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The Improvement of the Heisenberg Uncertainty Principle

L. M. Arévalo Aguilar, C. P. García Quijas and
Carlos Robledo-Sanchez

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1. Introduction

One of the fundamental cornerstone of quantum mechanics is the Heisenberg uncertainty principle. This principle is so fundamental to quantum theory that it is believed that if a single phenomenon that could violate it is found then the whole building of quantum mechanics will fall apart. However, since the formulation of the uncertainty principle until today there is not clear and universal agreement in its formulation or interpretation. Even Heisenberg was not clear about the exact meaning of p_1 and x_1 in their first formulation of the uncertainty relations [1]:

$$p_1 q_1 \sim h, \tag{1}$$

nor in the interpretation of the uncertainty principle. According to Heisenberg, in Eq. (1) q_1 represents "*the precision with which the value of q is know (q_1 is, say, the mean error of q), therefore here the wavelength of light. Let p_1 be the precision with which the value of p is determinable; that is, here, the discontinuous change of p in the Compton effect* [1]". He also thought the uncertainty principle in terms of disturbance produced on an observable when it is measured its canonical counterpart.

The relevance of the uncertainty principle to Physics is that it introduced for the first time the indeterminacy in a physical theory, which mean the end of the era of *certainty* in Physics. That is to say, what uncertainty principle made evident was the peculiar characteristic of quantum theory of not being able to predict with certainty a property of a physical system; in words of Heisenberg: "*...canonically conjugate quantities can be determined simultaneously only with a characteristic indeterminacy. This indeterminacy is the real basis for the occurrence of statistical relations in quantum mechanics* [1]".

Since now, you can perceive two different meanings of the Uncertainty Principle in the two quoted paragraphs above. In the first one, the uncertainty comes from a statistical property (according with Heisenberg, the mean error) of quantum theory; in the second meaning the uncertainty is a restriction to simultaneously measure two physical properties.

On the other hand, to elucidate the meaning of the time-energy uncertainty relation [1] $E_1 t_1 \sim h$ is quite difficult, for, contrary to the uncertainty relation given in Eq. (1), it is not possible to deduce it from the postulates of quantum mechanics, i. e. there is not an operator for time. In Heisenberg's paper the meaning of t_1 is the "*time during which the atoms are under the influence of the deflecting field*" and E_1 refers to the accuracy in the energy measurement. Heisenberg concludes that "*a precise determination of energy can only be obtained at the cost of a corresponding uncertainty in the time* [1]".

In this Chapter of the book, we will review the evolution of the Uncertainty Principle since its inception by Heisenberg until their application to measure entanglement. We will review some problems (usually untouched by quantum mechanic's textbooks) that the usual interpretation of the Uncertainty Principle have in terms of standard deviations and its dependence of the wave function. Also, we will review the efforts made to clarify the meaning of the Uncertainty Principle using uncertainty relations.

2. The relation between the Heisenberg Uncertainty Principle and the Uncertainty Relations

The uncertainty principle is one of the fundamental issues in which quantum theory differs from the classical theories, then since its formulation has attracted considerable attention, even from areas normally outside the scientific development. This has lead to create misunderstandings about the content of the principle. Thus, it is important to mention that when we say that there is a lower limit on irreducible uncertainty in the result of a measurement, what we mean is that the uncertainty is not due to experimental errors or to inaccuracies in the laboratory. Instead, the restriction attributed to the uncertainty principle is fundamental and inherent to the theory and is based on theoretical considerations in which it is assumed that all observations are ideal and perfectly accurate.

A reading of the original Heisenber's paper shows that he writes (i. e. believes) in some pharagraps that the indeterminacies comes from the observational procedures. For, in his original paper, Heisenberg stated [1] that the concepts of classical mechanics could be used analogously in quantum mechanics to describe a mechanical system, however, the use of such concepts are affected by an indeterminacy originated *purely* by the observational procedures used to determine *simultaneously* two canonically conjugate variables. This could be contrasted with the called *Statistical Interpretation* where it is tough that the wave function represents and ensemble of identical prepared system and, therefore, the indeterminacy comes form an intrinsic indeterminacy of the physical properties.

