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Convective Heat Transfer to Sisko Fluid over a Nonlinear Radially Stretching Sheet

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1. Introduction

Over the past several decades, the Navier-Stokes equations have been studied frequently in the literature. This is due to the fact that the use of the Newtonian fluid model in numerous industrial applications to predict the behavior of many real fluids has been adopted. However, there are many materials of industrial importance (e.g. polymeric liquids, molten plastics, lubricating oils, drilling muds, biological fluids, food products, personal care products, paints, greases and so forth) are non-Newtonian. That is, they might exhibit dramatic deviation from Newtonian behavior and display a range of non-Newtonian characteristics. A few points of non-Newtonian characteristic are the ability of the fluid to exhibit relaxation and retardation, shear dependent viscosity, shear thinning or shear thickening, yield stress, viscoelasticity and many more. Thus, it has been now well recognized in technology and industrial applications that non-Newtonian fluids has become an active field of research for the last few years.

Unlike, the Newtonian fluid, it is very difficult to provide a universal constitutive model for non-Newtonian fluids as they possess very complex structure. However, there are some classes of fluids that cannot be classified as Newtonian or purely non-Newtonian such as water-borne coating etc. This situation demands some more general models which can be utilized for analysis of both Newtonian and non-Newtonian behaviors. For this purpose, some models have been proposed in the literature including generalized Newtonian fluids. The Sisko fluid model [1] is a subclass of the generalized Newtonian fluids which is considered as the most appropriate model for lubricating oils and greases [2]. The Sisko fluid model is of much importance due to its adequate description of a few non-Newtonian fluids over the most important range of shear rates. The appropriateness of the Sisko fluid model has been successfully extended to the shear thinning rheological behavior of concentrated non-



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Newtonian slurries [3]. The three parameters Sisko fluid model, which can be considered as a generalized power-law model that includes Newtonian component, has not been given due attention in spite of its diverse industrial applications. A representative sample of the recent literature on the Sisko fluid is provided by references [4-10].

Investigations of the boundary layer flow and heat transfer of non-Newtonian fluids over a stretching surface are important due to immense applications in engineering and science. A great number of investigations concern the boundary layer behavior on a stretching surface. Many manufacturing processes involve the cooling of continuous sheets. To be more specific, examples of such applications are wire drawing, hot rolling, drawing of plastic films, paper production, and glass fiber etc. In all these situations, study of the flow and heat transfer is of significant importance as the quality of the final products depends to the large extent on the skin friction and heat transfer rate at the surface. In view of these, the boundary layer flows and heat transfer over a stretching surface have been studied extensively by many researchers. Crane [11] was first to investigate the boundary layer flow of a viscous fluid over a stretching sheet when the sheet is stretched in its own plane with velocity varies linearly with the distance from a fixed point on the sheet. Dutta et al. [12] examined the heat transfer in a viscous fluid over a stretching surface with uniform heat flux. Later on, this problem was extended by Chen and Char [13] by considering the variable heat flux. Grubka and Bobba [14] analyzed the heat transfer over a stretching surface by considering the non-isothermal wall that is varying as a power-law with the distance. Cortell [15] investigated the flow and heat transfer of a viscous fluid over nonlinear stretching sheet by considering the constant surface temperature and prescribed surface temperature. It seems that Schowalter [16] was the first who has obtained the similarity solutions for the boundary layer flow for power-law pseudoplastic fluids. Howel et al. [17] considered the laminar flow and heat transfer of a power-law fluid over a stretching sheet. Hassanien et al. [18] investigated the heat transfer to a power-law fluid flow over a nonisothermal stretching sheet. Abel et al. [19] studied the flow and heat transfer of a power-law fluid over a stretching sheet with variable thermal conductivity and non-uniform heat source. Prasad and Vajravelu [20] analyzed the heat transfer of a power-law fluid over a non-isothermal stretching sheet. Khan and Shahzad [21,22] have considered the boundary layer theory of the Sisko fluid over the planer and radially stretching sheets and found the analytic solutions; however, they only considered the integral values of the power-law index in their flow problems. The integral values of the power-law index are inadequate to completely comprehend the shear thinning and shear thickening effects of the Sisko fluid. Moreover, a literature survey also indicates that no work has so far been available with regards to heat transfer to Sisko fluid flow over a stretching sheet in presence of viscous dissipation.

The objective of this chapter is to analyze the flow and heat transfer characteristics of Sisko fluid over a radially stretching sheet with the stretching velocity cr^s in the presence of viscous dissipation. In the present work we have spanned the value of the power-law index from highly shear thinning to shear thickening Sisko fluid ($0.2 \le n \le 1.9$). The modeled partial differential equations are reduced to a system of nonlinear ordinary differential equations using the appropriate transformations. The resulting equations are then solved numerically by implicit finite difference method in the domain $[0, \infty)$. The numerical results for the velocity and

temperature fields are graphically depicted and effects of the relevant parameters are discussed in detail. In addition, the skin friction coefficient and the local Nusselt number for different values of the pertaining parameters are given in tabulated form. Moreover, numerical results are compared with exact solutions as special cases of the problem. Furthermore, the present results for the velocity field are also validated by comparison with the previous pertinent literature.

