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Emergent un-Quantum Mechanics

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1. Introduction

There is great interest in “emergent” dynamical systems and the possibility of quantum mechanics as emergent phenomena. We engage the topic by making a sharp distinction between models of microphysics, and the so-called quantum framework. We find the models have all the information. Given that the framework of quantum theory is mathematically self-consistent we propose it should be viewed as an information management tool not derived from physics nor depending on physics. That encourages practical applications of quantum-style information management to near arbitrary data systems. As part of developing the physics, we show there is no intrinsic distinction between quantum dynamics and classical dynamics in its general form, and there is no observable function for the unit converter known as Planck’s constant. The main accomplishment of quantum-style theory is a expanding the notion of probability. A map exists going from macroscopic information as “data” to quantum probability. The map allows a hidden variable description for quantum states, and broadens the scope of quantum information theory. Probabilities defined for mutually exclusive objects equal the classical ones, while probabilities of objects in more general equivalence classes yield the quantum values. Quantum probability is a remarkably efficient data processing device; the *Principle of Minimum Entropy* explains how it serves to construct order out of chaos.

2. Complexity and symmetry induce dynamics

The framework of quantum mechanics is intricately structured and thought the perfection of fundamental theory. It predicts an absolute and unvarying law of time evolution. There is a tightly defined space of possible states, upon which strictly prescribed operators act to produce crisp possibilities for observables. There is an unprecedented universal rule for predicting probabilities of observations. The general predictions of the framework are incompatible with hidden variables defined by distributions, and have been confirmed by every experiment so far conducted.

Meanwhile, the *particular realizations* of physical theories are widely believed to be *emergent*. That means they probably do not really represent fundamental physical law, but instead represent generic outputs of complicated systems not driven by the same laws. The Standard Model of particle physics is the most sophisticated prototype. It explains all data from all experiments done so far, except for a few outliers. Yet it is hard to find anyone involved that will argue the Standard Model is more than a generic derivative expansion embodying certain symmetries of some more complicated theory, of which numberless possibilities exist. Like Hooke's Law, Standard Model Laws are no longer imagined to be serious candidates for Laws, because they are so contrived and of the type that had to occur one way or the other. Nor does one really need the machinery of the Standard Model to understand most of the Universe. For most of what matters, the non-relativistic Schroedinger Equation is a "theory of everything"[1].

Then it is very reasonable to expect that quantum mechanics itself should be a generic, self-defining "emergent" feature of the Universe. In simple terms, an output, not an input.

2.0.1. Practical goals

What does that mean? Discussions of emergent quantum mechanics tend to become confused by discord over what we mean by quantum mechanics. Different writers will disagree about what is fundamental and even about what experimental data says. Not far behind is a superstition that quantum objects cannot possibly be understood, so that making them even more obscure and more difficult might be the intellectual high ground. We reject that holdover from the 1920's, but it's not clear whether it has died out, or might be coming back.

To skirt the morass we have a new point of view. We want physics to be practical and simple. In this century not many find the mathematics of quantum theory so intimidating. Perhaps physics "quantum-style" is not so profound after all. What is the evidence that the quantum *framework* itself is so meaningful and fundamental?

Most accomplishments of "quantum mechanics" come from the model details. For example, understanding the Hydrogen atom and calculating $g - 2$ of the electron are astounding achievements. However those accomplishments come directly from the model details, and systematic laborious tuning of theory to experimental facts, rather than from the framework of quantum mechanics itself. That is not always noticed while the framework lays claim for every accomplishment. The framework passes every test, especially when tested by thought-experiments set up for validation by pencil and typewriter challenges that recycle the framework. But the *framework* is rather hard to falsify experimentally. When it might have failed, a little ingenuity never fails to bring it back. That is quite unlike models challenging conventional theory, such as non-linear Schroedinger equations and so on, which have the decency to be able to fail.

Following this and other clues, we propose the quantum *framework* amounts to *descriptive* tools and *classification tools* that categorize data beautifully, but predict very little at all. When the "Laws" of quantum mechanics are considered as procedural and classification structures, it's much easier to guess how they would *emerge* as human-made bookkeeping.