Usually, the uncertainty principle is stated in terms of uncertainty relations. One of the first way to obtain this indeterminacy relation is due to Robertson [2]. Here, instead, we use the textbooks approach to deduce the uncertainty relations from the quantum postulates [3, 4]. This approach uses both the Schwarz inequality

$$\langle \phi | \phi \rangle \langle \varphi | \varphi \rangle \geq |\langle \phi | \varphi \rangle|^2, \tag{2}$$

and the following quantum postulates:

- The state of a quantum system is represented by a wave function $\Psi(x, t)$ ($|\Psi\rangle$, in Dirac notation).

- For every observable A there is a self-adjoint operator $\hat{A}$, its expectation value is given by $\langle \hat{A} \rangle = \int \Psi^*(x) \hat{A} \Psi(x) dx = \langle \Psi | \hat{A} | \Psi \rangle$.

Now, consider the following operators defined as [1]:

$$\Delta \hat{A} = \hat{A} - \langle \hat{A} \rangle$$
$$\Delta \hat{B} = \hat{B} - \langle \hat{B} \rangle . \tag{3}$$

Let them operate on an state $|\Psi\rangle$, given:

$$\Delta \hat{A} |\Psi\rangle = |\psi_a\rangle$$
$$\Delta \hat{B} |\Psi\rangle = |\psi_b\rangle . \tag{4}$$

Therefore, using the Schwarz inequality given in the Eq. (2),

$$\langle \psi_a | \psi_a \rangle \langle \psi_b | \psi_b \rangle \geq |\langle \psi_a | \psi_b \rangle|^2 \tag{5}$$

we arrive to:

$$\left\langle \Delta \hat{A}^2 \right\rangle \left\langle \Delta \hat{B}^2 \right\rangle \geq \left| \langle \Delta \hat{A} \Delta \hat{B} \rangle \right|^2 , \tag{6}$$

where $\left\langle \Delta \hat{A}^2 \right\rangle = \left\langle \hat{A}^2 \right\rangle - \left\langle \hat{A} \right\rangle^2 = \delta A^2$ is the variance, the same for the operator $\hat{B}$. From the Eq. (6), it is not difficult to show that:

$$\delta A \delta B \geq \sqrt{\left| \langle [\hat{A}, \hat{B}] \rangle \right|^2 + \left| \langle \{\hat{A}, \hat{B}\} \rangle \right|^2}, \tag{7}$$

where $\{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$, and δA and δB are the standard deviation. It is worth to notice that the association of the standard deviation whit the uncertainty relations was not proposed by Heisenberg, it was Kennard and Robertson [2] who made this association. Although Heisenberg endorse it later. As it was stated above, Heisenberg associates p_1 and q_1 with the mean error, also in the same paper he associates these quantities with the widths of Gaussian functions representing the quantum states of the system.

Some problems arises with the textbooks uncertainty relations: *i)* They are given in terms of the standard deviation, *ii)* They depend on the state of the system. Additionally, *iii)* They

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[1] There are others forms to obtain the uncertainty relations, this begin by defining an operator as $\hat{D} = \Delta \hat{A} + \lambda \Delta \hat{B}$ and, then, requiring that $\langle \hat{D}^\dagger \hat{D} \rangle \geq 0$.

does not represent the meaning of the impossibilities of simultaneous measurement of two observables, *iv)* They does not quantify the role of the disturbance in the state after the measurement process. Finally, *v)* They does not address the concept of complementarity. There have been proposed some criteria to solve this problems, we are going to review this proposals in the next sections.

3. Reformulations to the uncertainty principle

In this section we will review some proposed solutions to the problems stated in the last paragraph of the previous section.

