The Schwinger Action Principle and Its Applications to Quantum Mechanics

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1. Introduction

In physics it is generally of interest to understand the dynamics of a system. The way the dynamics is to be specified and studied invariably depends on the scale of the system, that is whether it is macroscopic or microscopic. The formal machinery with which the world is explained and understood depends at which of these two levels an experiment is conducted [1]. At the classical level, the dynamics of the system can be understood in terms of such things as trajectories in space or space-time.

In fact, classical mechanics can be formulated in terms of a principle of stationary action to obtain the Euler-Lagrange equations. To carry this out, an action functional has to be defined. It is written as S and given by

$$S[\mathbf{q}(t)] = \int_{t_1}^{t_2} L(q^i(t), \dot{q}^i(t), t)\, dt. \tag{1}$$

The action depends on the Lagrangian, written L in (1). It is to be emphasized that the action is a functional, which can be thought of as a function defined on a space of functions. For any given trajectory or path in space, the action works out to be a number, so S maps paths to real numbers.

One way to obtain equations of motion is by means of Hamilton's principle. Hamilton's principle states that the actual motion of a particle with Lagrangian L is such that the action functional is stationary. This means the action functional achieves a minimum or maximum value. To apply and use this principle, Stationary action must result in the Euler-Lagrange equations of motion. Conversely, if the Euler-Lagrange equations are imposed, the action functional should be stationary. As is well known, the Euler-Lagrange equations provide a system of second-order differential equations for the path. This in turn leads to other approaches to the same end.

As an illustration, the momentum canonically conjugate to the coordinate q^i is defined by

$$p_i = \frac{\partial L}{\partial \dot{q}^i}. \tag{2}$$

The dependence on the velocity components $\dot{q}^i$ can be eliminated in favor of the canonical momentum. This means that (2) must be solved for the $\dot{q}^i$ in terms of the q^i and p_i, and the inverse function theorem states this is possible if and only if $(\partial p_i/\partial \dot{q}^j) \neq 0$. Given a non-singular system all dependence on $\dot{q}^i$ can be eliminated by means of a Legendre transformation

$$H(\mathbf{q}, \mathbf{p}, t) = p_i \dot{q}^i - L(\mathbf{q}, \dot{\mathbf{q}}, t). \tag{3}$$

The Hamiltonian equations are obtained by considering the derivatives of the Hamiltonian with respect to q^i and p_i. The action in terms of H is written

$$S[\mathbf{q}, \mathbf{p}] = \int_{t_1}^{t_2} dt\, [p_i \dot{q}^i - H(\mathbf{q}, \mathbf{p}, t)], \tag{4}$$

such that the action now depends on both $\mathbf{q}$ and $\mathbf{p}$. Using this the principle of stationary action can then be modified so that Hamilton's equations result.

In passing to the quantum domain, the concept of path or trajectory is of less importance, largely because it has no meaning. In quantum physics it can no longer be assumed that the interaction between system and measuring device can be made arbitrarily small and that there are no restrictions on what measurements can be made on the system either in terms of type or in accuracy. Both these assumptions tend to break down at the scales of interest here, and one is much more interested in states and observables, which replace the classical idea of a trajectory with well defined properties [2,3].

Now let us follow Dirac and consider possible measurements on a system as observables. Suppose A_i denotes any observable with a_i as a possible outcome of any measurement of this observable. As much information as possible can be extracted with regard to a quantum mechanical system by measuring some set of observables $\{A_i\}_{i=1}^n$ without restriction. Thus, the observables should be mutually compatible because the measurement of any observable in the set does not affect the measurement of any of the other observables. The most information that can be assembled about a system is the collection of numbers $\{a_i\}_{i=1}^n$, which are possible values for the set of mutually compatible observables, and this set specifies the state of the system.

2. Schwinger's action principle

In Dirac's view of quantum mechanics, the state of a system is associated with a vector in a complex vector space V. The knowledge of the values for a complete set of mutually compatible observables gives the most information about a state. It can then be assumed that $\{|a\rangle\}$, where $|a\rangle = |a_1, \cdots, a_n\rangle$, the set of all possible states, forms a basis for V. Associated

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with any vector space V is the dual space V^* whose elements are referred to as bras in Dirac's terminology. A basis for V^* is denoted by $\{\langle a|\}$ and is dual to $\{|a\rangle\}$. The quantities satisfy

$$\langle a'|a''\rangle = \delta(a', a''), \tag{5}$$

where $\delta(a', a'')$ is the Kronecker delta if a' is a discrete set, and the Dirac delta if it is continuous. The choice of a complete set of mutually compatible observables is not unique. Suppose $\{|b\rangle\}$ also provides a basis for V relevant to another set of mutually compatible observables $B_1, B_2, \cdots$. Since $\{|a\rangle\}$ and $\{|b\rangle\}$ are both bases for V, this means that one set of basis vectors can be expressed in terms of the other set

$$|b\rangle = \sum_a |a\rangle\langle a|b\rangle, \tag{6}$$

and the $\langle a|b\rangle$ coefficients are some set of complex numbers, so that $\langle b|a\rangle^* = \langle a|b\rangle$. If there is a third basis for V provided by $\{|c\rangle\}$, then these complex numbers are related by means of

$$\langle a|c\rangle = \sum_b \langle a|b\rangle\langle b|c\rangle. \tag{7}$$

Schwinger's action principle is based on the types of transformation properties of the transformation functions which can be constructed from this basis set [4].

Suppose the transformation function is subjected to, as Schwinger asserted, any conceivable infinitesimal variation. Then, by performing an arbitrary variation of (7), it follows that

$$\delta\langle a|c\rangle = \sum_b [(\delta\langle a|b\rangle)\langle b|c\rangle + \langle a|b\rangle(\delta\langle b|c\rangle)], \tag{8}$$

and moreover,

$$\delta\langle a|b\rangle = \delta\langle b|a\rangle^*. \tag{9}$$

Now a new operator can be defined which evaluates this actual variation when it is placed between the relevant state vectors. Define this operator to be δW_{ab}, so that it has the following action between states

$$\delta\langle a|b\rangle = \frac{i}{\hbar}\langle a|\delta W_{ab}|b\rangle. \tag{10}$$

Including the factor of $\hbar$ gives the operator units of action. Using (10) in (8) produces,

$$\langle a|\delta W_{ac}|c\rangle = \sum_b [\langle a|\delta W_{ab}|b\rangle\langle b|c\rangle + \langle a|b\rangle\langle b|\delta W_{bc}|c\rangle] = \langle a|\delta W_{ab} + \delta W_{bc}|c\rangle, \tag{11}$$

using the completeness relation (7). Now it follows from (11) that

$$\delta W_{ac} = \delta W_{ab} + \delta W_{bc}. \tag{12}$$

In the case in which the a and b descriptions are identified then using $\delta\langle a|a'\rangle = 0$, there results

$$\delta W_{aa} = 0. \tag{13}$$

Identifying the a and c pictures in (11) gives,

$$\delta W_{ba} = -\delta W_{ab}. \tag{14}$$

The complex conjugate of (10) with a and b descriptions reversed implies

$$-\frac{i}{\hbar}\langle b|\delta W_{ba}|a\rangle^* = \frac{i}{\hbar}\langle a|\delta W_{ab}|b\rangle. \tag{15}$$

This has the equivalent form

$$-\langle a|\delta W_{ba}^{\dagger}|b\rangle = \langle a|\delta W_{ab}|b\rangle. \tag{16}$$

Using (14), this yields the property

$$\delta W_{ba}^{\dagger} = \delta W_{ab}. \tag{17}$$

The basic properties of the transformation function and the definition (10) have produced all of these additional properties [5,6].

