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Quantum Damped Harmonic Oscillator

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Additional information is available at the end of the chapter

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1. Introduction

In this chapter we introduce a toy model of Quantum Mechanics with Dissipation. Quantum Mechanics with Dissipation plays a crucial role to understand real world. However, it is not easy to master the theory for undergraduates. The target of this chapter is eager undergraduates in the world. Therefore, a good toy model to understand it deeply is required.

The quantum damped harmonic oscillator is just such a one because undergraduates must use (master) many fundamental techniques in Quantum Mechanics and Mathematics. That is, harmonic oscillator, density operator, Lindblad form, coherent state, squeezed state, tensor product, Lie algebra, representation theory, Baker–Campbell–Hausdorff formula, etc.

They are “jewels” in Quantum Mechanics and Mathematics. If undergraduates master this model, they will get a powerful weapon for Quantum Physics. I expect some of them will attack many hard problems of Quantum Mechanics with Dissipation.

The contents of this chapter are based on our two papers [3] and [6]. I will give a clear and fruitful explanation to them as much as I can.

2. Some preliminaries

In this section let us make some reviews from Physics and Mathematics within our necessity.

2.1. From physics

First we review the solution of classical damped harmonic oscillator, which is important to understand the text. For this topic see any textbook of Mathematical Physics.

The differential equation is given by

$$\ddot{x} + \omega^2 x = -\gamma \dot{x} \iff \ddot{x} + \gamma \dot{x} + \omega^2 x = 0 \quad (\gamma > 0) \quad (2.1)$$

where $x = x(t)$, $\dot{x} = dx/dt$ and the mass m is set to 1 for simplicity. In the following we treat only the case $\omega > \gamma/2$ (the case $\omega = \gamma/2$ may be interesting).

Solutions (with complex form) are well-known to be

$$x_{\pm}(t) = e^{-\left(\frac{\gamma}{2} \pm i\sqrt{\omega^2 - \left(\frac{\gamma}{2}\right)^2}\right)t},$$

so the general solution is given by

$$\begin{aligned} x(t) &= \alpha x_+(t) + \bar{\alpha} x_-(t) = \alpha e^{-\left(\frac{\gamma}{2} + i\sqrt{\omega^2 - \left(\frac{\gamma}{2}\right)^2}\right)t} + \bar{\alpha} e^{-\left(\frac{\gamma}{2} - i\sqrt{\omega^2 - \left(\frac{\gamma}{2}\right)^2}\right)t} \\ &= \alpha e^{-\left(\frac{\gamma}{2} + i\omega\sqrt{1 - \left(\frac{\gamma}{2\omega}\right)^2}\right)t} + \bar{\alpha} e^{-\left(\frac{\gamma}{2} - i\omega\sqrt{1 - \left(\frac{\gamma}{2\omega}\right)^2}\right)t} \end{aligned} \quad (2.2)$$

where α is a complex number. If $\gamma/2\omega$ is small enough we have an approximate solution

$$x(t) \approx \alpha e^{-\left(\frac{\gamma}{2} + i\omega\right)t} + \bar{\alpha} e^{-\left(\frac{\gamma}{2} - i\omega\right)t}. \quad (2.3)$$

Next, we consider the quantum harmonic oscillator. This is well-known in textbooks of Quantum Mechanics. As standard textbooks of Quantum Mechanics see [2] and [11] ([2] is particularly interesting).

For the Hamiltonian

$$H = H(q, p) = \frac{1}{2}(p^2 + \omega^2 q^2) \quad (2.4)$$

where $q = q(t)$, $p = p(t)$, the canonical equation of motion reads

$$\dot{q} \equiv \frac{\partial H}{\partial p} = p, \quad \dot{p} \equiv -\frac{\partial H}{\partial q} = -\omega^2 q.$$

From these we recover the equation

$$\ddot{q} = -\omega^2 q \iff \ddot{q} + \omega^2 q = 0.$$

See (2.1) with $q = x$ and $\lambda = 0$.

Next, we introduce the Poisson bracket. For $A = A(q, p)$, $B = B(q, p)$ it is defined as

$$\{A, B\}_c \equiv \frac{\partial A}{\partial q} \frac{\partial B}{\partial p} - \frac{\partial A}{\partial p} \frac{\partial B}{\partial q} \quad (2.5)$$

where $\{, \}_c$ means classical. Then it is easy to see

$$\{q, q\}_c = 0, \quad \{p, p\}_c = 0, \quad \{q, p\}_c = 1. \quad (2.6)$$

Now, we are in a position to give a quantization condition due to Dirac. Before that we prepare some notation from algebra.

Square matrices A and B don't commute in general, so we need the commutator

$$[A, B] = AB - BA.$$

Then Dirac gives an abstract correspondence $q \rightarrow \hat{q}$, $p \rightarrow \hat{p}$ which satisfies the condition

$$[\hat{q}, \hat{q}] = 0, \quad [\hat{p}, \hat{p}] = 0, \quad [\hat{q}, \hat{p}] = i\hbar \mathbf{1} \quad (2.7)$$

corresponding to (2.6). Here \hbar is the Plank constant, and \hat{q} and \hat{p} are both Hermite operators on some Fock space (a kind of Hilbert space) given in the latter and $\mathbf{1}$ is the identity on it. Therefore, our quantum Hamiltonian should be

$$H = H(\hat{q}, \hat{p}) = \frac{1}{2}(\hat{p}^2 + \omega^2 \hat{q}^2) \quad (2.8)$$

from (2.4). Note that a notation H instead of \hat{H} is used for simplicity. From now on we consider a complex version. From (2.4) and (2.8) we rewrite like

$$H(q, p) = \frac{1}{2}(p^2 + \omega^2 q^2) = \frac{\omega^2}{2}(q^2 + \frac{1}{\omega^2} p^2) = \frac{\omega^2}{2}(q - \frac{i}{\omega} p)(q + \frac{i}{\omega} p)$$

and

$$\begin{aligned} H(\hat{q}, \hat{p}) &= \frac{\omega^2}{2}(\hat{q}^2 + \frac{1}{\omega^2} \hat{p}^2) = \frac{\omega^2}{2} \left\{ (\hat{q} - \frac{i}{\omega} \hat{p})(\hat{q} + \frac{i}{\omega} \hat{p}) - \frac{i}{\omega} [\hat{q}, \hat{p}] \right\} \\ &= \frac{\omega^2}{2} \left\{ (\hat{q} - \frac{i}{\omega} \hat{p})(\hat{q} + \frac{i}{\omega} \hat{p}) + \frac{\hbar}{\omega} \right\} = \omega \hbar \left\{ \frac{\omega}{2\hbar} (\hat{q} - \frac{i}{\omega} \hat{p})(\hat{q} + \frac{i}{\omega} \hat{p}) + \frac{1}{2} \right\} \end{aligned}$$

by use of (2.7), and if we set

$$a^\dagger = \sqrt{\frac{\omega}{2\hbar}}(\hat{q} - \frac{i}{\omega} \hat{p}), \quad a = \sqrt{\frac{\omega}{2\hbar}}(\hat{q} + \frac{i}{\omega} \hat{p}) \quad (2.9)$$

we have easily

$$[a, a^\dagger] = \frac{\omega}{2\hbar} [\hat{q} + \frac{i}{\omega} \hat{p}, \hat{q} - \frac{i}{\omega} \hat{p}] = \frac{\omega}{2\hbar} \left\{ -\frac{2i}{\omega} [\hat{q}, \hat{p}] \right\} = \frac{\omega}{2\hbar} \left\{ -\frac{2i}{\omega} \times i\hbar \right\} = \mathbf{1}$$

by use of (2.7). As a result we obtain a well-known form

$$H = \omega \hbar (a^\dagger a + \frac{1}{2}), \quad [a, a^\dagger] = \mathbf{1}. \quad (2.10)$$

Here we used an abbreviation $1/2$ in place of $(1/2)\mathbf{1}$ for simplicity.