2. Governing equations

This section comprises the governing equations and the rheological model for the steady twodimensional flow and heat transfer of an incompressible and inelastic fluid Sisko fluid in the cylindrical polar coordinates. To derive the governing equations we make use of fundamental laws of fluid mechanics, namely conservations of mass, linear momentum and energy, including the viscous dissipation

$$\nabla \cdot \mathbf{V} = 0, \tag{1}$$

$$\rho(\mathbf{V}\cdot\nabla)\mathbf{V} = -\nabla p_1 + \nabla\cdot\mathbf{S},\tag{2}$$

$$\rho c_{v} (\mathbf{V} \cdot \nabla) T = -\nabla \cdot \mathbf{q} + \mathbf{S} \cdot \mathbf{L}.$$
(3)

In the above equations **V** is the velocity vector, ρ the density of fluid, c_p the specific heat at constant pressure, p_1 the pressure, T the temperature, **S** the extra stress tensor and **q** the heat flux given by

$$\mathbf{q} = -\kappa(\nabla T),\tag{4}$$

where κ is the thermal conductivity of the fluid and ∇ the gradient operator.

The extra stress tensor **S** for an incompressible fluid obeys the Sisko rheological model. This model mathematically can be expressed as [4]

$$\mathbf{S} = \left[a + b \left| \sqrt{\frac{1}{2} tr \left(\mathbf{A}_{1}^{2} \right)} \right|^{n-1} \right] \mathbf{A}_{1}, \tag{5}$$

where \mathbf{A}_1 is the rate of deformation tensor or the first Rivlin-Erickson tensor defined as

$$\mathbf{A}_{1} = \mathbf{L} + \mathbf{L}^{\mathrm{T}}, \ \mathbf{L} = \nabla \mathbf{V}, \tag{6}$$

with *a* the dynamic viscosity, *b* the Sisko fluid parameter or the flow consistency index, $(n \ge 0)$ the power-law index or the flow behavior index (a non-negative real number) and **T** stands for transpose.

The quantity

$$\mu_{eff} = \left[a + b \left| \sqrt{\frac{1}{2} tr(\mathbf{A}_1^2)} \right|^{n-1} \right],\tag{7}$$

represents an apparent or effective viscosity as a function of the shear rate. If a=0 and n=1 (or b=0) the equations for Newtonian fluid, a=0 for the power-law model and n=0 with b as yield stress for the Bingham plastic model are obtained.

For the steady two-dimensional axisymmetric flow, we assume the velocity, temperature and stress fields of the form

$$\mathbf{V} = \left[u(r,z), 0, w(r,z) \right], \ T = T(r,z), \ \mathbf{S} = \mathbf{S}(r,z),$$
(8)

when (r, z) denotes the cylindrical polar coordinates along the sheet and vertical to it, u and w the velocity components in the r – and z – directions, respectively.

The steady two-dimensional and incompressible equations of motion (2) including conservation of mass (1) and thermal energy (3) can be written as

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0, \tag{9}$$

$$\rho\left(u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p_1}{\partial r} + \frac{\partial S_{rr}}{\partial r} + \frac{\partial S_{rz}}{\partial z} + \frac{S_{rr} - S_{\phi\phi}}{r},\tag{10}$$

$$\rho\left(u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p_1}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(rS_{rz}\right) + \frac{\partial S_{zz}}{\partial z},\tag{11}$$

$$\rho c_{p} \left(u \frac{\partial T}{\partial r} + w \frac{\partial T}{\partial z} \right) = \kappa \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{\partial^{2} T}{\partial z^{2}} \right) + a \left[2 \left(\frac{\partial u}{\partial r} \right)^{2} + \left(\frac{\partial u}{\partial z} \right)^{2} + 2 \frac{\partial w}{\partial z} \frac{\partial w}{\partial r} + 2 \frac{u^{2}}{r^{2}} \right] \\ + b \left[2 \left(\frac{\partial u}{\partial r} \right)^{2} + \left(\frac{\partial u}{\partial z} \right)^{2} + 2 \frac{\partial u}{\partial z} \frac{\partial w}{\partial r} + 2 \frac{\partial w}{\partial z} \frac{\partial w}{\partial r} + 2 \frac{u^{2}}{r^{2}} \right] \left| \sqrt{\frac{1}{2} tr \mathbf{A}_{1}^{2}} \right|^{n-1},$$
(12)

where

$$\frac{1}{2}tr\left(\mathbf{A}_{1}^{2}\right) = 2\left(\frac{\partial u}{\partial r}\right)^{2} + 2\left(\frac{\partial w}{\partial z}\right)^{2} + 2\left(\frac{\partial u}{\partial z}\right)^{2} + \frac{\partial u}{\partial z}\frac{\partial w}{\partial r} + 2\frac{u^{2}}{r^{2}}.$$
(13)

