(t-h) + \delta u^T(t)H_{uv}(t)\delta u(t-h)\Big]\Big\}dt,$$
$$\delta u(\cdot) \in \tilde{C}^+\left(I_1, R^r\right), \delta u(t) = 0, t \in I_0,$$

where $\delta^1 s\left(u^0; \delta u(\cdot)\right)$ and $\delta^2 s\left(u^0; \delta u(\cdot)\right)$ are, respectively, the first and the second variations of the functional $S(u)$ at the point $u^0(\cdot)$; $H(\psi, x, u, v, t) := \psi^T f(x, u, v, t)$, $H(t) := H\left(\psi^0(t), x^0(t), u^0(t), v^0(t), t\right), H_\mu(t) := H_\mu\left(\psi^0(t), x^0(t), u^0(t), v^0(t), t\right)$, $H_{\mu\nu}(t) := H_{\mu\nu}\left(\psi^0(t), x^0(t), u^0(t), v^0(t), t\right), t \in I$, $\mu, \nu \in \{x, u, v\}$; $\delta u(\cdot)$ is the variation of the control $u^0(\cdot)$, while $\delta x(\cdot)$ is the corresponding variation of the trajectory $x^0(t)$, $t \in I$, which $\delta x(\cdot)$ is the solution of the system

$$\delta \dot{x}(t) = f_x(t)\delta x(t) + f_u(t)\delta u(t) + f_v(t)\delta u(t-h), t \in I,$$
$$\delta x(t_0) = 0, \delta u(t) = 0, t = I_0, \tag{2.4}$$

where $f_\mu(t) := f_\mu\left(x^0(t), u^0(t), u^0(t-h), t\right), t \in I$ and $\mu \in \{x, u, v\}$, while the vector function $\psi^0(\cdot)$ is the solution of the conjugate system

$$\dot{\psi}^0(t) = -H_x(t), t \in I, \ \psi^0(t_1) = -\varphi_x\left(x^0(t_1)\right). \tag{2.5}$$

Below, we consider that the following conditions are fulfilled:

$$H(t) = 0, \ H_\mu(t) = 0, \ H_{\mu\nu}(t) = 0, \text{ for } t > t_1 \text{ and } \mu, \nu \in \{x, u, v\}. \tag{2.6}$$

If $\left(u^0(\,\cdot\,), x^0(\,\cdot\,)\right)$ is an optimal process, then, by definition of an admissible control and taking into consideration (2.2)–(2.4) from (2.1), proceeding the same way as in [27, p. 53], we obtain the classical necessary conditions of optimality (analogs of the Euler equation and Legendre-Clebsch condition) [10, 43], that is, the following relations are valid:

a.
$$H_u(t) + \chi(t) H_v(t+h) = 0, \quad \forall t \in I; \tag{2.7}$$

b.
$$\tilde{u}^T \left[H_{uu}(t) + \chi(t) H_{vv}(t+h) \right] \tilde{u} \le 0, \quad \forall t \in I, \forall \tilde{u} \in R^r; \tag{2.8}$$

c. $H_v(t_1) = 0, \quad \tilde{u}^T \left[H_{uu}(t_1 - h) + H_{vv}(t_1) \right] \tilde{u} \le 0,$ for all $\tilde{u} \in R^r$, if optimal control $u^0(\,\cdot\,)$ is continuous at the points $t = t_1 - ih, i = 1, 2$. Here, $\chi(\,\cdot\,)$ is the characteristic function of the set $[t_0, t_1 - h)$.

It should be noted that the optimality condition (c) is the corollary of conditions (a) and (b).

Definition 2.1. An admissible control $u^0(t), \quad t \in I$, satisfying conditions (2.7) and (2.8), is called singular (in classical sense) if

$$\mathrm{rang}\left[H_{uu}(t) + \chi(t) H_{vv}(t+h) \right] = r_1 < r, \quad \forall t \in I.$$

In this case, the set I is called a singular plot for an admissible control $u^0(\,\cdot\,)$. The main goal of this chapter is to study such singular controls.

Let $u = (p, q)^T$, $v = (\tilde{p}, \tilde{q})^T$, where $p, \tilde{p} \in R^{r_0}$, $q, \tilde{q} \in R^{r_1}$, $r_0 + r_1 = r$. Without loss of generality [[27], p. 138], we assume that the singularity to the control $u^0(\,\cdot\,)$ is delivered by a vector component $p \in R^{r_0}$, that is,

$$H_{pp}(t) + \chi(t) H_{\tilde{p}\tilde{p}}(t+h) = 0, \quad t \in I. \tag{2.9}$$

Note that the general inequality (2.8) implies the equality-type optimality condition for a singular (in classical sense) control $u^0(\,\cdot\,)$:

$$H_{pq}(t) + \chi(t) H_{\tilde{p}\tilde{q}}(t+h) = 0, \quad t \in I. \tag{2.10}$$

Proposition 2.1. Let assumptions (A1) and (A2) be fulfilled, the admissible control $u^0(\,\cdot\,) = (p(\,\cdot\,), q(\,\cdot\,))^T$ be singular (in the classical sense) and condition (2.9) be fulfilled along it. Let also the variations $\delta u(t) = \left(\delta_0 p(t), \delta q(t) \right)^T \in \tilde{C}^+\left(I_1, R^r\right)$ be non-zero only on $[\theta, \theta + \varepsilon)$,

where $\theta \in [t_0, t_1)$ and $\varepsilon \in (0, \varepsilon_0)$, with the number $\varepsilon_0 \in (0, h)$ be such that (1) if $\theta \in [t_0, t_1 - h)$, then $\varepsilon_0 < t_1 - \theta - h$ and (2) if $\theta \in [t_1 - h, t_1)$, then $\varepsilon_0 < t_1 - \theta$. Then, (a) the variational system (2.4) becomes

$$\begin{cases} \delta\dot{x}(t) = f_x(t)\delta x(t) + f_p(t)\delta_0 p(t) + f_q(t)\delta q(t) + f_{\widetilde{p}}(t)\delta_0 p(t-h) \\ \qquad\qquad + f_{\widetilde{q}}(t)\delta q(t-h), \quad t \in [\theta, t_1], \\ \delta x(t) = 0, \ t \in [t_0, \theta], \ \delta_0 p(t) = 0, \ \delta q(t) = 0, \quad t \in [t_0 - h, \theta); \end{cases} \tag{2.11}$$

(b) the following representation is valid for the second variation (2.3):

$$\begin{aligned} \delta^2 S(u^0; \delta u(\cdot)) &= \delta x^T(t_1)\varphi_{xx}(x^0(t_1))\delta x(t_1) - \int_\theta^{t_1} \delta x^T(t)H_{xx}(t)\delta x(t)dt - \\ &\quad - 2\int_\theta^{\theta+\varepsilon} \left\{ \left[\delta x^T(t)H_{xp}(t) + \delta x^T(t+h)H_{x\widetilde{p}}(t+h) \right]\delta_0 p(t) \right. \\ &\quad + \left. \left[\delta x^T(t)H_{xq}(t) + \delta x^T(t+h)H_{x\widetilde{q}}(t+h) \right]\delta q(t) \right\}dt - \\ &\quad - 2\int_\theta^{\theta+\varepsilon} \delta_0 p^T(t)\left[H_{pq}(t) + H_{\widetilde{p}\widetilde{q}}(t+h) \right]\delta q(t)dt - \\ &\quad - \int_\theta^{\theta+\varepsilon} \delta q^T(t)\left[H_{qq}(t) + H_{\widetilde{q}\widetilde{q}}(t+h) \right]\delta q(t)dt, \ \forall \varepsilon \in (0, \varepsilon_0). \end{aligned} \tag{2.12}$$

Proof. To prove (a), it suffices to consider the definition of the variation $\delta u(\cdot) = (\delta_0 p(\cdot), \delta q(\cdot))^T$ in (2.4). The proof of (b) follows directly from (2.3), in view of (2.6), (2.9), (2.11), and the definition of the variation $\delta u(\cdot) = (\delta_0 p(\cdot), \delta q(\cdot))^T$.

3. Transformation of the second variation of the functional by means of modified variant of matrix impulse method (when studying singular (in the sense of Definition 2.1) of controls)

Let conditions (A1) and (A2) be fulfilled and along the singular control $u^0(\cdot)$ the equality (2.9) hold. Use Proposition 2.1. Let the variation $\delta u(t) = (\delta_0 p(t), \delta q(t))^T \in \sim C^+(I_1, R^r)$ have the form:

$$\delta_0 p(t) = \begin{cases} \xi, \ t \in [\theta, \theta + \varepsilon), \ \varepsilon \in (0, \varepsilon_0), \\ 0, \ t \in I_1 \setminus [\theta, \theta + \varepsilon), \end{cases} \qquad \delta q(t) = 0, t \in I, \tag{3.1}$$

where $\xi \in E^{r_0}$, $\theta \in [t_0, t_1)$, and the number ε_0 was defined in Proposition 2.1.

Along the singular control $u^0(\cdot) = (p(\cdot), q(\cdot))^T$ satisfying condition (2.9), taking into account (3.1), formula (2.12) takes the form:

$$\delta^2 S(u^0; \delta u(\cdot)) = \delta x^T(t_1) \varphi_{xx}(x^0(t_1)) \delta x(t_1) - \Delta_1^* - 2\Delta_2^*, \tag{3.2}$$

where

$$\Delta_1^* = \int_\theta^{t_1} \delta x^T(t) H_{xx}(t) \delta x(t) dt, \quad \Delta_2^* = \int_\theta^{\theta+\varepsilon} \left[\delta x^T(t) H_{xp}(t) + \delta x^T(t+h) H_{x\tilde{p}}(t+h) \right] \xi dt,$$

where $\delta x(t)$, $t \in I$ is the solution of the system (2.11).

By the Cauchy formula, we have

$$\delta x(t) = \begin{cases} \int_\theta^t \lambda(s,t) \left[f_p(s) \delta_0 p(s) + f_{\tilde{p}}(s) \delta_0 p(s-h) \right] ds, & t \in (\theta, t_1] \\ 0, & t \in [t_0, \theta], \end{cases} \tag{3.3}$$

where $\lambda(s,t)$, $(s,t) \in I \times I$ is the solution of the system

$$\lambda_t(s,t) = f_x(t) \lambda(s,t), \quad t_0 \le s < t \le t_1, \tag{3.4}$$

$\lambda(s,t) = 0$, $s > t$, $\lambda(s,s) = E$ (E is a unit $n \times n$ matrix).

As (A2) and $u^0(\cdot) \in \tilde{C}^+(I_1, R^r)$ are fulfilled, then by (3.1) and (3.4) and for all $\theta \in [t_0, t_1)$, from (3.3) we get

$$\delta x(t) = \begin{cases} 0, & t \in [t_0, \theta], \\ (t-\theta)\lambda(\theta,t) f_p(\theta)\xi + o(t-\theta), & t \in (\theta, \theta+\varepsilon), \\ \varepsilon\lambda(\theta,t) f_p(\theta)\xi + o(\varepsilon), & t \in [\theta+\varepsilon, \theta+h) \cap I, \\ \varepsilon\lambda(\theta,t) f_p(\theta)\xi + (t-\theta-h)\chi(\theta)\lambda(\theta+h,t) f_{\tilde{p}}(\theta+h)\xi \\ \quad + o(t-\theta-h), & t \in [\theta+h, \theta+h+\varepsilon) \cap I, \\ \varepsilon\left[\lambda(\theta,t) f_p(\theta) + \chi(\theta)\lambda(\theta+h,t) f_{\tilde{p}}(\theta+h)\right]\xi + o(\varepsilon), & t \in [\theta+h+\varepsilon, t_1] \cap I \end{cases} \tag{3.5}$$

where $\chi(\cdot)$ is the characteristic function of the set $[t_0, t_1 - h)$; $o(\tau)/\tau \to 0$, as $\tau \to 0$.