If we define an operator $N = a^\dagger a$ (which is called the number operator) then it is easy to see the relations

$$[N, a^\dagger] = a^\dagger, \quad [N, a] = -a, \quad [a, a^\dagger] = \mathbf{1}. \quad (2.11)$$

For the proof a well-known formula $[AB, C] = [A, C]B + A[B, C]$ is used. Note that $aa^\dagger = a^\dagger a + [a, a^\dagger] = N + 1$. The set $\{a^\dagger, a, N\}$ is just a generator of Heisenberg algebra and we can construct a Fock space based on this. Let us note that a, a^\dagger and N are called the annihilation operator, creation one and number one respectively.

First of all let us define a vacuum $|0\rangle$. This is defined by the equation $a|0\rangle = 0$. Based on this vacuum we construct the n state $|n\rangle$ like

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle \quad (0 \leq n).$$

Then we can easily prove

$$a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle, \quad a |n\rangle = \sqrt{n} |n-1\rangle, \quad N |n\rangle = n |n\rangle \quad (2.12)$$

and moreover can prove both the orthogonality condition and the resolution of unity

$$\langle m | n \rangle = \delta_{mn}, \quad \sum_{n=0}^{\infty} |n\rangle \langle n| = \mathbf{1}. \quad (2.13)$$

For the proof one can use for example

$$a^2 (a^\dagger)^2 = a(aa^\dagger)a^\dagger = a(N+1)a^\dagger = (N+2)aa^\dagger = (N+2)(N+1)$$

by (2.11), therefore we have

$$\langle 0 | a^2 (a^\dagger)^2 | 0 \rangle = \langle 0 | (N+2)(N+1) | 0 \rangle = 2! \implies \langle 2 | 2 \rangle = 1.$$

The proof of the resolution of unity may be not easy for readers (we omit it here).

As a result we can define a Fock space generated by the generator $\{a^\dagger, a, N\}$

$$\mathcal{F} = \left\{ \sum_{n=0}^{\infty} c_n |n\rangle \in \mathbf{C}^\infty \mid \sum_{n=0}^{\infty} |c_n|^2 < \infty \right\}. \quad (2.14)$$

This is just a kind of Hilbert space. On this space the operators (= infinite dimensional matrices) a^\dagger, a and N are represented as

$$a = \begin{pmatrix} 0 & 1 & & & \\ & 0 & \sqrt{2} & & \\ & & 0 & \sqrt{3} & \\ & & & 0 & \ddots \\ & & & & \ddots \end{pmatrix}, \quad a^\dagger = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & \sqrt{2} & 0 & & \\ & & \sqrt{3} & 0 & \\ & & & \ddots & \ddots \end{pmatrix},$$

$$N = a^\dagger a = \begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & 2 & & \\ & & & 3 & \\ & & & & \ddots \end{pmatrix} \quad (2.15)$$

by use of (2.12).

Note We can add a phase to $\{a, a^\dagger\}$ like

$$b = e^{i\theta} a, \quad b^\dagger = e^{-i\theta} a^\dagger, \quad N = b^\dagger b = a^\dagger a$$

where θ is constant. Then we have another Heisenberg algebra

$$[N, b^\dagger] = b^\dagger, \quad [N, b] = -b, \quad [b, b^\dagger] = 1.$$

Next, we introduce a coherent state which plays a central role in Quantum Optics or Quantum Computation. For $z \in \mathbb{C}$ the coherent state $|z\rangle \in \mathcal{F}$ is defined by the equation

$$a|z\rangle = z|z\rangle \quad \text{and} \quad \langle z|z\rangle = 1.$$

The annihilation operator a is not hermitian, so this equation is never trivial. For this state the following three equations are equivalent :

$$\begin{cases} (1) & a|z\rangle = z|z\rangle \text{ and } \langle z|z\rangle = 1, \\ (2) & |z\rangle = e^{za^\dagger - \bar{z}a} |0\rangle, \\ (3) & |z\rangle = e^{-\frac{|z|^2}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle. \end{cases} \quad (2.16)$$

The proof is as follows. From (1) to (2) we use a popular formula

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!} [A, [A, B]] + \dots$$

(A, B : operators) to prove

$$e^{-(za^\dagger - \bar{z}a)} a e^{za^\dagger - \bar{z}a} = a + z.$$

From (2) to (3) we use the Baker-Campbell-Hausdorff formula (see for example [17])

$$e^A e^B = e^{A+B + \frac{1}{2}[A,B] + \frac{1}{6}[A,[A,B]] + \frac{1}{6}[B,[A,B]] + \dots}.$$

If $[A, [A, B]] = 0 = [B, [A, B]]$ (namely, $[A, B]$ commutes with both A and B) then we have

$$e^A e^B = e^{A+B + \frac{1}{2}[A,B]} = e^{\frac{1}{2}[A,B]} e^{A+B} \implies e^{A+B} = e^{-\frac{1}{2}[A,B]} e^A e^B. \quad (2.17)$$

In our case the condition is satisfied because of $[a, a^\dagger] = 1$. Therefore we obtain a (famous) decomposition

$$e^{za^\dagger - \bar{z}a} = e^{-\frac{|z|^2}{2}} e^{za^\dagger} e^{-\bar{z}a}. \quad (2.18)$$

The remaining part of the proof is left to readers.

From the equation (3) in (2.16) we obtain the resolution of unity for coherent states

$$\int \int \frac{dx dy}{\pi} |z\rangle \langle z| = \sum_{n=0}^{\infty} |n\rangle \langle n| = \mathbf{1} \quad (z = x + iy). \quad (2.19)$$

The proof is reduced to the following formula

$$\int \int \frac{dx dy}{\pi} e^{-|z|^2} \bar{z}^m z^n = n! \delta_{mn} \quad (z = x + iy).$$

The proof is left to readers. See [14] for more general knowledge of coherent states.

2.2. From mathematics

We consider a simple matrix equation

$$\frac{d}{dt} X = AXB \quad (2.20)$$

where

$$X = X(t) = \begin{pmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

A standard form of linear differential equation which we usually treat is

$$\frac{d}{dt}\mathbf{x} = C\mathbf{x}$$

where $\mathbf{x} = \mathbf{x}(t)$ is a vector and C is a matrix associated to the vector. Therefore, we want to rewrite (2.20) into a standard form.

For the purpose we introduce the Kronecker product of matrices. For example, it is defined as

$$\begin{aligned} A \otimes B &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \otimes B \equiv \begin{pmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{pmatrix} \\ &= \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix} \end{aligned} \quad (2.21)$$

for A and B above. Note that recently we use the tensor product instead of the Kronecker product, so we use it in the following. Here, let us list some useful properties of the tensor product

$$\begin{aligned} (1) \quad & (A_1 \otimes B_1)(A_2 \otimes B_2) = A_1 A_2 \otimes B_1 B_2, \\ (2) \quad & (A \otimes E)(E \otimes B) = A \otimes B = (E \otimes B)(A \otimes E), \\ (3) \quad & e^{A \otimes E + E \otimes B} = e^{A \otimes E} e^{E \otimes B} = (e^A \otimes E)(E \otimes e^B) = e^A \otimes e^B, \\ (4) \quad & (A \otimes B)^\dagger = A^\dagger \otimes B^\dagger \end{aligned} \quad (2.22)$$

where E is the unit matrix. The proof is left to readers. [9] is recommended as a general introduction.