In the Heisenberg picture, the basis kets become time dependent. The transformation function relates states which are eigenstates of different complete sets of commuting observables at different times. Instead of using different letters, different subscripts $1, 2$ can be used to denote different complete sets of commuting observables. In this event, (10) takes the form

$$\delta\langle a_2', t_2|a_1', t_1\rangle = \frac{i}{\hbar}\langle a_2', t_2|\delta W_{21}|a_1', t_1\rangle. \tag{18}$$

The assumption at the heart of this approach is that the operator δW_{21} in (18) is obtained from the variation of a single operator W_{21}. This is referred to as the action operator.

To adapt the results of the previous notation to the case with subscripts, we should have

$$W_{31} = W_{32} + W_{21}, \qquad W_{11} = 0, \qquad W_{21} = -W_{12} = W_{21}^{\dagger}. \tag{19}$$

At this point, a correspondence between the Schwinger action principle and the classical principle of stationary action can be made. Suppose the members of a complete set of commuting observables A_1 which have eigenvectors $|a_1, t\rangle$ in the Heisenberg picture are deformed in some fashion at time t_1. For example, take the alteration in the observables to correspond to a unitary transformation $A \to U^{\dagger}AU$ such that $U^{\dagger} = U^{-1}$. To remain eigenstates of the transformed operator, it must be that states transform as $|a\rangle \to U^{\dagger}|a\rangle$. Thinking of the transformation as being infinitesimal in nature, the operator U can be written $U = I + \frac{i}{\hbar} G$, where G is Hermitean. It is then possible to define a variation

$$\delta |a_1, t_1\rangle = -\frac{i}{\hbar} G_1 |a_1, t_1\rangle. \tag{20}$$

Here operator G is a Hermitian operator and depends only on the observables A_1 at the time t_1. Similarly, if observables A_2 are altered at t_2, it is the case that

$$\delta \langle a_2, t_2| = \frac{i}{\hbar} \langle a_2, t_2| G_2, \tag{21}$$

and the operator G_2 depends only on observables A_2 at time t_2. If both sets A_1, A_2 are altered infinitesimally, then the change in the transformation function is given by

$$\delta \langle a_2, t_2 | a_1, t_1 \rangle = \frac{i}{\hbar} \langle a_2, t_2 | G_2 - G_1 | a_1, t_1 \rangle. \tag{22}$$

Comparing this with (18), it is concluded that

$$\delta W_{21} = G_2 - G_1. \tag{23}$$

If the time evolution from state $|a_1, t_1\rangle$ to $|a_2, t_2\rangle$ can be thought of as occurring continuously in time, then W_{21} can be expressed as

$$W_{21} = \int_{t_1}^{t_2} L(t)\, dt, \tag{24}$$

where $L(t)$ is called the Lagrange operator. As a consequence of (23), it follows that if the dynamical variables which enter $L(t)$ are altered during an arbitrary infinitesimal change between t_1 and t_2, then it must be that

$$\delta W_{21} = 0. \tag{25}$$

The operator equations of motion are implied in this result. The usual form for the Lagrange operator is

$$L(t) = \frac{1}{2}(p_i \dot{x}^i + \dot{x}^i p_i) - H(\mathbf{x}, \mathbf{p}, t). \tag{26}$$

The first term has been symmetrized to give a Hermitian $L(t)$, as the operators $\dot{x}^i$ and p_i do not as usual commute. It is assumed the Hamiltonian H is a Hermitian operator. The action operator (24) is used to calculate the variation δW_{21}. In order to vary the endpoints t_1, t_2, we follow Schwinger exactly and change the variable of integration from t to τ such that $t = t(\tau)$. This allows for the variation of the functional dependence of t to depend on τ with the variable of integration τ held fixed. Then W_{21} takes the form,

$$W_{21} = \int_{\tau_1}^{\tau_2} d\tau \, [\frac{1}{2}(P_i \frac{dy^i}{d\tau} + \frac{dy^i}{d\tau} P_i) - \tilde{H}(\mathbf{y}, \mathbf{P}, \tau) \, \frac{dt}{d\tau}], \tag{27}$$

where in (27),

$$\tilde{H}(\mathbf{y}, \mathbf{P}, \tau) = H(\mathbf{x}, \mathbf{p}, t). \tag{28}$$

Thus $y^i(\tau) = x^i(t)$ and $P_i(\tau) = p_i(t)$ when the transformation $t = t(\tau)$ is implemented. Evaluating the infinitesimal variation of (27), it is found that

$$\delta W_{21} = \int_{\tau_1}^{\tau_2} d\tau [\frac{1}{2} \delta P_i \frac{dy^i}{d\tau} + \frac{1}{2} P_i \delta(\frac{dy^i}{d\tau}) + \frac{1}{2} \delta(\frac{dy^i}{d\tau}) P_i + \frac{1}{2} \frac{dy^i}{d\tau} \delta P_i - \delta \tilde{H} \frac{dt}{d\tau} - \tilde{H} \delta(\frac{dt}{d\tau})]. \tag{29}$$

Moving the operator δ through the derivative, this becomes

$$\delta W_{21} = \int_{t_1}^{t_2} d\tau \, \{ \frac{1}{2}(\delta P_i \frac{dy^i}{d\tau} + \frac{dy^i}{d\tau} \delta P_i - \frac{dP_i}{d\tau} \delta y^i - \delta y^i \frac{dP_i}{d\tau}) - \delta \tilde{H} \frac{dt}{d\tau} + \frac{d\tilde{H}}{d\tau} \delta t$$

$$+ \frac{d}{d\tau} [\frac{1}{2}(P_i \delta x^i + \delta x^i P_i) - \tilde{H} \delta t] \}. \tag{30}$$

No assumptions with regard to the commutation properties of the variations with the dynamical variables have been made yet. It may be assumed that the variations are multiples of the identity operator, which commutes with everything. After returning to the variable t in the integral in δW_{21}, the result is

$$\delta W_{21} = \int_{t_1}^{t_2} dt (\delta p_i \dot{x}^i - \dot{p}_i \delta x^i + \frac{dH}{dt} \delta t - \delta H) + G_2 - G_1. \tag{31}$$

