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# On MU-MIMO Precoding Techniques for WiMAX

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Additional information is available at the end of the chapter

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## 1. Introduction

Future wireless communication systems will require reliable and spectrally efficient transmission techniques to support the emerging high-data-rate applications. The design of these systems necessitates the integration of various recent research outcomes of wireless communication disciplines. So far, the most recent tracks of investigations for the design of the spectrally efficient system are multiple antennas, cooperative networking, adaptive modulation and coding, advanced relaying and cross layer design. The original works of Telatar [1], Foschini [2] and the early idea of Winters [3] as well as the contribution of Alamouti[4] stress the high potential gain and spectral efficiency by using multiple antenna elements at both ends of the wireless link. This promising additional gain achieved by using Multiple Input Multiple Output (MIMO) technology has rejuvenated the field of wireless communication. Nowadays this technology is integrated into all recent wireless standards such as IEEE 802.11n, IEEE 802.16e and LTE [5, 6]. The main objective of this chapter is to provide a comprehensive overview of various MIMO Techniques and critical discussion on the recent advances in Multi-user MIMO precoding design and pointing to a new era of the precoding application in WiMAX systems.

The chapter opens with the basic preliminaries on MIMO channel characterization and MIMO gains in section 2. This is followed by theoretical overview of precoding for MU-MIMO channel in section 3. Section 4 describes a new precoding method and numerical simulation stressing the importance of the proposed precoding in the WiMAX context. Section 5 concludes the chapter.

## 2. Preliminaries

### 2.1. Single user MIMO (SU-MIMO) channel model

In wireless communications, channel modeling and link parameter design are the core problems in designing communication system. To understand the MIMO system, MIMO channel modeling and the related assumptions behind the practical system realization should therefore be first discussed and summarized. Basically, it all begins by the designer identifying the channel type from among the three types of wireless communication channel namely: direct path, frequency selective and frequency-flat channels. Direct path, also called Line-of-Sight (LOS) channel is the simplest model where the channel gain consists of only the free space path loss plus some complex Additive White Gaussian Noise (AWGN). This simplified model is typically used to design the various microwave communication links such as terrestrial, near space satellite, and deep space communication links. The other two types of wireless channel models are the frequency flat and frequency selective channels which both describes the channel gains due to the complex propagation environment where both the free space path loss, shadowing effects and multipath interference are obvious and the objective of the link model is to account for multipath and Doppler shift effects. In frequency selective channel model, the link gain between each transmit and receive antennas is represented by multiple and different impulse response sequences across the frequency band of operation. This is in contrast to the frequency flat fading which has single constant scalar channel gain across the band. Frequency selectivity has crucial Inter-Symbol Interference (ISI) effect on the high speed wireless communication system transmission. Technically there are three ways to mitigate the negative effect of ISI. Two of which are transmission techniques namely: spread spectrum transmission and Multicarrier modulation transmission while the third is the equalization techniques as a receiver side mitigation method. In a MIMO system, researchers generally make common assumption that the channel-frequency-response is flat between each pair of transmit and receive antennas. Thus, from the system design point of view, the system designer can alleviate the frequency selectivity effect in a wideband system by subdividing the wideband channel into a set of narrow sub-bands as in [7, 8] using Orthogonal Frequency Division Multiplexing (OFDM). Figure 1 shows point-to-point single user MIMO system with  $N_T$  transmit antennas and  $M_R$  receive antennas. The channel from the multiple transmit antenna to the multiple receiving antenna is described by the gain matrix  $\mathbf{H}$ . With the basic assumption of frequency-flat fading narrow band link between the transmitter and the receiver,  $\mathbf{H}$  will be given by:

$$\mathbf{H} = \begin{bmatrix} h_{11} & \cdots & h_{1,N_T} \\ \vdots & \ddots & \vdots \\ h_{M_R,1} & \cdots & h_{M_R,N_T} \end{bmatrix} \in \mathbf{C}^{M_R \times N_T} \quad (1)$$

where  $\mathbf{h}_{i,j}$  denotes the complex channel gain between the  $j^{th}$  transmit antenna to the  $i^{th}$  receive antenna, while  $1 \leq j \leq N_T$ ,  $1 \leq i \leq M_R$ . We further assume that the channel bandwidth

is equal to  $\omega$ . Thus, the received signal vector  $\mathbf{y}[n] \in \mathbb{C}^{M_R \times 1}$  at the time instants  $[n]$  can be given by:

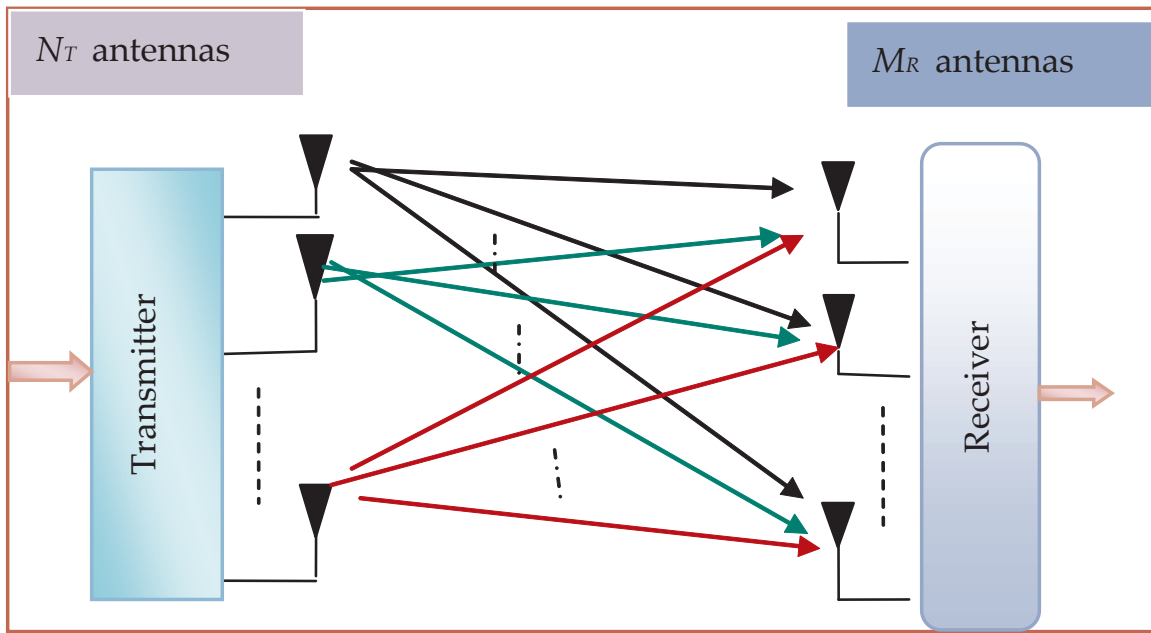
$$\mathbf{y}[n] = \mathbf{H}\mathbf{s}[n] + \mathbf{n}[n] \quad (2)$$

where  $\mathbf{s}[n] = [\mathbf{s}_1 \cdots \mathbf{s}_{N_T}]^T \in \mathbb{C}^{N_T \times 1}$  denotes the complex transmitted signal vector, and  $\mathbf{n}[n] \in \mathbb{C}^{M_R \times 1}$  denotes the AWGN vector which is assumed to have independent complex Gaussian elements with zero mean and variance  $\sigma_n^2 \mathbf{I}_{M_R}$  where  $\sigma_n^2 = \omega N_o$  and  $N_o$  is the noise power spectral density.

The channel matrix  $\mathbf{H}$  is assumed to have independent complex Gaussian random variables with zero mean and unit variance. This statistical distribution is very useful and reasonable assumption to model the effect of richly scattering environment where the angular bins are fully populated paths with sparsely spaced antennas. Justification of these MIMO link assumptions is a very important point for system design. On choosing system operation bandwidth  $\omega$ , the system designers and researchers always assume that the channel frequency response over the bandwidth of MIMO transmission is flat, as practically, it is very hard to implement equalizers for mitigating ISI across all the multiple antennas in a MIMO system. It is to be noted that the twin requirements of broadband transmission to support the high rate applications and narrow band transmission to facilitate the use of simple equalizers to mitigate ISI can be met by utilizing OFDM based physical layer transmission. In physically rich scattering environment (e.g. typical urban area signal propagation) with proper antenna array spacing, the common assumption is that the elements of the Channel matrix are independent identically distributed (i.i.d.). Though, beyond the scope of this study, in practice, if there is any kind of spatial correlation, this will reduce the degrees of freedom of the MIMO channel and consequently this would then result in a decrease in the MIMO channel capacity gain [3, 9]. The assumption of i.i.d channel is partially realizable by correctly separating the multiple element antennas. In some deployment scenarios, where there are not enough scatterers in the propagation environment, the i.i.d assumption is not practical to statistically model the MIMO fading correlation channel. A part of the last decade research was focused on the study of this kind of channel correlation effects [10]. In general, fading correlation between the elements of MIMO channel matrix  $\mathbf{H}$  can be separated into two independent components, namely: transmit correlation and receive correlation [11]. Accordingly, the MIMO channel model  $\mathbf{H}$  can be described by:

$$\mathbf{H} = \mathbf{R}_r^{1/2} \mathbf{H}_w \mathbf{R}_t^{1/2} \quad (3)$$

where  $\mathbf{H}_w$  is the channel matrix whose elements are i.i.d, and  $\mathbf{R}_r^{1/2}$  and  $\mathbf{R}_t^{1/2}$  are the receive and transmit correlation matrix respectively.



**Figure 1.** Block Diagram of single User MIMO System

### 2.1.1. MIMO channel gain

The key to the performance gain in MIMO systems lies in the additional degree-of-freedom provided by the spatial domain and associated with multiple antennas. These additional degree-of-freedom can be exploited and utilized in the same way as the frequency and time resources have been used in the classical Single Input Single Output (SISO) systems. The initial promise of an increase in capacity and spectral efficiency of MIMO systems ignited by the work of Telatar [1] and Foschini [2] has now been validated where by adding more antennas to the transmitter and receiver, the capacity of the system has been shown to increase linearly with the  $N_T$  or  $M_R$ , which is minimum, i.e. the  $\min(N_T, M_R)$  [12]. This capacity can be extracted by making use of three transmission techniques, namely: spatial multiplexing, spatial diversity, and beam-forming.

From classical communication and information theory, channel characteristics play a crucial role in the system design, in that both transmitter and receiver design are highly dependent on it [13, 14]. In MIMO system, the knowledge of the Channel State Information (CSI) is an important factor in system design. CSIT and CSIR refer to the CSI at the transmitter and receiver respectively. Basically, in the state-of-art communication system design, there is a common assumption that the receiver has perfect CSI. With this fair assumption, all MIMO performance gains are exploited. Further improvement in the performance is dependent on the availability and quality of CSI at the transmitter [8, 15]. Accordingly, the accessibility and utilization of CSI at the transmitter is one of the most important criteria of MIMO research classification in the last decade. Next sub-sections gives a brief overview of the most critical processing techniques and types of gains that we can extract from the single user point-to-point MIMO link in both open-loop systems (CSI is available at the receiver) and closed-loop systems (CSI is available at both transmitter and receiver).

### 2.1.2. Open-loop single user MIMO (SU-MIMO) transmission

When there is no CSI at the transmitter, this is called open-loop MIMO configuration. There are two types of performance gains that can be extracted - multiplexing gain and diversity gain [16]. Multiplexing gain is the increase in the transmission rate at no cost of power consumption. This type of gain is achieved through the use of multiple antennas at both transmitter and receiver. In a single user MIMO system with spatial multiplexing gain configuration, different data streams can be transmitted from the different transmit antennas simultaneously. At the receiver, both linear and nonlinear decoders are used to decode the transmitted data vector. Spatial multiplexing gain is very sensitive to long-deep channel fades. Thus, in such communication environment, the designer can solve this problem by resolving to system design that can extract MIMO diversity gain with the help of time or frequency domain.

Diversity gain is defined as the redundancy in the received signal [17]. It affects the probability distribution of received signal power favorably. In single user MIMO system, diversity gain can be extracted when replicas of information signals are received through independent fading channels. It increases the probability of successful transmission which, in turn increases the communication link reliability. In the single user MIMO system, there are two types of diversity methods that are popular, namely: transmit diversity and receive diversity.

Receive diversity is applied on a sub-category of MIMO system where there is only one transmit antenna and  $M_R$  receive antennas, also called Single Input Multiple Output (SIMO). In this case the MIMO channel  $\mathbf{H}$  is reduced to the vector of the form:

$$\mathbf{H} = \mathbf{h} = [h_1 h_2 \cdots h_{M_R}] \quad (4)$$

with  $\mathbf{s}$  denoting the transmitted signal with unit variance, the received signal  $\mathbf{y} \in \mathbb{C}^{M_R \times 1}$  can be expressed as:

$$\mathbf{y} = \mathbf{h}\mathbf{s} + \mathbf{n} \quad (5)$$

The received signal vector from all receiving antennas is combined using one of the many combining techniques like Selection Combining (SC), Maximal Ratio Combining (MRC) or Equal Gain Combining (EGC) to enhance the received Signal to Noise Ratio (SNR) [18]. The most notable drawback of these diversity techniques is that most of the computational burden is on the receiver which may lead to high power consumption on the receiver unit.