Then the equation (2.20) can be written in terms of components as

$$\begin{cases} \frac{dx_{11}}{dt} = a_{11}b_{11}x_{11} + a_{11}b_{21}x_{12} + a_{12}b_{11}x_{21} + a_{12}b_{21}x_{22}, \\ \frac{dx_{12}}{dt} = a_{11}b_{12}x_{11} + a_{11}b_{22}x_{12} + a_{12}b_{12}x_{21} + a_{12}b_{22}x_{22}, \\ \frac{dx_{21}}{dt} = a_{21}b_{11}x_{11} + a_{21}b_{21}x_{12} + a_{22}b_{11}x_{21} + a_{22}b_{21}x_{22}, \\ \frac{dx_{22}}{dt} = a_{21}b_{12}x_{11} + a_{21}b_{22}x_{12} + a_{22}b_{12}x_{21} + a_{22}b_{22}x_{22} \end{cases}$$

or in a matrix form

$$\frac{d}{dt} \begin{pmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{21} & a_{12}b_{11} & a_{12}b_{21} \\ a_{11}b_{12} & a_{11}b_{22} & a_{12}b_{12} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{21} & a_{22}b_{11} & a_{22}b_{21} \\ a_{21}b_{12} & a_{21}b_{22} & a_{22}b_{12} & a_{22}b_{22} \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \end{pmatrix}.$$

If we set

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \implies \hat{X} = (x_{11}, x_{12}, x_{21}, x_{22})^T$$

where T is the transpose, then we obtain a standard form

$$\frac{d}{dt}X = AXB \implies \frac{d}{dt}\hat{X} = (A \otimes B^T)\hat{X} \quad (2.23)$$

from (2.21).

Similarly we have a standard form

$$\frac{d}{dt}X = AX + XB \implies \frac{d}{dt}\hat{X} = (A \otimes E + E \otimes B^T)\hat{X} \quad (2.24)$$

where $E^T = E$ for the unit matrix E .

From these lessons there is no problem to generalize (2.23) and (2.24) based on 2×2 matrices to ones based on any (square) matrices or operators on \mathcal{F} . Namely, we have

$$\begin{cases} \frac{d}{dt}X = AXB \implies \frac{d}{dt}\hat{X} = (A \otimes B^T)\hat{X}, \\ \frac{d}{dt}X = AX + XB \implies \frac{d}{dt}\hat{X} = (A \otimes I + I \otimes B^T)\hat{X}. \end{cases} \quad (2.25)$$

where I is the identity E (matrices) or 1 (operators).

3. Quantum damped harmonic oscillator

In this section we treat the quantum damped harmonic oscillator. As a general introduction to this topic see [1] or [16].

3.1. Model

Before that we introduce the quantum harmonic oscillator. The Schrödinger equation is given by

$$i\hbar \frac{\partial}{\partial t} |\Psi(t)\rangle = H |\Psi(t)\rangle = \left(\omega \hbar \left(N + \frac{1}{2} \right) \right) |\Psi(t)\rangle$$

by (2.10) (note $N = a^\dagger a$). In the following we use $\frac{\partial}{\partial t}$ instead of $\frac{d}{dt}$.

Now we change from a wave-function to a density operator because we want to treat a mixed state, which is a well-known technique in Quantum Mechanics or Quantum Optics.

If we set $\rho(t) = |\Psi(t)\rangle \langle \Psi(t)|$, then a little algebra gives

$$i\hbar \frac{\partial}{\partial t} \rho = [H, \rho] = [\omega \hbar N, \rho] \implies \frac{\partial}{\partial t} \rho = -i[\omega N, \rho]. \quad (3.1)$$

This is called the quantum Liouville equation. With this form we can treat a mixed state like

$$\rho = \rho(t) = \sum_{j=1}^N u_j |\Psi_j(t)\rangle \langle \Psi_j(t)|$$

where $u_j \geq 0$ and $\sum_{j=1}^N u_j = 1$. Note that the general solution of (3.1) is given by

$$\rho(t) = e^{-i\omega N t} \rho(0) e^{i\omega N t}.$$

We are in a position to state the equation of quantum damped harmonic oscillator by use of (3.1).

Definition The equation is given by

$$\frac{\partial}{\partial t} \rho = -i[\omega a^\dagger a, \rho] - \frac{\mu}{2} (a^\dagger a \rho + \rho a^\dagger a - 2a \rho a^\dagger) - \frac{\nu}{2} (a a^\dagger \rho + \rho a a^\dagger - 2a^\dagger \rho a) \quad (3.2)$$

where μ, ν ($\mu > \nu \geq 0$) are some real constants depending on the system (for example, a damping rate of the cavity mode)¹.

Note that the extra term

$$-\frac{\mu}{2} (a^\dagger a \rho + \rho a^\dagger a - 2a \rho a^\dagger) - \frac{\nu}{2} (a a^\dagger \rho + \rho a a^\dagger - 2a^\dagger \rho a)$$

is called the Lindblad form (term). Such a term plays an essential role in **Decoherence**.

3.2. Method of solution

First we solve the Lindblad equation :

$$\frac{\partial}{\partial t} \rho = -\frac{\mu}{2} (a^\dagger a \rho + \rho a^\dagger a - 2a \rho a^\dagger) - \frac{\nu}{2} (a a^\dagger \rho + \rho a a^\dagger - 2a^\dagger \rho a). \quad (3.3)$$

Interesting enough, we can solve this equation completely.

¹ The aim of this chapter is not to drive this equation. In fact, its derivation is not easy for non-experts, so see for example the original papers [15] and [10], or [12] as a short review paper

Let us rewrite (3.3) more conveniently using the number operator $N \equiv a^\dagger a$

$$\frac{\partial}{\partial t} \rho = \mu a \rho a^\dagger + \nu a^\dagger \rho a - \frac{\mu + \nu}{2} (N \rho + \rho N + \rho) + \frac{\mu - \nu}{2} \rho \quad (3.4)$$

where we have used $aa^\dagger = N + 1$.

From here we use the method developed in Section 2.2. For a matrix $X = (x_{ij}) \in M(\mathcal{F})$ over \mathcal{F}

$$X = \begin{pmatrix} x_{00} & x_{01} & x_{02} & \cdots \\ x_{10} & x_{11} & x_{12} & \cdots \\ x_{20} & x_{21} & x_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

we correspond to the vector $\hat{X} \in \mathcal{F}^{\dim_c \mathcal{F}}$ as

$$X = (x_{ij}) \longrightarrow \hat{X} = (x_{00}, x_{01}, x_{02}, \dots; x_{10}, x_{11}, x_{12}, \dots; x_{20}, x_{21}, x_{22}, \dots; \dots)^T \quad (3.5)$$

where T means the transpose. The following formulas

$$\widehat{AXB} = (A \otimes B^T) \hat{X}, \quad (\widehat{AX + XB}) = (A \otimes \mathbf{1} + \mathbf{1} \otimes B^T) \hat{X} \quad (3.6)$$

hold for $A, B, X \in M(\mathcal{F})$, see (2.25).

Then (3.4) becomes

$$\begin{aligned} \frac{\partial}{\partial t} \hat{\rho} &= \left\{ \mu a \otimes (a^\dagger)^T + \nu a^\dagger \otimes a^T - \frac{\mu + \nu}{2} (N \otimes \mathbf{1} + \mathbf{1} \otimes N + \mathbf{1} \otimes \mathbf{1}) + \frac{\mu - \nu}{2} \mathbf{1} \otimes \mathbf{1} \right\} \hat{\rho} \\ &= \left\{ \frac{\mu - \nu}{2} \mathbf{1} \otimes \mathbf{1} + \nu a^\dagger \otimes a^\dagger + \mu a \otimes a - \frac{\mu + \nu}{2} (N \otimes \mathbf{1} + \mathbf{1} \otimes N + \mathbf{1} \otimes \mathbf{1}) \right\} \hat{\rho} \end{aligned} \quad (3.7)$$

where we have used $a^T = a^\dagger$ from the form (2.15), so that the solution is formally given by

$$\hat{\rho}(t) = e^{\frac{\mu - \nu}{2} t} e^{t \{ \nu a^\dagger \otimes a^\dagger + \mu a \otimes a - \frac{\mu + \nu}{2} (N \otimes \mathbf{1} + \mathbf{1} \otimes N + \mathbf{1} \otimes \mathbf{1}) \}} \hat{\rho}(0). \quad (3.8)$$

In order to use some techniques from Lie algebra we set

$$K_3 = \frac{1}{2} (N \otimes \mathbf{1} + \mathbf{1} \otimes N + \mathbf{1} \otimes \mathbf{1}), \quad K_+ = a^\dagger \otimes a^\dagger, \quad K_- = a \otimes a \quad (K_- = K_+^\dagger) \quad (3.9)$$

then we can show the relations

$$[K_3, K_+] = K_+, \quad [K_3, K_-] = -K_-, \quad [K_+, K_-] = -2K_3.$$

This is just the $su(1,1)$ algebra. The proof is very easy and is left to readers.