Here G_1 and G_2 denote the quantity

$$G = p_i \, \delta x^i - H \, \delta t, \tag{32}$$

when it is evaluated at the two endpoints $t = t_1$ and $t = t_2$. If we define,

$$\delta H = \delta x^i \, \frac{\partial H}{\partial x^i} + \delta p_i \frac{\partial H}{\partial p_i} + \delta t \frac{\partial H}{\partial t}, \tag{33}$$

then δW_{21} can be expressed in the form,

$$\delta W_{21} = \int_{t_1}^{t_2} dt\, \{\delta p_i(\dot{x}^i - \frac{\partial H}{\partial p_i}) - \delta x^i(\dot{p}_i + \frac{\partial H}{\partial x^i}) + (\frac{dH}{dt} - \frac{\partial H}{\partial t})\delta t\} + G_2 - G_1. \tag{34}$$

Taking the variations with endpoints fixed, it follows that $G_1 = G_2 = 0$. Consequently, the operator equations of motion which follow from equating δW_{21} in (34) to zero are then

$$\dot{x}^i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial x^i}, \qquad \frac{dH}{dt} = \frac{\partial H}{\partial t}. \tag{35}$$

The results produced in this way are exactly of the form of the classical Hamilton equations of motion, and the derivatives in the first two equations of (35) are with respect to operators.

3. Commutation relations

Let B represent any observable and consider the matrix element $\langle a|B|a'\rangle$. If the variables A are subjected to a unitary transformation $A \to \bar{A} = UAU^\dagger$, where U is a unitary operator, then the eigenstates $|a\rangle$ are transformed into $|\bar{a}\rangle = U|a\rangle$ having the eigenvalue a. Define the operator $\bar{B}$ to be

$$\bar{B} = UBU^\dagger. \tag{36}$$

These operators have the property that

$$\langle \bar{a}|\bar{B}|\bar{a}'\rangle = \langle a|B|a'\rangle. \tag{37}$$

Let U now be an infinitesimal unitary transformation which can be expressed in the form

$$U = I - \frac{i}{\hbar}G_a. \tag{38}$$

In (38), G_a is a Hermitian quantity and can depend on observables A. Consequently, if δ_{G_a} is the change produced by the canonical transformation whose generator is G_a, then

$$\delta_{G_a}|a\rangle = |\bar{a}\rangle - |a\rangle = -\frac{i}{\hbar}G_a|a\rangle, \tag{39}$$

and $\bar{B}$ is given by

$$\bar{B} = (I - \frac{i}{\hbar}G_a)B(I + \frac{i}{\hbar}G_a) = B - \frac{i}{\hbar}[G_a, B]. \tag{40}$$

The change in the matrix element $\langle a|B|a'\rangle$ can then be considered to be entirely due to the change in the state vector, with B held fixed. To first order in the operator G_a, there results

$$\delta_{G_a}\langle a|B|a'\rangle = \langle\bar{a}|B|\bar{a}'\rangle - \langle a|B|a'\rangle = \langle a|\frac{i}{\hbar}[G_a, B]|a'\rangle. \tag{41}$$

Alternatively, this can be approached by taking the change in $\langle a|B|a'\rangle$ to be due to a change in the operator B, but with the states held fixed. Define then the change in B as $\delta_{G_a}B$ such that

$$\delta_{G_a}\langle a|B|a'\rangle = \langle a|\delta_{G_a}B|a'\rangle. \tag{42}$$

If the result obtained for the change in $\langle a|B|a'\rangle$ is not to depend on which of these approaches is taken, by comparing (41) and (42) it is concluded that

$$\delta_{G_a}B = \frac{i}{\hbar}[G_a, B]. \tag{43}$$

As an example, let the change in the operators A at times t_1 and t_2 be due to a change in time $t \to t + \delta t$ with the x^i fixed at times t_1 and t_2 so that the generator for this tranformation is given by $G(t) = -H(t)\delta t$ and (43) in this case is

$$\delta_{G_a}B = \frac{i}{\hbar}[B, H]\delta t, \tag{44}$$

where B and H are evaluated at the same time. It is only the change in B resulting from a change in operators which is considered now. Specifically, if $B = B[A(t), t]$, then $\bar{B} = B[A(t+\delta t), t]$ and so

$$\delta B = -\delta t(\frac{dB}{dt} - \frac{\delta B}{\delta t}). \tag{45}$$

Using (45) in (44), a result in agreement with the Heisenberg picture equation of motion results,

$$\frac{dB}{dt} = \frac{\partial B}{\partial t} + \frac{i}{\hbar}[H, B]. \tag{46}$$

The Heisenberg equation of motion has resulted by this process. If B is taken to be the operator $B = x^i$, then (46) becomes,

$$\dot{x}^i = \frac{i}{\hbar}[H, x^i]. \tag{47}$$

If $B = p_i$ is used in (46), there results

$$\dot{p}_i = \frac{i}{\hbar}[H, p_i]. \tag{48}$$

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The general equations of motion have been produced [7].

The canonical commutation relations are also a consequence of the action principle. To see this, first fix δt at the times t_1 and t_2 but permit δx^i to vary. Then (32) gives the generator of this transformation

$$G = p_i \delta x^i. \tag{49}$$

Then for any operator B, equation (43) gives

$$\delta B = \frac{i}{\hbar} \delta x^i \, [p_i, B]. \tag{50}$$

Putting $B = x^i$ and $B = p_j$ respectively in (50) leads to the following pair of commutation relations

$$[x^i, p_j] = i\hbar \, \delta^i_j, \qquad [p_i, p_j] = 0. \tag{51}$$

If B has a dependence on x, then $\delta B = B[x(t)] - B[x(t) - \delta x(t)]$ implies

$$[p_i, B] = -\frac{i}{\hbar} \frac{\partial}{\partial x^i} B. \tag{52}$$

In order to obtain the commutator $[x^i, x^j]$, the freedom of altering the Lagrangian operator by the addition of a total time derivative can be used. Suppose that

$$F = \frac{1}{2}(p_i x^i + x^i p_i), \tag{53}$$

which satisfies $F^{\dagger} = F$, and consequently

$$\delta F = \frac{1}{2}(\delta p_i x^i + p_i \delta x^i + \delta x^i p_i + x^i \delta p_i) = \delta p_i x^i + p_i \delta x^i. \tag{54}$$

Then $\bar{G}$ is calculated to be

$$\bar{G} = G - \delta F = -\delta p_i x^i - H \, \delta t. \tag{55}$$

Taking $\delta t = 0$ so that G_p can be defined as $G_p = -\delta p_i \, x^i$, it follows that

$$\delta B = -\frac{i}{\hbar} \delta p_i \, [x^i, B], \tag{56}$$

for any operator B. Taking $B = p_j$, the bracket $[x^i, p_j] = i\hbar \delta^i_j$ is obtained from (56), and when $B = x^j$, using the independence of δx^i and δp_i there results the bracket

$$[x^i, x^j] = 0. \tag{57}$$

It has been shown that the set of canonical commutation relations can be obtained from this action principle.