Not everything in quantum mechanics is procedural, and some of its general workings contain clues to Nature. It is rarely noticed that quantum theory has infinities of hidden variables. They are not classical hidden variables of the usual kind. The "electron" and

“photon” of the 1920’s thought to be so fundamental and ultimate are not fundamental objects. They are approximations that never stop interacting with an infinite ocean of quantum fields, if not something more unknown and more interesting. The self-consistency of the *framework* suggests that living in peaceful co-existence with what it must ignore may be its main accomplishments.

2.1. What would emerge?

Many workers seek to derive quantum mechanics. An active movement suspects or maintains that it is emergent, not fundamental[2–4]. We think it would be a waste to obtain exactly the quantum framework already known. Progress requires new features and new viewpoints. Progress usually involves dropping obsolete views and clearing out deadwood: call it *un-Quantization*.

We start by considering what can and cannot be given up. Quantum mechanics wiped out the previous vision of a Newtonian universe made of point particles. It’s gone forever. There remains some confusion and disagreement about whether the theory is about point particles of some subtle magic kind. While point particles are loosely cited in press releases, and inaccurately associated with theories based on *local* space-time interactions, we cannot find any evidence for them, and give that up. We believe all the rest of early quantum lore also can be given up, especially those parts leading to pedagogical confusion. We can’t explain why it is unconventional or even scandalous to admit that the pre-history of quantum mechanics –meaning that period between 1900 and 1926 – was characterized by great theoretical advances that were all wrong or limiting in some way or other. Being wrong is normal in physics but covering it for generations is very strange.

Many professional physicists are still influenced by the cult of the “quantum of action,” forgetting it went away when quantum mechanics found action is not quantized. Physicists have been programmed from birth to hold Einstein’s relations $E = \hbar\omega$, $\vec{p} = \hbar\vec{k}$ as highly fundamental. They actually know these relations are not universal, but derived facts coming from special cases, yet downgrading them is taboo. Physicists often believe that classical Newtonian mechanics shall be a starting point to be “quantized” to predict quantum systems, forgetting this recipe only predicts textbook problems for training purposes. It would forbid the Standard Model of physics to exist at all. Physicists are also trained that Feynman’s path integral creates a quantum theory from a classical one, forgetting that what’s integrated over nowadays has no relation to the starting point young Mr. Feynman used. When we un-Quantize this, an integral representation of correlation functions is a math tool, not an independent principle, nor does it “come from physics.”

We must honor our forebears and we do. Yet why are historical misdirections kept around with special emphasis in quantum physics? The culture of quantum mechanics almost seems to maintain mistakes of pre-quantum history on purpose. Instead of giving them up, misconceptions are kept around as philosophical quandaries and paradoxes because there is no other way to perpetuate mistakes except as paradoxes. *All that can be given up.*

2.1.1. Quantum-Style Things Not to Give Up

Quantum mechanics is a misnomer held-over from the pre-quantum era. The Schroedinger equation explained how and when quantization of physical quantities occurs as an outcome

of *dynamics*, and quantization is not the primary new feature of the theory. We have coined the term “quantum-style” to describe things done in the organizational style of quantum mechanics that we don’t want to give up.

Quantum-style mechanics describes certain data of the Universe, and the ability to describe experimental data absolutely cannot be given up. But that does not mean in some future time we would interpret success the same way. A certain vagueness of description is probably tied to success. For example, it is rarely noticed that the dramatic demonstrations of beginning quantum probability, such as the Bell inequalities and EPR “paradoxes,” are realized experimentally only by virtuoso fiddling and selection of systems nice enough to make them work out. When an experiment produces nothing but the mundane predictions of ordinary classical probability, nobody notices that was also a retrospective quantum prediction of certain dirty density matrices, protecting the framework from falsification. But we have noticed, and it suggests that quantum mechanics must be a framework that is so broad and flexible it does not restrict much. For example, if a future civilization discovered a true and ultimate Newtonian particle, contradicting everything now believed, any competent theorist would find no great difficulty describing it with an appropriately constructed density matrix.