On the other hand, MIMO transmit-diversity gain can be extracted by using what is called Space Time Codes (STC) or Space Frequency Code (SFC) [12, 19, 20]. Unlike receive diversity, transmit diversity requires simple linear receive processing to decode the received signal. STC and SFC are almost similar in many aspects except that one of them uses the time domain while the other uses frequency domain. Space-time codes are further classified into Space-Time Block Codes (STBC) and Space-Time Trellis Codes (STTC) families. In general, STTC families achieve



better performance than STBC families at the cost of extra computational load. A well known example and starting point for understanding the STBC transmit diversity techniques is the basic method of Alamouti code [4] which has diversity gain of the order of  $2M_R$ . However, the main limitation of the basic Alamouti method is that it works only for two transmit antennas. However, latest advances in MIMO diversity techniques extends this method to the case of MIMO channel with more than two transmit antennas through what is known today as Orthogonal Space Time Block Codes (OSTBC) [21].

#### 2.1.2.1. Channel capacity of open-loop single user MIMO system

Without the CSI at the transmitter, the MIMO channel capacity is defined and obtained in [1, 22, 23]. Specifically, for the time-invariant communication channel, the capacity is defined as the maximum mutual information between the MIMO channel input and the channel output and is given by:

$$C = \omega \log_2 \left\| \mathbf{I} + \frac{1}{\sigma_n^2} \mathbf{H} \mathbf{R}_s \mathbf{H}^* \right\| \text{ bits/s} \quad (6)$$

where  $\omega$  is bandwidth in Hz and  $\mathbf{R}_s$  is the covariance matrix of the transmitted signal and  $P_T = \text{tr}(\mathbf{R}_s)$  is the total power-constraint. So, for the single user MIMO channel with a Gaussian random matrix with i.i.d elements, the channel capacity will be maximized by distributing the total transmit power over all transmit antennas equally. Thus, in this uniform power allocation scenario, the input covariance matrix  $\mathbf{R}_s$  must be selected such that:

$$\mathbf{R}_s = \frac{P_T}{N_T} \mathbf{I}_{N_T} \quad (7)$$

With power constraint inequality of the form  $\text{tr}(\mathbf{R}_s) \leq P_T$ , where:  $P_T$  is the total transmitting power, the substitution of the power constraint in the average capacity formula of equation (6) yields:

$$C = \omega \log_2 \left\| \mathbf{I} + \frac{1}{\sigma_n^2} \frac{P_T}{N_T} \mathbf{H} \mathbf{H}^* \right\| \text{ bit/s} \quad (8)$$

[18], and in the case of SIMO configuration (one transmit antenna and  $M_R$  receive antennas) the channel capacity reduces to:

$$C_{\text{SIMO}} = \omega \log_2 \left( 1 + \frac{P_T}{\sigma_n^2} \|\mathbf{h}\|_F^2 \right) \text{ bit/s} \quad (9)$$

And consequently, for the case of MISO configuration ( $N_T$  transmit antenna and one receive antennas) the channel capacity reduces to:

$$C_{SIMO} = \omega \log_2 \left( 1 + \frac{1}{\sigma_n^2} \frac{p_T}{N_T} \|\mathbf{h}\|_F^2 \right) \text{ bit/s} \quad (10)$$

Conversely, for the time varying communication channel, the capacity in equation (8) becomes random or ergodic [7] and is defined by:

$$C_{\text{ergodic}} = E \left\{ \omega \log_2 \left\| \mathbf{I} + \frac{1}{\sigma_n^2} \frac{p_T}{N_T} \mathbf{H} \mathbf{H}^* \right\| \right\} \text{ bit/s} \quad (11)$$

Unlike the capacity gains defined in equations (8-10) which can be extracted by spatial multiplexing or diversity, the system capacity in equation (11) is unidentified and it has no significant practical meaning. Thus, in such cases, the system designer can use some kind of system outage metric for the performance evaluation. The quantity called the outage capacity can be defined by the probability that the channel mutual information is less than some constant  $C$ :

$$pro_{\text{out}} = \text{prob} \left\{ \mathbf{H} : \omega \log_2 \left\| \mathbf{I} + \frac{1}{\sigma_n^2} \frac{p}{N_T} \mathbf{H} \mathbf{H}^* \right\| < C \right\} \quad (12)$$

### 2.1.3. Closed-loop single user MIMO transmission

When the CSI is available at both transmitter and receiver, all kinds of MIMO gains (diversity, spatial multiplexing and beam forming) can be extracted and optimized. In practice, CSI can be acquired at the transmitter either through feedback channels in Frequency Division Duplex (FDD) systems or just taking the dual transpose of the received channel in the case of time-invariant Time Division Duplex (TDD) systems [24]. To extract the maximum spatial multiplexing gain, transmission optimization should be done by what is called channel precoder and decoder [25, 26]. For single user MIMO channel, firstly the precoder is designed, multiplied with the user's data, and launched through  $N_T$  transmit antennas at the transmitter site. At the receiver, the received signal from the  $M_R$  receive antennas is processed by the optimized linear decoder. The general form of the precoded received signal is written as:

$$\mathbf{y} = \mathbf{H} \mathbf{F} \mathbf{s} + \mathbf{n} \quad (13)$$



where  $\mathbf{F}$  is the transmit precoding matrix. Different constraints and conditions are used to design the single user MIMO precoding matrix. Generalized method of joint optimum precoder and decoder for single user MIMO system based on Minimum Mean Square Error (MMSE) approach is proposed in [15]. In this method, minimum mean square error performance criteria is used. As the name suggests, the framework is general and leads to flexible solution for performance criterias such as minimum BER and maximum information rate. The main drawbacks of this method are its high computational complexity and the restrictions on the number of antennas. In addition, there are many other simple and linear methods of precoding such as zero forcing, Singular Value Decomposition (SVD) [6, 8] or code book based techniques [27, 28]. Although these methods are simple, they have quite acceptable performance. On the other hand, the spatial diversity gain can also be optimized by precoding when some kind of CSI is available at the transmitter. The precoding across the space-time block code in [19] or transmit antenna selection method in [29] are two other notable closed-loop spatial diversity gain optimization techniques.

#### 2.1.3.1. Channel capacity of the closed-loop single user MIMO

Consider the general capacity formula for MIMO system given in [2]:

$$C = \left\{ \log_2 \left\| \mathbf{I} + \frac{1}{\sigma_n^2} \mathbf{H} \mathbf{R}_s \mathbf{H}^* \right\| \right\} \text{ bit/s} \quad (14)$$

This capacity depends on the channel realization  $\mathbf{H}$  and the input covariance matrix  $\mathbf{R}_s$ . Taking into account the availability of the CSI at the transmitter, there exists for any practical channel realization, an optimum choice of the input covariance matrix  $\mathbf{R}_s$  such that the channel capacity is maximized subject to the transmit power constraint [7]. This capacity is calculated from the following optimization:

$$\begin{aligned} C &= \max \sum_i \omega \log_2 \left( 1 + \frac{\lambda_i P_i}{\sigma_n^2} \right) \\ &\text{subject to:} \\ &\sum_i P_i \leq P_T \end{aligned} \quad (15)$$

where  $\lambda_i$  is the  $i^{\text{th}}$  eigenvalue of the single user MIMO covariance matrix  $(\mathbf{H}\mathbf{H}^*)$  and  $P_i$  is the transmit power on the  $i^{\text{th}}$  channel. For the purpose of generalization, we assume that the rank of the covariance matrix  $(\mathbf{H}\mathbf{H}^*)$  is  $(r)$ , so  $i = 1 \dots r$ . The solution of the optimization problem given in equation (15) [7] shows that the maximum capacity is achieved by what is called the water-filling in space solution which is given by:

$$\frac{P_i}{P_T} = \begin{cases} \frac{1}{\gamma_o} - \frac{1}{\gamma_i} & \gamma_i \geq \gamma_o \\ 0 & \gamma_i < \gamma_o \end{cases} \quad (16)$$

given that :

$$\gamma_i = \frac{\lambda_i P}{\sigma_n^2}, \quad \gamma_o \text{ is some cut-off SNR}$$

In short this means that, technically we have to allocate more power to the strong eigen modes and less power to the weak ones. It is also clear that this capacity is proportional to the  $\min(N_T, M_R)$ .

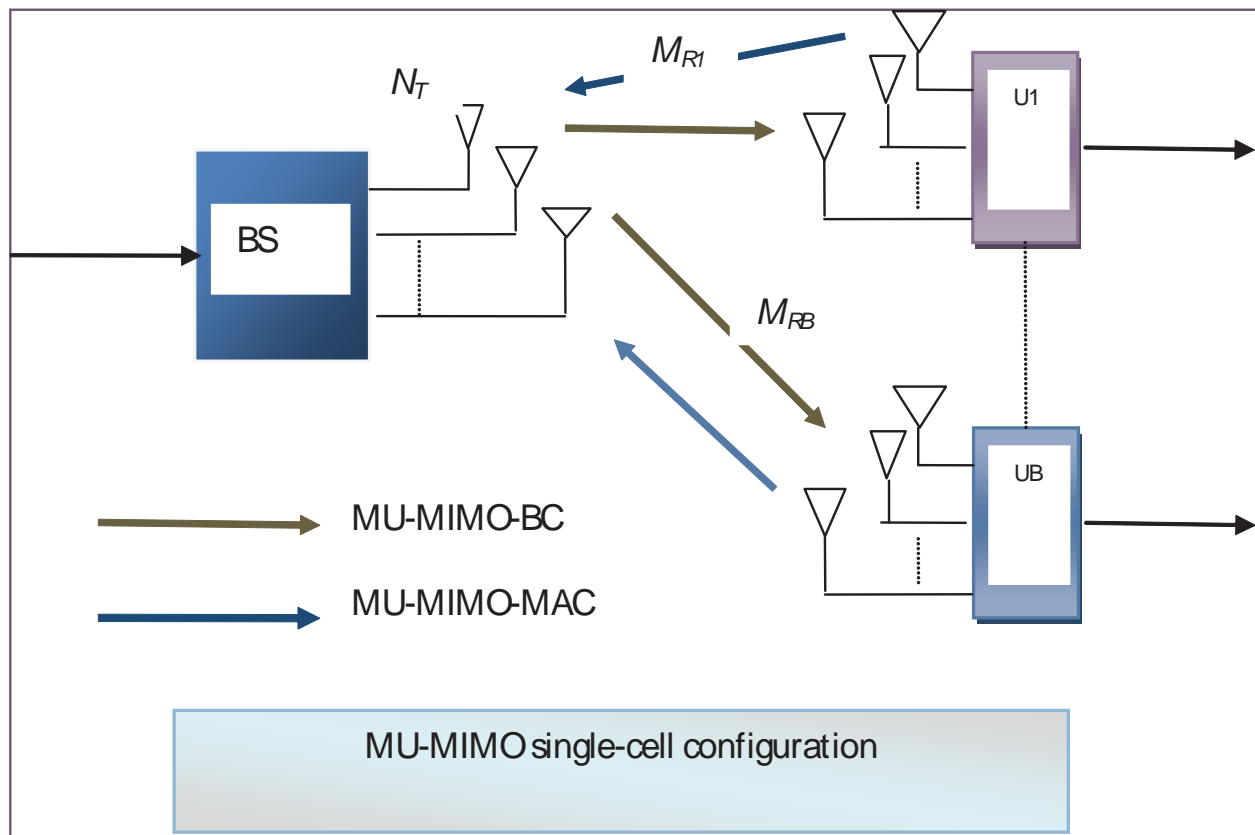
## 2.2. Multiuser MIMO channel

Unlike the simple SU-MIMO channel, Multi-user MIMO (MU-MIMO) channel is a union of a set of SU-MIMO channels. In MU-MIMO system configuration, there are two main communication links - the downlink channel (One-to-many transmission link) which is also known as MU-MIMO Broadcast Channel (MU-MIMO-BC) and the uplink channel (Many-to-One Transmission link) which is also known as MU-MIMO Multiple Access (MU-MIMO-MAC) channel. In addition to the conventional MIMO channel gains, in MU-MIMO we can make use of the multi-user diversity gain to send simultaneously to a group of users or receive data from multiple users at the same time and frequency. As depicted in figure 2, MU-MIMO system configuration can be described as follows: Central node/base station equipped with  $N_T$  transmit antennas transmitting simultaneously to  $B$  number of users in the downlink MU-MIMO-BC channel, where  $k^{th}$  user is equipped with  $M_k$  receive antennas,  $k=1, \dots, B$ . In the reverse uplink MU-MIMO-MAC the base station receiving data from the multiple users simultaneously.