Consider a matrix element $\langle a|F(A,B)|b\rangle$ where A and B each represent a complete set of mutually compatible observables and $F(A,B)$ is some function, and it is not assumed the observables from the two different sets commute with each other. If the commutator $[B,A]$ is known, it is always possible to order the operators in $F(A,B)$ so all A terms are to the left of B, which allows the matrix element to be evaluated. Let

$$\mathcal{F}(A,B) = F(A,B) \tag{58}$$

denote the operator where the commutation relation for $[A,B]$ has been used to move all occurrences of A in $F(A,B)$ to the left of all B, and (58) is said to be well-ordered

$$\langle a|F(A,B)|b\rangle = \mathcal{F}(a,b)\langle a|b\rangle. \tag{59}$$

The matrix element of $F(A,B)$ is directly related to the transformation function $\langle a|b\rangle$.

The idea of well-ordering operators can be used in the action principle. Define

$$\delta\mathcal{W}_{21} = \delta W_{21}. \tag{60}$$

be the well-ordered form of δW_{21}. Then

$$\delta\langle a_2, t_2|a_1, t_1\rangle = \frac{i}{\hbar}\delta\mathcal{W}_{21}\langle a_2, t_2|a_1, t_1\rangle, \tag{61}$$

where $\delta\mathcal{W}_{21}$ denotes the replacement of all operators with their eigenvalues. Equation (61) can be integrated to yield

$$\langle a_2, t_2|a_1, t_1\rangle = \exp(\frac{i}{\hbar}\mathcal{W}_{21}). \tag{62}$$

Using (23) and (32) in the action principle,

$$\delta\langle a_2, t_2|a_1, t_1\rangle = \frac{i}{\hbar}\delta x^i(t_2)\langle a_2, t_2|p(t_2)|a_1, t_1\rangle - \frac{i}{\hbar}\delta t_2\langle a_2, t_2|H(t_2)|a_1, t_1\rangle$$

$$-\frac{i}{\hbar}\delta x^i(t_1)\langle a_2, t_2|p_i(t_1)|a_1, t_1\rangle + \frac{i}{\hbar}\delta t_1\langle a_2, t_2|H(t_1)|a_1, t_1\rangle. \tag{63}$$

Since $\delta\mathcal{W}_{21}$ is well-ordered, it is possible to write

$$\delta\langle a_2, t_2|a_1, t_1\rangle = \frac{i}{\hbar}\langle a_2, t_2|\delta\mathcal{W}_{21}|a_1, t_1\rangle. \tag{64}$$

Comparing this with (63), it must be that

$$\frac{\partial\mathcal{W}_{21}}{\partial x^i(t_2)} = p_i(t_2), \quad \frac{\partial\mathcal{W}_{21}}{\partial t_2} = -H(t_2), \quad \frac{\partial\mathcal{W}_{21}}{\partial x^i(t_1)} = -p_i(t_1), \quad \frac{\partial\mathcal{W}_{21}}{\partial t_1} = H(t_1). \tag{65}$$

If we consider a matrix element $\langle x_2, t_2|x_1, t_1\rangle$, which could be regarded as the propagator, we then have a form of (62),

$$\langle \mathbf{x}_2, t_2|\mathbf{x}_1, t_1\rangle = \exp(\frac{i}{\hbar}\mathcal{W}_{21}). \tag{66}$$

The arbitrary integration constant is determined by requiring that $\langle \mathbf{x}_2, t_2|\mathbf{x}_1, t_1\rangle = \delta(\mathbf{x}_2, \mathbf{x}_1)$. Any other transformation function can be recovered from the propagator equation

$$\langle a_2, t_2|a_1, t_1\rangle = \int d^n x_2 \int d^n x_1 \, \langle a_2, t_2|\mathbf{x}_2, t_2\rangle\langle \mathbf{x}_2, t_2|\mathbf{x}_1, t_1\rangle\langle \mathbf{x}_1 t_1|a_1, t_1\rangle, \tag{67}$$

which follows from the completeness relation. Another way of formulating transition amplitudes will be seen when the path integral approach is formulated.

4. Action principle adapted to case of quantum fields

The Schwinger action principle, much like the Feynman path integral, concentrates on the transition amplitude between two quantum states. The action principle will be formulated here for a local field theory. Classically, a local field is a function which depends only on a single spacetime point, rather than on an extended region of spacetime. The theory can be quantized by replacing the classical fields $\phi^i(x)$ with field operators $\varphi^i(x)$.

Let Σ denote a spacelike hypersurface. This means that any two points on Σ have a spacelike separation and consequently must be causally disconnected. As a consequence of this, the values of the field at different points of the surface Σ must be independent. If x_1 and x_2 are two spacetime points of Σ, then

$$[\varphi^i(x_1), \varphi^j(x_2)] = 0. \tag{68}$$

This follows since it must be the case that a measurement at x_1 must not influence one at x_2. A fundamental assumption of local field theory is that a complete set of commuting observables can be constructed based on the fields and their derivatives on the surface Σ. Let τ denote such a complete set of commuting observables on Σ such that τ' represents the eigenvalues of the observables. A quantum state is then denoted by

$$|\tau', \Sigma\rangle. \tag{69}$$

Since causality properties are important in field theory, the surface Σ is written explicitly in the ket. The Heisenberg picture has been adopted here. The states are then time-independent with the time dependence located in the operators. This is necessary for manifest covariance, since the time and space arguments of the field are not treated differently.

Suppose that Σ_1 and Σ_2 are two spacelike hypersurfaces such that all points of Σ_2 are to the future of Σ_1. Let τ_1 be a complete set of commuting observables defined on Σ_1, and τ_2 a complete set of observables defined on Σ_2, such that these observables have the same eigenvalue spectrum. Then τ_1 and τ_2 should be related by a unitary transformation

$$\tau_2 = U_{12}\tau_1 U_{12}^{-1}. \tag{70}$$

The eigenstates are related by

$$|\tau_2', \Sigma_2\rangle = U_{12}|\tau_1', \Sigma_1\rangle. \tag{71}$$

In (71), U_{12} is a unitary operator giving the evolution of the state in the spacetime between the two spacelike hypersurfaces. The transition amplitude is defined by

$$\langle \tau_2', \Sigma_2|\tau_1', \Sigma_1\rangle = \langle \tau_1', \Sigma_1|U_{12}^{-1}|\tau_1', \Sigma_1\rangle. \tag{72}$$

