Chapter 1

# On Open Quantum Systems and Mathematical Modeling of Quantum Noise

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Additional information is available at the end of the chapter

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## 1. Introduction

This paper gives an introduction to some aspects of quantum processes described by quantum operations and quantum master equations. Quantum operations (quantum channels) and quantum master equations come into play whenever a mathematical description of irreversible time behaviour of quantum systems is investigated.

In modeling of physical systems for which time behaviour can be represented by stochastic processes one assumes that a system in question is described by certain mathematical model, for example, by random variables in classical case or by sets of non-commuting observables in quantum case, acting on an abstract probability space. In most cases our assumptions are not believed to be a fully realistic model of the physical reality, but nevertheless very often it turns out to be extremely useful.

In the algebraic formulation of quantum mechanics, a fixed quantum mechanical system is represented by an algebra $\mathcal{A}$ of operators acting on some Hilbert space $\mathcal{H}$. In this approach, the observables of the system are identified with self-adjoint elements in $\mathcal{A}$ and the physical states are given by positive unital functionals on $\mathcal{A}$. We will consider the case when $\mathcal{H}$ is finite-dimensional, then the set of states can be identified with the set of $\mathcal{S}(\mathcal{H})$ of density operators, that is, positive elements in $\mathcal{A}$ with unital trace. The evolutions of the system are described by transformations on the set $\mathcal{S}(\mathcal{H})$, or more generally, by linear maps between $\mathcal{S}(\mathcal{H})$ and $\mathcal{S}(\mathcal{K})$, where $\mathcal{H}$ and $\mathcal{K}$ represent two finite-dimensional Hilbert spaces.

Nonclassical correlations among subsystems of a composite quantum system, known as entanglement, can be precisely described mathematically but in a rather ineffective way. Here, by an effective way (effective procedure) we mean a method which uses only a finite number of arithmetic operations on elements of a given density matrix and allows us to formulate an answer to the question: is a given composite quantum system in a separable or entangled state?

It appears that the questions of the above type can be naturally connected with some problems of changes in the space of quantum states, that is with some aspects of quantum dynamics and properties of dynamical maps. Here, by dynamical maps we understand linear transformations that take one density operator to another. Moreover, this kind of connection between some linear transformations and states of composed quantum systems is a one-to-one type. In this context we will discuss the so-called Metzler operators which play a substantial role in quantum dynamics. By definition, an operator $\mathbb{K}$ acting on the space of hermitian matrices is said to be a Metzler operator if there exists a real number $\alpha_0$ such that for all $\alpha > \alpha_0$ the resolvent of $\mathbb{K}$ is positive in the sense that

$$R(\alpha, \mathbb{K})\mathcal{P}(\mathcal{H}) \subseteq \mathcal{P}(\mathcal{H}),$$

where $\mathcal{P}(\mathcal{H})$ denotes the set of semipositive matrices.

In this paper we will discuss some effective methods of study of certain properties of quantum operations and we will use some methods and results which are typical for the research area known as *noncommutative Perron-Frobenius theory*. In particular, in our study of dynamical maps we will use the natural identifications

$$M_k(\mathbb{C}) \otimes B(\mathcal{H}) \cong M_k(B(\mathcal{H})) \cong B\left(\bigoplus_{i=1}^{k} \mathcal{H}\right),$$

where $\mathcal{H}$ denotes a finite-dimensional Hilbert space, $B(\mathcal{H})$ represents the set of all linear operators on $\mathcal{H}$ and symbols $M_k(\mathcal{A})$ for $k, 2, 3, \ldots$ denote $k \times k$ matrices with elements from an algebra $\mathcal{A}$. In our approach we will concentrate on a quantum analogy of the classical theory of positive maps also known as Perron-Frobenius theory. The Perron theorem on entrywise positive matrices and its generalization by Frobenius on entrywise nonnegative matrices have interested mathematicians since the results appeared at the beginning of last century. Later these theorems have been generalized to operators on a partially ordered real Banach spaces. This has motivated several authors to consider linear maps on a finite dimensional space which leave a fixed cone invariant.

Now, let us fix the notation. Let $\mathcal{H}$ denote the Hilbert space associated with a given quantum system $\mathcal{S}$. By $B(\mathcal{H})$ we will denote the set of all linear continuous operators on $\mathcal{H}$. Then the set of states of the system $\mathcal{S}$ is, by definition, represented by all semi-positive elements of $B(\mathcal{H})$ with trace equal to one. This set of states we will denote by $\mathcal{S}(\mathcal{H})$. The mentioned above connection between linear maps on $\mathcal{S}(\mathcal{H})$ and some operators constructed on the tensor product $\mathcal{H} \otimes \mathcal{H}$, is a one-to-one type. One can say that there is a one-to-one correspondence between properties of the maps

$$\Phi : B(\mathcal{H}) \to B(\mathcal{H}), \tag{1}$$

such that

$$\Phi(\mathcal{S}(\mathcal{H})) \subseteq \mathcal{S}(\mathcal{H}), \tag{2}$$

and properties of linear operators $W := J(\Phi)$ on $\mathcal{H} \otimes \mathcal{H}$, where $J$ denotes this correspondence [1–4].

At first sight this seems strange that two entirely different issues, namely the inner structure and properties of operators on $\mathcal{H} \otimes \mathcal{H}$, and maps on the set of states are so strongly connected. But in fact there exists an intricate and strong link between them — there exists an isomorphism $J$ between maps on $B(\mathcal{H})$ (these maps very often are called superoperators) and some properties of operators on $\mathcal{H} \otimes \mathcal{H}$ defined in an appropriate way. At present, various methods of study of entangled systems based on properties of positive maps are discussed in hundreds of papers and many monographs and textbooks (cf. [4–8]).

In the beginning of seventies it appeared that some natural questions connected with fundamentals of quantum mechanics (more precisely, with the theory of open quantum systems) lead to investigations of linear maps in a real Banach space of self-adjoint operators on a fixed Hilbert space [9, 32]. This concept of a Banach space with the partial order defined by a specific cone, namely, the cone of positive semidefinite operators, constitutes a basic idea in the description of open quantum systems.

In concrete applications one distinguishes two approaches to describe time evolutions (changes) of an open quantum system. One of them starts from a fixed physical model defined by a given Hamiltonian which determines the Schrödinger equation (von Neumann equation) or the master equation with a given generator of time evolution which is, in general, time dependent.