Regardless of its implementation complexity, it is generally known that the minimum mean-square-error with successive interference cancelation (MMSE-SIC) multi-user detector is the best optimum receiver structure for the MU-MIMO-MAC channel [30]. To simultaneously transmit to multiple users in the downlink MU-MIMO-BC, Costa's Dirty-Paper Coding (DPC) or precoding is needed [31] to mitigate the Multi-User Interference (MUI). Both linear and nonlinear precoding transmission techniques have been heavily researched in the last decade with much preference given to the linear precoding methods owing to their simplicity [32-37]. In the downlink MU-MIMO-BC channel at some  $k^{th}$  user, the received signal is given by:

$$\mathbf{y}_k = \mathbf{H}_k \mathbf{F}_k \mathbf{s}_k + \mathbf{H}_k \sum_{\substack{i=1 \\ i \neq k}}^B \mathbf{F}_i \mathbf{s}_i + \mathbf{n}_k \quad (17)$$

where  $\mathbf{H}_k \in \mathbf{C}^{M_k \times N_T}$  is the channel from the base station to the  $k^{th}$  user.  $\mathbf{F}_k \in \mathbf{C}^{N_T \times M_k}$  is the  $k^{th}$  user precoding matrix, while  $\mathbf{s}_k \in \mathbf{C}^{M_k \times 1}$  is the  $k^{th}$  user transmitted data vector and  $\mathbf{n}_k$  is the received additive white noise vector at the  $k^{th}$  user antenna front end.



**Figure 2.** Block Diagram of MU-MIMO System Configuration

Before returning to the general question like how to design the precoder, what is the best Precoder, and what we can gain by incorporating multi-user mode in the WiMAX standards, we summarize the different MIMO system configurations in figure 3. Basically, there are two main modes, SU-MIMO and MU-MIMO. Essentially MU-MIMO is a closed-loop transmission system which means that the channel-state information is required at the transmitter for any transmission. There are two modes of operation for SU-MIMO configurations – closed-loop where the CSI is required at the transmitter and open-loop where the CSI is not required to be used at the transmitter. Also the diagram indicates at each end the type of MIMO gain that can be extracted by each mode of operation and specific configuration.

### 2.3. On MIMO receiver

MIMO receiver design is also one of the hot areas of wireless communication research and system development in the last decade. Many receiving techniques have been reported to decode these kinds of vector transmissions. For the linear transmission techniques, the decoders design complexities range from simple linear methods like Zero-Forcing (ZF) and Minimum Mean-Square-Error (MMSE) receivers to complex spherical sub-optimal decoding and optimal Maximum Likelihood Detections (MLD) [8, 10].

### 2.3.1. Zero-forcing MIMO receiver

Zero-Forcing (ZF) decoder is a simple linear transformation of the received signal to remove the inter-channel interference by multiplying the received signal vector by the inverse of the channel matrix [38]. In fact, if perfect CSI is available at the receiver, the zero-forcing estimate of the transmitted symbol vector can be written as

$$\bar{\mathbf{y}} = \mathbf{G}(\mathbf{H}\mathbf{s} + \mathbf{n}) = \mathbf{s} + \mathbf{G}\mathbf{n} \quad (18)$$

where the decoder is calculated from  $\mathbf{G} = (\mathbf{H}^* \mathbf{H})^{-1} \mathbf{H}^*$ , which is also known as the pseudo inverse of the MIMO channel matrix. In ZF, the complexity reduction comes at the expense of noise enhancement which results in some performance losses compared to other MIMO receiving methods.

### 2.3.2. Minimum mean-square-error MIMO receiver

Unlike ZF receiver which completely force the interference to zero, the MIMO MMSE receiver tries to balance between interference mitigation and noise enhancement [39]. Thus, at low SNR values the MMSE outperforms the ZF receiver. In the MMSE MIMO receiver the decoding factor  $\mathbf{G}$  is designed to maximize the expectation criteria of the form:

$$E\{[\mathbf{G}\mathbf{y} - \mathbf{s}][\mathbf{G}\mathbf{y} - \mathbf{s}]^*\} \quad (19)$$

By analytically solving this MMSE criterion for MIMO channel, the factor  $\mathbf{G}$  is found to be:

$$\mathbf{G} = (\mathbf{H}^* \mathbf{H} + \sigma_n^2 \mathbf{I})^{-1} \mathbf{H}^* \quad (20)$$

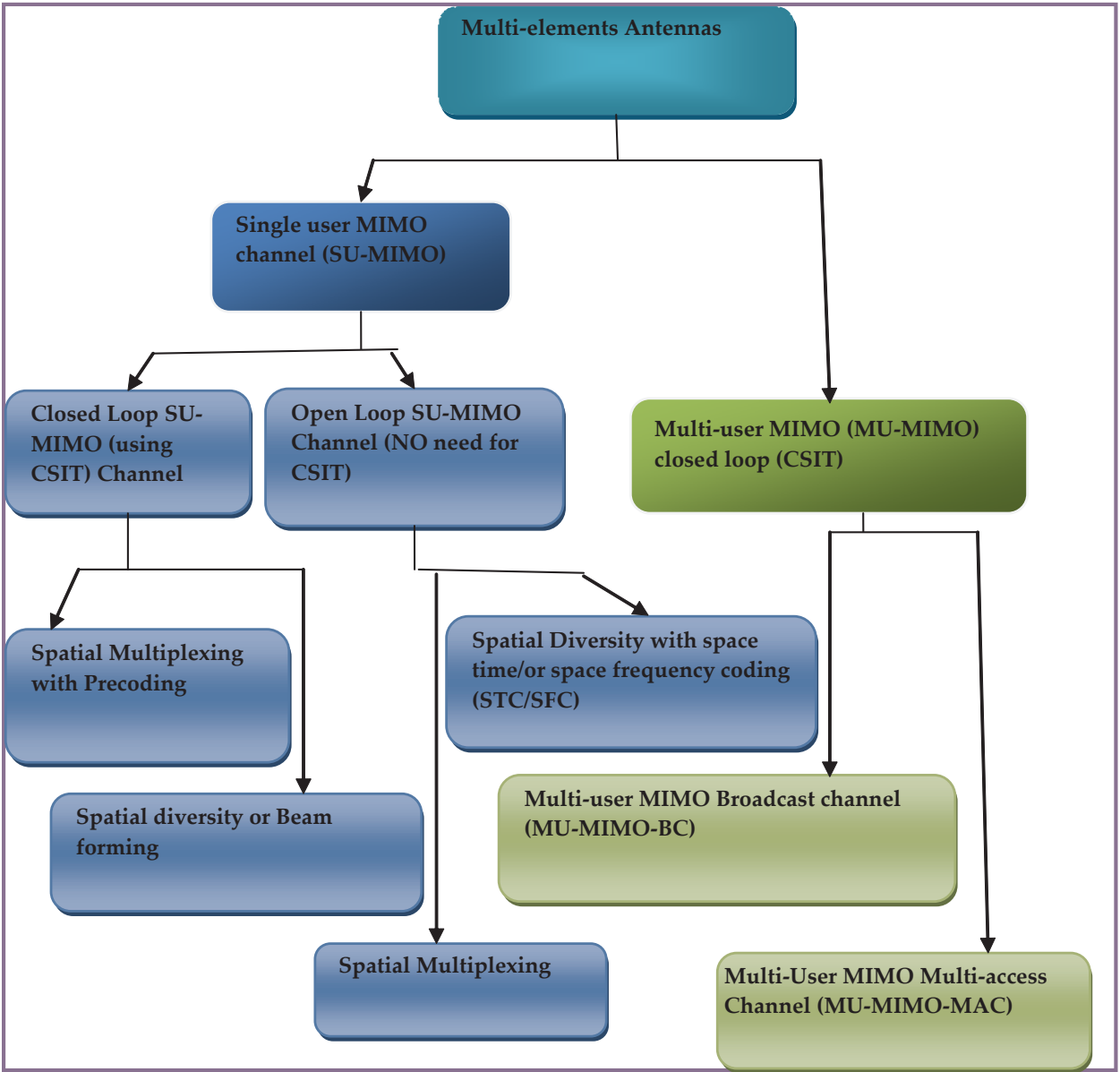
With successive Interference Cancellation (SIC), additional nonlinear steps are added to the original ZF and MMSE equalizers. The resulting versions are ZF-SIC and MMSE-SIC decoding methods. In short, in SIC, the data layer symbols are decoded and subtracted successively from the next received data symbol starting with the highest SINR received signal at each decoding stage. The main drawback of this kind of receive structure is however, the error propagation.

### 2.3.3. Maximum likelihood MIMO receiver

The Maximum Likelihood (ML) decoder is an optimum receiver that achieves the best BER performance among all other decoding techniques. In ML, the decoder searches for the input vector  $\mathbf{s}$  that minimizes the ML criteria of the form:

$$\|\mathbf{y} - \mathbf{H}\mathbf{s}\|_F^2 \quad (21)$$

where  $\| \cdot \|_F^2$  denotes the matrix/or vector Frobenius norm. The complexity of this decoder increases exponentially as the number of transmit and receive antennas increases. In spite of its good BER performance, ML decoding is however not used in any practical system.



**Figure 3.** Different forms of Multi-element Antennas Channel Configuration

2.3.4. Sphere decoding MIMO receiver

Sphere Decoding (SD) families are the new decoding techniques that aim to reduce the computational complexity of the ML decoding technique. In the sphere decoder, the received signal is compared to the closest lattice point, since each codeword is represented by a lattice point. The number of lattice points scanned in a sphere decoder depends on the initial radius

of the sphere. The correctness of the codeword is in turn dependent on the SNR of the system. The search in Sphere decoding is restricted by drawing a circle around the received signal in a way to encompass a small number of lattice points. This entails a search within sub-set of the codes-words in the constellation and allows only those code-words to be checked. All code-words outside the sphere are not taken into consideration for the decoding operation [8].

### 2.3.5. MIMO in the current WiMAX standard

MIMO techniques have been incorporated in all recent wireless standards including IEEE 802.16e, IEEE 802.16m, IEEE 802.11n, and the Long-Term Evaluation (LTE). The WiMAX profile IEEE 802.16e defines three different single user open loop transmission schemes in both uplink and downlink channel summarized as below:

- Scheme defined as matrix  $A$  which describes spatial multiplexing mode of operation for two different symbol streams through two different antennas.
- Scheme defined as matrix  $B$  which describes the spatial diversity mode of operation for two different symbol streams through two different antennas with the basic Alamouti Space-Time Block Code (STBC) [4].
- Scheme defined as matrix  $C$  which combines the respective advantages of diversity and spatial multiplexing modes of operation for two different symbol streams through two different antennas. More details of these schemes are given in [40-42].

In addition to the basic MIMO techniques supported by the IEEE 802.16e, the profile of IEEE 802.16m also supports several advanced MIMO techniques including more complex configurations of SU-MIMO and MU-MIMO (spatial Multiplexing and beam-forming) as well as a number of advanced transmit diversity [43, 44]. The profile also defines multi-mode capability to adapt between SU-MIMO and MU-MIMO in a predefined and flexible manner. Furthermore, flexible receiver decoding mode selection is also supported. Unitary precoding or beam-forming with code-book is also defined for both SU-MIMO and MU-MIMO configurations [45]. Code-book based MU-MIMO precoding techniques found to be effective for the FDD mode of operation because of the great amount of reduction on feedback channel provided, while they are ineffective in the TDD mode of operation [24]. In the next section, we will introduce non-unitary MU-MIMO precoding method to the area of WiMAX. It can be shown that the non-unitary precoding like our proposed method will be applicable and suitable to the TDD mode of operation as accurate CSI is available at the transmitter for the precoding design. In the next section, we will review the most recent researched precoding methods and extend them by proposing our new method.