The unitary operator U_{12} depends on a number of details of the quantum system, namely, the choice made for the commuting observables τ, and the spacelike hypersurfaces Σ_1 and Σ_2. A change in any of these quantities will induce a change in the transformation function according to

$$\delta\langle \tau_2', \Sigma_2|\tau_1', \Sigma_1\rangle = \langle \tau_1', \Sigma_1|\delta U_{12}^{-1}|\tau_1', \Sigma_1\rangle. \tag{73}$$

The unitary operator U_{12} can be expressed in the form

$$U_{12} = \exp(-\frac{i}{\hbar}S_{12}), \tag{74}$$

in which $S_{12}^{\dagger} = S_{12}$ is a Hermitian operator. Moreover, beginning with $U_{12}U_{12}^{-1} = I$ and using (74), it is found that

$$\delta U_{12}^{-1} = -U_{12}^{-1}\delta U_{12}U_{12}^{-1} = \frac{i}{\hbar}U_{12}^{-1}\delta S_{12}. \tag{75}$$

The change in the transformation function can be written in terms of the operator in (74) as

$$\delta\langle \tau_2', \Sigma_2|\tau_1', \Sigma_1\rangle = \frac{i}{\hbar}\langle \tau_2', \Sigma_2|\delta S_{12}|\tau_1', \Sigma_1\rangle. \tag{76}$$

Equation (76) can be regarded as a definition of δS_{12}. In order that δS_{12} be consistent with the basic requirement that

$$\langle \tau_2', \Sigma_2|\tau_1', \Sigma_1\rangle^* = \langle \tau_1', \Sigma_1|\tau_2', \Sigma_2\rangle,$$

it must be that δS_{12} is Hermitean. If Σ_3 is a spacelike hypersurface, all of whose points lie to the future of those on Σ_2, the basic law for composition of probability amplitudes is

$$\langle \tau_3', \Sigma_3 | \tau_1', \Sigma_1 \rangle = \sum_{\tau_2'} \langle \tau_3', \Sigma_3 | \tau_2', \Sigma_2 \rangle \langle \tau_2', \Sigma_2 | \tau_1', \Sigma_1 \rangle. \tag{77}$$

Varying both sides of the expression in (77),

$$\langle \tau_3', \Sigma_3 | \delta S_{13} | \tau_1', \Sigma_1 \rangle = \sum_{\tau_2'} \{ \langle \tau_3', \Sigma_3 | \delta S_{23} | \tau_2', \Sigma_2 \rangle \langle \tau_2', \Sigma_2 | \tau_1', \Sigma_1 \rangle + \langle \tau_3', \Sigma_3 | \tau_2', \Sigma_2 \rangle \langle \tau_2', \Sigma_2 | \delta S_{12} | \tau_1', \Sigma_1 \rangle$$

$$= \langle \tau_3', \Sigma_3 | \delta S_{23} + \delta S_{12} | \tau_1', \Sigma_1 \rangle. \tag{78}$$

Comparing both ends of the result in (78), it follows that

$$\delta S_{13} = \delta S_{23} + \delta S_{12}. \tag{79}$$

In the limit $\Sigma_2 \to \Sigma_1$, it must be that

$$\delta S_{12} = 0.$$

In the limit in which $\Sigma_3 \to \Sigma_1$, it follows that

$$\delta S_{21} = -\delta S_{12}.$$

If the operators in τ_1 and τ_2 undergo infinitesimal, unitary transformations on the hypersurfaces Σ_1 and Σ_2, respectively, and only on these two hypersurfaces, then the change in the transformation function has the form

$$\delta \langle \tau_2', \Sigma_2 | \tau_1', \Sigma_1 \rangle = \frac{i}{\hbar} \langle \tau_2', \Sigma_2 | F_2 - F_1 | \tau_1', \Sigma_1 \rangle. \tag{80}$$

Here F_1 and F_2 are Hermitean operators constructed from a knowledge of the fields and their derivatives on Σ_1 and Σ_2. The result (80) is of the form (76) provided that

$$\delta S_{12} = F_2 - F_1. \tag{81}$$

A generator F of this type on a spacelike hypersurface Σ should be expressible in the form,

$$F = \int_\Sigma d\sigma_x n^\mu F_\mu(x), \tag{82}$$

and $d\sigma_x$ is the area element on Σ, n^μ is the outward unit normal to Σ, and $F_\mu(x)$ may be put together based on a knowledge of the fields on the surface Σ.

The points of Σ are all spacelike separated, hence independent and so the result follows by adding up all of these independent contributions. Applying (82) to Σ_1 and Σ_2 assuming that $F_\mu(x)$ is defined throughout the spacetime region bounded by these two surfaces, δS_{12} can be expressed as

$$\delta_{12} S = \int_{\Sigma_2} d\sigma_x \, n^\mu F_\mu(x) - \int_{\Sigma_1} d\sigma_x \, n^\mu F_\mu(x) = \int_{\Omega_{12}} dv_x \, \nabla^\mu F_\mu(x). \tag{83}$$

In (83), Ω_{12} is the spacetime region bounded by Σ_1 and Σ_2, and dv_x is the invariant volume. This assumes that the operators are changes only on Σ_1 and Σ_2. However, suppose the operators are changed in the spacetime region between Σ_1 and Σ_2. Assume once more that δS_{12} can be expressed as a volume integral as,

$$\delta S_{12} = \int_{\Omega_{12}} dv_x \, \delta \mathcal{L}(x) \tag{84}$$

for some $\delta \mathcal{L}(x)$. Combining these two types of variation yields

$$\delta S_{12} = F_2 - F_1 + \int_{\Omega_{12}} dv_x \, \delta \mathcal{L}(x) = \int_{\Omega_{12}} dv_x [\delta \mathcal{L}(x) + \nabla^\mu F_\mu(x)]. \tag{85}$$

It is an important result of (85) that altering $\delta \mathcal{L}$ by the addition of the divergence of a vector field will result in a unitary transformation of the states on Σ_1 and Σ_2.

To summarize, the fundamental assumption of the Schwinger action principle is that δS_{12} may be obtained from a variation of

$$S_{12} = \int_{\Omega_{12}} dv_x \, \mathcal{L}(x), \tag{86}$$

where $\mathcal{L}(x)$ is a Lagrangian density. The density depends on the fields and their derivatives at a single spacetime point. Since δS_{12} is required to be Hermitian, S_{12} must be Hermitian and similarly, so must the Lagrangian density.

5. Correspondence with Feynman path integrals

Suppose a classical theory described by the action $S[\varphi]$ is altered by coupling the field to an external source $J_i = J_I(x)$. By external it is meant that it has no dependence on the field φ^i, and i stands for (I, x). For example, $F_{,i}[\varphi]\sigma^i$ is an abbreviation for

$$F_{,i}[\varphi]\sigma^i = \int d^n x' \, \frac{\delta F[\varphi(x)]}{\delta \varphi^I(x')} \, \sigma^i(x').$$

The idea of introducing external sources originates with Schwinger. As he states, causality and space-time uniformity are the creative principles of source theory. Uniformity in

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space-time also has a complementary momentum-energy implication, illustrated by the source idea [5,8]. Not only for the special balance of energy and momentum involved in the emission or absorption of a single particle is the source defined and meaningful. Given a sufficient excess of energy over momentum, or an excess of mass, several particles can be emitted or absorbed. For example, consider the emission of a pair of charged particles by an extended photon source. This process is represented as the conversion of a virtual photon into a pair of real particles. In ordinary scattering, particle-particle scattering, the particles persist while exchanging a space-like virtual photon. Another is an annihilation of the particle-antiparticle pair, producing a time-like virtual photon, which quickly decays back into particles.