In view of Eq. (8) the stress components are inserted into the equations of motion and the usual boundary layer approximations are made, the equations of motion characterizing the steady boundary layer flow and heat transfer take the form

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$$\rho\left(u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p_1}{\partial r} + a\frac{\partial^2 u}{\partial z^2} + b\frac{\partial}{\partial z}\left(\left|\frac{\partial u}{\partial z}\right|^{n-1}\frac{\partial u}{\partial z}\right),\tag{14}$$

$$0 = -\frac{\partial p_1}{\partial z}$$
(15)
$$u\frac{\partial T}{\partial r} + w\frac{\partial T}{\partial z} = \alpha \frac{\partial^2 T}{\partial z^2} + a\left(\frac{\partial u}{\partial z}\right)^2 + b\left(\frac{\partial u}{\partial z}\right)^2 \left|\frac{\partial u}{\partial z}\right|^{n-1},$$
(16)

where $\alpha = \frac{\kappa}{\rho c_n}$ is the thermal diffusivity with κ as the thermal conductivity.

3. Mathematical formulation

3.1. Flow analysis

Consider the steady, two-dimensional and incompressible flow of Sisko fluid over a nonlinear radially stretching sheet. The fluid is confined in the region z > 0, and flow is induced due to stretching of the sheet along the radial direction with velocity $U_w = cr^s$ with c and s are positive real numbers pertaining to stretching of the sheet. We assume that the constant temperature of the sheet is T_w , while T_∞ is the uniform ambient fluid temperature with $T_w > T_\infty$ For mathematical modeling we take the cylindrical polar coordinate system (r, ϕ, z) . Due to the rotational symmetry, all the physical quantities are independent of θ . Note that if the streamwise velocity component u increases with the distance z from the moving surface, the velocity gradient and therefore the shear rate are positive; however, if u decreases with increasing z the velocity gradient and therefore shear rate are negative. In the present problem within the boundary layer the shear rate is assumed to be negative since the streamwise velocity component u decreases monotonically with increasing z from the moving boundary (stretching sheet). Thus, under these assumptions, the flow is governed by the following equation:

$$\rho\left(u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z}\right) = a\frac{\partial^2 u}{\partial z^2} - b\frac{\partial}{\partial z}\left(-\frac{\partial u}{\partial z}\right)^n.$$
(17)

The boundary conditions associated to flow field are

$$u = cr^s, w = 0 \text{ at } z = 0,$$
 (18)

$$u \to 0 \text{ as } z \to \infty.$$
 (19)



Figure 1. Physical model and coordinate system.

We define the following variables

$$\eta = \frac{z}{r} \operatorname{Re}_{b}^{\frac{1}{n+1}} \text{ and } \psi(r, z) = -r^{2} U \operatorname{Re}_{b}^{\frac{-1}{n+1}} f(\eta), \qquad (20)$$

where $\psi(r, z)$ is the Stokes stream function defined by $u = -\frac{1}{r} \frac{\partial \psi}{\partial z}$ and $w = \frac{1}{r} \frac{\partial \psi}{\partial r}$ giving

$$u = Uf'(\eta) \text{ and } w = -U\operatorname{Re}_{b}^{\frac{-1}{n+1}} \left[\frac{s(2n-1)+n+2}{n+1} f(\eta) - \frac{s(n-2)+1}{n+1} \eta f'(\eta) \right].$$
(21)

On employing the above transformations, Eqs. (17) to (19) take the form [21]

$$Af''' + n(-f'')^{n-1}f''' + \left(\frac{s(2n-1)+n+2}{n+1}\right)ff'' - s(f')^2 = 0,$$
(22)

$$f(0) = 0, f'(0) = 1, f'(\infty) = 0,$$
 (23)

where prime denotes differentiation with respect to η and

$$\operatorname{Re}_{a} = \rho r U / a, \operatorname{Re}_{b} = \rho r^{n} U^{2-n} / b \text{ and } A = \operatorname{Re}_{b}^{\frac{2}{n+1}} / \operatorname{Re}_{a}$$
(24)

The physical quantity of major interest is the local skin friction coefficient and is given by [21]

$$\frac{1}{2} \operatorname{Re}_{b}^{\frac{1}{p+1}} C_{f} = A f''(0) - \left[-f''(0)\right]^{n}.$$
(25)

3.2. Heat transfer analysis

In the assumption of boundary layer flow, the energy equation for the non-Newtonian Sisko fluid taking into account the viscous dissipation effects and neglecting the heat generation effects for the temperature field T = T(r, z) is

$$u\frac{\partial T}{\partial r} + w\frac{\partial T}{\partial z} = \alpha \frac{\partial^2 T}{\partial z^2} + \frac{1}{\rho c_p} \left[a \left(\frac{\partial u}{\partial z} \right)^2 + b \left(-\frac{\partial u}{\partial z} \right)^{n+1} \right].$$
(26)