3.1. The dependence on the standard deviation

The principal criticism to the dependence of the uncertainty relation on the standard deviation comes from J. Hilgevoord and J. M. B. Uffink [5, 6]. Their argument is based on two reason, first, they argue that the standard deviation is an appropriate measure of the error of a measurement because errors usually follow a Gaussian distribution, and the standard deviation is an appropriate measurement of the spread of a Gaussian; however, this is not true for a general distribution. Secondly, they gave as a principal counter argument the fact that even for simple phenomenon as the single slit the standard deviation of momentum diverges. Their approach is inside the thinking that the uncertainty relations are the measure of the spread of the probability distribution, i. e. it is believed that Δx and Δp represents the probability distribution of the possibles properties of the system. In short, it represents the spread of values (of $\hat{x}$ or $\hat{p}$) that are intrinsic in the physical system that are available to appear after a measurement.

The principal counter argument with regard to the standard deviations comes from the single-slit experiment. In this case, it is supposed that the state of an income beam of particles is represented by plane waves. This plane wave represents a particle of precise momentum p_0. Then, the plane wave arrives at the single-slit and is diffracted by it. Therefore, the wave function at the screen, according to Hilgevoord and Uffink, is:

$$\psi(x) = \begin{cases} (2a)^{-1/2}, & \text{if } |x| \leq a; \\ 0, & \text{if } |x| < a. \end{cases}$$

and

$$\phi(p) = (a/\pi)^{1/2} \frac{\sin ap}{ap}. \tag{8}$$

Now, the problem with the standard deviation, as defined in quantum mechanics, in this case is that it diverges:

$$\Delta p = \left\langle \hat{p}^2 \right\rangle - \langle \hat{p} \rangle^2 \to \infty. \tag{9}$$

Therefore, these authors defined, instead of the standard deviation, the overall width (W_ψ) and the mean peak width of ψ as the smaller W and w that satisfies the following

equations [7]:

$$\int_{x_0-W/2}^{x_0+W/2} |\psi(x)|^2 \, dx = N$$

$$\left| \int \psi^*(x')\psi(x'-w)dx' \right|^2 = M^2 \tag{10}$$

These quantities, i. e. W and w, provides a better characterization of the spread of the possible values of $\hat{x}$ and $\hat{p}$, in particular there is not any divergence in these numbers. Based in these definitions Hilgevoord and Uffink give the following uncertainty relations, that they propose as a substitute to the uncertainty relation given by Kennard ($\Delta x \Delta p \geq 1/2$), [7]:

$$w_\phi W_\psi \geq \arccos\left(\frac{M+1-N}{N}\right)$$

$$w_\psi W_\phi \geq \arccos\left(\frac{M+1-N}{N}\right) \tag{11}$$

these uncertainty relations works well for the single-slit and double-slit experiments.

3.2. Entropic Uncertainty Relations

In the quantum literature, there are many defined Entropic Uncertainty Relations. Mostly, they are based in terms of Shannon entropy [8, 9], although in last ten years there has been extension to other forms of entropy, like Renyi entropy [10]. In reference [11] there is a recent review of this research area.

One of the important result in this area was the one found by Deutsch [8]. What Deutsch pursuit was a quantitative expression of the Heisenberg uncertainty principle, he notice that the customary generalization has the drawback that the lower limit depends on the quantum state, that is:

$$\Delta A \Delta B \geq \frac{1}{4}\left|\langle[\hat{A},\hat{B}]\rangle\right|^2. \tag{12}$$

Deutsch stress that the right hand side of the Equ. (12) does not has a lower bound but is a function of the state $|\psi\rangle$, even it vanishes for some choices of $|\psi\rangle$. So, in search of a quantity that could represent the uncertainty principle Deutsch propose some elementary properties, like for example that the lower limit must vanishes if the observables have an eigenstate in common. Based in this considerations he proposed the following entropic uncertainty relation:

$$S_{\hat{A}} + S_{\hat{B}} \geq 2Ln\left(\frac{2}{1+sup\{|\langle a|b\rangle|\}}\right), \tag{13}$$

where $S_{\hat{A}} = -\sum_a |\langle a|\psi\rangle|^2 Ln |\langle a|\psi\rangle|^2$ and $S_{\hat{B}} = -\sum_b |\langle b|\psi\rangle|^2 Ln |\langle b|\psi\rangle|^2$ are the Shanon entropy, and $|a\rangle$ and $|b\rangle$ are, respectively, the eigenstates of $\hat{A}$ and $\hat{B}$.