All of this suggests what a new vision of “quantum theory” should include. *The restriction of the subject to describing micro-physical objects of fundamental physical character is obsolete.* It seems the same mistake as thinking complex numbers have no practical use because they have an “imaginary” part. We believe the framework of quantum theory is mathematically self-consistent. And when there are self-consistent structures, they ought to have more uses than microphysics, and not depend on microphysics for their uses. Even more directly, asking a self-consistent framework to “emerge” means mainly to start using it without fear it could go wrong.

This idea appears radical because it contradicts a few existing ideas. Physicists are educated in the magical antics of quantum objects and convinced they are impossible to understand before they begin serious coursework. As a result, misidentifying mathematical relations as inexplicable Facts of Nature is very common. That is fatal to physics, because Facts and Behaviors of a special Universe are not supposed to work on other Universes. That is why it is generally considered stupid to apply quantum-style methods to non-quantum objects.

The degree of stupidity depends strongly one’s Bayesian priors. 100,000 years ago humans would know the concepts of integers but probably be unable to separate them from empirical facts of Nature. It would be considered absurd and dangerous to imagine integers disassociated from their experimental realization in the number of rocks or rabbits. Yet those humans were not stupid. They lacked the background about the Universe to decide whether integers came from physics or came from human thinking, with physics probably having the edge because it was real. In much the same way the occult mysteries of \hbar , $i = \sqrt{-1}$, and electrons “in all states at once” conditioned the physics community to think “quantum methods are for quantum objects.”

Anyone thoughtful ought to notice that is circular, and ought to be open to using mathematical structures more liberally. But thoughtful people are given false information by the physicists about what’s established, and what can be contradicted, and even about what experiments find. We feel it is significant that the early 20th century was the first time physics needed to seriously deal with the details of experiments involving a large number of degrees

of freedom, equilibrium thermodynamics notwithstanding. Whether or not microphysics had any new and spooky elements, we find that quantum-style methods would need to be invented to handle the complexity. It is very efficient: And then the classification system and information processing power of quantum data organization should be used wherever a useful result might come out.

2.1.2. *How to Proceed*

Towards that end, we have developed an approach which largely avoids the historical path. That path and its traditional presentation interweaves a little dynamics, measurement theory, microphysical facts, experimental claims and the prehistory of failed theories in an alternating web. It is designed not to be challengeable, which is a cheat. To make progress we must change something. We first separate the dynamics, meaning rules of time evolution, from the rest, and identify it as being trivial. This is developed in Section 3. For us the ordinary form of quantum dynamics is a “toy model”, which at first seems too simple to be realistic. This is not to say that the *models* describing microphysics are trivial or easy to solve. Actually the models are the real discovery, while claims that the dynamical framework is the real discovery have things reversed.

The great roadblock to using the organizational scheme of quantum theory more generally is Planck’s constant. Once the claim is made that Nature partitions itself into little cells determined by \hbar , one cannot do without \hbar , nor use quantum theory for anything else but issues of \hbar . But the claim is wrong. In Section 4 we discuss a quantum-style Universe without \hbar . We claim it is the ordinary Universe, but if that is too provocative, the step of never introducing \hbar is a part of *our approach* where it never appears.

We also dispense with needlessly obscure definitions of “observables.” We define observables (Section 5) as maps from the system coordinates to numbers. This is plain and unpretentious. Satisfying things, including relations of Poisson Brackets usually postulated as independent, can actually be derived using symmetry. In principle the map between system and observables is invertible: the system coordinates (wave function, density matrix) are observable. It is a non-trivial fact that real physical systems seem to have infinite complexity, making exhaustive measurements a bit out of reach, but this “bug” is a “feature.”

In the end the new thing that came out of quantum mechanics are new definitions of “probability”, Section 6. The working of quantum probability has always been explained by physicists using a self-validating logic that “it works” because “features of microphysical object make it work”. (And this is very mysterious and too profound for humans to grasp, etc.) Every time that thinking style is used we find it unprogressive and circular.

Instead of buying it, we seek a rational explanation why certain mathematical tools work *sometimes* and other rules work *other times* without depending on circular postulates. In our approach the information management of quantum theory is a topic of mathematical classification, and for that reason mostly devoid of physical content. Since it is mathematics, we can derive the quantum rules of probability as a bookkeeping system that does not need any special features from the objects they describe. And we do this to increase the scope and *utility* of the rules so we can use them in new applications.