## 3. Linear precoding for MU-MIMO system

Keeping in mind the computational complexity of the nonlinear DPC precoding methods, the research community, as we mentioned before, gives more preference to the investigations of computationally simple linear precoding techniques. Many design metrics and conditions are



used to develop these linear precoding methods that are hard to deeply survey in one chapter. Generally, one can divide the MU-MIMO linear precoding methods in the literature into two categories - methods that formulate the design objective function for both the precoder and decoder independently such as the methods in [30, 46, 47] and methods that jointly design both the precoding and decoding matrices at the transmitter site (also called iterative method), such as the work in [20, 47-52]. In spite of the good performance joint precoding design obtains relative to the independent formulation methods, the downlink channel overload and complexity are the main drawbacks of this kind of design. One more possible classification is to distinguish between formulations that lead to a closed-form solution expressions such as the works in [53-55] versus those that lead to iterative solutions such as the works in [47, 50, 56, 57]. For comparison, formulations leading to iterative solutions tends to have higher computational complexity than closed form solution methods that are linear. Among the state-of-the-art methods in recent research works, the precoding method originally proposed by Mirette M. Sadek in [34] and based on Per-User Signal to Leakage plus Noise Ratio- Generalized Eigenvalue Decomposition (SLNR-GEVD) and its computationally stable extended version that appeared in [58] which is based on Per-User Signal to Leakage plus Noise Ratio- Generalized Singular Value Decomposition (SLNR-GSVD) are the best in performance. In the next section, we will review these state-of-the-art linear precoding method that seek to maximize Per-user Signal to Leakage plus Noise Ratio (SLNR), which will then be followed by a detailed derivation of our proposed precoding method which is based on maximizing Per-Antenna Signal to Leakage plus Noise Ratio (PA-SLNR) followed by simulation results under WiMAX Physical layer assuming the TDD mode of operation.

### 3.1. Precoding by signal-to- leakage-plus-noise ratio maximization based on GEVD computation

This precoding method is based on maximizing Signal-to-Leakage-plus-Noise Ratio (SLNR) proposed by [34, 59]. In MU-MIMO-BC, recall the system description of section 2.2 and figure 2. The received signal at the  $k^{th}$  user is given by:

$$\mathbf{y}_k = \mathbf{H}_k \mathbf{F}_k \mathbf{s}_k + \mathbf{H}_k \sum_{\substack{j=1 \\ j \neq k}}^B \mathbf{F}_j \mathbf{s}_j + \mathbf{n}_k \quad (22)$$

In this received signal expression, the first term represents the desired signal to the  $k^{th}$  user, while the second term is the Multi-User Interference (MUI) from the other users to the  $k^{th}$  user and the third term is the additive white Gaussian noise at the  $k^{th}$  user antenna front end. In the per-user SLNR precoding method, various variables used in the method are depicted in figure 4. The objective function is formulated such that the desired signal component to the  $k^{th}$  user,  $\|\mathbf{H}_k \mathbf{F}_k\|_F^2$  is maximized with respect to both the signal leaked from the  $k^{th}$  user to all other

users in the system  $\sum_{\substack{j=1 \\ j \neq k}}^B \mathbf{H}_j \mathbf{F}_k$  plus the noise power at the  $k^{th}$  user front end which is given by  $M_{R_k} \sigma_n^2$ . Thus the SLNR objective function for the  $k^{th}$  user can be written as:

$$SLNR_k = \frac{\|\mathbf{H}_k \mathbf{F}_k\|_F^2}{M_{R_k} \sigma_n^2 + \sum_{\substack{j=1 \\ j \neq k}}^B \|\mathbf{H}_j \mathbf{F}_k\|_F^2} \quad (23)$$

By defining the  $k^{th}$  user auxiliary interference domain matrix  $\tilde{\mathbf{H}}_k$  as:

$$\tilde{\mathbf{H}}_k = [\mathbf{H}_1 \cdots \mathbf{H}_{k-1} \mathbf{H}_{k+1} \cdots \mathbf{H}_B]^T \quad (24)$$

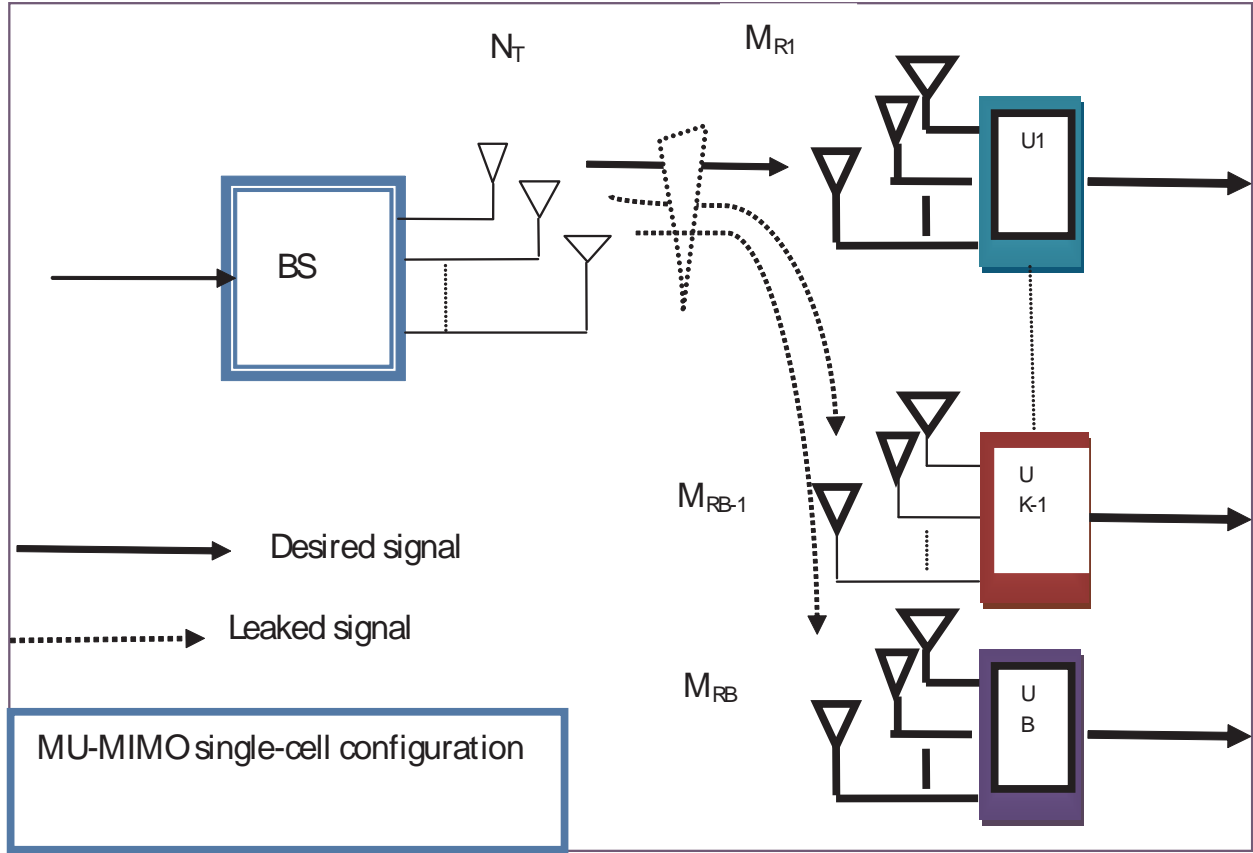
the precoding matrix  $\mathbf{F}_k$  obtained by per-user SLNR maximization is defined as:

$$\mathbf{F}_k = \arg \max_{\mathbf{F}_k} \frac{(\mathbf{F}_k^* \mathbf{H}_k^* \mathbf{H}_k \mathbf{F}_k)}{\mathbf{F}_k^* (\tilde{\mathbf{H}}_k^* \tilde{\mathbf{H}}_k + M_{R_k} \sigma_n^2 \mathbf{I}_{N_T}) \mathbf{F}_k} \quad (25)$$

Closed form solution is developed to solve this fractional rational mathematical optimization problem by making use of the Generalized Eigenvalue Decomposition (GEVD) technique. Unlike the conventional precoding formulations, this method relaxes the constraints on the number of transmitting antennas and has better BER performance.

### 3.2. Precoding by signal-to- leakage-plus-noise ratio maximization based on GSVD computation

One important point of observation in GEVD computation is that it is sensitive to the matrix singularity. Thus, the resulting computation accuracy is low. To resolve the singularity problem in the computation of the multi-user precoding matrices from the Per-user SLNR performance criteria, the work in [58, 60, 61] proposes a Generalized Singular Valve Decomposition (GSVD) and QR- Decomposition (QRD) based methods that both overcome the singularity problem and produce numerically better results. Basically, both the GSVD algorithm and QRD based methods optimize the same Per-user SLNR objective function of equation (25) but they handle the singularity problem in the covariance matrix differently. Thus, the final computation result is accurate and the calculated precoder is more efficient in inter-user interference mitigation. The reason behind the singularity problem in the leakage power plus the noise covariance matrix is that at the high SNR value, the power dominates across the matrix which reduces the degree of freedom.



**Figure 4.** The Definition of the Desired Signal and the Leaked Signal

Two algorithms to solve the objective function of equation (25) are given in [58, 60, 61]. Because of the singularity problem, both algorithms avoid matrix inversion to overcome the computational instability. The developed algorithms makes use of the GSVD analysis. Although there are several methods of GSVD formulations in the literature [62-64], the work in [60] and [58] makes use of the least restrictive form of GSVD algorithm due to Paige and Saunders [62] which is now summarized as follows:

**Theorem 1:** Paige and Saunders GSVD:

Consider any two matrices of the form:  $A_b \in \mathbb{C}^{D \times C}$  and  $A_w \in \mathbb{C}^{D \times N}$ . The GSVD is given by:

$$Y^T A_b^T Q = [\Sigma_b, \mathbf{0}] \text{ and } Z^T A_w^T Q = [\Sigma_w, \mathbf{0}] \quad (26)$$

$$\text{where } \Sigma_b = \begin{bmatrix} \mathbf{I}_b & & \\ & \mathbf{D}_b & \\ & & \mathbf{0}_b \end{bmatrix} \quad \text{and } \Sigma_w = \begin{bmatrix} \mathbf{I}_w & & \\ & \mathbf{D}_w & \\ & & \mathbf{0}_w \end{bmatrix} \quad (27)$$

$Y$  and  $Z$  are orthogonal matrices and  $Q$  is the non-singular eigenvectors matrix. Thus, we can write:

$$\mathbf{A}_b^T = \mathbf{Y}[\mathbf{\Sigma}_b, \mathbf{0}]\mathbf{Q}^{-1} \text{ and } \mathbf{A}_w^T = \mathbf{Z}[\mathbf{\Sigma}_w, \mathbf{0}]\mathbf{Q}^{-1} \quad (28)$$

where:  $\mathbf{D}_b = \text{diag}(\alpha_1 \alpha_2 \cdots \alpha_r)$  and  $\mathbf{D}_w = \text{diag}(\beta_1 \beta_2 \cdots \beta_r)$ , and  $\alpha_i + \beta_i = 1, i = 1, \dots, r$ .

The authors in [58, 60] made use of this theorem and developed an efficient precoding algorithm to calculate the precoding matrices for multiple users which is summarized in algorithm 1 as follows

**Algorithm 1:** The GSVD based Per-user SLNR Precoding algorithm [58, 60]

Assume that the combined channel matrix for MU-MIMO broadcast channel of  $B$  number of users is given by  $\mathbf{H}_{\text{com}}$  and the input noise power is given by  $\sigma_k^2$ .

1. Input:  $\mathbf{H}_{\text{com}} = [\mathbf{H}_1; \dots; \mathbf{H}_B]$ , and  $\sigma_k^2$
2. Output: The algorithm computes the precoding matrices for  $B$  users.
  - a Set  $\mathbf{\Psi} = [\mathbf{H}; \sqrt{M_k \sigma_k} \mathbf{I}_{N_T}] \in \mathbb{C}^{(BM_k + N_T) \times N_T}$
  - b Compute the reduced QRD of  $\mathbf{\Psi}$  i.e  $\mathbf{\Omega}^H \mathbf{\Psi} = \mathbf{R}$  where  $\mathbf{\Omega} \in \mathbb{C}^{(BM_k + N_T) \times N_T}$  orthonormal columns and  $\mathbf{R} \in \mathbb{C}^{N_T \times N_T}$  is upper triangular.
3. For  $k = 1 : B$
4. Compute  $\mathbf{V}_k$  from the SVD of  $\mathbf{\Omega}((k-1)M_k + 1 : kM_k, 1 : N_T)$
5. Solve the triangular system  $\mathbf{R}\mathbf{F}_k = \mathbf{V}_k(:, 1 : M_k)$
6. End

The Per-user SLNR precoding based on GSVD computation produces better results than all the conventional methods. However, the objective function based on per-user SLNR neglects to take the intra-user antenna interference into account. Hence, this formulation is sub-optimum for spatial multiplexing gain extraction.