The next step in this line of research, was quite soon given by Hossein Partovi [12], he points out that the above uncertainty relation does not take into account the measurement process. Then, considering that the measuring device realizes a partitioning of the spectrum of the observable and the assignation of their corresponding probabilities, he proposes the following definition of entropy [12]:
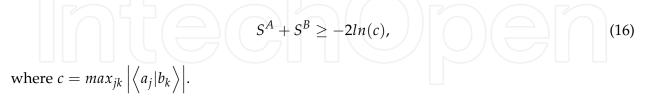
$$S_A = -\sum_i p_i ln\{p_i\}. \tag{14}$$

where $p_i = \langle \psi | \hat{\pi}_i^A | \psi \rangle / \langle \psi | \psi \rangle$ and $\hat{\pi}_i^A$ is the projection onto the subspaces spanned by the states corresponding to the partition induced by the measuring apparatus [12]. In this case, p_i gives the probability of obtaining the outcome of a measurement in a subset of the partition realized by the measuring apparatus. In this approach, the whole spectrum correspond to the observable $\hat{A}$ but its partitioning correspond to the measuring device. Using these considerations Hossein Patrovi proses the following lower bound for the uncertainty relation:

$$S_{\hat{A}} + S_{\hat{B}} \geq 2Ln\left(\frac{2}{1 + sup_{ij}\{||\hat{\pi}_i^A + \hat{\pi}_j^B||\}}\right). \tag{15}$$

In the special case where the partition realized by the measuring device includes only one point of the spectrum of $\hat{A}$, i. e. $\hat{\pi}_i^A = |a_i\rangle \langle a_i|$ and $\hat{B}$, i. e. $\hat{\pi}_j^B = |b_j\rangle \langle b_j|$, then Equ. (15) reduces to Equ. (13). Finally, it is worth to mention that the Patrovi's formulation requires a formulation of the details of the measuring devices, specifically, the kind of partition that induces (or could be used) in the spectrum of the observable.

There were two additional improvement on the lower bound of the entropic uncertainty relations defined above. The first one was due to Bialynicki-Birula who presented, based in his earlier wok [9], a lower bound for the angle-angular momentum pair [13] $S^\phi + S^{L_z} \geq -ln(\Delta\phi/2\pi)$ and an improved lower bound for the position-momentum pair $S^x + S^p \geq 1 - ln(2) - ln(\gamma)$, where $\gamma = \Delta x \Delta p / h$. The second one was proposed by Maasen and Uffink [14] who demonstrated, based on a previous work of Kraus [15], that

$$S^A + S^B \geq -2ln(c), \tag{16}$$

where $c = max_{jk} \left| \langle a_j | b_k \rangle \right|$.

3.3. Simultaneous measurement

Whereas in the previous two subsection we treated the face of the Uncertainty Principle that is related with the probability distribution of observables of a given wave function, in this subsection we talk a bout a second version of The Uncertainty Principle. This version is related with the fact that it is not possible to determine simultaneously, with precision, two canonically conjugate observable and usually called *joint measurement*. This is stated, generally, as: "*It is impossible to measure simultaneously two observables like, for example, position*

and momentum." So, this sub-research area is concerned with the simultaneous measurement of two observables.

One of the first work in this approach was that of Arthurs and Kelly [16], they analyze this problems as follows: First, they realize that as the problem is the measurement of two observables, then it is required two devices to perform the measurement. That is, the system is coupled to two devises. Then, they consider that as the two meter position commutes then it is possible to perform two simultaneous measurements of them. Therefore, the simultaneous measurement of the two meters constitutes a simultaneous measurement of two non-commuting observables of the system. As the two meters interacts with the quantum system, they consider the following Hamiltonian:

$$\hat{H}_{int} = K \left(\hat{q}\hat{P}_x + \hat{p}\hat{P}_y \right) \tag{17}$$

where $\hat{q}$ and $\hat{p}$ correspond to the position and momentum of the quantum system, respectively, and $\hat{P}_x$ and $\hat{P}_y$ are the momentum of the two independent meters. Using two Gaussian function as the initial wave function of the meters they arrive at the following uncertainty relation for the simultaneous measurement of two observables:

$$\sigma_x \sigma_p \geq 1. \tag{18}$$

Therefore, the uncertainty relation of the simultaneous measurement of $\hat{q}$ and $\hat{p}$ is greater (by a factor of two) than the uncertainty relations based on the probability distribution of the two observables, the topic of the previous two sub-sections.