The equation (3.8) can be written simply as

$$\hat{\rho}(t) = e^{\frac{\mu-\nu}{2}t} e^{t\{\nu K_+ + \mu K_- - (\mu+\nu)K_3\}} \hat{\rho}(0), \quad (3.10)$$

so we have only to calculate the term

$$e^{t\{\nu K_+ + \mu K_- - (\mu+\nu)K_3\}}, \quad (3.11)$$

which is of course not simple. Now the disentangling formula in [4] is helpful in calculating (3.11).

If we set $\{k_+, k_-, k_3\}$ as

$$k_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad k_- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad k_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (k_- \neq k_+^\dagger) \quad (3.12)$$

then it is very easy to check the relations

$$[k_3, k_+] = k_+, \quad [k_3, k_-] = -k_-, \quad [k_+, k_-] = -2k_3.$$

That is, $\{k_+, k_-, k_3\}$ are generators of the Lie algebra $su(1,1)$. Let us show by $SU(1,1)$ the corresponding Lie group, which is a typical noncompact group.

Since $SU(1,1)$ is contained in the special linear group $SL(2; \mathbb{C})$, we **assume** that there exists an infinite dimensional unitary representation $\rho : SL(2; \mathbb{C}) \rightarrow U(\mathcal{F} \otimes \mathcal{F})$ (group homomorphism) satisfying

$$d\rho(k_+) = K_+, \quad d\rho(k_-) = K_-, \quad d\rho(k_3) = K_3.$$

From (3.11) some algebra gives

$$\begin{aligned} e^{t\{\nu K_+ + \mu K_- - (\mu+\nu)K_3\}} &= e^{t\{\nu d\rho(k_+) + \mu d\rho(k_-) - (\mu+\nu)d\rho(k_3)\}} \\ &= e^{d\rho(t(\nu k_+ + \mu k_- - (\mu+\nu)k_3))} \\ &= \rho \left(e^{t(\nu k_+ + \mu k_- - (\mu+\nu)k_3)} \right) \quad (\Downarrow \text{ by definition}) \\ &\equiv \rho \left(e^{tA} \right) \end{aligned} \quad (3.13)$$

and we have

$$\begin{aligned}
 e^{tA} &= e^{t\{\nu k_+ + \mu k_- - (\mu + \nu)k_3\}} \\
 &= \exp \left\{ t \begin{pmatrix} -\frac{\mu + \nu}{2} & \nu \\ -\mu & \frac{\mu + \nu}{2} \end{pmatrix} \right\} \\
 &= \begin{pmatrix} \cosh\left(\frac{\mu - \nu}{2}t\right) - \frac{\mu + \nu}{\mu - \nu} \sinh\left(\frac{\mu - \nu}{2}t\right) & \frac{2\nu}{\mu - \nu} \sinh\left(\frac{\mu - \nu}{2}t\right) \\ -\frac{2\mu}{\mu - \nu} \sinh\left(\frac{\mu - \nu}{2}t\right) & \cosh\left(\frac{\mu - \nu}{2}t\right) + \frac{\mu + \nu}{\mu - \nu} \sinh\left(\frac{\mu - \nu}{2}t\right) \end{pmatrix}.
 \end{aligned}$$

The proof is based on the following two facts.

$$(tA)^2 = t^2 \begin{pmatrix} -\frac{\mu + \nu}{2} & \nu \\ -\mu & \frac{\mu + \nu}{2} \end{pmatrix}^2 = t^2 \begin{pmatrix} \left(\frac{\mu + \nu}{2}\right)^2 - \mu\nu & 0 \\ 0 & \left(\frac{\mu + \nu}{2}\right)^2 - \mu\nu \end{pmatrix} = \left(\frac{\mu - \nu}{2}t\right)^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$e^X = \sum_{n=0}^{\infty} \frac{1}{n!} X^n = \sum_{n=0}^{\infty} \frac{1}{(2n)!} X^{2n} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} X^{2n+1} \quad (X = tA).$$

Note that

$$\cosh(x) = \frac{e^x + e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \quad \text{and} \quad \sinh(x) = \frac{e^x - e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}.$$

The remainder is left to readers.

The Gauss decomposition formula (in $SL(2; \mathbb{C})$)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & \frac{b}{d} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{d} & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{c}{d} & 1 \end{pmatrix} \quad (ad - bc = 1)$$

gives the decomposition

$$e^{tA} = \begin{pmatrix} 1 & \frac{2\nu}{\mu-\nu} \sinh\left(\frac{\mu-\nu}{2}t\right) \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} \frac{1}{\cosh\left(\frac{\mu-\nu}{2}t\right) + \frac{\mu+\nu}{\mu-\nu} \sinh\left(\frac{\mu-\nu}{2}t\right)} & 0 \\ 0 & \cosh\left(\frac{\mu-\nu}{2}t\right) + \frac{\mu+\nu}{\mu-\nu} \sinh\left(\frac{\mu-\nu}{2}t\right) \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ -\frac{2\mu}{\mu-\nu} \sinh\left(\frac{\mu-\nu}{2}t\right) & 1 \end{pmatrix}$$

and moreover we have

$$\begin{aligned} e^{tA} &= \exp \begin{pmatrix} 0 & \frac{2\nu}{\mu-\nu} \sinh\left(\frac{\mu-\nu}{2}t\right) \\ 0 & 0 \end{pmatrix} \times \exp \begin{pmatrix} -\log\left(\cosh\left(\frac{\mu-\nu}{2}t\right) + \frac{\mu+\nu}{\mu-\nu} \sinh\left(\frac{\mu-\nu}{2}t\right)\right) & 0 \\ 0 & \log\left(\cosh\left(\frac{\mu-\nu}{2}t\right) + \frac{\mu+\nu}{\mu-\nu} \sinh\left(\frac{\mu-\nu}{2}t\right)\right) \end{pmatrix} \times \\ &\exp \begin{pmatrix} 0 & 0 \\ -\frac{2\mu}{\mu-\nu} \sinh\left(\frac{\mu-\nu}{2}t\right) & 0 \end{pmatrix} \\ &= \exp \left(\frac{\frac{2\nu}{\mu-\nu} \sinh\left(\frac{\mu-\nu}{2}t\right)}{\cosh\left(\frac{\mu-\nu}{2}t\right) + \frac{\mu+\nu}{\mu-\nu} \sinh\left(\frac{\mu-\nu}{2}t\right)} k_+ \right) \times \\ &\exp \left(-2 \log \left(\cosh\left(\frac{\mu-\nu}{2}t\right) + \frac{\mu+\nu}{\mu-\nu} \sinh\left(\frac{\mu-\nu}{2}t\right) \right) k_3 \right) \times \\ &\exp \left(\frac{\frac{2\mu}{\mu-\nu} \sinh\left(\frac{\mu-\nu}{2}t\right)}{\cosh\left(\frac{\mu-\nu}{2}t\right) + \frac{\mu+\nu}{\mu-\nu} \sinh\left(\frac{\mu-\nu}{2}t\right)} k_- \right). \end{aligned}$$