#### 4. The proposed precoding by maximization of per-antenna signal-to-leakage-plus-noise ratio

The precoding technique originally proposed in [65] maximizes the SLNR for each user, thus the precoder so designed just cancels the inter-user interference. The technique proposed herein, however, utilizes a new cost function that seeks to maximize the Per-Antenna Signal-to-Leakage-plus-Noise Ratio (PA-SLNR) which would help minimize even the intra-user antenna interference. Thus, the precoder so designed maximizes the overall SLNR per user more efficiently. This is justified because the PA-SLNR as explained in figure 5, takes into account the intra-user antenna interference cancelation. For  $j^{\text{th}}$  receive antenna of  $k^{\text{th}}$  user, the PA-SLNR given by  $\gamma_{k,j}^i$ , is defined as the ratio between the desired signal power of  $j^{\text{th}}$  receive

antenna to the interference introduced by the signal power intended for  $j^{th}$  antenna but leaked to all other antennas plus the noise power at that receiving antenna front end. So for the  $j^{th}$  receive antenna of  $k^{th}$  user, the PA-SLNR,  $\gamma_k^j$  is defined by:

$$\gamma_k^j = \frac{\|\mathbf{h}_k^j \mathbf{f}_k^j\|_F^2}{\sum_{\substack{i=1 \\ i \neq k}}^B \|\mathbf{H}_i \mathbf{f}_k^j\|_F^2 + \sum_{\substack{i=1 \\ i \neq j}}^{M_k} \|\mathbf{h}_k^i \mathbf{f}_k^j\|_F^2 + \sigma_{n_k}^2} \quad (29)$$

where  $\mathbf{h}_k^j \in \mathbb{C}^{1 \times N_T}$  is the  $k^{th}$  user,  $j^{th}$  antenna received row. If we define an auxiliary matrix  $\mathbf{H}_k^j$  as the matrix that contains all the received antennas rows of  $k^{th}$  user except the  $j^{th}$  row as follows:

$$\mathbf{H}_k^j = \begin{bmatrix} h_k^{(1,1)} & h_k^{(1,2)} & \dots & h_k^{(1,N_T)} \\ \vdots & \vdots & \vdots & \vdots \\ h_k^{(j-1,1)} & h_k^{(j-1,2)} & \dots & h_k^{(j-1,N_T)} \\ h_k^{(j+1,1)} & h_k^{(j+1,2)} & \dots & h_k^{(j+1,N_T)} \\ \vdots & \vdots & \vdots & \vdots \\ h_k^{(M_k,1)} & h_k^{(M_k,2)} & \dots & h_k^{(M_k,N_T)} \end{bmatrix} \in \mathbb{C}^{((M_k-1) \times N_T)} \quad (30)$$

and the combined channel matrices for all other systems receive antennas except the  $j^{th}$  desired receive antenna row as as:

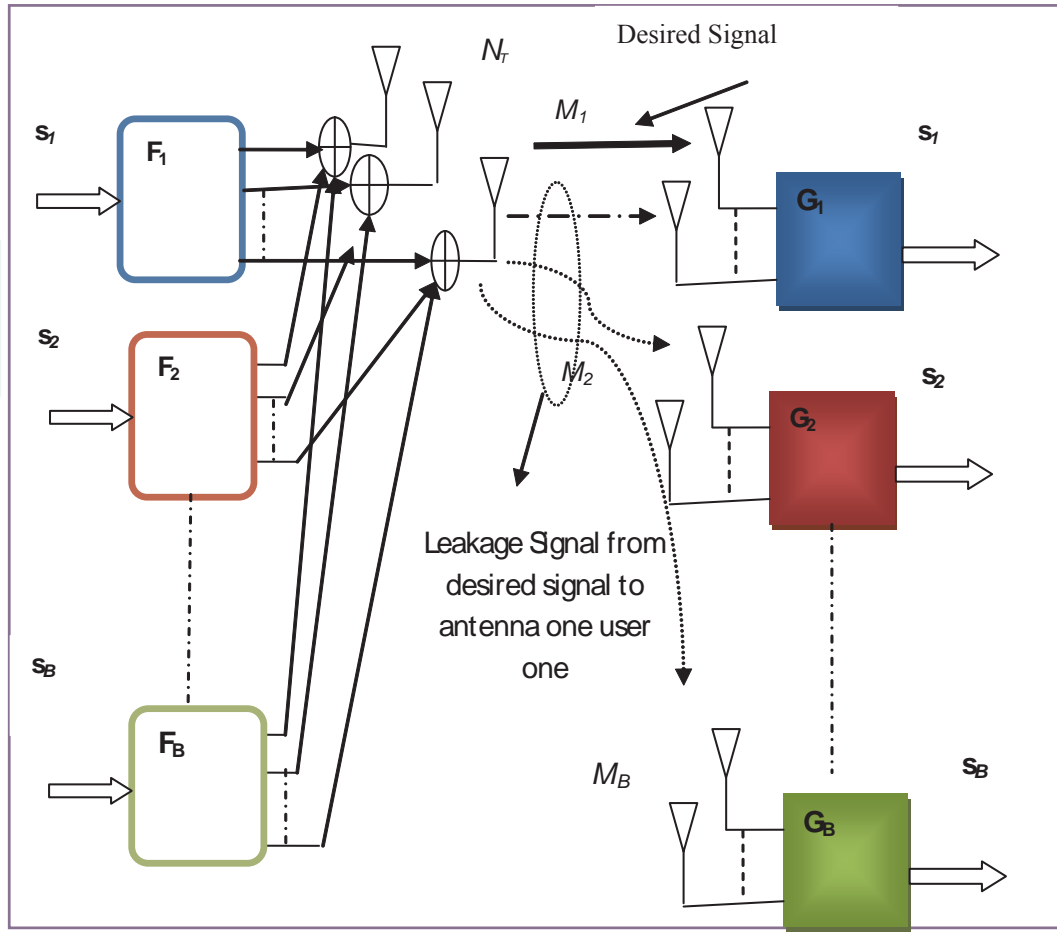
$$\tilde{\mathbf{H}}_k^j = [\mathbf{H}_k^j \mathbf{H}_1^T \mathbf{H}_1^T \dots \mathbf{H}_{k-1}^T \mathbf{H}_{k+1}^T \dots \mathbf{H}_B^T]^T \quad (31)$$

then from equation (31) and equation (30) the optimization expression in (29) can be rewritten as:

$$\gamma_k^j = \frac{\|\mathbf{h}_k^j \mathbf{f}_k^j\|_F^2}{\|\tilde{\mathbf{H}}_k^j \mathbf{f}_k^j\|_F^2 + \sigma_{v_k}^2} \quad (32)$$

**Problem Formulation:** For any  $j^{th}$  receive antenna of the  $k^{th}$  user, select the precoding vector  $\mathbf{f}_k^j$ , where  $k=1, \dots, B$ ,  $j=1, \dots, M_k$  such that the PA-SLNR ratio is maximized:

$$\begin{aligned} \mathbf{f}_k^j &= \arg \max_{\mathbf{f}_k^j \in \mathbb{C}^{N_T \times 1}} \frac{\mathbf{f}_k^{j*} (\mathbf{h}_k^j \mathbf{h}_k^j) \mathbf{f}_k^j}{\mathbf{f}_k^{j*} (\tilde{\mathbf{H}}_k^j \tilde{\mathbf{H}}_k^j + \sigma_{n_k}^2 \mathbf{I}_{N_T}) \mathbf{f}_k^j} \\ &\text{subject to:} \\ &\text{tr}(\mathbf{F}_k \mathbf{F}_k^*) = 1 \\ &\mathbf{F}_K = [\mathbf{f}_1^1, \dots, \mathbf{f}_k^{M_k}] \end{aligned} \quad (33)$$



**Figure 5.** System Model depicts all Variables.

The optimization problem in the equation (33) deals with the  $j^{th}$  antenna desired signal power in the numerator and a combination of total leaked power from desired signal to the  $j^{th}$  antenna to all other antennas plus noise power at the  $j^{th}$  antenna front end in the denominator. To calculate the precoding matrix for each user we need to calculate the precoding vector for each receive antenna independently. This requires solving the linear fractional optimization problem in the equation (33)  $M_k \times B$  times using either GEVD [65] or GSVD [66] which both leading-to high computational load at the base stations. In the next section, Fukunaga-Koontz Transform (FKT) based solution method for solving such series of linear fractional optimization problems is described and simple computational method for MU-MIMO precoding algorithm is developed.

#### 4.1. FKT and FKT based precoding algorithm

Fukunaga-Koontz Transform (FKT) is a normalization transform process which was first introduced in [67] to extract the important features for separating two pattern classes in pattern recognition. Since the time it was first introduced, FKT is used in many Linear Discriminant Analysis (LDA) applications notably in [68, 69]. Researchers in [70, 71] formulate the problem of recognition of two classes as follows: Given the data matrices  $\psi_1$  and  $\psi_2$ , then from these



two classes, the autocorrelation matrices  $\Pi_1 = \psi_1 \psi_1^T$  and  $\Pi_2 = \psi_2 \psi_2^T$  are positive semi-definite (p.s.d) and symmetric. For any given p.s.d autocorrelation matrices  $\Pi_1$  and  $\Pi_2$  the sum  $\Pi$  is still p.s.d and can be written as:

$$\Pi = \Pi_1 + \Pi_2 = [\mathbf{U} \ \mathbf{U}_\perp] \begin{bmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{U}^T \\ \mathbf{U}_\perp^T \end{bmatrix} \quad (34)$$

Without loss of generality the sum  $\Pi$  can be singular and  $r = \text{rank}(\Pi) \leq \text{Dim}(\Pi)$ , where  $\mathbf{D} = \text{diag}(\lambda_1, \dots, \lambda_r)$  and also  $\lambda_1 \geq \dots \geq \lambda_r > 0$ .  $\mathbf{U} \in \mathbb{C}^{\text{Dim}(\Pi) \times r}$  is the set of eigenvectors that correspond to the set of nonzero-eigenvalues and  $\mathbf{U}_\perp \in \mathbb{C}^{\text{Dim}(\Pi) \times (\text{Dim}(\Pi) - r)}$  is the orthogonal complement of  $\mathbf{U}$ . From the equation (34), the FKT transformation [71] matrix operator is defined as:

$$\mathbf{P} = \mathbf{U} \mathbf{D}^{-1/2} \quad (35)$$

By using this FKT transformation factor, the sum p.s.d matrix  $\Pi$  can be whitened such that the sum of the two Sub-matrices  $\tilde{\Pi}_1$  and  $\tilde{\Pi}_2$  gives the identity matrix as follows:

$$\mathbf{P}^T \Pi \mathbf{P} = \mathbf{P}^T (\Pi_1 + \Pi_2) \mathbf{P} = \tilde{\Pi}_1 + \tilde{\Pi}_2 = \mathbf{I}^{r \times r} \quad (36)$$

Where  $\tilde{\Pi}_1 = \mathbf{P}^T \Pi_1 \mathbf{P}$ ,  $\tilde{\Pi}_2 = \mathbf{P}^T \Pi_2 \mathbf{P}$  are the transformed covariance matrices for  $\Pi_1$  and  $\Pi_2$  respectively, and  $\mathbf{I}^{r \times r}$  is an identity matrix. Suppose that  $\mathbf{v}$  is an eigenvector of  $\tilde{\Pi}_1$  with corresponding eigenvalue  $\lambda_1$ , then  $\tilde{\Pi}_1 \mathbf{v} = \lambda_1 \mathbf{v}$  and from the equation (36) we have  $\tilde{\Pi}_1 = \mathbf{I} - \tilde{\Pi}_2$ . Thus, the following results can be pointed:

$$(\mathbf{I} - \tilde{\Pi}_2) \mathbf{v} = \lambda_1 \mathbf{v} \quad (37)$$

$$\tilde{\Pi}_2 \mathbf{v} = (1 - \lambda_1) \mathbf{v} \quad (38)$$

This means that  $\tilde{\Pi}_2$  has the same eigenvectors as  $\tilde{\Pi}_1$  with corresponding eigenvalues related as  $\lambda_2 = (1 - \lambda_1)$ . Thus, we can conclude that the dominant eigenvectors of  $\tilde{\Pi}_1$  is the weakest eigenvectors of  $\tilde{\Pi}_2$  and vice versa. Based on the FKT transform analysis we conclude that the transformed matrices  $\tilde{\Pi}_1$  and  $\tilde{\Pi}_2$  share the same eigenvectors and the sum of the two corresponding eigenvalues are equal to one. Thus, the following decomposition is valid for any positive definite and positive semi-definite matrices:

$$\tilde{\Pi}_1 = \mathbf{V}\Lambda_1\mathbf{V}^T \quad (39)$$

$$\tilde{\Pi}_2 = \mathbf{V}\Lambda_2\mathbf{V}^T \quad (40)$$

$$\mathbf{I} = \Lambda_1 + \Lambda_2 \quad (41)$$

Where  $\mathbf{V} \in \mathbb{C}^{r \times r}$  is the matrix that contains all the eigenvectors, and  $\Lambda_1, \Lambda_2$  are the corresponding eigenvalues matrices. Thus, from these analyses we conclude that FKT gives the best optimum solution for any fractional linear problem without going through any serious matrix inversion step. By relating FKT transform analysis of the two covariance matrices, and the precoding design problem for MU-MIMO (multiple linear fractional optimization problem), we can make direct mapping of the optimization variable from equation (33) to the FKT transform as follows:

$$\Pi_1 = \mathbf{h}_k^j \mathbf{h}_k^j \quad (42)$$

$$\Pi_2 = \tilde{\mathbf{H}}_k^j \tilde{\mathbf{H}}_k^j + \sigma_k^2 \mathbf{I}_{N_T} \quad (43)$$

and consequently the sum  $\Pi$  of the two covariance matrices becomes:

$$\Pi = \mathbf{H}_{com}^* \mathbf{H}_{com} + \sigma_k^2 \mathbf{I}_{N_T} \quad (44)$$

where  $\mathbf{H}_{com}$  is the combined channel matrix for all user which is given as  $\mathbf{H}_{com} = [\mathbf{H}_1^T \mathbf{H}_2^T \dots \mathbf{H}_B^T]^T$ .