The corresponding thermal boundary conditions are

$$T = T_{w} \quad \text{at} \quad z = 0, \tag{27}$$

$$T \to T_{\infty} \text{ as } z \to \infty.$$
 (28)

Using the transformations (20) the above problem reduces to

$$\theta'' + \Pr \frac{s(2n-1) + (n+2)}{n+1} f\theta' + Br(f'')^2 + \Pr Ec(-f'')^{n+1} = 0,$$
(29)

$$\theta(0) = 1, \text{ and } \theta \to 0 \text{ as } \eta \to \infty,$$
(30)

where $\theta(\eta) = \frac{T - T_{\infty}}{T_{w} - T_{\infty}}$, $Br = \frac{aU^{2}}{\kappa(T_{w} - T_{\infty})}$ the Brinkman number, $Ec = \frac{U^{2}}{c_{p}(T_{w} - T_{\infty})}$ the Eckert number and $\Pr = \frac{rUR}{a}$ the generalized Prandtl number.

The local Nusselt number Nu_r at the wall is defined as

$$Nu_{r} = \frac{rq_{w}}{\kappa(T_{w} - T_{\infty})}\Big|_{z=0},$$
(31)

where the wall heat flux at the wall is $q_w = -\kappa \left(\frac{\delta T}{\delta z}\right) \Big|_{z=0}$, which by virtue of Eq. (31) reduces to

$$\operatorname{Re}_{b}^{-1/n+1} N u_{r} = -\theta'(0). \tag{32}$$

4. Solution procedure

The two point boundary value problems comprising Eqs. (22) and (29) along with the associated boundary conditions are solved by implicit finite difference scheme along with Keller box scheme. To implement the scheme, Eqs. (22) and (29) are written as a system of first-order differential equations in η as follows:

$$f' = p, \tag{33}$$
$$p' = q, \tag{34}$$

$$Aq' + n(-q)^{n-1}q' + Dfq - sp^2 = 0,$$
(35)

$$\theta' = t, \tag{36}$$

$$t' + \Pr Ec(-q)^{n+1} + \Pr Dft + Brq^2 = 0,$$
(37)

where $D = \frac{s(2n-1) + (n+2)}{n+1}$.

The boundary conditions in terms of new variable are written as

$$f(0) = 0, \ p(0) = 1 \text{ and } \theta(0) = 1,$$
 (38)

$$p \to 0 \text{ and } \theta \to 0 \text{ as } \eta \to \infty.$$
 (39)

The functions and their derivatives are approximated by central difference at the midpoint $\eta_{j-\frac{1}{2}}$ of the segment $\eta_{j-1}\eta_{j'}$, where j=1, 2, ..., N.

$$\eta_0 = 0, \ \eta_j = \eta_{j-1} + h_j, \ \eta_N = \eta_{\infty}.$$
(40)

Using the finite difference approximations equations (33) to (37) can be written as

$$f_j - f_{j-1} - h_j p_{j-1/2} = 0, (41)$$

$$p_j - p_{j-1} - h_j q_{j-1/2} = 0, (42)$$

$$A(q_j - q_{j-1}) + nh_j(-q_{j-1/2})^{n-1}(q_j - q_{j-1}) - sh_j(p_{j-1/2})^2 + Dh_jf_{j-1/2}q_{j-1/2} = 0)$$
(43)

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$$\theta_{j} - \theta_{j-1} - h_{j} t_{j-1/2} = 0, \tag{44}$$

$$(t_j - t_{j-1}) + \Pr Ech_j (-q_{j-1/2})^{n+1} (q_j - q_{j-1}) + Brh_j (q_{j-1/2})^2 + D\Pr h_j f_{j-1/2} t_{j-1/2} = 0,$$
(45)

where
$$j=1, 2, 3, \dots N$$
, $f_{j-1/2} = \frac{f_j + f_{j-1}}{2}$, $p_{j-1/2} = \frac{p_j + p_{j-1}}{2}$, $q_{j-1/2} = \frac{q_j + q_{j-1}}{2}$, and $t_{j-1/2} = \frac{t_j + t_{j-1}}{2}$.
Boundary conditions (38) and (39) are written as
 $f_0 = 0, p_0 = 1, \theta_0 = 1$, (46)

$$p_N = 0 \text{ and } \theta_N = 0. \tag{47}$$

Eqs. (41) to (45) are system of nonlinear equations and these equations are linearized employing the Newton's method and using the expressions:

$$f_{j}^{(k+1)} = f_{j}^{(k)} + \delta f_{j}^{(k)}, \ p_{j}^{(k+1)} = p_{j}^{(k)} + \delta p_{j}^{(k)}, \ q_{j}^{(k+1)} = q_{j}^{(k)} + \delta q_{j}^{(k)},$$

$$\theta_{j}^{(k+1)} = \theta_{j}^{(k)} + \delta \theta_{j}^{(k)}, \ t_{j}^{(k+1)} = t_{j}^{(k)} + \delta t_{j}^{(k)},$$
(48)

where k = 1, 2, 3, ...