Since ρ is a group homomorphism ($\rho(XYZ) = \rho(X)\rho(Y)\rho(Z)$) and the formula $\rho(e^{Lk}) = e^{Ld\rho(k)}$ ($k = k_+, k_3, k_-$) holds we obtain

$$\rho(e^{tA}) \exp \left(\frac{\frac{2\nu}{\mu-\nu} \sinh \left(\frac{\mu-\nu}{2} t \right)}{\cosh \left(\frac{\mu-\nu}{2} t \right) + \frac{\mu+\nu}{\mu-\nu} \sinh \left(\frac{\mu-\nu}{2} t \right)} d\rho(k_+) \right) \times$$

$$\exp \left(-2 \log \left(\cosh \left(\frac{\mu-\nu}{2} t \right) + \frac{\mu+\nu}{\mu-\nu} \sinh \left(\frac{\mu-\nu}{2} t \right) \right) d\rho(k_3) \right) \times$$

$$\exp \left(\frac{\frac{2\mu}{\mu-\nu} \sinh \left(\frac{\mu-\nu}{2} t \right)}{\cosh \left(\frac{\mu-\nu}{2} t \right) + \frac{\mu+\nu}{\mu-\nu} \sinh \left(\frac{\mu-\nu}{2} t \right)} d\rho(k_-) \right).$$

As a result we have the disentangling formula

$$e^{t\{\nu K_+ + \mu K_- - (\mu+\nu)K_3\}} = \exp \left(\frac{\frac{2\nu}{\mu-\nu} \sinh \left(\frac{\mu-\nu}{2} t \right)}{\cosh \left(\frac{\mu-\nu}{2} t \right) + \frac{\mu+\nu}{\mu-\nu} \sinh \left(\frac{\mu-\nu}{2} t \right)} K_+ \right) \times$$

$$\exp \left(-2 \log \left(\cosh \left(\frac{\mu-\nu}{2} t \right) + \frac{\mu+\nu}{\mu-\nu} \sinh \left(\frac{\mu-\nu}{2} t \right) \right) K_3 \right) \times$$

$$\exp \left(\frac{\frac{2\mu}{\mu-\nu} \sinh \left(\frac{\mu-\nu}{2} t \right)}{\cosh \left(\frac{\mu-\nu}{2} t \right) + \frac{\mu+\nu}{\mu-\nu} \sinh \left(\frac{\mu-\nu}{2} t \right)} K_- \right) \quad (3.14)$$

by (3.13).

In the following we set for simplicity

$$E(t) = \frac{\frac{2\mu}{\mu-\nu} \sinh \left(\frac{\mu-\nu}{2} t \right)}{\cosh \left(\frac{\mu-\nu}{2} t \right) + \frac{\mu+\nu}{\mu-\nu} \sinh \left(\frac{\mu-\nu}{2} t \right)},$$

$$F(t) = \cosh \left(\frac{\mu-\nu}{2} t \right) + \frac{\mu+\nu}{\mu-\nu} \sinh \left(\frac{\mu-\nu}{2} t \right), \quad (3.15)$$

$$G(t) = \frac{\frac{2\nu}{\mu-\nu} \sinh \left(\frac{\mu-\nu}{2} t \right)}{\cosh \left(\frac{\mu-\nu}{2} t \right) + \frac{\mu+\nu}{\mu-\nu} \sinh \left(\frac{\mu-\nu}{2} t \right)}.$$

Readers should be careful of this “proof”, which is a heuristic method. In fact, it is incomplete because we have assumed a group homomorphism. In order to complete it we want to show a disentangling formula like

$$e^{t\{\nu K_+ + \mu K_- - (\mu+\nu)K_3\}} = e^{f(t)K_+} e^{g(t)K_3} e^{h(t)K_-}$$

with unknowns $f(t)$, $g(t)$, $h(t)$ satisfying $f(0) = g(0) = h(0) = 0$. For the purpose we set

$$A(t) = e^{t\{\nu K_+ + \mu K_- - (\mu + \nu)K_3\}}, \quad B(t) = e^{f(t)K_+} e^{g(t)K_3} e^{h(t)K_-}.$$

For $t = 0$ we have $A(0) = B(0) = \text{identity}$ and

$$\dot{A}(t) = \{\nu K_+ + \mu K_- - (\mu + \nu)K_3\}A(t).$$

Next, let us calculate $\dot{B}(t)$. By use of the Leibniz rule

$$\begin{aligned} \dot{B}(t) &= (\dot{f}K_+)e^{f(t)K_+}e^{g(t)K_3}e^{h(t)K_-} + e^{f(t)K_+}(\dot{g}K_3)e^{g(t)K_3}e^{h(t)K_-} + e^{f(t)K_+}e^{g(t)K_3}(\dot{h}K_-)e^{h(t)K_-} \\ &= \left\{ \dot{f}K_+ + \dot{g}e^{fK_+}K_3e^{-fK_+} + \dot{h}e^{fK_+}e^{gK_3}K_-e^{-gK_3}e^{-fK_+} \right\} e^{f(t)K_+}e^{g(t)K_3}e^{h(t)K_-} \\ &= \left\{ \dot{f}K_+ + \dot{g}(K_3 - fK_+) + \dot{h}e^{-g}(K_- - 2fK_3 + f^2K_+) \right\} B(t) \\ &= \left\{ (\dot{f} - \dot{g}f + \dot{h}e^{-g}f^2)K_+ + (\dot{g} - 2\dot{h}e^{-g}f)K_3 + \dot{h}e^{-g}K_- \right\} B(t) \end{aligned}$$

where we have used relations

$$\begin{aligned} e^{fK_+}K_3e^{-fK_+} &= K_3 - fK_+, \\ e^{gK_3}K_-e^{-gK_3} &= e^{-g}K_- \quad \text{and} \quad e^{fK_+}K_-e^{-fK_+} = K_- - 2fK_3 + f^2K_+. \end{aligned}$$

The proof is easy. By comparing coefficients of $\dot{A}(t)$ and $\dot{B}(t)$ we have

$$\begin{cases} \dot{h}e^{-g} = \mu, \\ \dot{g} - 2\dot{h}e^{-g}f = -(\mu + \nu), \\ \dot{f} - \dot{g}f + \dot{h}e^{-g}f^2 = \nu \end{cases} \implies \begin{cases} \dot{h}e^{-g} = \mu, \\ \dot{g} - 2\mu f = -(\mu + \nu), \\ \dot{f} - \dot{g}f + \mu f^2 = \nu \end{cases} \implies \begin{cases} \dot{h}e^{-g} = \mu, \\ \dot{g} - 2\mu f = -(\mu + \nu), \\ \dot{f} + (\mu + \nu)f - \mu f^2 = \nu. \end{cases}$$

Note that the equation

$$\dot{f} + (\mu + \nu)f - \mu f^2 = \nu$$

is a (famous) Riccati equation. If we can solve the equation then we obtain solutions like

$$f \implies g \implies h.$$

Unfortunately, it is not easy. However there is an ansatz for the solution, G , F and E . That is,

$$f(t) = G(t), \quad g(t) = -2\log(F(t)), \quad h(t) = E(t)$$

in (3.15). To check these equations is left to readers (as a good exercise). From this

$$A(0) = B(0), \quad \dot{A}(0) = \dot{B}(0) \implies A(t) = B(t) \quad \text{for all } t$$

and we finally obtain the disentangling formula

$$e^{t\{\nu K_+ + \mu K_- - (\mu + \nu)K_3\}} = e^{G(t)K_+} e^{-2\log(F(t))K_3} e^{E(t)K_-} \quad (3.16)$$

with (3.15).