Summing up, as the fundamental objects in modern quantum theory one considers the set of states

$$S(\mathcal{H}) := \{\rho : \mathcal{H} \to \mathcal{H}; \quad \rho \geq 0, \quad \mathrm{Tr}\rho = 1\}, \tag{3}$$

and the set of bounded hermitean (self-adjoint) operators

$$\mathcal{B}_{\star}(\mathcal{H}) := \{\mathrm{Q} : \mathcal{H} \to \mathcal{H}; \quad \mathrm{Q} = \mathrm{Q}^{\star}\}. \tag{4}$$

Time evolutions of systems are governed by linear master equations of the form (in the so-called Schrödinger picture)

$$\frac{d\rho(t)}{dt} = \mathbb{K}(t)\rho(t), \tag{5}$$

or in the dual form (in the so-called Heisenberg picture)

$$\frac{d\mathrm{Q}(t)}{dt} = \mathbb{L}(t)\mathrm{Q}(t), \tag{6}$$

where superoperators $\mathbb{K}$ and $\mathbb{L}$ act on operators from the sets $S(\mathcal{H})$ and $\mathcal{B}_{\star}(\mathcal{H})$, respectively. They represent dual forms of the same physical idea.

An alternative approach to the dynamics of an open quantum system relies on a stroboscopic picture and a discrete time evolution. We start from a mathematical construction of a quantum map on $\mathcal{S}(\mathcal{H})$ (in fact on $B_*(\mathcal{H})$) which is allowed by the general laws of quantum mechanics. Such approach is particulary useful if we want to investigate the set of all possible operators independently on whether the physical situation is exactly specified.

Both sets $\mathcal{S}(\mathcal{H})$ and $B_*(\mathcal{H})$ can be considered as subsets of the vector space $\mathcal{B}(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$ and they can be treated as scenes on which problems of quantum mechanical systems should be discussed. As in this paper we will consider finite-dimensional Hilbert spaces, so in fact $\mathcal{B}(\mathcal{H})$ denotes the set of *all* linear operators on $\mathcal{H}$, $\dim \mathcal{H} = d$. If we introduce the scalar product in $\mathcal{B}(\mathcal{H})$ by the equality

$$\langle A, B \rangle := \mathrm{Tr}\,(A^{\star}B)\,, \tag{7}$$

then $\mathcal{B}(\mathcal{H})$ can be regarded as yet another inner product space, namely the *Hilbert–Schmidt* space. It is not difficult to see that $\mathcal{B}_{\star}(\mathcal{H})$ with scalar product defined by (7) is a real vector space and $\dim \mathcal{B}(\mathcal{H}) = d^2$. It should be stressed that in this case we consider linear maps on $B(\mathcal{H})$ not only as maps on a vector space but also as maps on $B(\mathcal{H})$ equipped with a natural structure of an algebra. Such linear maps on the set of operators are called *superoperators* and their general form is well known. Namely, for a given superoperator $\Phi$, $\Phi : B(\mathcal{H}) \to B(\mathcal{H})$, there always exists an operator-sum representation given by

$$\Phi(X) = \sum_{i=1}^{\kappa} A_i X B_i\,, \tag{8}$$

where $A_i, B_i$ are elements of $B(\mathcal{H})$. A particular class of such maps, the so-called completely positive maps (or in physical terminology — *quantum operations*), plays a prominent role in formulation of *evolution* of open quantum systems and in the theory of *quantum measurements*. In the case of completely positive maps we have in the above formula $B_i = A_i^{\star}$ for all $i = 1, \ldots, \kappa$ (Kraus theorem). A comprehensive description of these problems from the physical point of view can be found in books [3, 14].

It is important to observe that most of the papers devoted to the classical Perron-Frobenius theory of nonnegative linear maps are concerned mainly with the existence problems. The purpose of this paper is, on the one hand, to connect the Frobenius theory of irreducible linear operators with the so-called Kraus representation of the completely positive linear maps and, on the other hand, to show that there exist some effective procedures which allow us to verify if a given quantum operation (i.e. a given completely positive map) is irreducible or not. Here by irreducibility we mean the natural generalization of this concept, introduced by Frobenius in his famous paper of 1912 year ([12]) and based on the specific block representation of nonnegative matrices, to a geometric approach formulated in terms of the invariance of faces of a fixed cone. In this context, by an effective procedure we understand a method which uses only a finite number of arithmetic operations on matrices $K_i$ (Kraus coefficients) which define a fixed quantum operation (completely positive map)

$$\Phi(X) = \sum_{i=1}^{\kappa} K_i X K_i^{*}\,. \tag{9}$$

Another important question may be formulated in the following way: for a fixed quantum operation (a superoperator) $\Phi$ defined by a set of Kraus operators $K_1, \ldots, K_\kappa$ there exists a decoherence-free subspace or not. By definition, a decoherence-free subspace (DFS) it is a subspace of the system's Hilbert space $\mathcal{H}$ which is invariant under non-unitary evolution. Alternatively formulated, DFS is this part of the system Hilbert space where the system is decoupled from the environment and thus its evolution is completely unitary. For a given quantum operation $\Phi$ defined by (9) DFS can exist or not. The questions is: can we check this property in an effective way or not?

The paper is organized as follows. In Section 2 we will review the concepts needed for our discussion based on geometric and operator theoretic formulation of the Perron-Frobenius theory and, on the other hand, we describe the structure of the cone of positive definite operators defined on a given Hilbert space. Moreover, some properties of faces of this cone are analyzed. Section 3 describes some properties of general positive maps (superoperators), and their representations as operators acting on doubled Hilbert space $\mathcal{H} \otimes \mathcal{H}$. The main results of the paper are contained in Sections 4 and 5. In Section 4 we formulate some effective methods of checking whether a given superoperator, i.e. a fixed quantum operation, is irreducible or not. A similar problem whether a given generator of time evolution in master equation description of evolution secures positivity of density operators or not, is discussed in Section 5.

## 2. Properties of cones

In order to start this section we recall some definitions and facts from the theory of cones and operators on a partially ordered vector space. A convex closed set $K$ of a real normed space $V$ is called a *wedge* iff

$$\alpha K + \beta K \subseteq K \tag{10}$$

for all $\alpha, \beta \geq 0$. A wedge $K$ is called a *cone* if, in addition, we have

$$K \cap (-K) = \{\mathbf{0}\}, \tag{11}$$

where $\mathbf{0}$ denotes the zero element of $V$. If we have the equality $V = K - K$, then the cone $K$ is called *generating* or *reproducing* (sometimes one also uses the term *full cone*).

If $\Phi$ is a linear transformation on $V$, $\Phi : V \to V$, then we denote by $r(\Phi)$ the *spectral radius* of $\Phi$, i.e.

$$r(\Phi) := \max\{|\lambda|\,;\, \lambda \in \sigma(\Phi)\}, \tag{12}$$

where $\sigma(\Phi)$ denotes the spectrum of $\Phi$. For any cone $K$ we let $K^\circ$ denote the interior of $K$ and by $\partial K$ we denote its boundary.