According to FKT analysis, we can calculate the FKT factor and consequently we use this transformation factor to generate the shared eigenspace matrices  $\tilde{\Pi}_1$  and  $\tilde{\Pi}_2$  using the facts from equations (42-44) for each receive antenna in the system. The shared eigen subspaces are complements of each other such that the best principal eigenvectors of the first transformed covariance matrix  $\tilde{\Pi}_1$  are the least principal eigenvector for the second transformed covariance matrix  $\tilde{\Pi}_2$  and vice-versa. Thus, we can find the receive antenna precoding vector by simply multiplying the FKT factor with the eigenvectors corresponding to the best eigenvalue of the transformed antenna covariance matrix or eigenvector corresponding to the least eigenvalue of the transformed leakage plus noise covariance matrix. The most notable observation is that, for the set of  $M_K \times B$  receiving antennas in the system, we need to compute the FKT transform factor only once, which cuts down the computation load sharply. Algorithm 2, summarizes the computation steps of the precoding matrix for multiple  $B$  users in the system using FKT.

**Algorithm 2:** PA-SLNR MU-MIMO precoding based on FKT for multiple  $B$  independent MU-MIMO users.

• Input: Combined channel matrix for all  $B$  users and the input noise variance

$$\mathbf{H}_{com} = [\mathbf{H}_1^T \mathbf{H}_2^T \dots \mathbf{H}_B^T]^T, \sigma_k^2$$

• Output: Precoding matrices  $\mathbf{F}_k$  for multiple  $B$  users such that,  $k = 1, \dots, B$

1. Compute the sum  $\mathbf{\Pi} = \mathbf{H}_{com}^* \mathbf{H}_{com} + \sigma_k^2 \mathbf{I}_{N_T}$

2. Compute FKT factor  $\mathbf{P} = \mathbf{U}\mathbf{D}^{-\frac{1}{2}}$  from  $SVD(\mathbf{\Pi})$

3. For  $k = 1$  to  $B$

4. For  $j = 1$  to  $M_k$

○ Transform the  $j^{th}$  receive antenna covariance matrix  $\mathbf{\Pi}_1$  using the FKT factor  $\mathbf{P}$  to  $\tilde{\mathbf{\Pi}}_1$  and select the first eigenvector  $\mathbf{v}_k^j$  of  $\tilde{\mathbf{\Pi}}_1$

○ The precoding vector corresponding to the  $j^{th}$  receive antenna at the  $k^{th}$  user is:  $\mathbf{f}_k^j = \mathbf{P}\mathbf{v}_k^j$

End

○ Synthesize the  $k^{th}$  user precoding matrix is  $\mathbf{F}_k = [\mathbf{f}_k^1 \dots \mathbf{f}_k^{M_k}]$

End

The algorithm takes the combined MU-MIMO channel matrix as well as the value of the noise variance as an input and outputs  $B$  users precoding matrices. It computes the FKT factor in step one and two and iterates  $B$  times (step three to six) to calculate the precoding matrices for  $B$  number of users. For each user, there are  $M_k$  sub-iteration operations (step four to five) to calculate each individual user precoding matrix in vector by vector basis.

#### 4.2. Performance evaluation

In this section we will highlight the importance of Precoding for MU-MIMO and showcase the performance of the proposed PA-SLNR-FKT scheme in two scenarios – scenario 1, single cell MU-MIMO where there is no interference from other cell and scenario 2 of multi-cell processing (MCP) where there is multicell interference and the objective is to make use of multiple antennas in all basestation cooperatively to improve the overall system performance. We use both basic assumptions and a typical simplified WiMAX physical layer Standard discrete channel Models for the Monte-Carlo simulation. Firstly, with basic assumptions we provide comparative performance evaluation results of the proposed MU-MIMO Precoder and the PU-SLNR maximization techniques using GEVD and GSVD proposed in [34, 58]. The comparison is done in terms of average received BER and output received SINR outage performance metrics. In each simulation setup, the entries of  $k^{th}$  user MIMO channel  $\mathbf{H}_k$  is generated as complex white Gaussian random variables with zero mean and unit variance. The users data

Parameter	Configuration
System configuration	MU-MIMO-BC
Each user channel	Matrix elements generated as zero-mean and unit-variance i.i.d complex Gaussian random variables
Modulation	4-QAM
Precoding methods	<ul style="list-style-type: none"> <li>• Proposed method (PA-SLNR-FKT)</li> <li>• Reference SLNR-GEVD, SLNR-GSVD</li> </ul>
Performance metrics	Received BER and Received SINR outage
MIMO Decoder	Matched filter.
Number of Iterations	<ul style="list-style-type: none"> <li>• 5000 System transmission for BER calculation</li> <li>• 2000 System transmission running for SINR outage calculation</li> </ul>

**Table 1.** Narrowband MU-MIMO System Configuration Summary

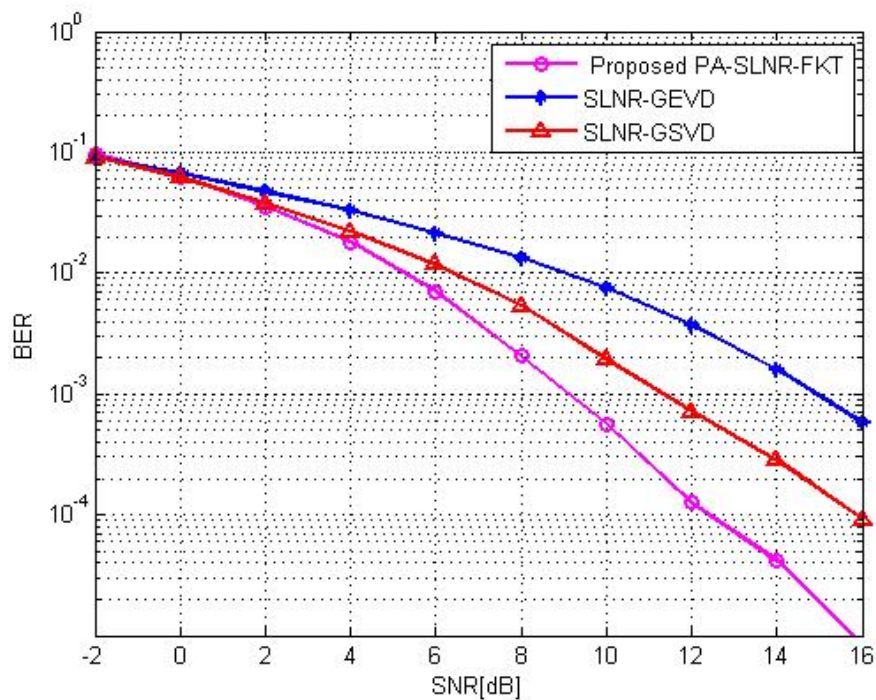
symbol vectors are modulated and spatially multiplexed at the base station. At the receivers, matched filter is used to decode each user's data. Detailed summary of the MU-MIMO-BC system configuration parameters are given in table (1).

#### 4.2.1. Scenario 1: Single cell MU-MIMO

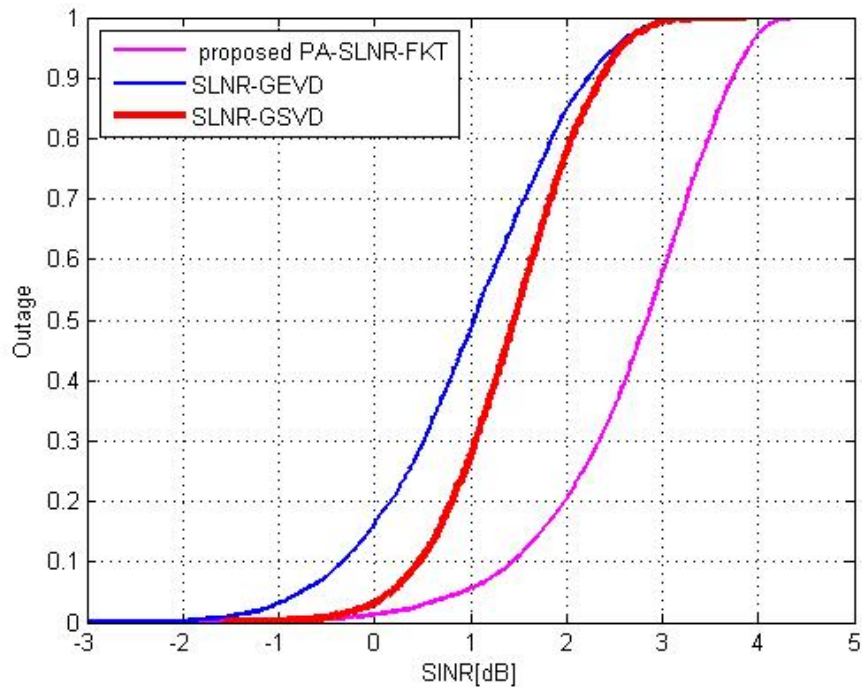
In this scenario we consider single cell transmission where implicitly we assume that the multi-cell interference is zero. Figure 6. shows the average received BER performance of the proposed PA-SLNR-FKT and the reference methods of SLNR-GSVD and SLNR-GEVD precoding schemes for the MU-MIMO-BC system configurations of  $N_T=14$ ,  $B=3M_k=4$ . In this configuration, the numbers of the base station antennas are more than the sum of all receiving antennas which also signifies more degree of freedom in MU-MIMO transmission. In this simulation also, the base station utilizes 4-QAM modulation to modulate, spatially multiplexes and precodes a vector of length 4 symbols to each user. The average BER is calculated over 5000 MU-MIMO channel realization for each algorithm. The proposed method outperforms SLNR-GSVD and SLNR-GEVD. At BER equal to  $10^{-4}$  there is approximately 4dB performance gain over SLNR-GSVD.

Figure 7 compares the received output SINR outage performance of the proposed PA-SLNR-FKT precoding and the reference SLNR-GSVD and SLNR-GEVD precoding methods. MU-MIMO system with full rate configuration of  $B=3$ ,  $M_k=4$ ,  $N_T=12$  and 2dB input SNR is considered in the simulation. The proposed method outperforms the SLNR-GSVD method by 1 dB gain at 10% received output SINR outage.

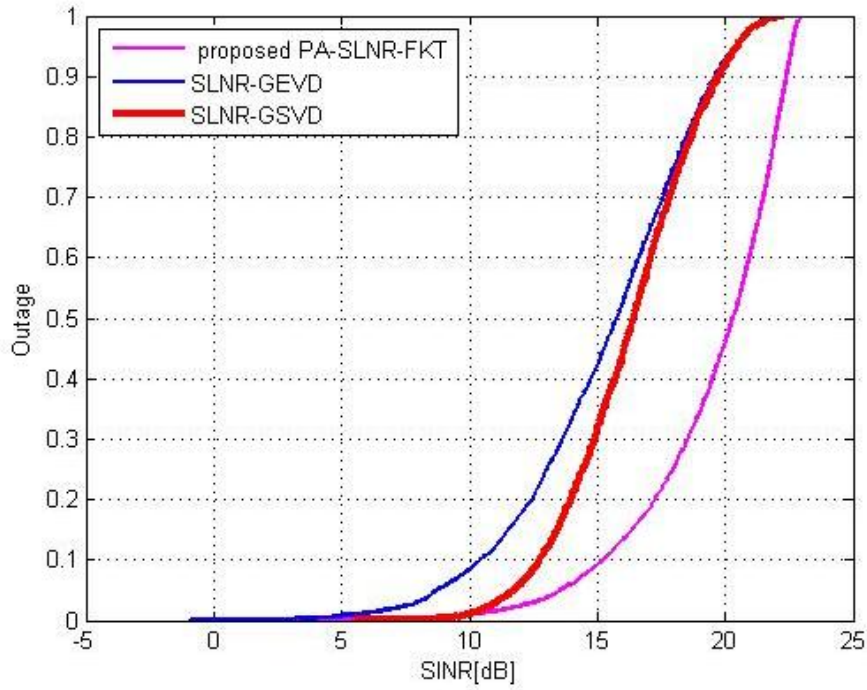
Figure 8 compares the received output SINR outage performance of the proposed PA-SLNR-FKT precoding and the reference SLNR-GSVD and SLNR-GEVD precoding methods. MU-MIMO system with full rate configuration of  $B=3$ ,  $M_k=4$ ,  $N_T=12$  and 10dB input SNR are considered in the simulation. The proposed method outperforms the SLNR-GSVD method by approximately 1.5 dB gain at 10% received output SINR outage.