The next step in this approach was given by Arthurs and Goodman [17]. In this case, the approach is as follow: To perform a measurement, the system observables, $\hat{C} = \hat{C}_1 \otimes \hat{I}_2$ and $\hat{D} = \hat{D}_1 \otimes \hat{I}_2$, must be coupled to a measuring apparatus which is represented by the operators $\hat{R} = \hat{I}_1 \otimes \hat{R}_2$ and $\hat{C} = \hat{C}_1 \otimes \hat{I}_2$. Then, if we consider that there is access only to the meter operators then there must exist an uncertainty relations for these operators that puts a limit to the available information. Based in this consideration, they prove what they call a generalized uncertainty relation. To prove it they defined a a noise operator by

$$\hat{N}_R = \hat{R} - G_R \hat{C}(0),$$
$$\hat{N}_S = \hat{S} - G_S \hat{D}(0) \tag{19}$$

where $\hat{C}(0)$ and $\hat{D}(0)$ are the system observables and $\hat{R}$ and $\hat{S}$ are the tracking apparatus observables, the latter obey the commutation rule $[\hat{R}, \hat{S}] = 0$. Also, it is required that the correlation between the system observables and the meter has, on average, a perfect match, that is:

$$Tr \left(\hat{\rho}\hat{N}_{R,S} \right) = \left\langle \hat{R} \right\rangle - G_R \left\langle \hat{C}(0) \right\rangle = 0. \tag{20}$$

Using the previous condition, i. e. Equ (20), it is possible to show that the noise operator is uncorrelated with all system operators like $\hat{C}$ and $\hat{D}$. Using all the previous properties of

the system, meter and noise operators they arrive to the following generalized Heisenberg uncertainty relation:

$$\sigma_\xi \sigma_\eta \geq \left| Tr \left(\hat\rho \left[\hat{C}, \hat{D} \right] \right) \right|, \tag{21}$$

where $\hat\rho$ is the state of the system, σ_ξ and σ_η are, respectively the standard deviation of the normalized operators $\xi = \hat{R}/G_R$ and $\eta = \hat{R}/G_R$. This uncertainty relation is four times the corresponding uncertainty relation for $\hat{C}$ and $\hat{D}$. Notice that in the left hand side of the Eq. (21) there is information of the meter operator whereas in the right hand side there is information of the system operators and that we have access only to the meter system. In reference [18] there was published an experimental verification of this uncertainty relation.

3.4. Disturbance due to measurement

The disturbance produced on an observable due to the measurement of another observable is, perhaps, the face of the uncertainty principled most talked about but the least studied. This comes from the fact that in quantum mechanics any measurement introduces an unforeseeable disturbance in the measured quantum system. It was only recently that there have been some research and understanding of this effect.

Originally, the idea that the measuring process disturb observables comes from Heisenberg's analysis of the observation of an electron by means of a microscope. This kind of uncertainty principle is written down, to use recent terminology, as [19]:

$$\epsilon(x)\eta(p) \geq \frac{1}{2} \left| \langle \psi | [\hat{x}, \hat{p}] | \psi \rangle \right|, \tag{22}$$

where $\epsilon(x)$ is the noise in the measurement in position and $\eta(p)$ is the disturbance caused by the apparatus [19]. Using a general description of measurement Ozawa demonstrated that the uncertainty relation for disturbance and noise given by the Eq. (22) does not accurately represent the disturbance process. He has show that this kind of uncertainty relation includes additional terms not present in Eq. (22). In the measurement process, the quantum system interacts with a measuring device. He considers that this devices measures observable A *precisely* if its experimental probabilty distribution coincides with the theoretical probability distribution of the observable. In the measurement process, when the interaction have been turned off, the device is subject to a measurement of an observable M. Then, $\hat{A}^{in} = \hat{A} \otimes \hat{I}$ is the input observable, $\hat{A}^{out} = \hat{U}^\dagger \left(\hat{A} \otimes \hat{I} \right) \hat{U}$ is the observable after the mesaurement, $\hat{M}^{in} = \hat{I} \otimes \hat{M}$ is the device observable when the interaction begin, $\hat{M}^{out} = \hat{U}^\dagger (\hat{I} \otimes \hat{M}) \hat{U}$ and $\hat{U}$ is the unitary time evolution operator