As is well known any fixed cone $K$ in $V$ determines a partial order in $V$. For this order we use the following terminology:

1. $x$ is *nonnegative*, $x \geq 0$, iff $x \in K$;
2. $x$ is *positive*, $x > 0$, iff $x \geq 0$ and $x \neq 0$;
3. $x$ is *strictly positive*, $x \gg 0$, iff $x \in K^\circ$.

Now, let us define the concept of face which plays a basic role in the theory of irreducible operators. Let $K$ be a cone in $V$. By a *face* $F$ of $K$ one understands a subset of $K$ which is a cone and satisfies an extra condition: if $0 \le y \le x$ and $x \in F$, then $y$ also belongs to $F$, $y \in F$.

Of course, if we fixed a basis in $V$, then we may regard vectors in $V$ as column vectors in $\mathbb{R}^n$. In this case the positive orthant $\mathbb{R}^n_+$ constitutes a cone in $\mathbb{R}^n$ and exact description of $\partial\mathbb{R}^n_+$ is obvious, namely,

$$F_M := \{x \in \mathbb{R}^n_+ ; \quad x_i = 0 \ \text{ if } \ i \notin M\}, \tag{13}$$

where $M \subseteq \{1, 2, \ldots, n\}$.

If $E \subset K$, then we will denote by $\Omega(E)$ the intersection of all faces containing $E$. It is easily seen that $\Omega(E)$ is a face. It is called the face *generated by* $E$.

The set of all operators $\Phi : V \to V$ such that $\Phi(K) \subseteq K$ we will denote by $\Pi(K)$. Let $B(V)$ be the set of all linear operators on $V$. Then we have

$$\Pi(K) := \{\Phi ; \Phi(K) \subseteq K\} \subseteq B(V) \tag{14}$$

and $\Pi(K)$ is a cone in $B(V)$. The elements of the set $\Pi(K)$ are said to be *K-nonnegative operators*. In particular, the operator $\Phi$ on $V$ is called *K-positive* in case $\Phi(K \setminus \{0\}) \subseteq K^o$. The set of all $K$-positive maps will be denoted by $\Pi^+(K)$.

Now we introduce one of the main ideas of the Perron-Frobenius theory both in commutative and non-commutative case. For a fixed $K$ in $V$ a natural generalization of the concept of an irreducible matrix is the following: $\Phi$ is *K-irreducible* if and only if $\Phi$ leaves invariant no face of $K$ except $\{\mathbf{0}\}$ and $K$ itself and $\kappa$ describes the minimal length of $\Phi$. In other words, an operator in $\Pi(K)$ is $K$-reducible iff it leaves invariant a nontrivial face of $K$.

Another, strictly equivalent, definition of $K$-irreducibility can be given by the following theorem: An operator $\Phi \in \Pi(K)$ is $K$-irreducible if and only if no eigenvector of $\Phi$ lies on the boundary of $K$. In fact, one can say even more: An operator $\Phi \in \Pi(K)$ is $K$-irreducible if and only if $\Phi$ has exactly one (up to scalar multiples) eigenvector in $K$ and this vector belongs to $K^o$. Moreover, for any proper cone $K$ we have

$$\Pi^+(K) \subseteq \widetilde{\Pi}(K) \subseteq \Pi(K), \tag{15}$$

where $\widetilde{\Pi}(K)$ denotes the set of all $K$-irreducible operators. If $K = \mathbb{R}^n_+$, then the both definitions, Frobenius one and the above, coincide. For details, see e.g. [13, 15]

Some important spectral properties of $K$-nonnegative operators are summarized in the following theorems, which one can consider as natural generalizations of well-known classical results.

**Theorem 1.** *Let* $\Phi \in \Pi^+(K)$*. Then we have*

**a)** *the spectral radius of the operator* $\Phi$ *is a simple eigenvalue of* $\Phi$*, greater than the magnitude of any other eigenvalue;*

**b)** *an eigenvector of* $\Phi$ *corresponding to* $r(\Phi)$ *belongs to* $K^o$*;*

**c)** *no other eigenvector of* $\Phi$ *(up to scalar multiples) belongs to K.*

**Theorem 2.** *Let* $\Phi \in \Pi(K)$*. Then the following hold*

**a)** $r(\Phi)$ *is an eigenvalue of* $\Phi$ *;*

**b)** *K contains an eigenvector of* $\Phi$ *corresponding to* $r(\Phi)$ *;*

**c)** *if* $\Phi \leq \Psi$*, then* $r(\Phi) \leq r(\Psi)$*.*

**Theorem 3.** *Let* $\Phi \in \widetilde{\Pi}(K)$*. Then the following hold*

**a)** $r(\Phi)$ *is a simple eigenvalue of* $\Phi$*;*

**b)** *no eigenvector of* $\Phi$ *lies on the boundary of* $K$ *;*

**c)** $\Phi$ *has exactly one (up to scalar multiples) eigenvector in K and this vector belongs to* $K^o$*;*

**d)** $(I + \Phi)^{n-1} \in \Pi^+(K)$*, where* $n = \dim V$*.*

For proofs of the above theorems consult [13, 17].

We will conclude this section with some comments on the two special cases which are important from the point of view of physics, namely, $K = \mathbb{R}^d_+$ and $K = M^+_d(\mathbb{C})$. Here, the last symbol denotes the set of all semipositive operators on the space $\mathbb{C}^d$, and $\mathbb{C}^d$ is regarded as a representation of a $d$-dimensional Hilbert space. For the case $K = \mathbb{R}^d_+$, the whole story reduces to the classical Perron-Frobenius theory and in statistical physics we use this theory for the so-called mesoscopic description of classical systems.

The case $K = M^+_d(\mathbb{C})$ plays a fundamental role in the description of representations of states for open quantum systems. In this case, $B(\mathcal{H})$ is a $d^2$-dimensional vector space. According to (4) we denote by $B_*(\mathcal{H})$ the set of all self-adjoint operators on $\mathcal{H}$ which can be naturally considered as a $d^2$-dimensional *real* Banach space. At the same time, $B(\mathcal{H})$ can be regarded as a Hilbert space with the scalar product defined by (7). The vector space $B_*(\mathcal{H})$ of all Hermitian (self-adjoint) operators on $\mathcal{H}$ constitutes an $d^2$-dimensional, real subspace of the Hilbert-Schmidt space. One can use the "internal structure" of vectors in $B_*(\mathcal{H})$ to define a positive cone. By definition, a *semipositive* element of $B_*(\mathcal{H})$ is an operator A on $\mathcal{H}$ such that $\langle\psi|A|\psi\rangle$ is real and nonnegative for all vectors $|\psi\rangle$ from $\mathcal{H}$. Of course, one can equivalently define a positive element of $B(\mathcal{H})$ as a self-adjoint operator with nonnegative eigenvalues. The set of all semipositive operators on $\mathcal{H}$ we will denote by $B^+_*(\mathcal{H})$ or $\mathcal{P}(\mathcal{H})$. In particular, if we have the inequality $\langle\psi|A|\psi\rangle > 0$ for all $|\psi\rangle$ from $\mathcal{H}$, then we say that $A$ is *positive*.