By (2.6) and (3.5) and taking into account $\lambda(s, s) = E$ and $\lambda(s, t) = 0$ for $s > t$, we calculate separate terms of (3.2). As a result, after simple reasoning, we get

$$
\begin{aligned}
\delta x^T(t_1)\varphi_{xx}\big(x^0(t)\big)\delta x(t_1) = \varepsilon^2 \xi^T \Big[& f_p^T(\theta)\lambda^T(\theta, t_1)\varphi_{xx}\big(x^0(t)\big)\lambda(\theta, t_1)f_p(\theta) + \\
& + 2\chi(\theta)f_p^T(\theta)\lambda^T(\theta, t_1)\varphi_{xx}\big(x^0(t_1)\big)\lambda(\theta + h, t_1)f_{\tilde{p}}(\theta + h) + \\
& + \chi(\theta)f_{\tilde{p}}^T(\theta + h)\lambda^T(\theta + h, t_1)\varphi_{xx}\big(x^0(t_1)\big)\lambda(\theta + h, t_1)f_{\tilde{p}}(\theta + h)\Big]\xi + o(\varepsilon^2),
\end{aligned}
\tag{3.6}
$$

$$
\begin{aligned}
\Delta_1^* = \varepsilon^2 \xi^T \int_\theta^{t_1} \Big[& f_p^T(\theta)\lambda^T(\theta, t)H_{xx}(t)\lambda(\theta, t)f_p(\theta) \\
& + 2\chi(\theta)f_p^T(\theta)\lambda^T(\theta, t)H_{xx}(t)\lambda(\theta + h, t)f_{\tilde{p}}(\theta + h) \\
& + \chi(\theta)f_{\tilde{p}}^T(\theta + h)\lambda^T(\theta + h, t)H_{xx}(t)\lambda(\theta + h, t)f_{\tilde{p}}(\theta + h) \Big]dt\,\xi + o(\varepsilon^2),
\end{aligned}
\tag{3.7}
$$

$$
\begin{aligned}
\Delta_2^* &= \xi^T \int_\theta^{\theta + \varepsilon} \Big[f_p^T(\theta)\lambda^T(\theta, t)H_{xp}(t)(t - \theta) + \chi(\theta)\big(\varepsilon f_p^T(\theta)\lambda^T(\theta, t + h)H_{x\tilde{p}}(t + h) \\
&\quad + f_{\tilde{p}}^T(\theta + h)\lambda^T(\theta + h, t + h)H_{x\tilde{p}}(t + h)(t - \theta)\big)\Big]dt\,\xi + o(\varepsilon^2) \\
&= \frac{\varepsilon^2}{2}\xi^T \Big[f_p^T(\theta)H_{xp}(\theta) + 2\chi(\theta)f_p^T(\theta)\lambda^T(\theta, \theta + h)H_{x\tilde{p}}(\theta + h) \\
&\quad + \chi(\theta)f_{\tilde{p}}^T(\theta + h)H_{x\tilde{p}}(\theta + h)\Big]\xi + o(\varepsilon^2).
\end{aligned}
\tag{3.8}
$$

Following [10, 14, 17], we consider the matrix functions

$$
\Psi(s, \tau) = \int_{t_0}^{t_1} \lambda^T(s, t)H_{xx}(t)\lambda(\tau, t)dt - \lambda^T(s, t_1)\varphi_{xx}\big(x^0(t_1)\big)\lambda(\tau, t_1), \quad (s, \tau) \in I \times I,
\tag{3.9}
$$

$$
M_0[p, \tilde{p}](s, \tau) = f_p^T(s)\lambda^T(s, \tau)H_{x\tilde{p}}(\tau) + f_p^T(s)\Psi(s, \tau)f_{\tilde{p}}(\tau), \quad (s, \tau) \in I \times I.
\tag{3.10}
$$

where $\lambda(\cdot, \cdot)$ is the solution of the system (3.4).

Thus, substituting (3.6)–(3.8) in (3.2), allowing for (3.9), (3.10) and equality $\lambda(s, t) = 0$, for $s > t$, $(s, t) \in I \times I$, we get the validity of the following statement.

Proposition 3.1. Let conditions (A1) and (A2) be fulfilled, and the admissible control $u^0(\cdot) = (p(\cdot), q(\cdot))^T$ be singular (in the classic sense) and the condition (2.9) be fulfilled along it. Then, for each $\theta \in [t_0, t_1)$ and for all $\xi \in R^{r_0}$ the following expansion is valid:

$$\delta^2 S\left(u^0; \delta u(\cdot)\right) = -\varepsilon^2 \xi^T \left\{ M_0[p, p](\theta, \theta) + 2\chi(\theta) M_0[p, \tilde{p}](\theta, \theta + h) \right.$$
$$\left. + \chi(\theta) M_0[\tilde{p}, \tilde{p}](\theta + h, \theta + h) \right\} \xi + o\left(\varepsilon^2\right), \ \forall \varepsilon \in (0, \varepsilon_0),$$

(3.11)

where the number ε_0 was defined above (see Proposition 2.1), $\chi(\cdot)$ is the characteristic function of the set $[t_0, t_1 - h)$ and matrix functions $M_0[p, p](\theta, \theta)$, $M_0[p, \tilde{p}](\theta, \theta + h)$, $M_0[\tilde{p}, \tilde{p}](\theta + h, \theta + h)$ that are defined by (3.10).

4. Transformation of the second variation of the functional by means of modified variant of variations transformation method

4.1. Expansion of the second variation $\delta^2 S\left(u^0; \delta u(\cdot)\right)$ in Kelley-type variation (first-order transformation)

Let $u^0(\cdot)$ be a singular control satisfying condition (2.9), and assumptions (A1), (A3), and (A4) be fulfilled. Now, we proceed to generalize and apply the variation transformation method [13].

Introduce the following set dependent on the admissible control $u^0(\cdot)$:

$$I^* := I\left(u^0(\cdot)\right) = \left\{ \theta \in [t_1 - h, t_1) : \text{ the derivative } \dot{u}^0(\cdot) \text{ is continuous} \right.$$

or continuous from the right at the point θ and $\theta - h\} \cup \{\theta \in [t_0, t_1 - h) : \text{ the derivative}$ (4.1)

$\dot{u}^0(\cdot)$ is continuous or continuous from the right at the points θ and $\theta \pm h\}$.

The following properties are obvious: (1) $I \backslash I^*$ is a finite set and $t_1 \bar{\in} I^*$; (2) for every $\theta \in I^*$, there exists a sufficiently small number $\hat{\varepsilon} > 0$ such that $[\theta, \theta + \hat{\varepsilon}) \cup [\theta + h, \theta + h + \hat{\varepsilon}) \cap I \subset I^*$; and (3) by (1.2), (1.3), and (2.5), the derivatives $\dot{x}^0(\cdot)$, $\dot{\psi}^0(\cdot)$ are continuous or continuous from the right at every $\theta \in I^*$. These properties are important for our further reasoning, and we call them properties of the set I^*.

Require that the variation $\delta u(\cdot) = \left(\delta_0 p(\cdot), \delta q(\cdot)\right)^T$ satisfies additionally the following conditions as well:

$$\int\limits_{\theta}^{\theta+\varepsilon} \delta_0 p(t)dt = 0, \theta \in I^*, \delta_0 p(t) = 0, \delta q(t) = 0, t \in I_1 \setminus [\theta, \theta+\varepsilon), \ \varepsilon \in (0, \varepsilon^*), \tag{4.2}$$

where $\varepsilon^* = \min\{\varepsilon_0, \hat{\varepsilon}\}$, and $\varepsilon_0, \hat{\varepsilon}$ were defined above.

Make a passage from the variation $\delta u(t) = (\delta_0 p(t), \delta q(t))^T$, $t \in I_1$, satisfying (4.2), to a new variation $\delta_1 u(t) = (\delta_1 p(t), \delta q(t))^T, t \in I_1$, where

$$\delta_1 p(t) = \int\limits_{\theta}^{t} \delta_0 p(\tau)d\tau, \ t \in I_1. \tag{4.3}$$

Obvious,

$$\delta_1 p(t) = 0, t \in I_1 \setminus (\theta, \theta+\varepsilon). \tag{4.4}$$

Transform the variation of the trajectory as well: in place of $\delta x(t)$, $t \in I$, consider the function $\delta_1 x(t)$, $t \in I$:

$$\delta_1 x(t) = \delta x(t) - g_0[p](t)\delta_1 p(t) - g_0[\tilde{p}](t)\delta_1 p(t-h), t \in I, \tag{4.5}$$

where

$$g_0[\mu](t) := f_\mu(t), t \in I, \ \mu \in \{p, \tilde{p}\}. \tag{4.6}$$

As assumptions (A3) and (A4) are fulfilled, then by virtue of property of the set I^* we easily have: the function $\delta_1 x(t)$, $t \in I$ is continuous and $\delta_1 \dot{x}(t) \in \tilde{C}(I, R^n)$.

By direct differentiation, allowing for (A3), (A4) and (2.11), (4.3), (4.4) from (4.5) we obtain that $\delta_1 x(t)$, $t \in I$ is the solution of the system

$$\delta_1 \dot{x}(t) = f_x(t)\delta_1 x(t) + g_1[p](t)\delta_1 p(t) + g_1[\tilde{p}](t)\delta_1 p(t-h)$$
$$+ f_q(t)\delta q(t) + f_{\tilde{q}}(t)\delta q(t-h), t \in [\theta, t_1], \tag{4.7}$$

$$\delta_1 x(t) = 0, t \in [t_0, \theta], \delta_1 p(t) = 0, \delta q(t) = 0, \ t \in [t_0 - h, \theta), \tag{4.8}$$

where

$$g_1[\mu](t) := f_x(t)g_0[\mu](t) - \frac{d}{dt}g_0[\mu](t), \ t \in I, \ \mu \in \{p, \tilde{p}\}. \tag{4.9}$$