Let us choose a simple scalar theory

$$S_J[\varphi] = S[\varphi] + \int J_I(x)\varphi^I(x)\, d^n x, \tag{87}$$

and try to establish a connection between the action principle introduced here and a path integral picture [9]. The transition amplitude $\langle \tau_2', \Sigma_2 | \tau_1', \Sigma_1 \rangle$ discussed in the previous section may be regarded as a functional of the source term $J_I(x)$. This will be denoted explicitly by writing $\langle \tau_2', \Sigma_2 | \tau_1', \Sigma_1 \rangle_J$. If the Schwinger action principle is applied to this modified theory

$$\delta \langle \tau_2', \Sigma_2 | \tau_1', \Sigma_1 \rangle_J = \frac{i}{\hbar} \langle \tau_2', \Sigma_2 | \delta S_J | \tau_1', \Sigma_1 \rangle_J. \tag{88}$$

The classical field $\varphi^I(x)$ has been replaced by the operator $\phi^I(x)$. Moreover, in addition to its previous meaning, δ now includes a possible change in the source. By considering the variation to be with respect to the dynamical variables which are held fixed on Σ_1 and Σ_2, the operator field equations are obtained as

$$\frac{\delta S[\phi]}{\delta \phi^I(x)} + J_I(x) = 0. \tag{89}$$

Assume that the variation in (88) is one in which the dynamical variables are held fixed and only the source is altered. Since the source enters in the simple way given in (87),

$$\delta \langle \tau_2', \Sigma_2 | \tau_1', \Sigma_1 \rangle = \frac{i}{\hbar} \int \delta J_I(x) \langle \tau_2', \Sigma_2 | \phi^I(x) | \tau_1', \Sigma_1 \rangle_J \, d^n x. \tag{90}$$

The result in (90) may be rewritten in the equivariant form,

$$\frac{\delta \langle \tau_2', \Sigma_2 | \tau_1', \Sigma_1 \rangle}{\delta J_i(x)} = \frac{i}{\hbar} \langle \tau_2', \Sigma_2 | \phi^i(x) | \tau_1', \Sigma_1 \rangle [J]. \tag{91}$$

To simplify this, it can be written in the alternate form,

$$\frac{\delta \langle 2|1 \rangle}{\delta J_i} = \frac{i}{\hbar} \langle 2 | \phi^i | 1 \rangle [J]. \tag{92}$$

An abbreviated notation for the initial and final state has been introduced. This result can be varied with respect to the source which gives

$$\delta \frac{\delta \langle 2|1\rangle [J]}{\delta J_i} = \frac{i}{\hbar} \delta \langle 2|\phi^i|1\rangle [J]. \tag{93}$$

To evaluate (93), a spatial hypersurface Σ' is introduced which resides to the future of Σ_1 and the past of Σ_2 and contains the spacetime point on which the $\phi^i(x)$ depend. Any source variation can be represented as the sum of a variation which vanishes to the future of Σ', but is non-zero to the past, and one which vanishes to the past of Σ' but is nonzero to the future. Consider the case where δJ_i vanishes to the future. In this event, any amplitude of the form $\langle 2|\phi^i|\tau'\rangle [J]$, where $|\tau'\rangle$ represents a state on Σ', can not be affected by the variation of the source since δJ_i will vanish to the future of Σ'. By using the completeness relation

$$\langle 2|\phi^i|1\rangle [J] = \sum_{\tau'} \langle 2|\phi^i|\tau'\rangle \langle \tau'|1\rangle [J],$$

it follows that the right-hand side of (93) may be reexpressed with the use of

$$\delta \langle 2|\phi^i|1\rangle [J] = \sum_{\tau'} \langle 2|\phi^i|\tau'\rangle \delta \langle \tau'|1\rangle [J]. \tag{94}$$

The Schwinger action principle then implies that

$$\delta \langle \tau'|1\rangle [J] = \frac{i}{\hbar} \delta J_k \langle \tau'|\phi^k|1\rangle [J]. \tag{95}$$

Substituting (95) into (94) leads to the conclusion that

$$\delta \langle 2|\phi^i|1\rangle [J] = \frac{i}{\hbar} \delta J_j \sum_{\tau'} \langle 2|\phi^i|\tau'\rangle \langle \tau'|\phi^j|1\rangle [J] = \frac{i}{\hbar} \delta J_j \, \langle 2|\phi^i \phi^j|1\rangle [J]. \tag{96}$$

Since it can be said that δJ_j vanishes to the future of Σ', which contains the spacetime point of ϕ^i, the spacetime point of ϕ^j must lie to the past of the former.

Consider the case in which δJ_j vanishes to the past of Σ'. A similar argument yields the same conclusion as (96), but with ϕ^j to the left of ϕ^i. Combining this set of results produces the following conclusion

$$\frac{\delta \langle 2|\phi^i|1\rangle}{\delta J_j} = \frac{i}{\hbar} \langle 2|T(\phi^i \phi^j)|1\rangle [J]. \tag{97}$$

In (97), T is the chronological, or time, ordering operator, which orders any product of fields in the sequence of increasing time, with those furthest to the past to the very right.

Differentiating (91) and using the result (97), we get

$$\frac{\delta^2 \langle 2|1\rangle[J]}{\delta J_i \delta J_j} = (\frac{i}{\hbar})\langle 2|T(\phi^i \phi^j)|1\rangle[J]. \tag{98}$$

This can be generalized, and omitting details,

$$\frac{\delta^n \langle 2|1\rangle[J]}{\delta J_{i_1} \cdots \delta J_{i_n}} = (\frac{i}{\hbar})^n \langle 2|T(\phi^{i_1} \cdots \phi^{i_n})|1\rangle[J]. \tag{99}$$

The amplitude $\langle 2|1\rangle[J]$ may be defined by a Taylor expansion about $J_i = 0$, so using the previous result

$$\langle 2|1\rangle[J] = \sum_{n=0}^{\infty} \frac{1}{n!}(\frac{i}{\hbar})^n J_{i_1} \cdots J_{i_n} \langle 2|T(\phi^{i_1} \cdots \phi^{i_n})|1\rangle[J=0]. \tag{100}$$

The series may be formally summed to yield

$$\langle 2|1\rangle[J] = \langle 2|T(\exp(\frac{i}{\hbar} J_i \phi^i))|1\rangle[J=0] \tag{101}$$

and J_i is set to zero everywhere on the right-hand side except in the exponential.