Let $\mathcal{P}_n$ denote the set of all orthogonal projections, i.e. $A \in \mathcal{P}_n$ if and only if $A \in B_*(\mathcal{H})$ and $A^2 = A$. With the natural order on projections, namely, $A \leq B$ iff $\mathrm{Im}(A) \subseteq \mathrm{Im}(B)$, the mapping from $A \in \mathcal{P}_n$ to $F(\mathcal{P}(\mathcal{H}))$, where $F(K)$ denotes the set of all faces of $K$, is an order preserving isomorphism.

Now, we formulate an important characterization of all faces of any cone $\mathcal{P}(\mathcal{H})$. It appears that each face of $\mathcal{P}(\mathcal{H})_d$, where the suffix $d$ denotes $\dim \mathcal{H}$, is isomorphic to $\mathcal{P}(\mathcal{H})_m$ for some $m$, where $0 \leq m \leq d$.

**Theorem 4.** *If $B \in \mathcal{P}(\mathcal{H})$ is of rank $m$, then there exists a unitary $U$ such that*

$$\Omega(B) = U^*(\mathcal{P}(\mathcal{H})_m \oplus 0_{d-m})U. \tag{16}$$

*Conversely, if $U$ denotes a unitary operator on $\mathcal{H}$, then $U^*(\mathcal{P}(\mathcal{H})_m \oplus 0_{d-m})U$ for $m$ = 0,1,...,$d$ are faces of $\mathcal{P}(\mathcal{H})_d$.*

It is well known that if $K$ is a polyhedral cone, then for all faces of $K$ we have

$$\operatorname{span} F + \operatorname{span} F^{\triangleleft} = V, \tag{17}$$

where $F^{\triangleleft}$ denotes the so-called complementary face of $F$ defined by

$$F^{\triangleleft} := \{z \in K^*; \langle z, x\rangle = 0 \text{ for all } x \in F\} \tag{18}$$

and $V$ denotes the ambient space of $K$. The *residual subspace* of $F$ is meant to measure "to what extent $F$ is nonpolyhedral". It is defined as

$$\operatorname{res}(F) := (\operatorname{span} F + \operatorname{span} F^{\triangleleft})^{\perp}. \tag{19}$$

A list of several examples of cones, along with the description of their faces and residual subspaces, is contained in [18].

## 3. Positive maps on $B_*(\mathcal{H})$

It is well known that if a linear map $\Phi : B(\mathcal{H}) \to B(\mathcal{H})$ sends the set $B_*(\mathcal{H})$ of all hermitian elements of $B(\mathcal{H})$ into itself, then $\Phi$ can be represented in the form

$$\Phi(X) = \sum_{i=1}^{\kappa} a_i A_i X A_i^{\star}, \tag{20}$$

where $A_i \in B(\mathcal{H})$, and $a_i$ for $i = 1, 2, ..., \kappa$ are real numbers (cf. eg. [3, 19]). In general, all maps of the above form are hermitian-preserving. However, the representation (20) is not unique. In general, for a given $\Phi$ , there exist many possible representations of the form (20). For a given $\Phi$ the smallest $\kappa$ in (20) is called the *minimal length* of $\Phi$ and this minimal length is always smaller or equal to $(\dim \mathcal{H})^2$. If we assume that the operators $A_i$ for $i = 1, 2, \ldots, \kappa$ are linearly independent, then $\kappa$ in (20) must be minimal.

According to the general definition introduced in Section 2 a *positive map* $\Phi$ is a linear map from $B(\mathcal{H})$ into itself, which leaves $\mathcal{P}(\mathcal{H})$ invariant. Now, $\Phi$ is called *k-positive* if its *k-amplification* $\Phi_{(k)} := \mathbb{I}_k \otimes \Phi$ that is the map

$$\mathbb{I}_k \otimes \Phi : M_k(\mathbb{C}) \otimes B(\mathcal{H}) \to M_k(\mathbb{C}) \otimes B(\mathcal{H}) \tag{21}$$

is positive. Here $M_k(\mathbb{C})$ denotes as usual the set of all $k \times k$ complex matrices. It is not difficult to observe that we can identify the set $M_k(\mathbb{C}) \otimes \mathcal{A}$, where for simplicity we denote the algebra $B(\mathcal{H})$ by $\mathcal{A}$, with the set of all $k \times k$ matrices $M_k(\mathcal{A})$ with entries from $\mathcal{A}$.

The map $\Phi$ is called completely positive if it is $k$-positive for all $k = 1, 2, \ldots$. This terminology goes back to Stinespring [20], cf. also [4]. It is well known that for $d$-dimensional Hilbert space $\mathcal{H}$, $d$-positive maps on $B(\mathcal{H})$ are already completely positive [21].

Let us observe that all hermitian-preserving maps which are not only positive but completely positive can be written in the form (20) with positive $a_i$, $i = 1, \ldots, \kappa$, i.e. by

$$\Phi(X) = \sum_{i=1}^{\kappa} K_i X K_i^\star, \tag{22}$$

where $K_i := \sqrt{a_i} A_i$, and $\kappa \leq d^2$. The above expression is called the Kraus representation of a completely positive map $\Phi$ and, in case of the finite-dimensional Hilbert space $\mathcal{H}$, can be regarded also as a definition of the completely positive map. This representation is very useful in quantum information theory. In particular, completely positive maps are used to describe all quantum operations, quantum channels and to model quantum devices.

Let us observe that an equivalent representation of the evolution described by expression (22) can be made in terms of the operator $W$ mentioned in Section 1. This operator is connected in a one-to-one way with the superoperator $\Phi$ (quantum map on $\mathcal{S}(\mathcal{H})$) by the formula

$$W(\Phi) := (\mathbb{I} \otimes \Phi)\left(|\psi_+\rangle\langle\psi_+|\right), \tag{23}$$

where

$$|\psi_+\rangle := \sum_{i=1}^{d} |e_i\rangle \otimes |e_i\rangle, \tag{24}$$

and $\{|e_i\rangle\}$ denotes a basis in $\mathcal{H}$. The operator $W$, acting on the doubled space $\mathcal{H} \otimes \mathcal{H}$ is, after normalization, called $J$-state and the correspondence defined by (23) is called $J$-isomorphism. According to the best knowledge of the author this relationship between $\Phi$ and $W(\Phi)$ was applied to problems of evolution of quantum systems for the first time in papers cited in [1] and [2] in the begining of seventies. More details about definition (23) is given in the next section.