Now, let us write down the second variation (2.12) in terms of new variables. By (4.4) from (4.5), we have $\delta x(t_1) = \delta_1 x(t_1)$. According to this property and (4.2)–(4.6), for any $\varepsilon \in (0, \varepsilon^*)$ the second variation (2.12), after simple reasoning takes a new form

$$\delta^2 S(u^0; \delta u(\cdot)) = \sum_{i=1}^{4} \Delta_i, \tag{4.10}$$

where

$$
\begin{aligned}
\Delta_1 := {}& \delta_1 x^T(t_1)\varphi_{xx}\big(x^0(t_1)\delta_1 x(t_1)\big) - \int_\theta^{t_1} \delta_1 x^T(t)H_{xx}(t)\delta_1 x(t)\,dt \\
& -2\int_\theta^{\theta+\varepsilon} \big[\delta_1 x^T(x)H_{xx}(t)g_0[p](t) + \delta_1 x^T(t+h)H_{xx}(t+h)g_0[\tilde{p}](t+h)\big]\delta_1 p(t)\,dt - \\
& -2\int_\theta^{\theta+\varepsilon} \big[\delta_1 x^T(x)H_{xq}(t) + \delta_1 x^T(t+h)H_{x\tilde{q}}(t+h)\big]\delta q(t)\,dt,
\end{aligned}
\tag{4.11}
$$

$$
\begin{aligned}
\Delta_2 := {}& -\int_\theta^{\theta+\varepsilon} \Big\{ \delta_1 p^T(t)\big[g_0^T[p](t)H_{xx}(t)g_0[p](t) + g_0^T[\tilde{p}](t+h)H_{xx}(t+h)g_0[\tilde{p}](t+h)\big]\delta_1 p(t) \\
& +2\delta_1 p^T(t)\big[g_0^T[p](t)H_{xq}(t) + g_0^T[\tilde{p}](t+h)H_{x\tilde{q}}(t+h)\big]\delta q(t) \\
& +2\delta_0 p^T(t)\big[H_{pq}(t) + H_{\tilde{p}\tilde{q}}(t+h)\big]\delta q(t) + \delta q^T(t)\big[H_{qq}(t) + H_{\tilde{q}\tilde{q}}(t+h)\big]\delta q(t) \Big\}\,dt,
\end{aligned}
\tag{4.12}
$$

$$\Delta_3 := -2\int_\theta^{\theta+\varepsilon} \big[\delta_1 x^T(t)H_{xp}(t) + \delta_1 x^T(t+h)H_{x\tilde{p}}(t+h)\big]\delta_0 p(t)\,dt, \tag{4.13}$$

$$\Delta_4 := -2\int_\theta^{\theta+\varepsilon} \delta_1 p^T(t)\big[g_0^T[p](t)H_{xp}(t) + g_0^T[\tilde{p}](t+h)H_{x\tilde{p}}(t+h)\big]\delta_0 p(t)\,dt. \tag{4.14}$$

In the obtained representation, taking into account (A3), (A4), (4.2), (4.3), (4.7), (4.8), (4.13), (4.14) and the property of the set I^*, we transform Δ_3, Δ_4 by integration by parts. Then, we have

$$\Delta_3 := 2 \int_{\theta}^{\theta+\varepsilon} \left\{ \delta_1 x^T(t) \left[\frac{d}{dt} \left(H_{xp}(t) \right) + f_x^T(t) H_{xp}(t) \right] \right.$$

$$+ \delta_1 x^T(t+h) \left[\frac{d}{dt} \left(H_{x\tilde{p}}(t+h) \right) + f_x^T(t+h) H_{x\tilde{p}}(t+h) \right] \right\} \delta_1 p(t) dt$$

$$+ 2 \int_{\theta}^{\theta+\varepsilon} \delta_1 p^T(t) \left[g_1^T[p](t) H_{xp}(t) + g_1^T[\tilde{p}](t+h) H_{x\tilde{p}}(t+h) \right] \delta_1 p(t) dt$$

$$+ 2 \int_{\theta}^{\theta+\varepsilon} \delta_1 p^T(t) \left[H_{xp}(t) f_q(t) + H_{x\tilde{p}}^T(t+h) f_{\tilde{q}}(t+h) \right] \delta q(t) dt,$$

$$\Delta_4 := \int_{\theta}^{\theta+\varepsilon} \delta_0 p^T(t) \left[Q_0[p](t) + Q_0[\tilde{p}](t+h) \right] \delta_1 p(t) dt$$

$$+ \int_{\theta}^{\theta+\varepsilon} \delta_1 p^T(t) \left[\frac{d}{dt} \left(g_0^T[p](t) H_{xp}(t) \right) + \frac{d}{dt} \left(g_0^T[\tilde{p}](t+h) H_{x\tilde{p}}(t+h) \right) \right] \delta_1 p(t) dt,$$

where $g_1[\mu](\cdot), \mu \in \{p, \tilde{p}\}$ is defined by (2.19),

$$Q_0[\mu](t) := g_0^T[\mu](t) H_{x\mu}(t) - H_{x\mu}^T(t) g_0[\mu](t), \quad t \in I, \ \mu \in \{p, \tilde{p}\}. \tag{4.15}$$

By substituting these relations in (4.10), after elementary transformations considering (4.11) and (4.12), we arrive at the validity of the following statement.

Proposition 4.1. Let assumptions (A1), (A3), (A4), and conditions (2.6) be fulfilled. Also, let the functions $g_0[\mu](\cdot)$, $g_1[\mu](\cdot)$, $Q_0[\mu](\cdot)$ be defined by (4.6), (4.9), and (4.15), respectively, and $\delta_1 x(t)$, $t \in I$ be the solution of the system (4.7) and (4.8). Then along the singular control $u^0(\cdot)$, satisfying condition (2.9), and on the variations $\delta u(t) = \left(\delta_0 p(t), \delta q(t) \right)^T$, $t \in I_1$ satisfying (4.2), (4.3), the following representation (first-order transformation) is valid:

$$\delta^2 S\left(u^0; \delta u(\cdot)\right) = \Delta_1^{(2)} S\left(u^0; \delta_1 p, \delta q, \delta_1 x, \varepsilon\right) + \Delta_2^{(2)} S\left(u^0; \delta_0 p, \delta_1 p, \delta q, \varepsilon\right), \forall \varepsilon \in \left(0, \varepsilon^*\right). \tag{4.16}$$

Here

$$\Delta_1^{(2)} S\left(u^0; \delta_1 p, \delta q, \delta_1 x, \varepsilon\right) = \delta_1 x^T(t_1) \varphi_{xx}\left(x^0(t_1)\right) \delta_1 x(t_1) - \int_{\theta}^{t_1} \delta_1 x^T(t) H_{xx}(t) \delta_1 x(t) dt$$

$$- 2 \int_{\theta}^{\theta+\varepsilon} \left\{ \left[\delta_1 x^T(t) G_1[p](t) + \delta_1^T x(t+h) G_1[\tilde{p}](t+h) \right] \delta_1 p(t) \right. \tag{4.17}$$

$$+ \left[\delta_1 x^T(t) H_{xq}(t) + \delta_1 x^T(t+h) H_{x\tilde{q}}(t+h) \right] \delta q(t) \right\} dt$$

$$\Delta_2^{(2)} S\left(u^0; \delta_0 p, \delta_1 p, \delta q, \varepsilon\right) = \int_\theta^{\theta+\varepsilon} \left\{ \delta_1 p^T(t)\left[L_1[p](t) + L_1[\tilde{p}](t+h)\right]\delta_1 p(t) \right.$$
$$+ 2\delta_1 p^T(t)\left[P_1[p,q](t) + P_1[\tilde{p},\tilde{q}](t+h)\right]\delta q(t) +$$
$$\delta_0 p^T(t)\left[Q_0[p](t) + Q_0[\tilde{p}](t+h)\right]\delta_1 p(t) \tag{4.18}$$
$$- 2\delta_0 p^T(t)\left[H_{pq}(t) + H_{\tilde{p}\tilde{q}}(t+h)\right]\delta q(t)$$
$$\left. - \delta q^T(t)\left[H_{qq}(t) + H_{\tilde{q}\tilde{q}}(t+h)\right]\delta q(t)\right\} dt,$$

where ε^* was defined above (see (4.2)),

$$G_1[\mu](t) := H_{xx}(t)g_0[\mu](t) - f_x^T(x)H_{x\mu}(t) - \frac{d}{dt}H_{x\mu}(t), \ t\in I, \mu\in\{p,\tilde{p}\}, \tag{4.19}$$

$$P_1[p,q](t) := H_{xp}^T(t)f_q(t) - g_0^T[p](t)H_{xq}(t), t\in I,$$
$$P_1[\tilde{p},\tilde{q}](t) := H_{x\tilde{p}}^T(t)f_{\tilde{q}}(t) - g_0^T[\tilde{p}](t)H_{x\tilde{q}}(t), t\in I, \tag{4.20}$$

$$L_1[\mu](t) := -g_0^T[\mu](t)H_{xx}(t)g_0[\mu](t) + 2g_1^T[\mu](t)H_{x\mu}(t)$$
$$+ \frac{d}{dt}\left(g_0^T[\mu](t)H_{x\mu}(t)\right), t\in I, \mu\in\{p,\tilde{p}\}. \tag{4.21}$$

4.2. Higher-order transformation

Let $\left(u^0(\cdot), x^0(\cdot)\right)$ be some process, where $u^0(\cdot)$ is a singular control satisfying condition (2.9), and assumptions (A1), (A5), and (A6) be fulfilled. Introduce the matrix functions calculated along the process $\left(u^0(\cdot), x^0(\cdot)\right)$ and determined by the following recurrent formulas:

$$g_{i+1}[\mu](t) = f_x(t)g_i[\mu](t) - \frac{d}{dt}g_i[\mu](t),$$
$$g_0[\mu](t) := f_\mu(t), \ t\in I, \ \mu\in\{p,\tilde{p}\}, i = 0,1,\ldots, \tag{4.22}$$

$$G_{1+1}[\mu](t) = H_{xx}(t)g_i[\mu](t) - f_x^T(t)G_i[\mu](t) - \frac{d}{dt}G_i[\mu](t),$$
$$G_0[\mu](t) := H_{x\mu}(t), \ t\in I, \mu\in\{p,\tilde{p}\}, i = 0,1,\ldots \tag{4.23}$$

Furthermore, similar to (4.15), (4.20), (4.21), and (3.10), consider the functions

$$P_{i+1}[p,q](t) = G_i^T[p](t)f_q(t) - g_i^T[p](t)H_{xq}(t), P_0[p,q](t) := H_{pq}(t), t \in I, i = 0,1,\dots,$$
$$P_{i+1}[\tilde{p},\tilde{q}](t) = G_i^T[\tilde{p}](t)f_{\tilde{q}}(t) - g_i^T[\tilde{p}](t)H_{x\tilde{q}}(t), P_0[\tilde{p},\tilde{q}](t) := H_{\tilde{p}\tilde{q}}(t), t \in I,$$

(4.24)

$$Q_i[\mu](t) = g_i^T[\mu](t)G_i[\mu](t) - G_i^T[\mu](t)g_i[\mu](t), \ \mu \in \{p,\tilde{p}\}, t \in I, i = 0,1,\dots,$$

(4.25)

$$L_{i+1}[\mu](t) = -g_i^T[\mu](t)H_{xx}(t)g_i[\mu](t) + 2g_{i+1}^T[\mu](t)G_i[\mu](t) + \frac{d}{dt}\left(g_i^T[\mu](t)G_i[\mu](t)\right),$$
$$L_0[\mu](t) := H_{\mu\mu}(t), \ \mu \in \{p,\tilde{p}\}, t \in I, i = 0,1,\dots,$$

(4.26)

$$M_i[p,\tilde{p}](s,\tau) := g_i^T[p](s)\lambda^T(s,\tau)G_i[\tilde{p}](\tau)$$
$$+ g_i^T[p](s)\Psi(s,\tau)g_i[\tilde{p}](\tau), (s,\tau) \in I \times I, \ i = 0,1,\dots,$$

(4.27)

where $\lambda(\cdot)$ and $\Psi(\cdot)$ are determined by (3.4) and (3.9), respectively.