Putting the left hand side of the above expressions into Eqs. (41) to (45) and dropping the quadratic terms in $\delta f^{(k)}$, $\delta p^{(k)}$, $\delta q^{(k)}$, $\delta \theta^{(k)}$ and $\delta t^{(k)}$, the following linear equations are obtained:

$$\delta f_{j} - \delta f_{j-1} - h_{j} \delta p_{j-1/2} = (r_{1})_{j-1/2},$$

$$(49)$$

$$(\xi_{1})_{j} \delta q_{j} + (\xi_{2})_{j} \delta q_{j-1} + (\xi_{3})_{j} \delta f_{j} + (\xi_{4})_{j} \delta f_{j-1} +$$

$$(\xi_5)_j \delta p_j + (\xi_6)_j \delta p_{j-1} + (\xi_7)_j \delta \theta_j + (\xi_8)_j \delta \theta_{j-1} = (r_2)_{j-1/2},$$
(50)

$$(\eta_1)_j \delta z_j + (\eta_2)_j \delta z_{j-1} + (\eta_3)_j \delta f_j + (\eta_4)_j \delta f_{j-1} +$$

$$(\eta_5)_j \delta p_j + (\eta_6)_j \delta p_{j-1} + (\eta_7)_j \delta \theta_j + (\eta_8)_j \delta \theta_{j-1} = (r_3)_{j-1/2},$$
(51)

$$\delta p_j - \delta p_{j-1} - h_j \delta q_{j-1/2} = (r_4)_{j-1/2}, \tag{52}$$

$$\delta\theta_{j} - \delta\theta_{j-1} - h_{j}\delta t_{j-1/2} = (r_{5})_{j-1/2},$$
(53)

where

$$\xi_{1} = \left[A - n(n-1) \left(\frac{q_{j} - q_{j-1}}{2} \right) \left(-q_{j-1/2} \right)^{n-2} + n \left(-q_{j-1/2} \right)^{n-1} + \frac{Dh_{j}}{2} f_{j-1/2} \right], \quad (54)$$

$$\xi_{2} = \left[-A - n(n-1) \left(\frac{q_{j} - q_{j-1}}{2} \right) \left(-q_{j-1/2} \right)^{n-2} + n \left(-q_{j-1/2} \right)^{n-1} + \frac{Dh_{j}}{2} f_{j-1/2} \right], \tag{55}$$

$$\xi_3 = \xi_4 = \frac{Dh_j}{2} q_{j-1/2}, \ \xi_5 = \xi_6 = -sh_j p_{j-1/2}, \ \xi_7 = \xi_8 = 0,$$
(56)

and

$$\eta_1 = 1 + \frac{D\Pr}{2} h_j f_{j-1/2}, \ \eta_2 = \eta_1 - 2, \ \eta_3 = \eta_4 = \frac{D\Pr}{2} h_j t_{j-1/2},$$
(57)

$$\eta_5 = \eta_6 = Brh_j q_{j-1/2} - \frac{\Pr Ec}{2} (n+1)h_j \left(-q_{j-1/2}\right)^n.$$
(58)

The right hand sides of Eqs. (49) to (53) are given by

$$r_{1} = -(f_{j} - f_{j-1}) + h_{j}p_{j-1/2},$$
(59)

$$r_{2} = -\left[A\left(q_{j} - q_{j-1}\right) + Dh_{j}f_{j-1/2}q_{j-1/2} - sh_{j}p_{j-1/2}^{2}\right] - n\left(q_{j} - q_{j-1}\right)\left(-q_{j-1/2}\right)^{n-1},\tag{60}$$

$$r_{3} = -\left[\left(t_{j} - t_{j-1}\right) + D\Pr h_{j}f_{j-1/2}t_{j-1/2} + Brh_{j}q_{j-1/2}^{2} + \Pr Ec\left(-q_{j-1/2}\right)^{n+1}\right],\tag{61}$$

$$r_4 = -(p_j - p_{j-1}) + h_j q_{j-1/2}, \ r_5 = -(t_j - t_{j-1}) + h_j t_{j-1/2}.$$
(62)

The boundary conditions (46) and (47) become

$$\delta f_0 = 0, \ \delta p_0 = 0, \ \delta q_0 = 0, \ \delta \theta_0 = 0, \ \delta q_N = 0 \text{ and } \delta \theta_N = 0.$$
 (63)

The linearized Eqs. (49) to (53) can be solved by using block elimination method as outlined by Na [23]. The iterative procedure is stopped when the difference in computing the velocity and temperature in the next iteration is less than 10⁻⁵. The present method is unconditionally stable and has second-order accuracy.