Three decades later it was shown by R. Timoney [22] that a positive map $\Phi$ (positive superoperator) which is $m$-positive, where $m = \lfloor \sqrt{\kappa} \rfloor$ must be completely positive. Here $\lfloor \sqrt{\kappa} \rfloor$ denotes the integer part of the number $\sqrt{\kappa}$. In other words, if a positive map $\Phi : B(\mathcal{H}) \to B(\mathcal{H})$, $\dim \mathcal{H} = d$, has a minimal length $\kappa$ and $\Phi$ is $m$-positive, for some $m < n$ such that $(m+1)^2 > \kappa$, then $\Phi$ is already completely positive (cf. also [19]).

Now, let us observe that one can apply the results stated in Theorems 1 – 3 from Section 2, to the particular cone $\mathcal{P}(\mathcal{H})$ in $B(\mathcal{H})$, $\dim \mathcal{H} = d$. In particular, Theorem 3 describes properties of irreducible superoperators and according to this theorem we have: for irreducible $\Phi$, the spectral radius of $\Phi$ is a simple eigenvalue of the superoperator and $\Phi$ has exactly one (up to scalar coefficient) eigenvector in $\mathcal{P}(\mathcal{H})$ and this vector belongs to $\mathcal{P}(\mathcal{H}) \setminus \partial\mathcal{P}(\mathcal{H})$. One can say even more. Let $\mathcal{P}_d$ denote the set of all orthogonal projections, i.e. $A \in \mathcal{P}_d$ iff $A^2 = A$ and $A = A^*$. Then we have [23]

**Theorem 5.** *The following statements are equivalent for a positive map on $\mathcal{P}(\mathcal{H})$.*

**1.)** *There is a nontrivial (that is different from $\{0\}$ and $\mathcal{P}(\mathcal{H})$) face of $\mathcal{P}(\mathcal{H})$ that is invariant under $\Phi$;*

**2.)** *There is nontrivial projection $P \in \mathcal{P}_d$ and a positive real number $\lambda > 0$ such that $\Phi(P) \leq \lambda P$;*

**3.)** *There is a nontrivial projection $P \in \mathcal{P}_d$ such that the subalgebra $P(\mathcal{P}(\mathcal{H}))P$ is invariant under $\Phi$.*

In order to produce nontrivial examples of irreducible positive maps and in certain cases to characterize all irreducible maps within the class of completely positive maps we will use the following consequences of the Kraus representations of completely positive maps.

A family of closed subspaces of a given Hilbert space is called trivial if this family contains only $\{0\}$ and $\mathcal{H}$. For a fixed operator $X \in B(\mathcal{H})$ we will denote by $\mathrm{Inv}\,(X)$ the set of all invariant subspaces of $X$. Now, we can state the following theorem which is a reformulation of some results from [23].

**Theorem 6** (Farenick)**.** *Let $\Phi$ denote a superoperator on $B(\mathcal{H})$ which is $\mathcal{P}(\mathcal{H})$-positive. If $\Phi$ is completely positive, then there exist some operators $A_1, \ldots, A_\kappa$ such that $\Phi(X) = \sum_j A_j X A_j^\star$. Completely positive $\Phi$ is irreducible if and only if the Kraus operators $A_j$ do not have a nontrivial common invariant subspace in $\mathcal{H}$.*

To better understand the above theorem, let us observe that if Kraus operators do not have a common invariant subspace, i.e., are such that $\cap_j \mathrm{Inv}(A_j)$ is trivial and $\Phi(P) \leq \lambda P$ for some $\lambda \geq 0$ and $P \in \mathcal{P}_n$, then we have

$$\langle \Phi(P)\psi|\psi\rangle = \sum_{j=1}^{\kappa} \langle PA_j\psi|A_j\psi\rangle \leq \lambda\langle P\psi|\psi\rangle\,. \tag{25}$$

The left-hand side of the above equality is nonnegative for all $\psi \in \mathcal{H}$. On the other hand for $\psi \in \ker P$, we have $\langle P\psi|\psi\rangle = 0$ on the right-hand side. In this way the equality (25) implies that $\langle PA_j\psi|A_j\psi\rangle = 0$ for each $j = 1, \ldots, \kappa$. This means that $\langle PA_j\psi|PA_j\psi\rangle = 0$, if we remember that $P^2 = P$, $P^* = P$. In consequence, $\ker P \in \cap_j \mathrm{Inv}(A_j)$, that is $\ker P$ is either $\{0\}$ or $\mathcal{H}$.

## 4. Some effective methods in the study of positive maps

If we compare the set of all linear maps $\Phi : B(\mathcal{H}) \to B(\mathcal{H})$ and the set of all linear operators on $\mathcal{H} \otimes \mathcal{H}$, i.e. $\mathcal{B}(\mathcal{H} \otimes \mathcal{H})$, where $\mathcal{H}$ is a $d$-dimensional Hilbert space, then it is easy to see that these sets represent isomorphic spaces (as two vector spaces with the same dimension). Because of linearity of maps $\Phi$, any fixed map is fully defined if we know the values of $\Phi$ on elements of arbitrary basis of the space $B(\mathcal{H})$. For example, we know $\Phi$ if we know the values of $\Phi$ on elements of $B(\mathcal{H})$ of the form $|e_i\rangle\langle e_j|$

$$\Phi(E_{ij}) := \Phi(|e_i\rangle\langle e_j|)\,, \tag{26}$$

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where $|e_i\rangle$, for $i = 1, \ldots, d$ are orthonormal vectors in $\mathcal{H}$. Usually, we assume that this is the so-called natural zero-one basis in $\mathbb{C}^d$; $|e_i\rangle$ represents vector with 1 on $i$-th position and zeros on other places. Let us observe that operators of the form $|e_i\rangle\langle e_j|$ which act on vectors from $\mathbb{C}^d$ by the rule

$$(|e_i\rangle\langle e_j|)|\psi\rangle = \langle e_j|\psi\rangle|e_i\rangle \tag{27}$$

constitute a basis in the space $B(\mathcal{H})$ and are projectors in directions $|e_i\rangle$.

Now, an important question is: how to relate, for a given map $\Phi : B(\mathcal{H}) \to B(\mathcal{H})$, an operator on $\mathcal{H} \otimes \mathcal{H}$ (that is an element from $\mathcal{B}(\mathcal{H} \otimes \mathcal{H})$) in such a way in order to have a representation of $\Phi$ which will be useful for description of properties of this map. In other words, we look for a specific isomorphism between maps $\Phi$ on $B(\mathcal{H})$ and elements from $\mathcal{B}(\mathcal{H} \otimes \mathcal{H})$. It appears that this isomorphism can be defined in the following way: we take values $\Phi(E_{ij})$ from (26) and define an element $W(\Phi)$ from $\mathcal{B}(\mathcal{H} \otimes \mathcal{H}) \equiv B(\mathcal{H}) \otimes B(\mathcal{H})$ by the formula

$$W(\Phi) := \sum_{i,j=1}^{d} E_{ij} \otimes \Phi(E_{ij}) \tag{28}$$

which is equivalent to (23).