Similar to I^*, we introduce the set I^{**} when assumption (A6) is fulfilled:

$I^{**} := I(u^0(\cdot)) = \{\theta \in [t_1 - h, t_1): $ the admissible control $u^0(\cdot)$ is sufficiently smooth or sufficiently smooth from the right at the points θ and $\theta - h\} \cup \{\theta \in [t_0, t_1 - h): $ the admissible control $u^0(\cdot)$ is sufficiently smooth or sufficiently smooth

$$\text{from the right at the points } \theta \text{ and } \theta \pm h\}.$$

(4.28)

The following obvious properties hold: (1) $I \backslash I^{**}$ is a finite set, and $t_1 \overline{\in} I^{**}$, also $I^{**} \subset I^*$; (2) for every $\theta \in I^{**}$ there exists a sufficiently small number $\tilde{\varepsilon} > 0$, such that $[\theta, \theta + \hat{\varepsilon}) \cup [\theta + h, \theta + h + \hat{\varepsilon}) \cap I \subset I^*$, furthermore, (3) by (A5), (A6), (1.2), (1.3), and (2.5), the functions $x^0(\cdot), \psi^0(\cdot)$ are continuous and sufficiently smooth or sufficiently smooth from the right at every point $\theta \in I^{**}$. These properties are important at the next reasoning and we call them the properties of the set I^{**}.

Let us consider a variation $\delta u(t) = (\delta_0 p(t), \delta q(t))^T, \ t \in I_1$ that in addition satisfies the following conditions as well:

$$\delta_0 p(t) = 0, \ \delta q(t) = 0, t \in I_1 \backslash [\theta, \theta + \varepsilon),$$
$$\delta_1 p(t) = \dots = \delta_k p(t) = 0, t \in I_1 \backslash (\theta, \theta + \varepsilon),$$

(4.29)

where

$$\delta_i p(t) = \int_\theta^t \delta_{i-1} p(\tau) d\tau, \ t \in I_1, \ i = 1, 2, ..., k, \ k \in \{1, 2, ...\}, \tag{4.30}$$

$\theta \in I^{**}, \quad \varepsilon \in (0, \varepsilon^{**}), \quad \varepsilon^{**} = \min\{\varepsilon_0, \hat{\varepsilon}, \tilde{\varepsilon}\}$ ($\varepsilon_0, \hat{\varepsilon}, \tilde{\varepsilon}$ were defined above).

According to (4.30), we have

$$\delta_i p(t) = \int_\theta^t \frac{(t-\tau)^{i-1}}{(i-1)!} \delta_0 p(\tau) d\tau, \theta \in I^{**}, t \in I_1, \ i = 1, 2, ..., k, \ k \in \{1, 2, ...\}. \tag{4.31}$$

The following statement is valid.

Proposition 4.2. Let assumptions (A1), (A5), (A6), and condition (2.6) be fulfilled. Furthermore, let the functions $g_i[\mu](\cdot)$, $G_i[\mu](\cdot)$, $P_i[p, q](\cdot)$, $P_i[\tilde{p}, \tilde{q}](\cdot)$, $Q_i[\mu](\cdot)$ and $L_i[\mu](\cdot)$, where $\mu \in \{p, \tilde{p}\}$, $i = 0, 1, ...$, be defined by (4.22)–(4.26), and the set I^{**} be defined by (4.28). Then along the singular control $u^0(\cdot)$, satisfying condition (2.9), and on the variations $\delta u(t) = (\delta_0 p(t), \delta q(t))^T$, $t \in I_1$ satisfying (4.29) and (4.30), the following representation (k-th order transformation, where $k \in \{1, 2, ...\}$) is valid:

$$\delta^2 S(u^0; \delta u) = \Delta_1^2 S(u^0; \delta_k p, \delta q, \delta_k x, \varepsilon) + \Delta_2^2 S(u^0; \delta_0 p, ..., \delta_k p, \delta q, \varepsilon). \tag{4.32}$$

Here

$$\begin{aligned}
\Delta_1^2 S(\cdot) = {} & \delta_k x^T(t_1) \varphi_{xx}(x^0(t_1)) \delta_k x(t_1) - \int_\theta^{t_1} \delta_k x^T(t) H_{xx}(t) \delta_k x(t) dt \\
& -2 \int_\theta^{\theta+\varepsilon} \Big\{ \big[\delta_k x^T(t) G_k[p](t) + \delta_k x^T(t+h) G_k[\tilde{p}](t+h) \big] \delta_k p(t) \\
& + \big[\delta_k x^T(t) H_{xq}(t) + \delta_k x^T(t+h) H_{x\tilde{q}}(t+h) \big] \delta q(t) \Big\} dt,
\end{aligned} \tag{4.33}$$

$$\begin{aligned}
\Delta_2^2 S(\cdot) = {} & \int_\theta^{\theta+\varepsilon} \Big\{ \sum_{i=0}^{k-1} \big[\delta_{i+1} p^T(t) \big(L_{i+1}[p](t) + L_{i+1}[\tilde{p}](t+h) \big) \delta_{i+1} p(t) dt \\
& + 2\delta_{i+1} p(t) \big(P_{i+1}[p, q](t) + P_{i+1}[\tilde{p}, \tilde{q}](t+h) \big) \delta q(t) \\
& + \delta_i p^T(t) \big(Q_i[p](t) + Q_i[\tilde{p}](t+h) \big) \delta_{i+1} p(t) \big] \\
& - 2\delta_0 p^T(t) \big(P_0[p, q](t) + P_0[\tilde{p}, \tilde{q}](t+h) \big) \delta q(t) \\
& - \delta q^T(t) \big(H_{qq}(t) + H_{\tilde{p}, \tilde{q}}(t+h) \big) \delta q(t) \Big\} dt,
\end{aligned} \tag{4.34}$$

where $\theta \in I^{**}$, $\varepsilon \in (0, \varepsilon^{**})$ (the number ε^{**} was defined above), $\delta_k x(t)$, $t \in I$ is the solution of the system

$$\delta_k \dot{x}(t) = f_x(t)\delta_k x(t) + g_k[p](t)\delta_k p(t) + g_k[\tilde{p}](t)\delta_k p(t-h)$$
$$+ f_q(t)\delta q(t) + f_{\tilde{q}}(t)\delta q(t-h), t \in [\theta, t_1] \tag{4.35}$$
$$\delta_k x(t) = 0, \ t \in [t_0, \theta], \ \delta_k p(t) = 0, \ \delta q(t) = 0, \ t \in [t_0 - h, \theta), \ k \in \{1, 2, \dots\},$$

Proof. We carry out the proof of Proposition 4.2 by induction. For $k = 1$, Proposition 4.2 was completely proved at item 4 (see Proposition 4.1). Assume that Proposition 4.2 is valid for all the cases to $(k-1)$ inclusively, $(k \geq 2)$. We prove the validity of representation (4.32) for the case k. Let the variation $\delta u(t) = \left(\delta_0 p(t), \delta q(t)\right)^T, t \in I_1$ satisfies the conditions (4.29) and (4.30). Then by assumption the following representation is valid:

$$\delta^2 S(u^0; \delta u) = \Delta_1^2 S(u^0; \delta_{k-1}p, \delta q, \delta_{k-1}x, \varepsilon) + \Delta_2^2 S(u^0; \delta_0 p, \dots, \delta_{k-1}p, \delta q, \varepsilon). \tag{4.36}$$

Here

$$\Delta_1^2 S(u^0; \delta_{k-1}p, \delta q, \delta_{k-1}x, \varepsilon) = \delta_{k-1}x^T(t_1)\varphi_{xx}(x^0(t_1))\delta_{k-1}x(t_1) - \int_\theta^{t_1} \delta_{k-1}x^T(t)H_{xx}(t)\delta_{k-1}x(t)dt$$
$$-2\int_\theta^{\theta+\varepsilon} \left\{ \left[\delta_{k-1}x^T(t)G_{k-1}[p](t) + \delta_{k-1}x^T(t+h)G_{k-1}[\tilde{p}](t+h) \right]\delta_{k-1}p(t) \right. \tag{4.37}$$
$$\left. + \left[\delta_{k-1}x^T(t)H_{xq}(t) + \delta_{k-1}x^T(t+h)H_{x\tilde{q}}(t+h) \right]\delta q(t) \right\} dt,$$

$$\Delta_2^2 S(u^0; \delta_0 p, \dots, \delta_{k-1}p, \delta q, \varepsilon) =$$
$$\int_\theta^{\theta+\varepsilon} \left\{ \sum_{i=0}^{k-2} \left[\delta_{i+1}p^T(t)\left(L_{i+1}[p](t) + L_{i+1}[\tilde{p}](t+h)\right)\delta_{i+1}p(t) \right. \right.$$
$$+ 2\delta_{i+1}p^T(t)\left(P_{i+1}[p,q](t) + P_{i+1}[\tilde{p},\tilde{q}](t+h)\right)\delta q(t) \tag{4.38}$$
$$\left. + \delta_i p^T(t)\left(Q_i[p](t) + Q_i[\tilde{p}](t+h)\right)\delta_{i+1}p(t) \right] -$$
$$2\delta_0 p^T(t)\left(P_0[p,q](t) + P_0[\tilde{p},\tilde{q}](t+h)\right)\delta q(t)$$
$$\left. - \delta q^T(t)\left(H_{qq}(t) + H_{\tilde{q}\tilde{q}}(t+h)\right)\delta q(t) \right\} dt,$$

where $G_i[\mu](\cdot)$, $P_i[p,q](\cdot)$, $P_i[\tilde{p},\tilde{q}](\cdot)$, $Q_i[\mu](\cdot)$, $L_i[\mu](\cdot)$, $\mu \in \{p, \tilde{p}\}$, $i = 0, 1, \dots$ are defined by (4.23)-(4.26), and $\delta_{k-1}x(t), t \in I$ is the solution of the system:

$$\delta_{k-1}\dot{x}(t) = f_x(t)\delta_{k-1}x(t) + g_{k-1}[p](t)\delta_{k-1}p(t) + g_{k-1}[\tilde{p}](t)\delta_{k-1}p(t-h)$$
$$+ f_q(t)\delta q(t) + f_{\tilde{q}}(t)\delta q(t-h),$$
$$\delta_{k-1}x(t) = 0,\ t \in [t_0,\theta],\ \delta_{k-1}p(t) = 0,\ \delta q(t) = 0,\ t \in [t_0-h,\theta),\ k \geq 2. \tag{4.39}$$

Apply the modified variant of variations transformations method [13] to the system for $\delta_{k-1}x(t), t \in I$ and representation (4.36). According to the technique of the previous item (see item 4.1), we introduce a new variation in the following way:

$$\delta_k x(t) = \delta_{k-1}x(t) - g_{k-1}[p](t)\delta_k p(t) - g_{k-1}[\tilde{p}](t)\delta_k p(t-h), t \in I. \tag{4.40}$$