The action $S[\phi]$ can be expanded in a Taylor series about $\phi^i = 0$, indicating differentiation with a comma,

$$S[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!} S_{,i_1 \cdots i_n}[\phi = 0]\phi^{i_1} \cdots \phi^{i_n}. \tag{102}$$

Similarly, the derivative of S with respect to the field has the expansion [7],

$$S_{,i}[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!} S_{,ii_1 \cdots i_n}[\phi = 0]\phi^{i_1} \cdots \phi^{i_n}. \tag{103}$$

If ϕ^i is replaced by $\frac{\hbar}{i}\frac{\delta}{\delta J_i}$ in this expression and then operate on $\langle 2|1\rangle[J]$ with $S_{,i}[\frac{\hbar}{i}\frac{\delta}{\delta J_i}]$, and use (102), the following differential equation arises

$$S_{,i}[\frac{\hbar}{i}\frac{\delta}{\delta J_i}]\langle 2|1\rangle[J] = \langle 2|T(S_{,i}[\phi]\exp(\frac{i}{\hbar} J_i \phi^i))|1\rangle[J=0]. \tag{104}$$

The operator equation of motion (89) implies

$$S_{,i}[\frac{\hbar}{i}\frac{\delta}{\delta J_i}]\langle 2|1\rangle[J] = -J_i \langle 2|1\rangle[J]. \tag{105}$$

This results in a differential equation for the transition amplitude. In order to solve equation (105), the functional analogue of a Fourier transform may be used

$$\langle 2|1\rangle = \int (\prod_I d\varphi^I(x)) F[\varphi] \exp(\frac{i}{\hbar} \int J_I(x')\varphi^I(x')\, d^n x'). \tag{106}$$

The integration in (106) extends over all fields which correspond to the choice of states described by $|1\rangle$ and $|2\rangle$. The functional $F[\varphi]$ which is to be thought of as the Fourier transform of the transformation function, is to be determined by requiring (106) satisfy (105)

$$0 = \int (\prod_{I,x} d\varphi^I(x))\{S_{,i}[\varphi] + J_i\} F[\varphi] \exp(\frac{i}{\hbar} \int J_I(x')\varphi^I(x')\, d^n x')$$

$$= \int (\prod_I d\varphi^I(x))\{S_{,i}[\varphi]F[\varphi] + \frac{\hbar}{i} F[\varphi] \frac{\delta}{\delta\varphi^i}\} \exp(\frac{i}{\hbar} \int J_I(x')\varphi^I(x')\, d^n x').$$

Upon carrying out an integration by parts on the second term here

$$0 = \int (\prod_i d\varphi^i)\{S_{,i}[\varphi]F[\varphi] - \frac{\hbar}{i} F_{,i}[\varphi]\} \exp(\frac{i}{\hbar} \int J_I(x')\varphi^I(x')\, d^n x')$$

$$+ \frac{\hbar}{i} F[\varphi] \exp(\frac{i}{\hbar} \int J_I(x')\varphi^I(x')\, d^n x')|_{\varphi_1}^{\varphi_2}. \tag{107}$$

Assuming the surface term at the end vanishes, it follows from (107) that

$$F[\varphi] = \mathcal{N} \exp(\frac{i}{\hbar} S[\varphi]), \tag{108}$$

where $\mathcal{N}$ is any field-independent constant. The condition for the surface term to vanish is that the action $S[\varphi]$ be the same on both surfaces Σ_1 and Σ_2. This condition is usually fulfilled in field theory by assuming that the fields are in the vacuum state on the initial and final hypersurface.

The transformation function can then be summarized as

$$\langle 2|1\rangle[J] = \mathcal{N} \int (\prod_i d\varphi^i) \exp(\frac{i}{\hbar}\{S[\varphi] + J_i\varphi^i\}). \tag{109}$$

This is one form of the Feynman path integral, or functional integral, which represents the transformation function. This technique turns out to be very effective with further modifications applied to the quantization of gauge theories. These theories have been particularly successful in understanding the weak and strong interactions [10].

6. QED - A physical example and summary

A given elementary interaction implies a system of coupled field equations. Thus for the photon and the charged spin 1/2 particle as in quantum electrodynamics [5],

$$[\gamma(-i\partial - eqA(x)) + m]\psi(x) = \eta^A(x),$$

$$-\partial^2 A^\mu(x) + \partial^\mu \partial A(x) = J^\mu(x) + \frac{1}{2}\psi(x)\gamma^0\gamma^\mu eq\psi(x) - \int dx' f^\mu(x-x')\psi(x')\gamma^0 ieq\eta^A(x'). \tag{110}$$

Since this is a nonlinear system, the construction of the fields in terms of the sources will be given by doubly infinite power series. The succesive terms of this series $W_{n\nu}$ with n paricles and ν photon sources represent increasingly complicated physical processes. One of the simplest terms in the interaction skeleton will be discussed below to the point of obtaining experimental consequences.

There are two asymmetrical ways to eliminate the fields. First, introduce the formal solution of the particle field equation

$$\psi^A(x) = \int dx' G_+^A(x,x')\eta^A(x),$$

and $G_+^A(x,x')$ is the Green function

$$[\gamma(-i\partial - eqA(x)) + m]G_+^A(x,x') = \delta(x-x').$$

This gives the partial action

$$W = \int dx[J^\mu A_\mu - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}] + \frac{1}{2}\int dxdx'\, \eta^A(x)\gamma^0 G_+^A(x,x')\eta^A(x').$$

The stationarity requirement on variations of A_μ recovers the Maxwell equation above. If we eliminate the vector potential

$$A_\mu^f(x) = \int dx' D_+(x-x')[J^\mu(x) + j^\mu_{cons}] + \partial_\mu\lambda(x).$$

$$j^\mu_{cons}(x) = j^\mu(x) - \int dx'\, f^\mu(x-x')\partial'_\nu(x'),$$

and the gauge condition determines $\lambda(x)$.