Now, it can be shown that $\Phi$ preserves the cone $\mathcal{P}(\mathcal{H})$ of nonnegative operators if and only if [1,2]

$$\langle \varphi| \otimes \langle \psi|W(\Phi)|\varphi\rangle \otimes |\psi\rangle \geq 0 \tag{29}$$

for all $|\varphi\rangle$, $|\psi\rangle$ from $\mathcal{H}$, and $\Phi$ is completely positive if and only if operator $W(\Phi)$ is semipositive on $\mathcal{H} \otimes \mathcal{H}$. We will also use this representation in Section 5, to discuss some properties of master equations.

On the other hand, we know that any completely positive map (superoperator) on $B(\mathcal{H})$ can be represented by a set of Kraus operators $K_1, \ldots, K_\kappa$ and the superoperator $\Phi$ is irreducible if the operators $K_j$ ($j = 1, \ldots, \kappa$) do not have a nontrivial common invariant subspace in the space $\mathcal{H}$. The question connected with the existence of DFS with a fixed dimension can be also related to the following problem: is it possible to verify whether operators $K_1, \ldots, K_\kappa$ have — or do not have — a common invariant subspace of dimension $m < d$, by an effective procedure? For $m = 1$ an answer to this question was given by Dan Shemesh in 1984 [24].

**Theorem 7.** *Let $K_1$ and $K_2$ denote two matrices acting on $\mathcal{H} = \mathbb{C}^d$. A common eigenvector of $K_1$ and $K_2$ exists if and only if the subspace defined by*

$$\mathcal{M}_1 := \bigcap_{\alpha,\beta}^{d-1} \ker \left[K_1^{\alpha}, K_2^{\beta}\right] \tag{30}$$

*is nontrivial, that is $\mathcal{M}_1 \neq \{\mathbf{O}\}$ or, in other words, $\dim \mathcal{M}_1 > 0$. Here the symbol $[\cdot,\cdot]$ denotes the commutator of the matrices and $\alpha, \beta \in [1, \ldots, d-1]$.*

The above theorem is connected in natural way with the concept of partially commutative operators. Two operators $K_1$ and $K_2$ (complex matrices $d \times d$) which do not commute, $[K_1, K_2] \neq 0$, are said to be partially commuting if $K_1$ and $K_2$ have a common invariant subspace (at least one common eigenvector). The reason for introducing this term is obvious: if $|x\rangle \in \mathcal{H}$ is a nonzero vector such that

$$K_1|x\rangle = \lambda_1|x\rangle \quad \text{and} \quad K_2|x\rangle = \lambda_2|x\rangle\,, \tag{31}$$

then there exists a nontrivial common invariant subspace of $K_1$, $K_2$ on which these operators (matrices) commute.

As it was stressed in [25], the genuine meaning of the subspace $\mathcal{M}_1$, can be stated as follows.

**Theorem 8.** *A subspace $\mathcal{M}_1$ is invariant with respect to both matrices $K_1$ and $K_2$, and moreover, $K_1$ and $K_2$ commute on $\mathcal{M}_1$. Every subspace of $\mathcal{H}$, which is invariant under $K_1$ and $K_2$ and on which $K_1$ and $K_2$ commute is contained in $\mathcal{M}_1$.*

The condition of Theorem 8, that is $\dim \mathcal{M}_1 > 0$, can be formulated in a constructive form. To this end let us define the matrix

$$\mathcal{O} := \sum_{\alpha,\beta}^{d-1} \left[K_1^{\alpha}, K_2^{\beta}\right]^* \left[K_1^{\alpha}, K_2^{\beta}\right]. \tag{32}$$

The matrices $K_1$ and $K_2$ have common eigenvectors if and only if the matrix $\mathcal{O}$ is singular, i.e. $\det \mathcal{O} = 0$.

It is not difficult to check that if one of the operators $K_1$, $K_2$ has distinct eigenvalues (let say $K_1$ has this property), then the last expression for $\mathcal{O}$ reduces to

$$\mathcal{O} := \sum_{\alpha}^{d-1} \left[K_1^{\alpha}, K_2\right]^* \left[K_1^{\alpha}, K_2\right], \tag{33}$$

and the condition $\det \mathcal{O} = 0$ is simplified from the point of view of calculations.

One can say even more. If one of operators $K_j$, $j = 1, \ldots, \kappa$, has distinct eigenvalues (once again, let say $K_1$ has this property) then the appropriate operator (matrix) $\mathcal{O}$ takes the form

$$\mathcal{O} := \sum_{\alpha=1}^{d-1} \sum_{j=2}^{\kappa} \left[K_1^{\alpha}, K_j\right]^* \left[K_1^{\alpha}, K_j\right], \tag{34}$$

and the condition $\det \mathcal{O} = 0$ tells us that operators $K_1, \ldots, K_\kappa$ have a common eigenvector (Jamiołkowski, in preparation).

Now, using the concept of the so-called standard polynomials and the Amitsur-Levitzki theorem [27, 28], we can generalize the criterion of Shemesh in another way. Recall that the standard polynomial of degree $n$ is the polynomial in non-commuting variables $X_1, \ldots, X_n$ of the form

$$S_r(X_1, \ldots, X_n) := \sum \operatorname{sign}(\sigma) X_{\sigma(1)} \cdots X_{\sigma(n)}. \tag{35}$$

The summation here is assumed over all permutations of $1, \ldots, n$. The importance of the standard polynomials is underlined by the following Amitsur-Levitzki theorem.

**Theorem 9.** *The full matrix algebra $\mathbb{M}_d(\mathbb{C})$ satisfies the standard identity $S_{2d}(X_1, \ldots, X_{2d}) \equiv 0$. Moreover, the algebra $\mathbb{M}_d(\mathbb{C})$ does not satisfy any polynomial of degree less than $2d$.*

Let us observe that according to the above theorem, the algebra $\mathbb{M}_{d+1}(\mathbb{C})$ cannot satisfy the identity for $n = 2d$. In other words, the algebra $\mathbb{M}_k(\mathbb{C})$ satisfies the identity $S_{2d} = 0$ for $k \leq d$ and does not satisfy this identity for $k \geq d + 1$.