According to (4.22), (4.30), (4.31), and (4.39) from (4.40) by direct differentiation, we get the system (4.35) for $\delta_k x(t), t \in I$. Furthermore, as $\theta \in I^{**}$, then by (4.40) we get $\delta_k x(t_1) = \delta_{k-1}x(t_1)$. Taking into account this equality and by (4.29), (4.30), and (4.40) in (4.37), let us transform the representation (4.36) into new variables $\delta_k p(\cdot)$, $\delta q(\cdot)$, $\delta_k x(\cdot)$. Then,

$$\delta^2 S(u^0;\delta u) = \delta_k x^T(t_1)\varphi_{xx}(x^0(t_1))\delta_k x(t_1)$$
$$-\Delta_{11} - \Delta_{12} - \Delta_{13} + \Delta_2^2 S(u^0;\delta_0 p,...,\delta_{k-1}p,\delta q,\varepsilon), \tag{4.41}$$

where $\Delta_2^2 S(\cdot)$ is determined by formula (4.38) as well as Δ_{i1}, $i = 1,2,3$ by (4.22), (4.29), (4.30), (4.35), (2.6) are calculated in the following way:

$$\Delta_{11} = \int_\theta^{t_1} \delta_k x^T(t)H_{xx}(t)\delta_k x(t)dt$$
$$+2\int_\theta^{\theta+\varepsilon}\left[\delta_k x^T(t)H_{xx}(t)g_{k-1}[p](t) + \delta_k x^T(t+h)H_{xx}(t+h)g_{k-1}[\tilde{p}](t+h)\right]\delta_k p(t)$$
$$+\int_\theta^{\theta+\varepsilon}\delta_k p^T(t)\left[g_{k-1}^T[p](t)H_{xx}(t)g_{k-1}[p](t)\right.$$
$$\left. + g_{k-1}^T[\tilde{p}](t+h)H_{xx}(t+h)g_{k-1}[\tilde{p}](t+h)\right]\delta_k p(t)dt, \tag{4.42}$$

$$\Delta_{12} = 2\int_\theta^{\theta+\varepsilon}\left\{\left[\delta_k x^T(t) + \delta_k p^T(t)g_{k-1}^T[p](t)\right]G_{k-1}[p](t)\right.$$
$$\left. + \left[\delta_k x^T(t+h) + \delta_k p^T(t)g_{k-1}^T[\tilde{p}](t+h)\right]G_{k-1}[\tilde{p}](t+h)\right\}\delta_{k-1}p(t)dt =: \Delta_{12}^* + \Delta_{12}^{**}, \tag{4.43}$$

where

$$\Delta_{12}^{*} := 2\int_{\theta}^{\theta+\varepsilon} \Big[\delta_k x^T(t)G_{k-1}[p](t) + \delta_k x(t+h)G_{k-1}[\tilde{p}](t+h)\Big]\delta_{k-1}p(t)dt,$$

$$\Delta_{12}^{**} := 2\int_{\theta}^{\theta+\varepsilon} \delta_k p^T(t)\Big[g_{k-1}^T[p]G_{k-1}[p](t) + g_{k-1}^T[\tilde{p}](t+h)G_{k-1}[\tilde{p}](t+h)\Big]\delta_{k-1}p(t)dt,$$

$$\Delta_{13} = 2\int_{\theta}^{\theta+\varepsilon} \Big\{\Big[\delta_k x^T(t) + \delta_k p^T(t)g_{k-1}^T[p](t)\Big]H_{xq}(t)$$

$$+\Big[\delta_k x^T(t+h) + \delta_k p^T(t)g_{k-1}^T[\tilde{p}](t+h)\Big]H_{x\tilde{q}}(t+h)\Big\}\delta q(t)dt \tag{4.44}$$

$$= 2\int_{\theta}^{\theta+\varepsilon} \Big[\delta_k x^T(t)H_{xq}(t) + \delta_k x^T(t+h)H_{x\tilde{q}}(t+h)\Big]\delta q(t)dt$$

$$+2\int_{\theta}^{\theta+\varepsilon} \delta_k p^T(t)\Big[g_{k-1}^T[p](t)H_{xq}(t) + g_{k-1}^T[\tilde{p}](t+h)H_{x\tilde{q}}(t+h)\Big]\delta q(t)dt.$$

Taking into account (A5), (A6), (4.29), (4.30), (4.35), and the properties of the set $I**$, let us calculate $\Delta_{12}^{*}, \Delta_{12}^{**}$. Then, applying the method of integration by parts, we have

$$\Delta_{12}^{*} = -2\int_{\theta}^{\theta+\varepsilon} \Big\{\Big[\delta_k x^T(t)\Big(f_x^T(t)G_{k-1}[p](t) + \frac{d}{dt}\big(G_{k-1}[p](t)\big)\Big)$$

$$+\delta_k x^T(t+h)\Big(f_x^T(t+h)G_{k-1}[\tilde{p}](t+h) + \frac{d}{dt}\big(G_{k-1}[\tilde{p}](t+h)\big)\Big)\Big]\delta_k p(t)$$

$$+\delta_k p^T(t)\Big[g_k^T[p](t)G_{k-1}[p](t) + g_k^T[\tilde{p}](t+h)G_{k-1}[\tilde{p}](t+h)\Big]\delta_k p(t)$$

$$+\delta_k p^T(t)\Big[G_{k-1}^T[p](t)f_q(t) + G_{k-1}^T[\tilde{p}](t+h)f_{\tilde{q}}(t+h)\Big]\delta q(t)\Big\}dt;$$

$$\Delta_{12}^{**} = -\int_{\theta}^{\theta+\varepsilon} \delta_{k-1}p^T(t)\Big[Q_{k-1}[p](t) + Q_{k-1}[\tilde{p}](t+h)\Big]\delta_k p(t)dt$$

$$-\int_{\theta}^{\theta+\varepsilon} \delta_k p^T(t)\Big[\frac{d}{dt}\big(g_{k-1}^T[p](t)G_{k-1}[p](t)\big) + \frac{d}{dt}\big(g_{k-1}^T[\tilde{p}](t+h)G_{k-1}[\tilde{p}](t+h)\big)\Big]\delta_k p(t)dt.$$

At first, we substitute the last expression Δ_{12}^{*}, Δ_{12}^{**} in (4.43), and then (4.42)–(4.44) in (4.41). Then by (4.23)–(4.26), (4.33), (4.34), and (4.38), it is easy to get representation (4.32). Consequently, we get the proof for k. This completes the proof of Proposition 4.2.

5. Optimality conditions

Based on Propositions 3.1, 4.1, and 4.2, we prove the following theorem.

Theorem 5.1. Let conditions (A1), (A5), and (A6) be fulfilled, and the matrix functions $P_i[p,q](\cdot), P_i[\tilde{p},\tilde{q}](\cdot),$ $Q_i[\mu](\cdot), L_i[\mu](\cdot), M_i[p,\tilde{p}](\cdot),$ $\mu \in \{p,\tilde{p}\},$ $i = 0,1,\ldots$ be defined as in

(4.24)–(4.27). Let also the set I^{**} be defined as in (4.28) and along the singular (in the classical sense) control $u^0(\cdot)$ the following equalities be fulfilled:

$$L_i[p](t) + \chi(t)L_i[\tilde{p}](t+h) = 0, \ \forall t \in I^{**}, i = 0,1,...,k, \ k \in \{0,1,...\}, \tag{5.1}$$

where $\chi(\cdot)$ is the characteristic function of the set $[t_0, t_1 - h)$.

Then for the optimality of the admissible control $u^0(\cdot)$, it is necessary that the relations

$$P_i[p,q](\theta) + \chi(t)P_i[\tilde{p},\tilde{q}](\theta + h) = 0, \ i = 0,1,...,k, \tag{5.2}$$

$$\begin{aligned}
\xi^T \{ M_i[p,p](\theta,\theta) + 2\chi(t)M_i[p,\tilde{p}](\theta,\theta+h) \\
+ \chi(\theta)M_i[\tilde{p},\tilde{p}](\theta+h,\theta+h) \} \xi \le 0, \ i = 0,1,...,k,
\end{aligned} \tag{5.3}$$

$$Q_i[p](\theta) + \chi(\theta)Q_i[\tilde{p}](\theta + h) = 0, \ i = 0,1,...,k, \tag{5.4}$$

$$\begin{aligned}
\mathcal{L}_{k+1}(\theta,\xi,\eta) &:= \xi^T \left(L_{k+1}[p](\theta) + \chi(\theta)L_{k+1}[\tilde{p}](\theta+h) \right)\xi + \\
&+ 2\xi^T \left(P_{k+1}[p,q](\theta) + \chi(\theta)P_{k+1}[\tilde{p},\tilde{q}](\theta+h) \right)\eta - \\
&\quad - \eta^T \left(H_{qq}(\theta) + \chi(\theta)H_{\tilde{q}\tilde{q}}(\theta+h) \right)\eta \ge 0,
\end{aligned} \tag{5.5}$$

be fulfilled for all $\theta \in I^{**}$, $\xi \in R^{r_0}$ and $\eta \in R^{r_1}$.

Proof. Let $u^0(\cdot)$ be an optimal control. We will prove the theorem by induction. Let $k = 0$, that is, $i = 0$. Then, according to (4.24) and (2.10) we get the proof of optimality condition (5.2) for $k = 0$. The proof of optimality condition (5.3) for $k = 0$ directly follows from (3.11) allowing for (2.1) (see Proposition 3.1). Now, based on Proposition 4.1 prove the optimality conditions (5.4) and (5.5) for $k = 0$.

We first prove the validity of (5.4) for $k=0$.

Suppose that

$$\delta_0 p_m(t) = 0, \ \forall t \in I_1, \forall m \in \{1,2,...,r_0\} \setminus \{i,j\} \tag{5.6}$$

$$\delta_0 p_i(t) = \begin{cases} \alpha l_1\left(\dfrac{2(t-\theta)}{\varepsilon}-1\right), & t \in [\theta, \theta+\varepsilon), \ \varepsilon \in \left(0, \varepsilon^{**}\right), \\ 0, & t \in I_1 \setminus [\theta, \theta+\varepsilon), \end{cases}$$

$$\delta_0 p_j(t) = \begin{cases} \beta l_2\left(\dfrac{2(t-\theta)}{\varepsilon}-1\right), & t \in [\theta, \theta+\varepsilon), \ \varepsilon \in \left(0, \varepsilon^{**}\right), \\ 0, & t \in I_1 \setminus [\theta, \theta+\varepsilon), \end{cases}$$

$$\delta q(t) = 0, \ t \in I_1$$

where i, j $(i \neq j)$ are arbitrary fixed points of the set $\{1, 2, ..., r_0\}$ and $\delta_0 p_k(\cdot)$ is the k-th coordinate of the vector $\delta_0 p(\cdot)$; $\alpha, \beta \in R$ and $\theta \in I^{**}$ are arbitrary fixed points, the functions $l_1(s) = s$, $l_2(s) = \frac{3}{2}s^2 - \frac{1}{2}$, $s \in [-1, 1]$ are the Legendre polynomials.

It is clear that the variation $\delta u(t) = (\delta_0 p(t), \delta q(t))^T$, $t \in I_0 \cup I =: I_1$, defined by (5.6) satisfies the condition (4.2) and, according to (5.6) the function $\delta_1 p(t)$, $t \in I_1$, defined by (4.3) is of order ε, and the solution $\delta_1 x(t)$, $t \in I$ of the system (4.7), (4.8) is of order ε^2. Also, according to (4.15) it is easy to see that for every $t \in I$ the matrix $Q_0[p](t) + \chi(t)Q_0[\tilde{p}](t+h)$ is skew-symmetric. Therefore, by Proposition 4.1 and condition (2.6), considering (2.1), (4.3), (4.17), (4.18), and the properties of the set I^{**}, along the singular optimal control $u^0(\cdot)$, we have

$$\delta^2 S\left(u^0; \delta u(\cdot)\right) = \int_{\theta}^{\theta+\varepsilon} \delta_0 p^T(t)\left[Q_0[p](t) + \chi(t)Q_0[\tilde{p}](t+h)\right]\delta_1 p(t)\,dt + o\left(\varepsilon^2\right)$$

$$= \int_{\theta}^{\theta+\varepsilon}\left[q_{ij}^{(0)}(t)\delta_0 p_i(t)\delta_1 p_j(t) + q_{ji}^{(0)}(t)\delta_0 p_j(t)\delta_1 p_i(t)\right]dt + o\left(\varepsilon^2\right)$$

$$= \frac{\varepsilon^2}{4}\alpha\beta\left[q_{ij}^{(0)}(\theta) - q_{ji}^{(0)}(\theta)\right]\int_{-1}^{1} l_1(s)\int_{-1}^{s} l_2(\tau)\,d\tau\,ds + o\left(\varepsilon^2\right)$$

$$= -\frac{\varepsilon^2 \alpha\beta}{30}\left[q_{ij}^{(0)}(\theta) - q_{ji}^{(0)}(\theta)\right] + o\left(\varepsilon^2\right) \geq 0, \ \forall \varepsilon \in \left(0, \varepsilon^{**}\right)$$

where $q_{ij}^{(0)}(\theta)$, $q_{ji}^{(0)}(\theta)$ are the elements of the matrix $Q_0[p](\theta) + \chi(\theta)Q_0[\tilde{p}](\theta + h)$.