**Figure 6.** Shows The Compression of The Un-coded BER Performance for PA-SLNR-FKT, SLNR-GSVD and SLNR-GEVD Precoding Methods Under System Configuration of  $B=3$ ,  $M_k=4$ ,  $N_T=14$ , 4QAM Modulated Signal.



**Figure 7.** Shows The Comparison of The Output SINR Outage Performance of The PA-SLNR-FKT, SLNR-GSVD and SLNR-GEVD Precoding Methods Under System Configuration of  $B=3$ ,  $M_k=4$ ,  $N_T=12$ , and 2 dB Input SNR.



**Figure 8.** Shows The Comparison of the Output SINR Outage Performance PA-SLNR-FKT, SLNR-GSVD and SLNR-GEVD Precoding Methods Under System Configuration of  $B=3$ ,  $M_k=4$ ,  $N_T=12$ , and 10 dB Input SNR.

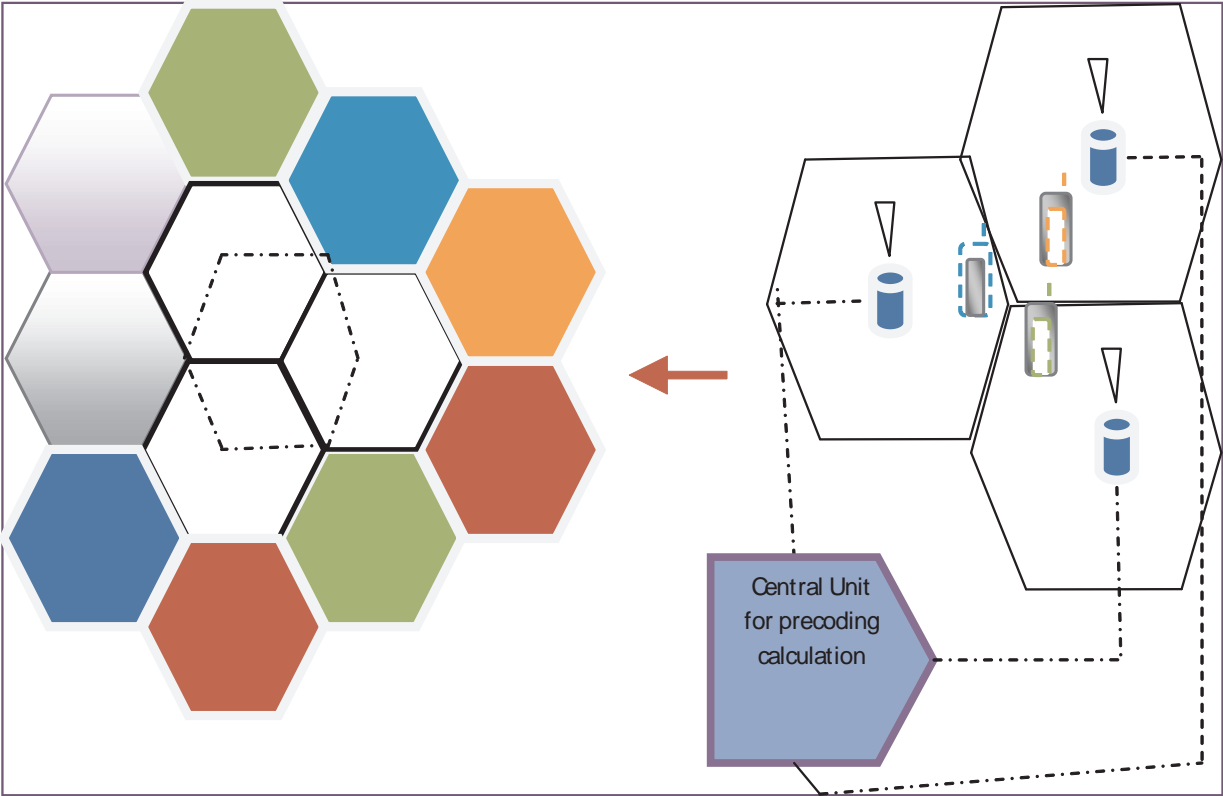
#### 4.2.2. Scenario 2: MU-MIMO with Multi Cell Processing (MCP)

To appreciate the importance of precoding to cancel inter-cell interference in the MCP configuration, geometrically we consider three cooperating BSs as illustrated in figure 9. Basically we consider micro-cellular setup of BS to BS distance equal to 1000 m. As shown in figure 9. we also consider a simplified as well as an extreme case where there are three users at the edges of the three cooperating cells, so that we just allow each user to uniformly position within the last 50 m of its anchor BS. In each transmission, a WiMAX standard channel model is used. Thus each entry of the  $k^{th}$  user MIMO channel matrix is generated according to pre-specified wireless communication channel model which include mean path loss, shadowing and slow fading discrete components as follows.

$$\mathbf{H}_e^k = (\phi_1 \phi_2)^{1/2} \tilde{\mathbf{H}}_e^k \quad (45)$$

where  $\tilde{\mathbf{H}}_e^k \in \mathbf{C}^{N_{R_k} \times N_{T_e}}$  represent the fast fading channel discrete component between the  $k^{th}$  user and the  $e^{th}$  BS and in this system simulation we use the WiMAX discrete channel values as given in [72] and  $\phi_1$  denotes the channel path loss component while  $\phi_2$  is the lognormal shadowing fading component. In each step of simulation fixed least square filter is used to decode the received data and unless specified otherwise, the following values listed in table 2. are used.





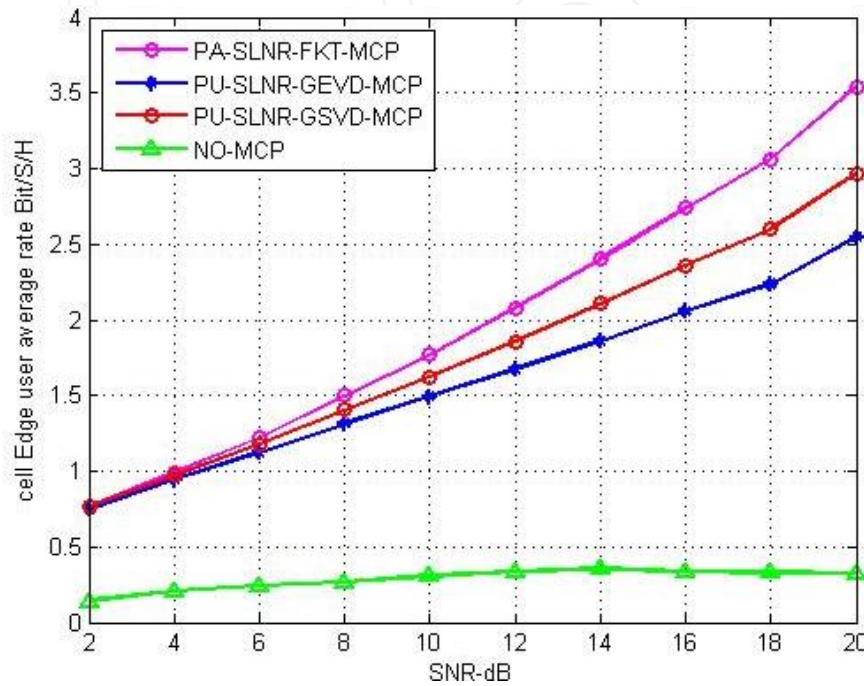
**Figure 9.** Example of Tidy Three Multi-cells Cooperation (Exchanging Both the Data and the CSI)

Channel Parameter	Parameter Value
$\phi_1=\varepsilon d^\alpha$ where $\varepsilon$ , $d$ , $\alpha$ are the intercept, BS to MS distance and path loss exponent [72, 73].	$\varepsilon=1.35\times10^7$ , $\alpha=-3$
$\phi_2=10^{\xi/10}$	$\xi$ is generated as zero mean real value Gaussian random variable with standard deviation equal to 8dB
Signal to noise ratio at the cell edge	(2-20) dB
The number of simulation runs	1000 simulations running
System configuration notations	$E$ denotes the number of cooperated cells. $N_T$ denotes the number of transmit antennas at each cells. $N_R$ denotes the number of receive antennas at each end user. $B$ denotes the number of user.

**Table 2.** System Simulation Parameters.

Figure 10. shows the simulated average user rate performance for the MU-MIMO MCP. In this figure, the proposed algorithm result is denoted by PA-SLNR-FK-MCP and the conventional precoding methods of PU-SLNR-GEVD-MCP and PU-SLNR-GSVD-MCP and as a benchmark

we also consider the case of no cooperation denoted as NO-MCP. The simulated configuration is for  $E=3$ ,  $B=3$ ,  $N_T=5$ ,  $N_R=4$ , with 1000 WiMAX discrete channel realizations. It is also clearly shown that without MCP there is no valuable rate for the cell edge user. Partially these results also support the claim that the proposed method has better performance than the related works at 18dB input SNR, there is approximately 0.5 bits/s/Hz sum rate gain over SLNR-GSVD.



**Figure 10.** Average Cell Edge User Rate Performance of MCP Precoding Methods for the Configuration ( $E=3$ ,  $N_T=5$ ,  $N_R=4$ ,  $B=3$ ), WiMAX Channel Model Precoding for Multi-cell Processing (Networked MIMO)

As shown before, the simulation result reveals the unique opportunities arising from MU-MIMO transmission optimization of antenna spatial multiplexing/spatial diversity techniques with Multi-cell Processing (Inter-cell interference cancelation). Furthermore, it also clearly indicates that MU-MIMO precoding approaches provide significant multiplexing (on the order of the number of antennas used at the transmitter) and diversity gains while resolving some of the issues associated with conventional cellular systems. Particularly, it brings precoding robustness with MU-MIMO gain and turn the inter-cell interference into diversity.

## 5. Conclusion and future research directions

To conclude, this chapter has introduced the principles of MIMO techniques, reviewed various MU-MIMO precoding methods, and extended the knowledge by proposing a new method that outperformed the methods available in the literature. The results as shown in in figure10, demonstrated conclusively the significant role of precoding in inter-cell interference cancelation in MCP scenario. There are still some interesting open issues and topics for future

research related to this application. For example related to the data and CSI sharing, basically we assume perfect system interconnection in TDD mode of operation but how much system pilot is needed for the multi-cell cooperation is an open problem. Also we assume that there is no system error or delay. This assumption is an ideal assumption for typical system deployment. The real conditions are non-ideal, therefore, modeling and investigation of the effects of system errors and delay are also an open issue that need to be researched.

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## References

- [1] Telatar, "Capacity of Multi-antenna Gaussian Channel," *European Transactions on Telecommunications*, vol. 10, pp. 585-595, October 1995.
- [2] G. J. Foschini, "Layered space-time architecture for wireless communication in a fading environment when using multi-element antennas," *Bell Labs Technical Journals*, vol. 1, pp. 41-59, 14 AUG 1996.
- [3] J. Winters, "On the capacity of radio communication systems with diversity in rayleigh fading environment," *IEEE Journal on Selected Areas in Communications*, vol. 5, pp. 871-878, June 1987.
- [4] S. M. Alamouti, "A simple transmit diversity scheme for wireless communications," *IEEE Journal on Selected Areas in Communications*, vol. 16, pp. 1451-1458, 06 August 1998.
- [5] P. DeBeasi. (2008, 20 Aug). 802.11n: Enterprise Deployment Considerations. Available: <http://www.wi-fi.org/>
- [6] F. Khalid and J. Speidel, "Advances in MIMO techniques for mobile communications- Asurvey," *Int'l J. of Communications, Network and System Sciences*, vol. 3, pp. 213-252, March 2010.
- [7] A. Goldsmith, *Wireless Communications*, First ed. Cambridge: Cambridge University Press., 2005.
- [8] D. Tse and P. Viswanath, *Fundamentals of Wireless Communication*, First ed. Cambridge: Cambridge university Press, 2005.