Now, we are ready to discuss a generalization of the Shemesh theorem. Namely, we introduce the family of the subspaces

$$\mathcal{M}_k := \bigcap \ker\left[S_{2k}(N_1, \ldots, N_{2k}) N_{2k+1}\right], \tag{36}$$

where $S_{2k}$ denotes the standard polynomial of degree $2k$ and the intersection is taken over all $(2k+1)$-tuples of matrices from the algebra $\mathcal{A}$ generated by two elements (Kraus operators $K_1$ and $K_2$). One can prove:

**Theorem 10.** *Assume that $\mathcal{M}_k$ satisfies $\dim \mathcal{M}_k > 0$. Then $\mathcal{M}_k$ is an invariant subspace of $\mathcal{A}$ and elements of this algebra restricted to $\mathcal{M}_k$ satisfy the identity $S_{2k} \equiv 0$; i.e. for all $N_1, \ldots, N_{2k}$ from $\mathcal{A}$ and $X \in \mathcal{M}_k$ we have*

$$S_{2k}(N_1, \ldots, N_{2k}) X = 0. \tag{37}$$

It is not obvious from (36) that $\mathcal{M}_k$ can be constructed by an effective procedure; however, it is possible to show that these subspaces can be constructed by a finite number of arithmetic operations [28].

As a conclusion of this section we can say that if in the Kraus representation of any fixed completely positive map $\Phi$,

$$\Phi(X) = \sum_{i=1}^{\kappa} K_i X K_i^*, \tag{38}$$

at least two Kraus operators do not have a common eigenvector, then the map $\Phi$ is irreducible. Moreover, using the generalization of Shemesh's theorem we can check in an effective way whether the algebra $\mathcal{A}$ generated by Kraus operators has a decoherence-free subspace of dimension $m \geq 2$, or not. In particular, according to (34), if one of Kraus' operators has distinct eigenvalues, the condition that $\det \mathcal{O} = 0$ is simplified from the point of view of efficiency of calculations.

## 5. Master equations and Metzler operators

In quantum physics, quantum chemistry and related fields by master equation one understands linear differential equation of the form

$$\frac{d\rho(t)}{dt} = \mathbb{K}(t)\rho(t) \tag{39}$$

with the following property: if $\rho(t_0)$ belongs to $\mathcal{P}(\mathcal{H}) = B_*^+(\mathcal{H})$ and satisfies the condition $\mathrm{Tr}(\rho(t_0)) = 1$, that is $\rho(t_0) \in \mathcal{S}(\mathcal{H})$, then the trajectory emanating from $\rho(t_0)$ stays in $\mathcal{S}(\mathcal{H})$ for all $t \geq t_0$. Here by trajectory we understand the solution $t \to \rho(t)$ to equation (39) with initial condition $\rho(t_0)$. In other words, a question of considerable physical, as well as, theoretical interest is the following. Under what conditions on $\mathbb{K}(t)$ does every solution of (39) which originates in $\mathcal{S}(\mathcal{H})$ remains in $\mathcal{S}(\mathcal{H})$ for all $t \geq 0$. In order to formulate the answer to this question in general form we say a few words on general solutions of equations in $B_*(\mathcal{H})$ with time dependent generators.

Let $\mathcal{D} \subset \mathbb{R}^1$ be an interval and let $\mathbb{K} : \mathcal{D} \to B_*(\mathcal{H})$ be a continuous operator-valued function with domain $\mathcal{D}$. Consider the linear differential equation

$$\frac{d\gamma(t)}{dt} = \mathbb{K}(t)\gamma(t)\,. \tag{40}$$

It is well known that for each $\gamma(t_0)$, equation (40) has a unique solution $\mathcal{D} \ni t \to \gamma(t) \in B_*(\mathcal{H})$ which is dependent in a linear way on $\gamma(t_0)$. Therefore for each pair $t, t_0$ from $\mathcal{D}$ one can define a linear operator $\Phi(t, t_0)$, $t \geq t_0$, by the formula

$$\Phi(t, t_0)\gamma(t_0) := \gamma(t, t_0, \gamma(t_0))\,, \tag{41}$$

where $\gamma(t, t_0, \gamma(t_0))$ satisfies (40), with the initial condition $\gamma(t_0)$.

The operator $\Phi(t, t_0)$ is usually called the *evolution operator* or the *propagator* of the time evolution defined in $B_*(\mathcal{H})$ by the generator $\mathbb{K}(t)$. The following properties characterize the propagator $\Phi(t, t_0)$:

1) $\Phi(t, u) \cdot \Phi(u, s) = \Phi(t, s)$ for all $t, u, s \in \mathcal{D}$,
2) $\Phi(t, s)$ is differentiable with respect to $t$ and

$$\frac{d\Phi(t,s)}{dt} = \mathbb{K}(t)\Phi(t, s)\,, \tag{42}$$

3) $\Phi(t, t) = \mathbb{I}$ for all $t \in \mathcal{D}$, where $\mathbb{I}$ denotes the identity operator,
4) if $\mathbb{K}(t) = \mathbb{K}$ for all $t \in \mathcal{D}$, then

$$\Phi(t, t_0) = \exp\{(t - t_0)\mathbb{K}\} = \sum_{m=0}^{d-1} \alpha_m(t - t_0)\mathbb{K}^m\,,$$

where $\alpha_m(t - t_0)$ for $m = 0, \ldots, d - 1$ are some analytic functions defined by the structure of $\mathbb{K}$ (cf. eg. [29]).

We say that $\mathbb{K}(t)$ defines a positive evolution if

$$\Phi(t, t_0) \geq 0 \tag{43}$$

for all $t \geq t_0,\ t, t_0 \in \mathcal{D}$. Here, the positivity of $\Phi(t, t_0)$ is understood with respect to the cone $\mathcal{P}(\mathcal{H}) = B_*^+(\mathcal{H})$.

An operator $\mathbb{K}$ (superoperator with respect to elements of $B_*(\mathcal{H})$) is characterized by its spectrum $\sigma(\mathbb{K})$ and the resolvent set $\omega(\mathbb{K})$. Now, a closed operator $\mathbb{K}$ is said to be a Metzler operator if there exists a real number $\alpha_0$ such that for all $\alpha > \alpha_0$ the resolvent of the operator $\mathbb{K}$ is positive, that is,

$$R(\alpha, \mathbb{K})\mathcal{P}(\mathcal{H}) \subseteq \mathcal{P}(\mathcal{H})\,. \tag{44}$$

One uses this terminology since Metzler operators are straightforward generalization of Metzler matrices. Indeed, let us take the space $\mathbb{R}^d$ and a cone $\mathbb{R}^d_+$. Suppose $\mathbb{K} \in M_d(\mathbb{R})$ is a Metzler matrix, that is there exists $b \in \mathbb{R}^1$ such that $\mathbb{K} + b\mathbb{I}$ is positive, which means that all entries of the matrix are nonnegative real numbers.