Therefore (3.8) becomes

$$\hat{\rho}(t) = e^{\frac{\mu-\nu}{2}t} \exp\left(G(t)a^\dagger \otimes a^\dagger\right) \exp\left(-\log(F(t))(N \otimes \mathbf{1} + \mathbf{1} \otimes N + \mathbf{1} \otimes \mathbf{1})\right) \exp(E(t)a \otimes a) \hat{\rho}(0)$$

with (3.15). Some calculation by use of (2.22) gives

$$\begin{aligned} \hat{\rho}(t) &= \frac{e^{\frac{\mu-\nu}{2}t}}{F(t)} \exp\left(G(t)a^\dagger \otimes a^\dagger\right) \left\{ \exp(-\log(F(t))N) \otimes \exp(-\log(F(t))N)^T \right\} \times \\ &\quad \exp\left(E(t)a \otimes (a^\dagger)^T\right) \hat{\rho}(0) \end{aligned} \quad (3.17)$$

where we have used $N^T = N$ and $a^\dagger = a^T$. By coming back to matrix form by use of (3.6) like

$$\begin{aligned} \exp\left(E(t)a \otimes (a^\dagger)^T\right) \hat{\rho}(0) &= \sum_{m=0}^{\infty} \frac{E(t)^m}{m!} \left(a \otimes (a^\dagger)^T\right)^m \hat{\rho}(0) \\ &= \sum_{m=0}^{\infty} \frac{E(t)^m}{m!} \left(a^m \otimes ((a^\dagger)^m)^T\right) \hat{\rho}(0) \longrightarrow \sum_{m=0}^{\infty} \frac{E(t)^m}{m!} a^m \rho(0) (a^\dagger)^m \end{aligned}$$

we finally obtain

$$\begin{aligned} \rho(t) &= \frac{e^{\frac{\mu-\nu}{2}t}}{F(t)} \times \\ &\sum_{n=0}^{\infty} \frac{G(t)^n}{n!} (a^\dagger)^n \left[\exp(-\log(F(t))N) \left\{ \sum_{m=0}^{\infty} \frac{E(t)^m}{m!} a^m \rho(0) (a^\dagger)^m \right\} \exp(-\log(F(t))N) \right] a^n. \end{aligned} \quad (3.18)$$

This form is very beautiful but complicated !

3.3. General solution

Last, we treat the full equation (3.2)

$$\frac{\partial}{\partial t} \rho = -i\omega(a^\dagger a \rho - \rho a^\dagger a) - \frac{\mu}{2} (a^\dagger a \rho + \rho a^\dagger a - 2a \rho a^\dagger) - \frac{\nu}{2} (a a^\dagger \rho + \rho a a^\dagger - 2a^\dagger \rho a).$$

From the lesson in the preceding subsection it is easy to rewrite this as

$$\frac{\partial}{\partial t} \hat{\rho} = \left\{ -i\omega K_0 + \nu K_+ + \mu K_- - (\mu + \nu) K_3 + \frac{\mu - \nu}{2} \mathbf{1} \otimes \mathbf{1} \right\} \hat{\rho} \quad (3.19)$$

in terms of $K_0 = N \otimes \mathbf{1} - \mathbf{1} \otimes N$ (note that $N^T = N$). Then it is easy to see

$$[K_0, K_+] = [K_0, K_3] = [K_0, K_-] = 0 \quad (3.20)$$

from (3.9), which is left to readers. That is, K_0 commutes with all $\{K_+, K_3, K_-\}$. Therefore

$$\begin{aligned} \hat{\rho}(t) &= e^{-i\omega t K_0} e^{t\{\nu K_+ + \mu K_- - (\mu + \nu) K_3 + \frac{\mu - \nu}{2} \mathbf{1} \otimes \mathbf{1}\}} \hat{\rho}(0) \\ &= e^{\frac{\mu - \nu}{2} t} \exp(-i\omega t K_0) \exp(G(t) K_+) \exp(-2 \log(F(t)) K_3) \exp(E(t) K_-) \hat{\rho}(0) \\ &= e^{\frac{\mu - \nu}{2} t} \exp(G(t) K_+) \exp(\{-i\omega t K_0 - 2 \log(F(t)) K_3\}) \exp(E(t) K_-) \hat{\rho}(0), \end{aligned}$$

so that the general solution that we are looking for is just given by

$$\begin{aligned} \rho(t) &= \frac{e^{\frac{\mu - \nu}{2} t}}{F(t)} \sum_{n=0}^{\infty} \frac{G(t)^n}{n!} (a^\dagger)^n [\exp(\{-i\omega t - \log(F(t))\} N) \times \\ &\quad \left\{ \sum_{m=0}^{\infty} \frac{E(t)^m}{m!} a^m \rho(0) (a^\dagger)^m \right\} \exp(\{i\omega t - \log(F(t))\} N)] a^n \end{aligned} \quad (3.21)$$

by use of (3.17) and (3.18).

Particularly, if $\nu = 0$ then

$$\begin{aligned} E(t) &= \frac{2 \sinh\left(\frac{\mu}{2} t\right)}{\cosh\left(\frac{\mu}{2} t\right) + \sinh\left(\frac{\mu}{2} t\right)} = 1 - e^{-\mu t}, \\ F(t) &= \cosh\left(\frac{\mu}{2} t\right) + \sinh\left(\frac{\mu}{2} t\right) = e^{\frac{\mu}{2} t}, \\ G(t) &= 0 \end{aligned}$$

from (3.15), so that we have

$$\rho(t) = e^{-\left(\frac{\mu}{2} + i\omega\right)tN} \left\{ \sum_{m=0}^{\infty} \frac{(1 - e^{-\mu t})^m}{m!} a^m \rho(0) (a^\dagger)^m \right\} e^{-\left(\frac{\mu}{2} - i\omega\right)tN} \quad (3.22)$$

from (3.21).

4. Quantum counterpart

In this section we explicitly calculate $\rho(t)$ for the initial value $\rho(0)$ given in the following.

4.1. Case of $\rho(0) = |0\rangle\langle 0|$

Noting $a|0\rangle = 0$ ($\Leftrightarrow 0 = \langle 0|a^\dagger$), this case is very easy and we have

$$\rho(t) = \frac{e^{\frac{\mu-\nu}{2}t}}{F(t)} \sum_{n=0}^{\infty} \frac{G(t)^n}{n!} (a^\dagger)^n |0\rangle\langle 0| a^n = \frac{e^{\frac{\mu-\nu}{2}t}}{F(t)} \sum_{n=0}^{\infty} G(t)^n |n\rangle\langle n| = \frac{e^{\frac{\mu-\nu}{2}t}}{F(t)} e^{\log G(t)N} \quad (4.1)$$

because the number operator $N (= a^\dagger a)$ is written as

$$N = \sum_{n=0}^{\infty} n |n\rangle\langle n| \implies N|n\rangle = n|n\rangle$$

, see for example (2.15). To check the last equality of (4.1) is left to readers. Moreover, $\rho(t)$ can be written as

$$\rho(t) = (1 - G(t)) e^{\log G(t)N} = e^{\log(1-G(t))} e^{\log G(t)N}, \quad (4.2)$$

see the next subsection.

4.2. Case of $\rho(0) = |\alpha\rangle\langle\alpha|$ ($\alpha \in \mathbf{C}$)

Remind that $|\alpha\rangle$ is a coherent state given by (2.16) $(a|\alpha\rangle = \alpha|\alpha\rangle \Leftrightarrow \langle\alpha|a^\dagger = \langle\alpha|\bar{\alpha})$. First of all let us write down the result :

$$\rho(t) = e^{|\alpha|^2 e^{-(\mu-\nu)t} \log G(t) + \log(1-G(t))} \exp \left\{ -\log G(t) \left(\alpha e^{-\left(\frac{\mu-\nu}{2} + i\omega\right)t} a^\dagger + \bar{\alpha} e^{-\left(\frac{\mu-\nu}{2} - i\omega\right)t} a - N \right) \right\} \quad (4.3)$$

with $G(t)$ in (3.15). Here we again meet a term like (2.3)

$$\alpha e^{-\left(\frac{\mu-\nu}{2} + i\omega\right)t} a^\dagger + \bar{\alpha} e^{-\left(\frac{\mu-\nu}{2} - i\omega\right)t} a$$

with $\lambda = \frac{\mu-\nu}{2}$.

Therefore, (4.3) is just our quantum counterpart of the classical damped harmonic oscillator.

The proof is divided into four parts.