Finally, the first few successive $W_{2\nu}$ are written out, noting each particle source is multiplied by a propagation function $G_+(x,x')$ to form the field ψ,

$$W_{21} = \frac{1}{2}\int d^4x\psi(x)\gamma^0 eq\gamma A(x)\psi(2),$$

$$W_{22} = \frac{1}{2}\int dx dx' \psi(x)\gamma^0 eq\gamma A(x) G_+(x-x') eq\gamma A(x')\psi(x').$$

As a brief introduction to how this formalism can lead to important physical results, let us look at a specific term like W_{21}, the interaction energy of an electron with a static electromagnetic field A_μ^{ext}

$$E = \int d^3x\, j_\mu A^\mu_{ext} = e\int d^3x \bar{\psi}_{p'}(\gamma_\mu + \Gamma^R_\mu(p',p) + \frac{i}{4\pi}\Pi^R_{\mu\nu} iD^{\nu\sigma}\gamma_\sigma)\psi_p A^\mu_{ext}. \tag{111}$$

These terms include the bare electron-photon term, the electron-photon correction terms and then the photon vacuum-polarization correction term, R means a renormalized quantity and γ_μ denote Dirac matrices. The self-energy correction is left out, because for free particles, it contributes only to charge and mass renormalization. The polarization tensor $\Pi_{\mu\nu}(q^2)$ is given by

$$\Pi_{\mu\nu}(q^2) = (q^2 g_{\mu\nu} - q_\mu q_\nu)\Pi(q^2), \tag{112}$$

where $\Pi(q^2)$ is the polarization function. A simple result is obtained in the limit of low momentum transfer, $q^2 \to 0$, which is also of special physical significance and the case of interest here. The renormalized polarization function is

$$\Pi^R(q^2) = -\frac{e^2}{\pi}\frac{q^2}{m^2}(\frac{1}{15} + \frac{1}{140}\frac{q^2}{m^2} + \cdots). \tag{113}$$

The regularized vertex function is

$$\Gamma^R_\mu(p',p) = \gamma_\mu F_1(q^2) + \frac{i}{2m}\sigma_{\mu\nu}q^\nu F_2(q^2). \tag{114}$$

The functions $F_1(q^2)$ and $F_2(q^2)$ are called form factors. The electron gets an apparent internal structure by the interaction with the virtual radiation field which alters it from a pure Dirac particle. In the limit, $q^2 \to 0$, these functions can be calculated to be

$$F_1(q^2) = \frac{\alpha}{3\pi}\frac{q^2}{m^2}(\ln(\frac{m}{\mu}) - \frac{3}{8}), \qquad F_2(q^2) = \frac{\alpha}{2\pi}. \tag{115}$$

Substituting all of these factors and $D_F^{\mu\nu}(q^2) = -4\pi g^{\mu\nu}/q^2$ into (111) yields for small values of q^2,

$$E = e\int d^3x\, \bar{\psi}_{p'}\,\{\gamma_\mu[1 + \frac{\alpha}{3\pi}\frac{q^2}{m^2}(\ln(\frac{m}{\mu}) - \frac{3}{8} - \frac{1}{5}) + \frac{\alpha}{2\pi}\frac{i}{2m}\sigma_{\mu\nu}q^\nu\}\psi_p A^\mu_{ext}. \tag{116}$$

Note μ appears in (115) as an elementary attempt to regularize a photon propagator in one of the terms and does not interfere further with the application at this level and α is the fine

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structure constant. The Gordon decomposition allows this to be written

$$E = e\int d^3x\bar{\psi}_{p'}\{\frac{1}{2m}(p+p')_\mu[1+\frac{\alpha}{3\pi}\frac{q^2}{m^2}(\ln(\frac{m}{\mu})-\frac{3}{8}-\frac{1}{5})]+(1+\frac{\alpha}{2\pi})\frac{i}{2m}\sigma_{\mu\nu}q^\nu\}\psi_p A^\mu_{ext}. \tag{117}$$

The momentum factors can be transformed into gradients in configuration space, thus $q_\mu \to i\partial_\mu$ acts on the photon field and $p'_\mu = -i\overleftarrow{\partial}_\mu$ $p_\mu = i\partial_\mu$ act on the spinor field to the left and right respectively. Then (117) becomes

$$E = e\int d^3x\{\frac{i}{2m}\bar{\psi}_{p'}(x)(\partial_\mu - \overleftarrow{\partial})\psi_p[1-\frac{\alpha}{3\pi}\frac{1}{m^2}(\ln(\frac{m}{\mu})-\frac{3}{8}-\frac{1}{5})]A^\mu_{ext}$$

$$-(1+\frac{\alpha}{2\pi})\frac{1}{2m}\bar{\psi}_p(x)\sigma_{\mu\nu}\psi_p(x)\partial^\nu A^\mu_{ext}\}. \tag{118}$$

The first term contains the convection current of the electron which interacts with the potential. In the special case of a purely magnetic field the second part can be identified as the dipole energy. By introducing the electromagnetic field strength tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ and using the antisymmetry of $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$, the second part is

$$W_{mag} = e(1+\frac{\alpha}{2\pi})\frac{1}{4m}\int d^3x\bar{\psi}(x)\sigma_{\mu\nu}\psi(x)F^{\mu\nu}. \tag{119}$$

When $F^{\mu\nu}$ represents a pure magnetic field, $F^{12} = -B^3$, $\sigma_{12} = \Sigma_3$ with cyclic permutations and the interaction energy becomes

$$W_{mag} = -\frac{e}{4m}(1+\frac{\alpha}{2\pi})2\int d^3x\,\bar{\psi}(x)\vec{\Sigma}\psi(x)\cdot\vec{B} = -\langle\vec{\mu}\rangle\cdot\vec{B}. \tag{120}$$

The magnetic moment is given by

$$\langle\vec{\mu}\rangle = \frac{e\hbar}{2mc}(1+\frac{\alpha}{2\pi})2\langle\vec{S}\rangle = g\mu_B\langle\vec{S}\rangle. \tag{121}$$

The magnetic moment is thus proportional to the spin expectation value of the electron. The proportionality factor is the so called g-factor

$$g = 2(1+\frac{\alpha}{2\pi}) = 2(1+0.00116141). \tag{122}$$

The first point to note is that the value of the g-factor obtained including quantum mechanics differs from the classical value of 2. The result in (122) was first calculated by Schwinger and it has been measured to remarkable accuracy many times. A modern experimental value for the g-factor is

$$g_{exp} = 2(1+0.001159652193), \tag{123}$$

and only the last digit is uncertain.

Of course Schwinger's calculation has been carried out to much further accuracy and is the subject of continuing work. At around order α^4, further corrections must be included, such as such effects as virtual hadron creation. The pure-QED contributions are represented by coefficients C_i as a power series in powers of α/π, which acts as a natural expansion parameter for the calculation

$$g_{theo} = 2[1 + C_1(\frac{\alpha}{\pi}) + C_2(\frac{\alpha}{\pi})^2 + C_3(\frac{\alpha}{\pi})^3 + \cdots]. \tag{124}$$

Not all of the assumptions made in classical physics apply in quantum physics. In particular, the assumption that it is possible, at least in principle to perform a measurement on a given system in a way in which the interaction between the measured and measuring device can be made as small as desired. In the absence of concepts which follow from observation, principles such as the action principle discussed here are extremely important in providing a direction in which to proceed to formulate a picture of reality which is valid at the microscopic level, given that many assumptions at the normal level of perception no longer apply. These ideas such as the action principle touched on here have led to wide ranging conclusions about the quantum world and resulted in a way to produce useful tools for calculation and results such as transition amplitudes, interaction energies and the result concerning the g factor given here.

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