To show that the original uncertainty relation need additional terms, he introduces the following noise $N(\hat{A})$ and disturbance $D(\hat{B})$ operators:

$$\begin{aligned} N(\hat{A}) &= \hat{M}^{out} - \hat{A}^{in}, \\ D(\hat{B}) &= \hat{B}^{out} - \hat{B}^{in}. \end{aligned} \tag{23}$$

Using this operators, and considering that $\left[\hat{M}^{out}, \hat{B}^{out}\right] = 0$, Ozawa was able to show the following uncertainty relation [19]:

$$\epsilon(A)\eta(B) + \frac{1}{2}\left|\left\langle\left[N(\hat{A}), \hat{B}^{in}\right]\right\rangle + \left\langle\left[\hat{A}^{in}, D(\hat{B})\right]\right\rangle\right| \geq \frac{1}{2}\left|\langle\psi|[\hat{A}, \hat{B}]|\psi\rangle\right|. \tag{24}$$

where the noise $\epsilon(A)$ was defined by Ozawa as the root-mean-square deviation of the experimental variable $\hat{M}^{out}$ from the theoretical variable $\hat{A}^{in}$:

$$\epsilon(A) = \left\langle\left(\hat{M}^{out} - \hat{A}^{in}\right)^{1/2}\right\rangle \tag{25}$$

and the disturbance $\eta(B)$ on observable $\hat{B}$ is the change in the observable caused by the measurement process:

$$\eta(B) = \left\langle\left(\hat{B}^{out} - \hat{B}^{in}\right)^{1/2}\right\rangle. \tag{26}$$

This uncertainty relation has been recently experimentally tested, see reference [20]

4. Entanglement determination using entropic uncertainty relations

Nowadays entanglement is considered as an useful resource to make non-clasical task. As a resource it is convenient to have adequate measures to quantify how much entanglement are in a given entangled state. However, until recently the most known proposed measures have the unwanted fact of being difficult to apply in experimental settings. Therefore, it was necessary to find out new ways of entanglement determination that enable that the amount of entanglement in a quantum state could be experimentally tested.

Recently there has been much research to proposed new entanglement determination based, mostly, in uncertainty relations. In this case, the entropic uncertainty relations helps to realize this task. Recently, Berta et. al. [21] have proposed a new uncertainty relation (based on that proposed in references [8, 14]) to test the entanglement:

$$S(R|B) + S(S|B)log2c + S(A|B) \tag{27}$$

to propose this equation Berta et. al. consider that the system, with observables S and R, is entangled with a memori, with observable B, so in equation $S(R, B)$ is the von Neumann entropy and gives the uncertainty about the measurement of R given information stored in a quantum memory, B. The term S(A | B) quantifies the amount of entanglement between the particle and the memory. This relation was experimentally tested in reference [22].

5. Conclusions

In this chapter we review some of the most important improvements of the Heisenberg uncertainty relation. Although there are advances in their understanding and formulation, it remains yet as an open research area, specially in the quantification of entanglement.

Acknowledgements

Author details

L. M. Arévalo Aguilar[1,*],
C. P. García Quijas[2] and Carlos Robledo-Sanchez[3]

* Address all correspondence to: olareva@yahoo.com.mx

1 Facultad de Ciencias Físico Matemáticas, Benemérita Universidad Autónoma de Puebla, Puebla, México
2 Departamento de Fśica, Universidad de Guadalajara, Guadalajara, Jalisco, México
3 Facultad de Ciencias Fisico Matemáticas, Benemérita Universidad Autónoma de Puebla, Puebla, México

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