5. Exact solutions for particular cases

It is pertinent to mention that Eq. (22) has simple exact solution to special cases, namely (i) n = 0 and s = 1 [22] and (*ii*) n = 1 and s = 3 [24]. For case (*i*), with Br = Ec = 0, Eq. (29) reduces to

$$\theta'' + \Pr f \theta' = 0. \tag{64}$$

The exact solution to Eq. (64) in terms of the incomplete Gamma function, satisfying boundary conditions (30), is

$$\theta(\eta) = \frac{\Gamma(\frac{\Pr}{\beta^2}, 0) - \Gamma(\frac{\Pr}{\beta^2}, \frac{\Pr}{\beta^2}e^{-\beta\eta})}{\Gamma(\frac{\Pr}{\beta^2}, 0) - \Gamma(\frac{\Pr}{\beta^2}, \frac{\Pr}{\beta^2})},$$
(65)

where $\beta = \frac{1}{\sqrt{4}}$ and $\Gamma(\cdot)$ is the incomplete Gamma function.

For case (*ii*), with Br = Ec = 0, Eq. (29) reduces to



Here the exact solution of Eq. (66) in terms of incomplete Gamma function, satisfying boundary conditions (30), is

$$\theta(\eta) = \frac{\Gamma(\frac{3\Pr}{a^2}, 0) - \Gamma(\frac{3\Pr}{a^2}, \frac{3\Pr}{a^2}e^{-\alpha\eta})}{\Gamma(\frac{3\Pr}{a^2}, 0) - \Gamma(\frac{3\Pr}{a^2}, \frac{3\Pr}{a^2})},$$
(67)

where $\alpha = \frac{\sqrt{3}}{\sqrt{1 + A}}$.

6. Validation of numerical results

The validation of present results is essential to check the credibility of the numerical solution methodology. The presently computed results are compared with the exact solutions obtained for some limiting cases of the problem. Figures 2 and 3 compare these results, and an excellent correspondence is seen to exist between the two sets of data. In addition, table 1 shows the comparison values of the local skin friction coefficient with those reported by of Khan and Shahzad [22]. It is seen that the comparison is in very good agreement, and thus gives us confidence to the accuracy of the numerical results.

7. Results and discussion

The main focus of the present chapter is to study the flow and heat transfer characteristics of a Sisko fluid over a nonlinear radially stretching sheet. To obtain physical insight of the flow and heat transfer, Eqs. (22) and (29) subject to boundary conditions (23) and (30) are solved numerically and the results are illustrated graphically. During the ensuing discussion, the assumption of incompressibility and isotropy of fluid is implicit. The influence of the flow behavior index n, the material parameter A, stretching parameter s, and Eckert number Ec on flow and heat transfer is the main interest of the study. Further, the effect of variation of Prandtl number Pr on heat transfer is also analyzed in depth. A comparison amongst the flow and heat transfer aspects of the Bingham, Newtonian, and Sisko fluids is also precisely depicted. Moreover, the flow and heat transfer characteristics are also discussed in terms of the local skin friction coefficient and local Nusselt number.

Figure 4 depicts the influence of the power-law index n on the velocity profile for pseudoplastic (n < 1) and dilatant (n > 1) regimes for nonlinearly stretching sheet. The lower is the value of n, greater is the degree of shear thinning. Figure 4(a) shows that the velocity profile and momentum boundary layer thickness decrease with an increase in the value of the power-law index n, owing to increase in apparent viscosity. The shear thickening behavior (n > 1) is illustrated in figure 4(b). This figure reveals that as the value of the power-law index n is progressively incremented, the velocity profile and the corresponding momentum boundary layer thickness decrease due to gradual strengthening of viscous effects.

The heat transfer aspects of the Sisko fluid over a constant surface temperature stretching sheet for shear thinning (n < 1) and thickening (n > 1) fluids for different values of the power-law index n with nonlinear stretching is illustrated in figure 5. Figure 5(a) depicts that the temperature profile and thermal boundary layer reduce with incrementing the value of the power-law index n. The effect on the temperature profile is marginalized when the power-law index approaches unity. Figure 5(b) reveals that the power-law index n does not affect the temperature profile strongly for n > 1. However, a slight decrease in the thermal boundary layer thickness is observed.

The stretching parameter *s* affects the temperature distribution and thermal boundary layer due to the influence of momentum transfer. Its effect on heat transfer is illustrated in figures

6 (a-c). For the power-law index (n < 1), stretching parameter s does not affects the heat transfer in the Sisko fluid very strongly. The thermal boundary layer thickness increases when value of s is incremented progressively (figure 6(a)). Although, for n=1, the effect of s on the temperature profile is significant and a contrary behavior is noticed as figure 6(b) elucidates. Figure 6(c) represents the temperature profiles for n=1.5. The temperature profile seems to decrease as value of the stretching parameter is incremented. Moreover, it is also noticed that larger the value of the power-law index n the more is decrease in the temperature profile.

The effect of the material parameter *A* on the temperature profile for nonlinear stretching is presented through figures 7(a-c). These figures also make a comparison amongst the temperature profiles of the Newtonian fluid (A=0 and n=1) and the power-law fluid (A=0 and $n\neq1$) with those of the Sisko fluid ($A\neq0$ and $n\neq0$). Qualitatively, figures 7(a-c) reveal that the temperature profile and the corresponding thermal boundary layer thickness reduce in each case with increasing value of the material parameter *A*.