Then, we conclude from the arbitrariness of $\alpha, \beta \in R$, $\theta \in I^{**}$ and $i, j \in \{1, 2, ..., r_0\}$, $i \neq j$ that the skew-symmetric matrix $Q_0[p](\theta) + \chi(\theta)Q_0[\tilde{p}](\theta + h)$ is also symmetric. Consequently, for

every $t \in I^{**}$ we have $Q_0[p](t) + \chi(t)Q_0[\tilde{p}](t+h) = 0$. This completes the proof of the optimality condition (5.4) for $k = 0$.

To prove statement (5.5) for $k = 0$, under the conditions (4.2) and (4.3), we write down the vector components of the variation $\delta u(\cdot) = (\delta_0 p(\cdot), \delta q(\cdot))^T$ in the following form:

$$
\delta_0 p(t) = \begin{cases} \xi l_1 \left(\dfrac{2(t-\theta)}{\varepsilon} - 1 \right), & t \in [\theta, \theta+\varepsilon), \\ 0 & t \in I_1 \setminus [\theta, \theta+\varepsilon), \ \varepsilon \in (0, \varepsilon^{**}), \end{cases}
$$
$$
\delta q(t) = \begin{cases} \eta \displaystyle\int_\theta^t l_1 \left(\dfrac{2(s-\theta)}{\varepsilon} - 1 \right) ds, & t \in [\theta, \theta+\varepsilon), \\ 0, & t \in I_1 \setminus [\theta, \theta+\varepsilon), \ \varepsilon \in (0, \varepsilon^{**}), \end{cases} \tag{5.7}
$$

where $l_1(\tau) = \tau$, $\tau \in [-1, 1]$ is a Legendre polynomial, $\xi \in R^{r_0}$, $\eta \in R^{r_1}$, $\theta \in I^{**}$ are arbitrary fixed points.

According to (4.2), (4.3), (4.7), (4.8), and (5.7), it is easy to prove that

$$
\delta_1 p(t) \sim \varepsilon, \delta q(t) \sim \varepsilon, t \in I_1, \delta_1 x(t) \sim \varepsilon^2, t \in I.
$$

In view of the last relations and above proved condition (5.4) (for the case $k = 0$) taking into account the properties of the set I^{**} and the relations (2.1), (4.3), (4.17), (4.18), and (5.7) from (4.16), we obtain the following relation along the singular optimal control $u^0(t), t \in I_1$:

$$
\begin{aligned}
\delta^2 S(u^0; \delta u(\cdot)) &= \int_\theta^{\theta+\varepsilon} \left\{ \delta_1 p^T(t) \left[L_1[p](t) + \chi(t)L_1[\tilde{p}](t+h) \right] \delta_1 p(t) \right. \\
&\quad + 2\delta_1 p^T(t) \left[P_1[p,q](t) + \chi(t)P_1[\tilde{p},\tilde{q}](t+h) \right] \delta q(t) \\
&\quad \left. - \delta q^T(t) \left[H_{qq}(t) + \chi(t) H_{\tilde{q}\tilde{q}}(t+h) \right] \delta q(t) \right\} dt + o(\varepsilon^3) \\
&= \frac{\varepsilon^3}{8} \left\{ \xi^T \left[L_1[p](\theta) + \chi(\theta)L_1[\tilde{p}](\theta+h) \right] \xi + 2\xi^T \left[P_1[p,q](\theta) + \chi(\theta)P_1[\tilde{p},\tilde{q}](\theta+h) \right] \eta \right. \\
&\quad \left. - \eta^T \left[H_{qq}(\theta) + \chi(\theta) H_{\tilde{q}\tilde{q}}(\theta+h) \right] \eta \right\} \int_{-1}^1 \left(\int_{-1}^t l_1(\tau) d\tau \right)^2 dt + o(\varepsilon^3) \geq 0, \forall \varepsilon \in (0, \varepsilon^*).
\end{aligned}
$$

Hence, taking into account the arbitrariness of $\theta \in I^*$, $\xi \in R^{r_0}$ and $\eta \in R^{r_1}$, we easily get the validity of the optimality condition (5.5) for $k = 0$.

Now suppose that all the statements of Theorem 5.1 are valid for $i = 1, 2, ..., k - 1$ ($k \geq 2$) as well. Prove statements (5.2)–(5.5), for $i = k$. By assumption, the inequality $\mathcal{L}_k(\theta, \xi, \eta) \geq 0$ (see (5.5) for the case k-1) is valid for all $\theta \in I^{**}$, $\xi \in R^{r_0}$ and $\eta \in R^{r_1}$. Hence, taking into account (5.1), we have

$$2\xi\left(P_k[p,q](\theta) + \chi(\theta)P_k[\tilde{p}, \tilde{q}](\theta + h)\right)\eta$$
$$-\eta^T\left(H_{qq}(\theta) + \chi(\theta)H_{\tilde{q}\tilde{q}}(\theta + h)\right)\eta \geq 0, \forall \theta \in I^{**}, \forall \xi \in R^{r_0}, \forall \eta \in R^{r_1}.$$

From this inequality, we easily get that $P_k[p, q](\theta) + \chi(\theta)P_k[\tilde{p}, \tilde{q}](\theta + h) = 0$, that is, we get the validity of optimality condition (5.2) for $i = k$.

Now, prove the validity of condition (5.3) for $i = k$. In formula (4.32), we put

$$\delta_0 p(t) = \begin{cases} \xi l_k\left(\dfrac{2(t - \theta)}{\varepsilon} - 1\right), & t \in [\theta, \theta + \varepsilon), \\ 0 & , t \in I_1 \setminus [\theta + \varepsilon), \end{cases} \qquad \delta q(t) = 0, t \in I_1, \tag{5.8}$$

where $l_k(\tau)$, $\tau \in [-1, 1]$ is the k-th Legendre polynomial, $\varepsilon \in (0, \varepsilon^{**})$ which the number ε^{**} is defined above (see (4.30)) and $\theta \in I^{**}$, $\xi \in R^{r_0}$.

Obviously, conditions (4.29) and (4.30) are fulfilled for variation (5.8).

As the conditions $L_i[p](t) + \chi(t)L_i[\tilde{p}](t) = 0$, $t \in I^{**}$, $i = \overline{0, k}$ and $Q_i[p](t) + \chi Q_i[\tilde{p}](t + h) = 0$, $t \in I^{**}, i = \overline{0, k - 1}$, are fulfilled, then by (4.33), (4.34), and (5.8), formula (4.32) takes the form:

$$\delta^2 S(u^0; \delta u) = \delta_k x^T(t_1)\varphi_{xx}(x^0(t_1))\delta_k x(t_1) - \Delta^*_{1k} - 2\Delta^{**}_{1k}, \tag{5.9}$$

where

$$\Delta^*_{1k} = \int_\theta^{t_1} \delta_k x^T(t)H_{xx}(t)\delta_k x(t)dt, \tag{5.10}$$

$$\Delta^{**}_{1k} = \int_\theta^{\theta + \varepsilon} \left[\delta_k x^T(t)G_k[p](t) + \delta_k x^T(t + h)G_k[\tilde{p}](t + h)\right]\delta_k p(t)dt. \tag{5.11}$$

Here, by (4.31), (4.35), (5.8), and the Cauchy formula, $\delta_k p(\cdot)$ and $\delta_k x(\cdot)$ are determined as follows:

$$\delta_k p(t) = \begin{cases} \xi \int\limits_\theta^t \dfrac{(t-s)^{k-1}}{(k-1)!} l_k\left(\dfrac{2(s-\theta)}{\varepsilon} - 1\right) ds, \ t \in [\theta, \theta + \varepsilon), \\ 0, \ t \in I_1 \setminus [\theta, \theta + \varepsilon), \ \varepsilon \in (0, \varepsilon^{**}), \end{cases} \tag{5.12}$$

$$\delta_k x(t) = \begin{cases} \int\limits_\theta^t \lambda(\tau, t) \big[g_k[p](\tau) \delta_k p(\tau) + g_k[\tilde{p}](\tau) \delta_k p(\tau - h) \big] d\tau, \ t \in (\theta, t_1], \\ 0, \ t \in [t_0, \theta], \end{cases} \tag{5.13}$$

where $\lambda(\cdot)$ is the solution of the system (3.4).

By considering (5.12) in (5.13), we calculate $\delta_k x(t)$, $t \in I$. As $\theta \in I^{**}$, then by the properties of the set I^{**}, we have

$$\delta_k x(t) = \begin{cases} 0, & t \in [t_0, \theta], \\ \lambda(\theta, t) g_k[p](\theta) \xi \int\limits_\theta^t c_k(\tau) d\tau + o(\varepsilon^{k+1}), & t \in (\theta, \theta + \varepsilon), \\ \lambda(\theta, t) g_k[p](\theta) \xi \int\limits_\theta^{\theta+\varepsilon} c_k(\tau) d\tau + o(\varepsilon^{k+1}), & t \in [\theta + \varepsilon, \theta + h) \cap I, \\ \lambda(\theta, t) g_k[p](\theta) \xi \int\limits_\theta^{\theta+\varepsilon} c_k(\tau) d\tau + \chi(\theta) \lambda(\theta + h, t) g_k[\tilde{p}](\theta + h) \xi \times \\ \int\limits_\theta^{t-h} c_k(\tau) d\tau + o(\varepsilon^{k+1}), & t \in [\theta + h, \theta + h + \varepsilon) \cap I, \\ \big[\lambda(\theta, t) g_k[p](\theta) + \chi(\theta) \lambda(\theta + h, t) g_k[\tilde{p}](\theta + h) \big] \xi \\ \times \int\limits_\theta^{\theta+\varepsilon} c_k(\tau) d\tau + o(\varepsilon^{k+1}), & t \in [\theta + h + \varepsilon, t_1], \end{cases} \tag{5.14}$$

where

$$c_k(\tau) = \int\limits_\theta^\tau \frac{(\tau - s)^{k-1}}{(k-1)!} l_k\left(\frac{2(s-\theta)}{\varepsilon} - 1\right) ds, \ \tau \in [\theta, \theta + \varepsilon], \ \varepsilon \in (0, \varepsilon^{**}). \tag{5.15}$$