[First Step] From (3.21) it is easy to see

$$\sum_{m=0}^{\infty} \frac{E(t)^m}{m!} a^m |\alpha\rangle\langle\alpha| (a^\dagger)^m = \sum_{m=0}^{\infty} \frac{E(t)^m}{m!} a^m |\alpha\rangle\langle\alpha| \bar{\alpha}^m = \sum_{m=0}^{\infty} \frac{(E(t)|\alpha|^2)^m}{m!} |\alpha\rangle\langle\alpha| = e^{E(t)|\alpha|^2} |\alpha\rangle\langle\alpha|.$$

[Second Step] From (3.21) we must calculate the term

$$e^{\gamma N} |\alpha\rangle \langle \alpha| e^{\bar{\gamma} N} = e^{\gamma N} e^{\alpha a^\dagger - \bar{\alpha} a} |0\rangle \langle 0| e^{-(\alpha a^\dagger - \bar{\alpha} a)} e^{\bar{\gamma} N}$$

where $\gamma = -i\omega t - \log(F(t))$ (note $\bar{\gamma} \neq -\gamma$). It is easy to see

$$e^{\gamma N} e^{\alpha a^\dagger - \bar{\alpha} a} |0\rangle = e^{\gamma N} e^{\alpha a^\dagger - \bar{\alpha} a} e^{-\gamma N} e^{\gamma N} |0\rangle = e^{e^{\gamma N} (\alpha a^\dagger - \bar{\alpha} a) e^{-\gamma N}} |0\rangle = e^{\alpha e^{\gamma} a^\dagger - \bar{\alpha} e^{-\gamma} a} |0\rangle$$

where we have used

$$e^{\gamma N} a^\dagger e^{-\gamma N} = e^{\gamma} a^\dagger \quad \text{and} \quad e^{\gamma N} a e^{-\gamma N} = e^{-\gamma} a.$$

The proof is easy and left to readers. Therefore, by use of Baker–Campbell–Hausdorff formula (2.17) two times

$$\begin{aligned} e^{\alpha e^{\gamma} a^\dagger - \bar{\alpha} e^{-\gamma} a} |0\rangle &= e^{-\frac{|\alpha|^2}{2}} e^{\alpha e^{\gamma} a^\dagger} e^{-\bar{\alpha} e^{-\gamma} a} |0\rangle = e^{-\frac{|\alpha|^2}{2}} e^{\alpha e^{\gamma} a^\dagger} |0\rangle \\ &= e^{-\frac{|\alpha|^2}{2}} e^{\frac{|\alpha|^2}{2} e^{\gamma + \bar{\gamma}}} e^{\alpha e^{\gamma} a^\dagger - \bar{\alpha} e^{\bar{\gamma}} a} |0\rangle = e^{-\frac{|\alpha|^2}{2} (1 - e^{\gamma + \bar{\gamma}})} |\alpha e^{\gamma}\rangle \end{aligned}$$

and we obtain

$$e^{\gamma N} |\alpha\rangle \langle \alpha| e^{\bar{\gamma} N} = e^{-|\alpha|^2 (1 - e^{\gamma + \bar{\gamma}})} |\alpha e^{\gamma}\rangle \langle \alpha e^{\bar{\gamma}}|$$

with $\gamma = -i\omega t - \log(F(t))$.

[Third Step] Under two steps above the equation (3.21) becomes

$$\rho(t) = \frac{e^{\frac{\mu - \nu}{2} t} e^{|\alpha|^2 (E(t) - 1 + e^{\gamma + \bar{\gamma}})}}{F(t)} \sum_{n=0}^{\infty} \frac{G(t)^n}{n!} (a^\dagger)^n |\alpha e^{\gamma}\rangle \langle \alpha e^{\bar{\gamma}}| a^n.$$

For simplicity we set $z = \alpha e^{\gamma}$ and calculate the term

$$(\sharp) = \sum_{n=0}^{\infty} \frac{G(t)^n}{n!} (a^\dagger)^n |z\rangle \langle z| a^n.$$

Since $|z\rangle = e^{-|z|^2/2} e^{za^\dagger} |0\rangle$ we have

$$\begin{aligned}
(\sharp) &= e^{-|z|^2} \sum_{n=0}^{\infty} \frac{G(t)^n}{n!} (a^\dagger)^n e^{za^\dagger} |0\rangle \langle 0| e^{\bar{z}a} a^n \\
&= e^{-|z|^2} e^{za^\dagger} \left\{ \sum_{n=0}^{\infty} \frac{G(t)^n}{n!} (a^\dagger)^n |0\rangle \langle 0| a^n \right\} e^{\bar{z}a} \\
&= e^{-|z|^2} e^{za^\dagger} \left\{ \sum_{n=0}^{\infty} G(t)^n |n\rangle \langle n| \right\} e^{\bar{z}a} \\
&= e^{-|z|^2} e^{za^\dagger} e^{\log G(t)N} e^{\bar{z}a}
\end{aligned}$$

by (4.1). Namely, this form is a kind of disentangling formula, so we want to restore an entangling formula.

For the purpose we use the **disentangling formula**

$$e^{\alpha a^\dagger + \beta a + \gamma N} = e^{\alpha \beta \frac{e^\gamma - (1+\gamma)}{\gamma^2}} e^{\alpha \frac{e^\gamma - 1}{\gamma} a^\dagger} e^{\gamma N} e^{\beta \frac{e^\gamma - 1}{\gamma} a} \quad (4.4)$$

where α, β, γ are usual numbers. The proof is given in the fourth step. From this it is easy to see

$$e^{ua^\dagger} e^{vN} e^{wa} = e^{-\frac{uw(e^v - (1+v))}{(e^v - 1)^2}} e^{\frac{uv}{e^v - 1} a^\dagger + \frac{vw}{e^v - 1} a + vN}. \quad (4.5)$$

Therefore ($u \rightarrow z, v \rightarrow \log G(t), w \rightarrow \bar{z}$)

$$(\sharp) = e^{-|z|^2} e^{\frac{|z|^2(1+\log G(t)-G(t))}{(1-G(t))^2}} e^{\frac{\log G(t)}{G(t)-1} za^\dagger + \frac{\log G(t)}{G(t)-1} \bar{z}a + \log G(t)N},$$

so by noting

$$z = \alpha e^\gamma = \alpha \frac{e^{-i\omega t}}{F(t)} \quad \text{and} \quad |z|^2 = |\alpha|^2 e^{\gamma + \bar{\gamma}} = |\alpha|^2 \frac{1}{F(t)^2}$$

we have

$$\begin{aligned}
\rho(t) &= \frac{e^{\frac{\mu-v}{2}t}}{F(t)} e^{|\alpha|^2(E(t)-1)} e^{|\alpha|^2 \frac{1+\log G(t)-G(t)}{F(t)^2(1-G(t))^2}} e^{\frac{\log G(t)}{F(t)(G(t)-1)} \alpha e^{-i\omega t} a^\dagger + \frac{\log G(t)}{F(t)(G(t)-1)} \bar{\alpha} e^{i\omega t} a + \log G(t)N} \\
&= \frac{e^{\frac{\mu-v}{2}t}}{F(t)} e^{|\alpha|^2 \left\{ E(t)-1 + \frac{1+\log G(t)-G(t)}{F(t)^2(1-G(t))^2} \right\}} e^{\frac{\log G(t)}{F(t)(G(t)-1)} \alpha e^{-i\omega t} a^\dagger + \frac{\log G(t)}{F(t)(G(t)-1)} \bar{\alpha} e^{i\omega t} a + \log G(t)N}.
\end{aligned}$$