The Prandtl number Pr of a fluid plays a significant role in forced convective heat transfer. Figures 8(a,b) present its effect on heat transfer to Sisko fluid for pseudo-plastic (n < 1) and dilatant (n > 1) regimes. These figures depict that the Pr affects the heat transfer process strongly by thinning the thermal boundary layer thickness. It in turn augments the heat transfer at the wall. The augmentation can be ascribed to the enhanced momentum diffusivity for larger Prandtl number. The temperature profile is slightly lower for fluids with shear thickness behavior than that of the shear thinning, for the same Prandtl number.

Eckert number *Ec* measures the transformation of kinetic energy into heat by viscous dissipation. The variation of temperature with increasing *Ec* is given in figures 9(a,b). Figure 9(a) describes the effect of increasing *Ec* on highly shear thinning (n=0.2) and moderately shear thinning (n=0.6) Sisko fluids. These figures clearly elucidate that the temperature profile increases with increasing *Ec*. Further the strongly shear thinning fluids are dominantly affected by *Ec* as compared to that of the moderately shear thinning regime. Figure 9(b) describes the same phenomenon for shear thickening fluid (n>1), but here the effects are less prominent.

Figures 10(a,b) compare the velocity and temperature profiles of the Bingham (n=0 and $A\neq 0$) and Newtonian (n=1 and A=0) fluids with those of the Sisko fluid ($n\neq 0$ and $A\neq 0$). While sketching figure 10(a), the value of material parameter A is adjusted at 1.5. This figure clearly shows that for the particular value of variable η , the velocity profile of the Bingham fluid substantially higher than those of Newtonian and Sisko fluids. This figure also delineates that the velocity variation of the Bingham fluid approaches free stream velocity for larger values of η as compared to those the Newtonian and Sisko fluids. The temperature profiles of three different fluids are demonstrated in figure 10(b). The Bingham fluid shows larger thermal boundary layer thickness, whilst that of the Sisko fluid least, resulting better heat transfer. A wider variation in the concomitant heat transfer slopes at wall is observed in this figure, with least slop for the Bingham fluid, showing minimum heat transfer at the wall.

Adiabatic Eckert number Ec_a is a measure of direction of heat flux q_x ; the heat flows from heated sheet to fluid ($q_x > 0$), when $Ec < Ec_a$, and vice versa. Figures 11(a,b) shows a series of adiabatic Eckert numbers evaluated numerically for shear thinning (n=0.5) and shear thick-

ening (n=1.5) Sisko fluids. It is clearly noticed from these figures that the adiabatic Eckert number increases at an accelerated pace for smaller values of Prandtl number. Further inspection of these figures reveal that the values of Ec_a decrease at a rapid rate for shear thinning fluid as compared to that of the shear thickening fluid.

Table 2 summarizes the overall trends of the skin friction coefficient for shear thinning and thickening fluids when the material parameter A, stretching parameter s and Eckert number Ec are varied. This table reveals that the value of the skin friction increases with each increment in the value of the material parameter A for linear as well as nonlinear stretching of the sheet, which results in increased drag to the Sisko fluid. The drag is slightly lower for the higher value of the power-law index. Further, this table also depicts the variation in the local Nusselt number with the increasing value of A for linear and nonlinear stretching. The Nusselt number shows improvement with each increment of A. It is also clear that the improvement is better for the power-law index n > 1. Moreover, a decrease in the local Nusselt number is observed with an increase in Ec. It is further noticed that the decrease in heat transfer from wall to fluid is about 24% for shear thinning and 13% for shear thickening fluids. Table 3 demonstrates the effects of the Prandtl number Pr on the local Nusselt number for different values of the power-law index n. An increase in the Nusselt number is larger for fluids with medium power-law index n.

The stream function appearing in Eq. (20) is plotted in figure 12 for several values of ψ and different values of the power-law index *n*. The streamlines are symmetrical about the midway vertical axis, owing to the fact that sheet is being stretched radially by applying equal force in each direction. Moreover, the flow seems nearly straight down at large values of η (away from the sheet) and tends to horizontal at small values of η (vicinity of the stretching sheet). This figure further describes that the flow become identical for each value of the power-law index *n* close to the sheet.



Figure 2. A comparison of the exact and numerical results (solid line exact results and open circles numerical results) when n=0 and s=1 are fixed.



Figure 3. A comparison of the exact and numerical results (solid line exact results and open circles numerical results) when n = 1 and s = 3 are fixed.



Figure 4. The velocity profile $f'(\eta)$ for different values of the power-law index *n* when *s* = 1.5 and *A* = 1.5 are fixed.



Figure 5. The temperature profile $\theta(\eta)$ for different values of the power-law index *n* when *s* = 1.5, *A* = 1.5, *P* = 2.0, *Ec* = 0.1 and *Br* = 0.1 are fixed.



Figure 6. The temperature profile $\theta(\eta)$ for different values of the stretching parameter *s* when *A*=1.5, Pr=2.0, *Ec*=0.1 and *Br*=0.1 are fixed.