- [9] S. Da-Shan, *et al.*, "Fading correlation and its effect on the capacity of multielement antenna systems," *IEEE Transactions on Communications*, vol. 48, pp. 502-513, Mar 2000.
- [10] A. B. Gershman and N. D. Sidiropoulos, *Space-time processing for MIMO communications*, First ed.: John Wiley & Sons, Ltd, 2005.
- [11] D.-s. Shiu, *et al.*, "fading correlation and Its effect on the capacity of multielement antenna systems," *IEEE TRANSACTION On CommunicationS*, vol. Vol. 48, pp. 502-512, March 2000.
- [12] A. Paulraj, *et al.*, *Introduction to Space-Time Wireless Communications*, First ed. New York: Cambridge university press, 2003.
- [13] R. Gallager. course materials for 6.450 Principles of Digital Communications I, [Online]. Available: <http://ocw.mit.edu>
- [14] T. M. Cover and J. A. Thomas, *Elements of Information Theory*, Second ed.: JOHN WILEY & SONS, INC., 2006.
- [15] H. Sampath, *et al.*, "Generalized Linear Precoder and Decoder Design for MIMO Channels Using The weighted MMSE Criterion," *IEEE TRANSACTION On Communications*, vol. 49, pp. 2198-2206, DEC 2001.
- [16] A. Lozano and N. Jindal, "Transmit Diversity vs. Spatial Multiplexing in Modern MIMO Systems," *IEEE TRANSACTIONS ON WIRELESS COMMUNICATIONS*, vol. 9, pp. 186-197, 08 January 2010.
- [17] E. A. Lee and D. G. Messerschmitt, *Digital Communication*, second ed. Boston: Kluwer, 1994.
- [18] Y. S. Cho, *et al.*, *MIMO-OFDM Wireless Communications With Matlab*, First ed.: John Wiley & Sons, 2010.
- [19] J. W. Huang, *et al.*, "Precoder Design for Space-Time Coded MIMO Systems with Imperfect Channel State Information," *IEEE Transactions on Wireless Communications*, vol. 7, pp. 1977-1981, 2008.
- [20] L. Zhi-Quan, *et al.*, "Transceiver optimization for block-based multiple access through ISI channels," *IEEE Transactions on Signal Processing*, vol. 52, pp. 1037-1052, 2004.
- [21] P. Mary, *et al.*, "Symbol Error Outage Analysis of MIMO OSTBC Systems over Rice Fading Channels in Shadowing Environments," *IEEE transactions on wireless communications*, vol. 10, pp. 1009 - 1014 15 April 2011.
- [22] A. Goldsmith, *et al.*, "Capacity limits of MIMO channels," *IEEE Journal on Selected Areas in Communications*, vol. 21, pp. 684-702, 2003.

- [23] A. Saad, *et al.*, "Capacity of MIMO channels at different antenna configurations," *Journal of Applied Sciences*, vol. 8, pp. 4595-4602, 2008.
- [24] M. Trivellato, *et al.*, "On channel quantization and feedback strategies for multiuser MIMO-OFDM downlink systems," *IEEE Transactions ON Comm*, vol. 57, pp. 2645-2654, Sept 2009.
- [25] M. Costa, "Writing on dirt paper," *IEEE TRANSACTIONS ON Information theory*, vol. 29, pp. 439-441, May 1983.
- [26] H. Sampath, *et al.*, "Generalized linear precoder and decoder design for MIMO channels using the weighted MMSE criterion," *IEEE Transactions on Communications*, vol. 49, pp. 2198-2206, 2001.
- [27] D. J. Love, *et al.*, "Grassmannian beamforming for multiple-input multiple-output wireless systems," *IEEE Transactions on Information Theory*, vol. 49, pp. 2735-2747, 2003.
- [28] Z. Jianchi, *et al.*, "Investigation on precoding techniques in E-UTRA and proposed adaptive precoding scheme for MIMO systems," in *14th Asia-Pacific Conference on Communications, 2008. APCC 2008.*, 2008, pp. 1-5.
- [29] R. Narasimhan, "Spatial multiplexing with transmit antenna and constellation selection for correlated MIMO fading channels," *IEEE Transactions on Signal Processing*, vol. 51, pp. 2829-2838, 2003.
- [30] V. Stankovic, "Multi-user MIMO wireless communication," PhD, Technischen Universität Ilmenau, Ilmenau, 2006.
- [31] M. Costa, "Writing on dirty paper (Corresp.)," *IEEE Transactions on Information Theory*, vol. 29, pp. 439-441, 1983.
- [32] E. Saeid, *et al.*, "Efficient Per-antenna Signal to Leakage Plus noise Ratio Precoding for Multiuser Multiple Input Multiple Output System," *Research Journal of Applied Sciences, Engineering and Technology*, vol. 4, pp. 2489-2495,, August 2012.
- [33] V. Stankovic and M. Haardt, "Generalized Design of Multi-User MIMO Precoding Matrices," *Wireless Communications, IEEE Transactions on*, vol. 7, pp. 953-961, 2008.
- [34] M. Sadek, *et al.*, "A Leakage-Based Precoding Scheme for Downlink Multi-User MIMO Channels," *IEEE TRANSACTION On Communications*, vol. 6, pp. 1711-1721, May 2007.
- [35] C. B. Peel, *et al.*, "A vector-perturbation technique for near-capacity multiantenna multiuser communication-part I: channel inversion and regularization," *IEEE Transactions on Communications*, vol. 53, pp. 195-202, 2005.
- [36] V. Mai and A. Paulraj, "MIMO Wireless Linear Precoding," *Signal Processing Magazine, IEEE*, vol. 24, pp. 86-105, 2007.



- [37] A. Kurve, "Multi-user MIMO systems: the future in the making," *IEEE Potentials*, vol. 28, pp. 37-42, 2009.
- [38] C. Wang, *et al.*, "On the Performance of the MIMO Zero-Forcing Receiver in the Presence of Channel Estimation Error," presented at the Information Sciences and Interaction Sciences, Chengdu, China 2007.
- [39] Y. Jiang, *et al.*, "Performance Analysis of ZF and MMSE Equalizers for MIMO Systems: An In-Depth Study of the High SNR Regime," *IEEE TRANSACTIONS ON Information theory*, vol. 57, pp. 2008 - 2026 april 2011.
- [40] A. Sibille, *et al.*, *MIMO From Theory to Implementation*, First ed.: Imprint: Academic Press, 2011.
- [41] L. Zheng, "Diversity-Multiplexing Tradeo: A Comprehensive View of Multiple Antenna Systems," PhD, Electrical Engineering and Computer Sciences, CALIFORNIA at BERKELEY, 2002.
- [42] Z. Lizhong and D. N. C. Tse, "Diversity and multiplexing: a fundamental tradeoff in multiple-antenna channels," *Information Theory, IEEE Transactions on*, vol. 49, pp. 1073-1096, 2003.
- [43] S. Ahmadi, "An overview of next-generation mobile WiMAX technology," *Communications Magazine, IEEE*, vol. 47, pp. 84-98, 2009.
- [44] K. Etemad, "Overview of mobile WiMAX technology and evolution," *Communications Magazine, IEEE*, vol. 46, pp. 31-40, 2008.
- [45] C. Swannach and G. W. Wornel, "Channel State Quantization IN MIMO Broadcast System: Architectures and Codes," PhD, Electrical Engineering and Computer Science, MIT, 2010.
- [46] S. Fang, *et al.*, "Multi-User MIMO Linear Precoding with Grassmannian Codebook," in *WRI International Conference on Communications and Mobile Computing*, 2009. CMC '09. , 2009, pp. 250-255.
- [47] M. C. H. Lim, *et al.*, "Spatial Multiplexing in the Multi-User MIMO Downlink Based on Signal-to-Leakage Ratios," in *Global Telecommunications Conference, 2007. GLOBE-COM '07. IEEE*, 2007, pp. 3634-3638.
- [48] S. Shi, "Transceiver Design for Multiuser MIMO Systems," Phd, Electrical and Electronic Engineering, Berlin, Berlin, 2009.
- [49] S. S. Christensen, *et al.*, "Weighted sum-rate maximization using weighted MMSE For MIMO-BC Beamforming Design," *IEEE TRANS . On . wireless Comm.*, vol. vol. 7 pp. 1-7, Dec 2008.
- [50] X. Gao, *et al.*, "A successive iterative optimization precoding method for downlink multi-user MIMO system," in *International Conference on Wireless Communications and Signal Processing (WCSP)*, 2010, pp. 1-5.



- [51] A. J. Tenenbaum and R. S. Adve, "Linear Processing and sum throughput in the Multiuser MIMO downlink," *IEEE Trans On wireless Comm.*, vol. 8, pp. 2652-2660, May 2009.
- [52] V. Stankovic, "Iterative Successive MMSE Multi-User MIMO Transmit Filtering," *ELEC. ENERG*, vol. 20, pp. 45-55, April 2007.
- [53] M. Joham, *et al.*, "Linear transmit processing in MIMO communications systems," *IEEE Transactions on Signal Processing*, vol. 53, pp. 2700-2712, 2005.
- [54] M. Joham, *et al.*, "Transmit Wiener filter for the downlink of TDDDS-CDMA systems," in *IEEE Seventh International Symposium on Spread Spectrum Techniques and Applications* 2002, pp. 9-13
- [55] L. Min and O. Seong Keun, "A Per-User Successive MMSE Precoding Technique in Multiuser MIMO Systems," in *Vehicular Technology Conference, 2007. VTC2007-Spring. IEEE 65th*, 2007, pp. 2374-2378.
- [56] H. Karaa, *et al.*, "Linear Precoding for Multiuser MIMO-OFDM Systems," in *IEEE International Conference on Communications*, 2007, pp. 2797-2802.
- [57] V. Sharma and S. Lambotharan, "Interference Suppression in Multiuser Downlink MIMO Beamforming using an Iterative Optimization Approach," presented at the 14th European Signal Processing Conference, Florence, Italy, 2006.
- [58] J. Park, *et al.*, "Generalised singular value decomposition-based algorithm for multiuser multiple-input multiple-output linear precoding and antenna selection," *IET Communcation*, vol. 4, pp. 1899-1907 5 November 2010.
- [59] M. Sadek, "Transmission Techniques for Multi-user MIMO communications," Phd, Electrical Engineering, University of California, LA, 2006.
- [60] P. Jaehyun, *et al.*, "Efficient GSVD Based Multi-User MIMO Linear Precoding and Antenna Selection Scheme," in *IEEE International Conference on Communications* 2009, pp. 1-6.
- [61] C. Peng, *et al.*, "A New SLNR-Based Linear Precoding for Downlink Multi-User Multi-Stream MIMO Systems," *Communications Letters, IEEE*, vol. 14, pp. 1008-1010, 2010.
- [62] C. Paige and M. A. Saunders, "Towards a Generalized Singular Value Decomposition," *SIAM Journal on Numerical Analysis*, vol. 18, pp. 398-405, 1981.
- [63] C. F. and V. Loan, "Generalized Singular value Decomposition," *SIAM Journal on Numerical Analysis* vol. 13, March 1976.
- [64] G. H. Golub and C. F. v. V. Loan, *Matrix Computations*, 3rd Edition ed. Baltimore and London The Johns Hopkins University Press 1996.
- [65] M. Sadek, *et al.*, "Active antenna selection in multiuser MIMO communications," *IEEE TRANS ON SIGNAL PROCESSING*, vol. 44, pp. 1498-1510, April 2007.

- [66] J. Park, *et al.*, "Efficient GSVD Based Multi-user MIMO Linear Precoding and Antenna Selection Scheme," in *IEEE ICC 2009*, Dresden 2009, pp. 1-6.
- [67] K. Fukunaga and W. L. G. Koontz, "Application of the Karhunen-Loève expansion to feature selection and ordering," *IEEE Transactions on Computers*, vol. 19, pp. 311-317, April 1970.
- [68] A. A. Miranday and P. F. Whelan, "Fukunaga-Koontz transform for small sample size problems," in *ISSC 2005 - IEE Irish Signals and Systems Conference*, Dublin., 2005, pp. 1-6.
- [69] W. Cao and R. Haralick, "Affine feature extraction: A Generalization of the Fukunaga-Koontz transformation," in *international conference on Machine Learning and Data Mining in Pattern Recognition Heidelberg*, 2007, pp. 163-173.
- [70] Z. Sheng and T. Sim, "Discriminant Subspace Analysis: A Fukunaga-Koontz Approach," *Pattern Analysis and Machine Intelligence, IEEE Transactions on*, vol. 29, pp. 1732-1745, 2007.
- [71] S. Zhang and T. Sim, "When Fisher meets Fukunaga-Koontz: A New Look at Linear Discriminants," in *IEEE Computer Society Conference on Computer Vision and Pattern Recognition (CVPR'06)*, New York, 2006, pp. 323 - 329
- [72] G. L. Stüber, *Principles of Mobile Communication*, Second ed.: Kluwer Academic Publishers, 2002.
- [73] V. Erceg, *et al.*, "An empirically based path loss model for wireless channels in suburban environments," *Selected Areas in Communications, IEEE Journal on*, vol. 17, pp. 1205-1211, 1999.