As $l_k(\tau)$, $\tau \in [-1, 1]$ is the k-th Legendre polynomial, then it is easy to get

$$\int_{\theta}^{\theta+\varepsilon} c_k(\tau)\,d\tau = \frac{\varepsilon^{k+1}}{k!\,2^{k+1}}\int_{-1}^{1}(1-\tau)^k l_k(\tau)\,d\tau = \frac{\varepsilon^{k+1}(-1)^k}{k!\,2^{k+1}}\int_{-1}^{1}\tau^k l_k(\tau)\,d\tau \neq 0. \tag{5.16}$$

Taking into account (5.12)–(5.16) and the fact that $\lambda(s,t)=0$ for $s>t$ we calculate separately each terms of (5.9). As a result, after simple reasoning we get

$$\begin{aligned}
\delta_k x^T(t_1)\varphi_{xx}\big(x^0(t_1)\big)\delta_k x(t_1) &= \xi^T\Big[g_k^T[p](\theta)\lambda^T(\theta,t_1)\varphi_{xx}\big(x^0(t_1)\big)\lambda(\theta,t_1)g_k[p](\theta)\\
&+ 2\chi(\theta)g_k^T[p](\theta)\lambda^T(\theta,t_1)\varphi_{xx}\big(x^0(t_1)\big)\lambda(\theta+h,t_1)g_k[\tilde{p}](\theta+h)\\
&+ \chi(\theta)g^T[\tilde{p}](\theta+h)\lambda^T(\theta+h,t_1)\varphi_{xx}\big(x^0(t_1)\big)\\
&\times \lambda(\theta+h,t_1)g_k[\tilde{p}](\theta+h)\Big]\Bigg(\xi\int_{\theta}^{\theta+\varepsilon}c_k(\tau)\,d\tau\Bigg)^2 + o\big(\varepsilon^{2k+2}\big),
\end{aligned}$$

$$\begin{aligned}
\Delta_{1k}^{*} &= \xi^T\int_{\theta}^{t_1}\Big[g_k^T[p](\theta)\lambda^T(\theta,t)H_{xx}(t)\lambda(\theta,t)g_k[p](\theta)\\
&+ 2\chi(\theta)g_k^T[p](\theta)\lambda^T(\theta,t)H_{xx}(t)\lambda(\theta+h,t)g_k[\tilde{p}](\theta+h)\\
&+ \chi(\theta)g_k^T[\tilde{p}](\theta+h)\lambda^T(\theta+h,t)H_{xx}(t)\lambda(\theta+h,t)\,g_k[\tilde{p}](\theta+h)\Big]dt\,\xi\Bigg(\int_{\theta}^{\theta+\varepsilon}c_k(\tau)\,d\tau\Bigg)^2 + o\big(\varepsilon^{2k+2}\big),
\end{aligned} \tag{5.17}$$

$$\begin{aligned}
\Delta_{1k}^{**} &= \frac{1}{2}\xi^T\Big[g_k^T[p](\theta)G_k[p](\theta) + 2\chi(\theta)g_k^T[p](\theta)\lambda^T(\theta,\theta+h)G_k[\tilde{p}](\theta+h)\\
&+ \chi(\theta)g_k^T[\tilde{p}](\theta+h)G_k[\tilde{p}](\theta+h)\Big]\xi\Bigg(\int_{\theta}^{\theta+\varepsilon}c_k(\tau)\,d\tau\Bigg)^2 + o\big(\varepsilon^{2k+2}\big)
\end{aligned}$$

Substitute (5.15)–(5.17) in (5.9). Then by (3.9), (4.27), and (5.14), we have

$$\begin{aligned}
\delta^2 S\big(u^0;\delta u\big) &= -\xi^T\Big\{ M_k[p,p](\theta,\theta) + 2\chi(\theta)M_k[p,\tilde{p}](\theta,\theta+h)\\
&+ \chi(\theta)M_k[\tilde{p},\tilde{p}](\theta+h,\theta+h)\Big\}\xi\frac{\varepsilon^{2k+2}}{(k!)^2\,4^{k+1}}\Bigg(\int_{-1}^{1}\tau^k l_k(\tau)\,d\tau\Bigg)^2\\
&+ o\big(\varepsilon^{2k+2}\big),\ \forall\theta\in I^{*},\ \forall\xi\in R^{r_0}.
\end{aligned}$$

Hence, taking into account the inequality in (2.1), it is easy to complete the proof of optimality condition (5.3) for $i=k$.

Continuing the proof of Theorem 5.1, we prove also the validity of optimality condition (5.4) for $i=k$. Based on Proposition 4.2, let us consider the $(k+1)$-th order transformation. As the equalities

$$L_i[p](t) + \chi(t)L_i[\tilde{p}](t+h) = 0, \ P_i[p,q](t) + \chi(t)P_i[\tilde{p},\tilde{q}](t+h) = 0, \ t \in I^{**}, \ i = \overline{0,k},$$
$$Q_i[p](t) + \chi(t)Q_i[\tilde{p}](t+h) = 0, \ t \in I^{**}, \ i = \overline{0,k-1}$$

taking into account (2.6), we have

$$
\begin{aligned}
\delta^2 S(u^0;\delta u) &= \Delta_1^2 S(u^0;\delta_{k+1}p,\delta q,\delta_{k+1}x,\varepsilon) + \int_{\theta}^{\theta+\varepsilon}\Big[\delta_{k+1}p^T(t)\big(L_{k+1}[p](t) + L_{k+1}[\tilde{p}](t+h)\big) \\
&\times \delta_{k+1}p(t) + 2\delta_{k+1}p^T(t)\big(P_{k+1}[p,q](t) + P_{k+1}[\tilde{p},\tilde{q}](t+h)\big)\delta q(t) \\
&+ \delta_k p^T(t)\big(Q_k[p](t) + Q_k[\tilde{p}](t+h)\big)\delta_{k+1}p(t) \\
&- \delta q^T(t)\big(H_{qq}(t) + H_{\tilde{q}\tilde{q}}(t+h)\big)\delta q(t)\Big]dt, \varepsilon \in (0,\varepsilon^{**}),
\end{aligned}
\tag{5.18}
$$

where $\Delta_1^2 S\big(u^0(\,\cdot\,); \delta_{k+1}p, \delta q, \delta_{k+1}x, \varepsilon\big)$ are determined similarly to (4.33) by changing the index k by $k+1$, and $\delta_{k+1}x(t)$ is the solution of the system (similar to (4.35))

$$
\begin{aligned}
\delta_{k+1}\dot{x}(t) &= f_x(t)\delta_{k+1}x(t) + g_{k+1}[p](t)\delta_{k+1}p(t) + g_{k+1}[\tilde{p}](t)\delta_{k+1}p(t-h) \\
&+ f_q(t)\delta q(t) + f_{\tilde{q}}(t)\delta q(t-h), \ t \in [\theta,t_1], \\
\delta_{k+1}x(t) &= 0, \ t \in [t_0,\theta], \ \delta_{k+1}p(t) = 0, \ \delta q(t) = 0, \ t \in [t_0 - h,\theta].
\end{aligned}
\tag{5.19}
$$

Choose the variation $\delta u(t) = \big(\delta_0 p(t), \delta q(t)\big)^T, \ t \in I_1$ in the following way:

$$
\begin{aligned}
\delta_0 p_m(t) &= 0, \ t \in I_1, \ m \in \{1,2,\ldots,r_0\}\setminus\{i,j\}, \ i,j \in \{1,2,\ldots,r_0\}, \ i \neq j, \\
\delta_0 p_i(t) &= \begin{cases} \alpha l_{k+1}\left(\dfrac{2(t-\theta)}{\varepsilon} - 1\right), & t \in [\theta,\theta+\varepsilon), \\[2mm] 0, & t \in I_1 \setminus [\theta,\theta+\varepsilon), \end{cases}
\end{aligned}
\tag{5.20}
$$

$$
\begin{aligned}
\delta_0 p_j(t) &= \begin{cases} \beta l_{k+2}\left(\dfrac{2(t-\theta)}{\varepsilon} - 1\right), & t \in [\theta,\theta+\varepsilon), \\[2mm] 0, & t \in I_1 \setminus [\theta,\theta+\varepsilon), \end{cases} \\
\delta q(t) &= 0, \ t \in I_1,
\end{aligned}
$$

where $l_z(\tau) = \dfrac{1}{z!}\dfrac{1}{2^z}\dfrac{d^z}{d\tau^z}\big[\tau^2 - 1\big]^z$, $z \in \{k+1, k+2\}$ is a Legendre polynomials $\alpha, \beta \in R, \ \theta \in I^{**}, \ \varepsilon \in (0, \varepsilon^{**})$.

Obviously, by (5.20), the variation $\delta u(\cdot) = \left(\delta_0 p(\cdot), \delta q(\cdot)\right)^T$ defined in (5.20) satisfies conditions (4.29), (4.30) for $k + 1$. Taking into account (5.20), by means of (4.30), (4.31), (4.33), and (5.19), it is easy to calculate

$$\delta_k p(t) \sim \varepsilon^k, \ \delta_{k+1} p(t) \sim \varepsilon^{k+1}, \ t \in I_1, \ \delta_{k+1} x(t) \sim \varepsilon^{k+2}, \ t \in I,$$
$$\Delta_1^2 S\left(u^0; \delta_{k+1} p, \delta q, \delta_{k+1} x, \varepsilon\right) \sim \varepsilon^{2k+4}. \tag{5.21}$$

By (5.20) and (5.21), from (5.18) we get

$$\delta^2 S\left(u^0; \delta u\right) = \int_\theta^{\theta+\varepsilon} \delta_k p^T(t)\left(Q_k[p](t) + Q_k[\tilde{p}](t+h)\right)\delta_{k+1} p(t)\,dt + o\left(\varepsilon^{2k+2}\right),$$

where $Q_k[\mu](\cdot)$, $\mu \in \{p, \tilde{p}\}$ is determined in (4.25).

Hence, taking into account the skew symmetry of the matrix $Q_k[p](t) + \chi(t)Q_k[\tilde{p}](t+h)$, $t \in I$ and the properties of the set I^{**}, and also by (2.1), (4.30), and (5.20), we have

$$\delta^2 S\left(u^0; \delta u\right) = \left[q_{ij}^{(k)}(\theta) - q_{ij}^{(k)}(\theta)\right] \int_\theta^{\theta+\varepsilon} \delta_k p_i(\tau)\delta_{k+1} p_j(\tau)\,\delta\tau + o\left(\varepsilon^{2k+2}\right)$$
$$= 4(k+1)(k+2)\left(\frac{\varepsilon}{2}\right)^{2k+2}\alpha\beta ab\left[q_{ij}^{(k)}(\theta) - q_{ji}^{(k)}(\theta)\right]\int_{-1}^1 \tau^2\left(\tau^2 - 1\right)^{2k+1}d\tau + o\left(\varepsilon^{2k+2}\right) \geq 0$$

where $\theta \in I^{**}$, $a = \dfrac{1}{(k+1)!\,2^{k+1}}$, $b = \dfrac{1}{(k+2)!\,2^{k+2}}$, and $q_{ij}^{(k)}(\theta)$, $q_{ji}^{(k)}(\theta)$ are the elements of the matrix $Q_k[p](\theta) + \chi(\theta)Q_k[\tilde{p}](\theta + h)$.

From the last inequality, by arbitrariness of $\theta \in I^{**}$, $\alpha, \beta \in R$ and $i, j \in \{1, 2, ..., r_0\}$ $(i \neq j)$ it follows that for each $\theta \in I^{**}$, the skew-symmetric matrix $Q_k[p](\theta) + \chi(\theta)Q_k[\tilde{p}](\theta + h)$ is also symmetric. Consequently, $Q_k[p](\theta) + \chi(\theta)Q_k[\tilde{p}](\theta + h) = 0$, that is, condition (5.4) is proved for $i=k$.