By the way, from (3.15)

$$G(t) - 1 = -\frac{e^{\frac{\mu-\nu}{2}t}}{F(t)}, \quad \frac{1}{F(t)(G(t)-1)} = -e^{-\frac{\mu-\nu}{2}t}, \quad E(t) - 1 = -\frac{e^{-\frac{\mu-\nu}{2}t}}{F(t)}$$

and

$$\begin{aligned} \frac{1-G(t)+\log G(t)}{F(t)^2(G(t)-1)^2} &= e^{-(\mu-\nu)t} \left\{ \frac{e^{\frac{\mu-\nu}{2}t}}{F(t)} + \log G(t) \right\} \\ &= \frac{e^{-\frac{\mu-\nu}{2}t}}{F(t)} + e^{-(\mu-\nu)t} \log G(t) \\ &= -(E(t)-1) + e^{-(\mu-\nu)t} \log G(t) \end{aligned}$$

we finally obtain

$$\begin{aligned} \rho(t) &= (1-G(t))e^{|\alpha|^2 e^{-(\mu-\nu)t} \log G(t)} e^{-\log G(t)} \left\{ \alpha e^{-i\omega t} e^{-\frac{\mu-\nu}{2}t} a^\dagger + \bar{\alpha} e^{i\omega t} e^{-\frac{\mu-\nu}{2}t} a - N \right\} \\ &= e^{|\alpha|^2 e^{-(\mu-\nu)t} \log G(t) + \log(1-G(t))} e^{-\log G(t)} \left\{ \alpha e^{-\left(\frac{\mu-\nu}{2} + i\omega\right)t} a^\dagger + \bar{\alpha} e^{-\left(\frac{\mu-\nu}{2} - i\omega\right)t} a - N \right\}. \end{aligned}$$

[Fourth Step] In last, let us give the proof to the disentangling formula (4.4) because it is not so popular as far as we know. From (4.4)

$$\begin{aligned} \alpha a^\dagger + \beta a + \gamma N &= \gamma a^\dagger a + \alpha a^\dagger + \beta a \\ &= \gamma \left\{ \left(a^\dagger + \frac{\beta}{\gamma} \right) \left(a + \frac{\alpha}{\gamma} \right) - \frac{\alpha\beta}{\gamma^2} \right\} \\ &= \gamma \left(a^\dagger + \frac{\beta}{\gamma} \right) \left(a + \frac{\alpha}{\gamma} \right) - \frac{\alpha\beta}{\gamma} \end{aligned}$$

we have

$$\begin{aligned} e^{\alpha a^\dagger + \beta a + \gamma N} &= e^{-\frac{\alpha\beta}{\gamma}} e^{\gamma \left(a^\dagger + \frac{\beta}{\gamma} \right) \left(a + \frac{\alpha}{\gamma} \right)} \\ &= e^{-\frac{\alpha\beta}{\gamma}} e^{\frac{\beta}{\gamma} a} e^{\gamma a^\dagger \left(a + \frac{\alpha}{\gamma} \right)} e^{-\frac{\beta}{\gamma} a} \\ &= e^{-\frac{\alpha\beta}{\gamma}} e^{\frac{\beta}{\gamma} a} e^{-\frac{\alpha}{\gamma} a^\dagger} e^{\gamma a^\dagger a} e^{\frac{\alpha}{\gamma} a^\dagger} e^{-\frac{\beta}{\gamma} a}. \end{aligned}$$

Then, careful calculation gives the disentangling formula (4.4) ($N = a^\dagger a$)

$$\begin{aligned}
 e^{-\frac{\alpha\beta}{\gamma}} e^{\frac{\beta}{\gamma}a} e^{-\frac{\alpha}{\gamma}a^\dagger} e^{\gamma N} e^{\frac{\alpha}{\gamma}a^\dagger} e^{-\frac{\beta}{\gamma}a} &= e^{-\frac{\alpha\beta}{\gamma}} e^{-\frac{\alpha\beta}{\gamma^2}} e^{-\frac{\alpha}{\gamma}a^\dagger} e^{\frac{\beta}{\gamma}a} e^{\gamma N} e^{\frac{\alpha}{\gamma}a^\dagger} e^{-\frac{\beta}{\gamma}a} \\
 &= e^{-(\frac{\alpha\beta}{\gamma} + \frac{\alpha\beta}{\gamma^2})} e^{-\frac{\alpha}{\gamma}a^\dagger} e^{\frac{\beta}{\gamma}a} e^{\gamma N} e^{\frac{\alpha}{\gamma}a^\dagger} e^{-\frac{\beta}{\gamma}a} \\
 &= e^{-(\frac{\alpha\beta}{\gamma} + \frac{\alpha\beta}{\gamma^2})} e^{-\frac{\alpha}{\gamma}a^\dagger} e^{\gamma N} e^{\frac{\beta}{\gamma}e^{\gamma}a} e^{\frac{\alpha}{\gamma}a^\dagger} e^{-\frac{\beta}{\gamma}a} \\
 &= e^{-(\frac{\alpha\beta}{\gamma} + \frac{\alpha\beta}{\gamma^2}) + \frac{\alpha\beta}{\gamma^2}e^{\gamma}} e^{-\frac{\alpha}{\gamma}a^\dagger} e^{\gamma N} e^{\frac{\alpha}{\gamma}a^\dagger} e^{\frac{\beta}{\gamma}e^{\gamma}a} e^{-\frac{\beta}{\gamma}a} \\
 &= e^{\alpha\beta\frac{e^{\gamma}-1-\gamma}{\gamma^2}} e^{-\frac{\alpha}{\gamma}a^\dagger} e^{\frac{\alpha}{\gamma}e^{\gamma}a^\dagger} e^{\gamma N} e^{\beta\frac{e^{\gamma}-1}{\gamma}a} \\
 &= e^{\alpha\beta\frac{e^{\gamma}-1-\gamma}{\gamma^2}} e^{\alpha\frac{e^{\gamma}-1}{\gamma}a^\dagger} e^{\gamma N} e^{\beta\frac{e^{\gamma}-1}{\gamma}a}
 \end{aligned}$$

by use of some commutation relations

$$e^{sa} e^{ta^\dagger} = e^{st} e^{ta^\dagger} e^{sa}, \quad e^{sa} e^{tN} = e^{tN} e^{se^t a}, \quad e^{tN} e^{sa^\dagger} = e^{se^t a^\dagger} e^{tN}.$$

The proof is simple. For example,

$$e^{sa} e^{ta^\dagger} = e^{sa} e^{ta^\dagger} e^{-sa} e^{sa} = e^{te^{sa} a^\dagger e^{-sa}} e^{sa} = e^{t(a^\dagger + s)} e^{sa} = e^{st} e^{ta^\dagger} e^{sa}.$$

The remainder is left to readers.

We finished the proof. The formula (4.3) is both compact and clear-cut and has not been known as far as we know. See [1] and [16] for some applications.

In last, let us present a challenging problem. A squeezed state $|\beta\rangle$ ($\beta \in \mathbb{C}$) is defined as

$$|\beta\rangle = e^{\frac{1}{2}(\beta(a^\dagger)^2 - \bar{\beta}a^2)} |0\rangle. \quad (4.6)$$

See for example [4]. For the initial value $\rho(0) = |\beta\rangle\langle\beta|$ we want to calculate $\rho(t)$ in (3.21) like in the text. However, we cannot sum up it in a compact form like (4.3) at the present time, so we propose the problem,

Problem sum up $\rho(t)$ in a compact form.

5. Concluding remarks

In this chapter we treated the quantum damped harmonic oscillator, and studied mathematical structure of the model, and constructed general solution with any initial condition, and gave a quantum counterpart in the case of taking coherent state as an initial condition. It is in my opinion perfect.

However, readers should pay attention to the fact that this is not a goal but a starting point. Our real target is to construct general theory of **Quantum Mechanics with Dissipation**.

In the papers [7] and [8] (see also [13]) we studied a more realistic model entitled "Jaynes–Cummings Model with Dissipation" and constructed some approximate solutions under any initial condition. In the paper [5] we studied "Superluminal Group Velocity of Neutrinos" from the point of view of Quantum Mechanics with Dissipation.

Unfortunately, there is no space to introduce them. It is a good challenge for readers to read them carefully and attack the problems.

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