Figure 7. The temperature profile $\theta(\eta)$ for different values of the material parameter *A* when *s*=1.5, Pr=2.0, *Ec*=0.1 and *Br*=0.1 are fixed.



Figure 8. Temperature profile $\theta(\eta)$ for different values of the Prandtl number Pr when s = A = 1.5, and Ec = Br = 0.1 are fixed.



Figure 9. Temperature profiles $\theta(\eta)$ for different values of the Eckert number *Ec* when *s*=1.5, *A*=1.5, *Br*=0.1, and Pr=2.0 are fixed.



Figure 10. A comparison among the velocity and temperature profiles for different fluids when s = 1.5, Ec = 0.1, Br = 0.1, and Pr = 2.0 are fixed.



Figure 11. Adiabatic Eckert number Ec_a with variation of Prandtl number Pr when s = 1.5, Br = 0.1, and A = 1.0 are fixed.



Figure 12. The streamlines for different values of the power-law index *n* when s = 1.5 and A = 1.5 are fixed.

	$\frac{1}{2}\operatorname{Re}_{b}^{\frac{1}{n+1}}C_{f}$					
	<i>n</i> =1			<i>n</i> =2		
Α	s	Numerical results	HAM results	Numerical results	HAM results	
0	1	-1.173721	-1.173721	-1.189598	-1.189567	
1	1	-1.659892	-1.659891	-1.605002	-1.605010	
2	1	-2.032953	-2.032945	-1.973087	-1.973092	
3	1	-2.347441	-2.347451	-2.297733	-2.297713	
1	2	-2.090755	-2.090753	-2.152145	-2.152153	
1	3	-2.449490	-2.449491	-2.621176	-2.621182	

Table 1. A tabulation of the local skin friction coefficient in terms of the comparison between the present results and the HAM results (ref. [22])

			<i>n</i> =0.5		<i>n</i> =1.5	
А	S	Ec	$\frac{1}{2}\operatorname{Re}_{b}^{\frac{1}{n+1}}C_{f}$	$\operatorname{Re}_{b}^{-\frac{1}{n+1}}Nu_{r}$	$\frac{1}{2}\operatorname{Re}_{b}^{\frac{1}{n+1}}C_{f}$	$\operatorname{Re}_{b}^{-\frac{1}{n+1}}Nu_{r}$
1.0	1.0	0.1	-1.1738050	-1.177395	-1.62676	-1.39104
2.0	1.0	0.1	-2.116299	-1.233161	-2.00146	-1.45209
3.0	1.0	0.1	-2.427310	-1.264815	-2.33186	-1.48665
1.0	2.0	0.1	-2.075148	-1.125830	-2.12359	-1.60519
2.0	2.0	0.1	-2.544792	-1.191590	-2.58828	-1.67792
3.0	2.0	0.1	-2.928524	-1.228861	-2.99557	-1.72052
1.5	1.5	0.1	-2.140943	-1.186123	-2.107053	-1.54061
1.5	1.5	0.3	-2.140943	-1.043431	-2.107053	-1.44104
1.5	1.5	0.5	-2.140943	-0.900738	-2.107053	-1.34148

Table 2. A tabulation of the local skin friction and the local Nusselt number.

	$\operatorname{Re}_{b}^{-\frac{1}{n+1}}Nu_{r}$	
Pr	<i>n</i> =0.5	<i>n</i> =1.5
1	-0.789183	-1.019614
2	-1.186123	-1.540612
3	-1.487571	-1.935358
4	-1.739727	-2.265692
5	-1.960520	-2.555230

Table 3. A tabulation of the local Nusselt number for different values of Prandtl number when s = 1.5 and A = 1.5 are fixed.

8. Concluding remarks

In this chapter, a theoretical framework for analyzing the boundary layer flow and heat transfer with viscous dissipation to Sisko fluid over a nonlinear radially stretching sheet has been formulated. The governing partial differential equations were transformed into a system of nonlinear ordinary differential equations. The transformed ordinary differential equations were then solved numerically using implicit finite difference scheme along with Keller-box scheme. The results were presented graphically and the effects of the power-law index n, the material parameter A, the stretching parameter s, the Prandtl number Pr, and the Eckert number were discussed. It is pertinent to mention that the analysis in ref. [22] for velocity field was restricted to integer value of the power-law index n. However, the investigations in the present work were upgraded by adding the non-integral values of the power-law index n for the flow and temperature fields.

Our computations have indicated that the momentum and thermal boundary layers thickness were decreased by increasing the power-law index and the material parameter. Further it was noticed that the effects of the Prandtl and Eckert numbers on the temperature and thermal boundary layer were quite opposite. However, both the Prandtl and Eckert numbers were affected dominantly for shear thinning fluid as compared to that of the shear thickening fluid. Additionally, the Bingham fluid had the thickest momentum and thermal boundary layers as compared to those of the Sisko and Newtonian fluids.

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