One can say immediately that for $\alpha > r(\mathbb{K} + b\mathbb{I})$ the resolvent

$$R(\alpha, \mathbb{K} + b\mathbb{I}) = (a\mathbb{I} - (\mathbb{K} + b\mathbb{I}))^{-1} = [(a-b)\mathbb{I} - \mathbb{K}]^{-1} \tag{45}$$

is a positive operator, that is according to the above definition, Metzler matrix represents a Metzler operator. Vice versa, if $\mathbb{K}$ is a Metzler operator, the off-diagonal elements of $\mathbb{K} \in M_d(R)$ are all nonnegative, that is, $\mathbb{K}$ is a Metzler matrix.

It follows directly from our definition of Metzler operators that a fixed closed operator $\mathbb{K}$ is a Metzler operator if and only if also $\mathbb{K} + b\mathbb{I}$ is a Metzler operator for some $b \in \mathbb{R}^1$. Now, using the Neumann representation of $R(\alpha, \mathbb{K})$ for large $\alpha > 0$, one can see that every operator on finite-dimensional $B_*(\mathcal{H})$ for which there exists $b \geq 0$ such that $\mathbb{K} + b\mathbb{I}$ is positive, is a Metzler operator on $B(\mathcal{H})$.

It was proved by Elsner [30] that the following conditions for an operator $\mathbb{K}$ on any finite-dimensional Banach space $V$ ordered by a closed, solid, convex cone $\mathcal{C}$ are equivalent:

1) $\mathbb{K}$ is a Metzler operator,
2) $\exp(t\mathbb{K})$ is positive for all $t \geq 0$,
3) $|x\rangle \in \mathcal{C}, \langle v| \in \mathcal{C}^*, \langle v|x\rangle = 0 \Rightarrow \langle v|\mathbb{K}x\rangle \geq 0$.

The condition 2) is often called exponential positivity or exponential nonnegativity. From 3) one can see that the Metzler operators constitute a convex cone. However, this cone in not pointed — it contains all scalar multiples of the identity operator. In fact we can say that the set of all Metzler operators constitutes a wedge.

Now, let us return to our system $B_*(\mathcal{H})$ and cones $\mathcal{P}(\mathcal{H}) = B_*^+(\mathcal{H})$. As we know, in this case $\rho(t) \in \mathcal{P}(\mathcal{H})$ for all $t \geq 0$ if

$$\langle\psi| \otimes \langle\varphi|W(\mathbb{K})|\psi\rangle \otimes |\varphi\rangle \geq 0 \tag{46}$$

([1] and [2], cf. also [3]). This means that the last inequality (the so-called block positivity [3]) is a sufficient condition for preservation of positivity by $\Phi(t, t_0) = \exp[(t - t_0)\mathbb{K}]$. But in fact the condition (46) is to strong — it is a sufficient condition but not a necessary. It is enough to assume that $\mathbb{K}$ satisfies the weaker condition — $\mathbb{K}$ should be a Metzler operator.

It appears that one can extend the conditions discussed above to the time dependent operator $\mathbb{K}(t)$. Namely, we can state that the operator family $\mathbb{K}(t)$ generates a positive evolution of the interval $\mathcal{D}$ if and only if $\mathbb{K}(t)$ is Metzler operator for all $t \in \mathcal{D}$ (cf. [31]).

It is important to observe that the property of being a Metzler operator can be expressed using the defined in (28) isomorphism $W = J(\mathbb{K})$. Namely, an operator $\mathbb{K}$ is a Metzler operator if and only if for all $|\psi\rangle$, $|\varphi\rangle$ such that they are orthogonal, we have inequality

$$\langle \psi| \otimes \langle \varphi| W(\mathbb{K}) |\psi\rangle \otimes |\varphi\rangle \geq 0\,, \qquad |\psi\rangle \perp |\varphi\rangle\,. \tag{47}$$

Indeed, as we know the superoperator $\mathbb{K}$ is a Metzler operator if $\langle A|\mathbb{K}(B)\rangle \geq 0$ for all $A, B$ from $B_*^+(\mathcal{H})$, such that $\langle A|B\rangle = \mathrm{Tr}(AB) = 0$. Since $\mathbb{K}$ is linear it is enough to take $A = |\psi\rangle\langle\psi|$ on $B = |\varphi\rangle\langle\varphi|$. Now we obtain $\langle\, |\psi\rangle\langle\psi|, |\varphi\rangle\langle\varphi|\,\rangle = 0$ iff $\langle\varphi|\psi\rangle\langle\psi|\varphi\rangle = 0$, that is if $|\psi\rangle \perp |\varphi\rangle$, so we obtain

$$\langle\, |\psi\rangle\langle\psi|, \mathbb{K}(|\varphi\rangle\langle\varphi|)\,\rangle = \langle\varphi|\,(\mathbb{K}|\psi\rangle\langle\psi|)\,|\varphi\rangle \geq 0 \tag{48}$$

for all $|\psi\rangle \perp |\varphi\rangle$. Using the results from [1] and [2] we obtain the condition (47).

As it was shown in [32], some necessary and sufficient conditions for a generator of a quantum dynamical semi-group can also be formulated using the concept of dissipative operators in the sense of Lumer and Phillips [33]. Namely, if a sequence $\{\mathcal{P}_1, \mathcal{P}_2, \ldots\} \equiv \pi$ of projection operators on closed subspaces of the Hilbert space $\mathcal{H}$ constitutes a discrete resolution of identity, that is, if

$$\mathcal{P}_i \mathcal{P}_j = \delta_{ij} \mathcal{P}_j\,, \quad \text{and} \quad \sum \mathcal{P}_i = \mathbb{I}\,, \tag{49}$$

then one can describe the properties of the operator $\mathbb{K}$ in the following: a linear operator $\mathbb{K}$ generates a dynamical semigroup iff for every discrete reduction of identity we have

$$a_{ij}(\pi) \geq 0 \quad \text{for} \quad i \neq j, \qquad a_{ii}(\pi) \leq 0\,, \tag{50}$$

and

$$\sum_i a_{ij}(\pi) = 0 \tag{51}$$

where

$$a_{ij}(\pi) = \mathrm{Tr}(\mathcal{P}_i(\mathbb{K}\mathcal{P}_j)) \tag{52}$$

for $i, j = 1, 2, \ldots$. The conditions (50) are quantum analogs of Kolmogorov conditions (cf. [34]) for discrete Markov processes.

## 6. Summary

In Sections 4 and 5 it was shown that one can analyze the properties of superoperators which are important in modeling of open quantum systems using the natural isomorphisms defined by relation (28). In some situations this approach gives more effective results than other methods. On the other hand, other approaches (for example, the method proposed by Kossakowski, which is based on dissipative operators) give us a beautiful similarity to classical results of Kolmogorov.

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