At last, let us prove optimality condition (5.5). Choose the variation $\delta u(t) = \left(\delta_0 p(t), \delta q(t)\right)^T$, $t \in I_1$ in the following way:

$$\delta_0 p(t) = \begin{cases} \xi \, l_{k+1}\left(\dfrac{2(t-\theta)}{\varepsilon} - 1\right), \; t \in [\theta, \theta+\varepsilon), \\ 0, \quad t \in I_1 \setminus [\theta, \theta+\varepsilon), \end{cases} \tag{5.22}$$

$$\delta q(t) = \begin{cases} \eta \displaystyle\int_\theta^t \dfrac{(t-s)^k}{k!} \, l_{k+1}\left(\dfrac{2(s-\theta)}{\varepsilon} - 1\right) ds, \; t \in [\theta, \theta+\varepsilon), \\ 0, \quad t \in I_1 \setminus [\theta, \theta+\varepsilon), \end{cases} \tag{5.23}$$

where $l_{k+1}(\tau)$, $\tau \in [-1, 1]$ is the $(1+k)$-th Legendre polynomial, $\xi \in R^{r_0}$, $\eta \in R^{r_1}$, $\theta \in I^{**}$, $\varepsilon \in (o, \varepsilon^{**})$.

Obviously, the variation $\delta u(t) = \left(\delta_0 p(t), \delta q(t)\right)^T$, $t \in I_1$ defined in (5.22) satisfies the conditions (4.29) and (4.30) for $i = 1, 2 \dots k+1$

By (4.30), (4.31), (5.12), (5.19), (5.22), and (5.23), the following relations hold:

$$\delta_{k+1} p(t) = \begin{cases} \xi \displaystyle\int_\theta^t \dfrac{(t-s)^k}{k!} \, l_{k+1}\left(\dfrac{2(s-\theta)}{\varepsilon} - 1\right) ds, \; t \in [\theta, \theta+\varepsilon), \\ 0, \quad t \in I_1 \setminus [\theta, \theta+\varepsilon), \end{cases} \tag{5.24}$$

$$\delta_{k+1} p(t) \sim \varepsilon^{k+1}, t \in I_1, \; \delta q(t) \sim \varepsilon^{k+1}, t \in I_1, \; \delta_{k+1} x(t) \sim \varepsilon^{k+2}, t \in I,$$
$$\Delta_1^2 S\left(u^0; \delta_{k+1} p, \delta q, \delta_{k+1} x, \varepsilon\right) \sim \varepsilon^{2k+4}. \tag{5.25}$$

Taking into account (5.23)–(5.25) and validity of the equality $Q_k[p](t) + \chi(t) Q_k[\tilde{p}](t+h) = 0$, $t \in I^{**}$ (see (5.4)), from (5.18), we get

$$\delta^2 S\left(u^0; \delta u\right) = \left(\dfrac{\varepsilon}{2}\right)^{2k+3} \left[\xi^T \left(L_{k+1}[p](\theta) + \chi(\theta) L_{k+1}[\tilde{p}](\theta+h)\right)\xi \right.$$
$$+ 2\xi^T \left(P_{k+1}[p,q](\theta) + \chi(\theta) P_{k+1}[\tilde{p}, \tilde{q}](\theta+h)\right)\eta$$
$$\left. - \eta^T \left(H_{qq}(\theta) + \chi(\theta) H_{\tilde{q}\tilde{q}}(\theta+h)\right)\eta\right] \dfrac{1}{(k!)^2} \int_{-1}^1 \left(\int_{-1}^t (t-s)^k \, p_{k+1}(s) ds\right)^2 dt + o\left(\varepsilon^{2k+3}\right).$$

From this expansion, taking into account (2.1), it follows inequality (5.5).

Therefore, Theorem 5.1 is completely proved.

Corollary 5.1. Let all the conditions of Theorem 5.1 be fulfilled. Let, in addition, the following equalities hold:

$$L_i[p](t) + \chi(t)L_i[\tilde{p}](t+h) = 0, \ \forall t \in I^{**}, \ i = 0,1,\dots$$

Then, for optimality of the singular control $u^0(\cdot)$, it is necessary that the relations

$$P_i[p,q](\theta) + \chi(\theta)P_i[\tilde{p},\tilde{q}](\theta+h) = 0, \ i = 0,1,\dots;$$
$$\xi^T\{M_i[p,p](\theta,\theta) + 2\chi(\theta)M_i[p,\tilde{p}](\theta,\theta+h)$$
$$+\chi(\theta)M_i[\tilde{p},\tilde{p}](\theta+h,\theta+h)\}\xi \leq 0, \ i = 0,1,\dots;$$
$$Q_i[p](\theta) + \chi(\theta)Q_i[\tilde{p}](\theta+h) = 0, \ i = 0,1,\dots$$

be fulfilled for all $\theta \in I^{**}$, $\xi \in R^{r_0}$.

The proof of the corollary follows immediately from Theorem 5.1.

Remark 5.1. As is seen (see Proposition 3.1 and (4.6), (4.15), and (4.24)), for validity of optimality conditions (5.2)–(5.4), for $k = 0$ it is sufficient that assumptions (A1) and (A2) be fulfilled.

Remark 5.2. It is clear that (see Proposition 4.1) for validity of optimality conditions (5.5), for $k = 0$ it is sufficient that assumptions (A1), (A3), and (A4) be fulfilled.

Remark 5.3. If in Definition 2.1 a special plot is some interval $(\bar{t},\hat{t}) \subset I$, then very easily similar to the proof of Theorem (5.1) we can prove that conditions (5.2)–(5.5) as optimality conditions are valid for all $\theta \in (\bar{t},\hat{t}) \cap I^{**}$ and $\xi \in R^{r_0}$, $\eta \in R^{r_1}$.

6. Conclusion

As is seen, systems (1.2) and (1.3) are not the most general among all the systems with retarded control. We have chosen it only for definiteness, just to demonstrate the essentials of our method. Nevertheless, the optimality conditions (5.2)–(5.5) can be generalized to the case for more general systems with retarded control.

It should be noted that (1) optimality conditions (5.4) and (5.5), for $k = 0$, are actually the analogs of the equality-type conditions and the Kelly [12] condition, while optimality condition (5.3) is the analog of the Gabasov [11] condition for the considered problem (1.1)–(1.3); (2) optimality condition (5.5), for $k = 1$ is the analog of the Koppa-Mayer [33] condition. Conditions (5.3)–(5.5) were obtained in [10] only for singular controls with complete degree of degeneracy, that is, for the case when $r_1 = 0$ (see Definition 2.1).

We also note that (1) the analog of the Kelly condition and equality-type condition was obtained in [24] by another method for systems with retarded state; (2) optimality-type conditions (5.2)–(5.5) for system with retarded state were obtained in [[31, 32], p. 119]; (3) optimality conditions of type (5.4), (5.5) for systems without retardation were obtained in the papers [[23, 26, 27], p. 145, [29, 30, 33, 34, 39–41], etc.].

The proof of Theorem 5.1 shows that the optimality conditions (5.3)–(5.5) are independent. Also, it is clear that, unlike (5.2), (5.3), and (5.5), the optimality condition (5.4) for $r_1 = r - 1$ (see Definition 2.1) becomes ineffective, though it is effective in the general case for $r_1 < r - 1$. To illustrate the rich content of condition (5.4), we consider a concrete example:

Example. $\dot{x}_1(t) = u_2(t) + u_1^2(t - 1) - u_3(t - 1)$, $\dot{x}_2(t) = u_1(t) - u_2(t)$,

$$\dot{x}_3(t) = \left(u_1(t) + u_2(t)\right) x_2(t) + u_3^2(t) + u_3^2(t - 1), \ t \in I: = [0, 2], \ x_i(0) = 0, \ u_i(t) = 0, \ t \in [-1, 0),$$

$\left|u_i\right| < 2, \ i = 1, 2, 3, h = 1, \ \phi(x(2)) = x_3(2) + \frac{1}{2}x_1^2(2) \to \min.$

Check for optimality of the control $u^0(t) = (0, 0, 0)^T, \ t \in [-1, 2]$. In this control according to (2.7), (2.8), (3.9), (3.10), (4.6), (4.9), (4.15), (4.21), and (4.24), we have

$x_i^0(t) = 0, \ i = 1, 2, 3, \ \psi_i^0(t) = 0, i = 1, 2, \ \psi_3^0(t) = -1, \ t \in I,$

$H\left(\psi^0(t), x, u, v, t\right) = -\left(u_1 + u_2\right)x_2 - u_3^2 - v_3^2, \quad H_{uu}(t): = \left(h_{ij}(t)\right), t \in I, \quad \text{where} \quad h_{ij}(t) = 0,$

$i, j \in \{1, 2, 3\}, \qquad (i, j) \neq (3, 3), \qquad h_{33}(t) = -2; H_{vv}(t + 1) = \left(\tilde{h}_{ij}(t)\right), \ t \in [0, 1], \qquad \text{where}$

$\tilde{h}_{ij}(t) = 0, i, j \in \{1, 2, 3\}, \qquad (i, j) \neq (3, 3), \qquad \tilde{h}_{33}(t) = -2; \qquad H_{vv}(t + 1) = 0, t \in (1, 2];$

$g_0^T[p](t) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix}, t \in I, \quad g_0^T[\tilde{p}](t) = 0, \quad t \in I, \quad \text{where} \quad p: = \left(u_1, u_2\right), \quad \tilde{p}: = \left(v_1, v_2\right);$

$g_1[p](t) = g_1[\tilde{p}](t) = 0, \ t \in I, \qquad Q_0[p](t) = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}, \qquad t \in I, \qquad Q_0[\tilde{p}](t + 1) = 0, \qquad t \in I,$

$L_1[p](t) = L_1[\tilde{p}](t + 1) = 0, \quad t \in I, \quad P_0[p, q](t) = P_0[\tilde{p}, \tilde{q}](t) = 0, \quad P_1[p, q](t) = P_1[\tilde{p}, \tilde{q}](t) = 0,$

$M_0[p, p](t, t) = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, t \in I, \qquad\qquad M_0[p, \tilde{p}](\cdot) = 0, \qquad\qquad M_0[\tilde{p}, \tilde{p}](\cdot) = 0,$

$H_{qq}(t) + H_{\tilde{q}\tilde{q}}(t + 1) = \begin{cases} -4, & t \in [0, 1), \\ -2, & t \in [1, 2], \end{cases}$ where $q = u_3, \ \tilde{q} = v_3.$

Hence, we have the following: (1) admissible control $u^0(t) = (0, 0, 0)^T, \ t \in [-1, 2]$ is singular (in the sense of Definition 2.1) and singularity to it is delivered by the vector component $p = \left(u_1, u_2\right)^T$, that is, equality (5.1) is fulfilled only $k = 0$; (2) optimality conditions (5.2), (5.3), (5.5), and the results of the papers [1–3, 6, 9, 10] cannot say that whether the control $u^0(\cdot)$ is an optimal or not. However, optimality condition (5.4) for $k = 0$ is not fulfilled

$(Q_0[p](t) + \chi(t)Q_0[\tilde{p}](t+1) = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} = 0, \ t \in I)$, that is, by condition (5.4) (for $k = 0$) we conclude that the control $u^0(t) = (0,0,0)^T, \ t \in [-1,2]$ cannot be optimal.

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