2.1.3. Question From the Bottom Up

It is not always helpful to put the framework of quantum mechanics on a high pedestal. It is sometimes assumed that quantum mechanics might only be “explained” by progress at the far edge of the research frontier involving quantum gravity, foamy space-time, strings, and so on[5]. But if true that would put our topic among those not seeking to deal with what is observable and testable. We are only interested in what is observable and testable.

Progress needs to come from revising the bottom. Successful work at the bottom revises basic notions that are actually harder to challenge than advanced work, because the whole system rests on the base. The mathematics of our discussion is not advanced. It is little more than linear algebra, and deliberate choices not to use mathematics that is more advanced than needs to be used.

There is a very elementary point often overlooked. Mathematical subjects can be reduced to a definite minimal number of axioms, which might be swapped around, but not decreased. Early on quantum theory looked ripe for axiomization, and it tends to be accepted today. Yet every effort to make physics into axioms fails because we don’t know what the Universe is. However physics can *often* reduce the number of postulates, axioms or guesses by swapping around the order, which actually changes their meaning and power. The everyday assumption that this was optimized long ago is not true. Thus, it is a form of progress to explore how *post-quantum* physics-axioms can be eliminated by re-ordering and re-interpreting the logic. The process will help quantum mechanics “emerge” more clearly from its own tangles.

3. Dynamics

3.0.4. Where to Start?

By very curious structuring, the usual approach to quantum mechanics starts with the doctrine of measurement postulates. What is out of order in those approaches is failing to first define the system and its dynamics. For example the Stern-Gerlach experiment is traditionally developed as a raw mystery of 1922 involving point-like electrons and two spots, then requiring a new principle[6]. If one were given from the start that a two-component wave function was involved the separation into two spots does not really need any new principles. It was known to Fresnel from calcite crystals and explained without requiring a new principle. And if one knew the particular two-component wave functions of electrons were expected from representations of the rotation group none of it would be a terrible surprise. While ordering things to make physics more mysterious and inexplicable was an early promotional tool, we lack any interest in it. That is why we will start with the dynamics, because it can be explained. We will discuss how there is nothing new contributed by quantum mechanics to its own framework of dynamics: *at least in our approach*.

3.1. Hamilton’s equations in Schroedinger’s notation

Physics predicts little more than evolution of systems with time, symbol t . By the end of this work we will argue the predictions (above and beyond the empirically-found model details!) originate in symmetry.

We assume the reader knows how to get equations by varying an action S , expressed using a Lagrangian $L(q_i, \dot{q}_i)$:

$$\delta S = \delta \int dt L(q_i, \dot{q}_i).$$

Symbols q_i are generalized coordinates, namely numbers describing a system, labeled by $i = 1 \dots N$, and the dot indicates a time derivative. We are not going to suggest that the “action principle” will be our foundation postulate. *When and if* an emergent quantum system has sufficiently nice dynamics that it gets noticed as an experimental regularity, the action is a fine invariant notation to express it.

By familiar steps, finding the extrema of the action produces Lagrange’s equations. Define the Hamiltonian H by

$$H(q_i, p_i) = p_i \dot{q}_i - L(q_i, \dot{q}_i).$$

Repeated indices are summed. When these transformations can be done, then Lagrange’s equations are equivalent to Hamilton’s:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}; \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}. \quad (1)$$

We will pause at this point to repeat that (q_i, p_i) are real-valued *numbers*, that everything above was known by (say) 1850, and that our subject is nevertheless quantum dynamics. We will never confuse a classical coordinate with an operator, we will never use the abusive term “quantum particle” except to reject it, and when an operator is intended it will be indicated by a *hat*.

The thing for our discussion not known in 1850 lies in the number of dynamical degrees of freedom (*dof*) we intend to use. A Newtonian particle has three *dof* usually taken to be its Cartesian coordinates. The Newtonian particle is not a valid prototype and (unlike the early history) we base nothing on making contact with it. In our approach we have no advance information on the number of *dof* describing a system, because that is an arbitrary defining feature of a system. For N *dof* the phase-space of (q_i, p_i) is $2N$ dimensional. We also pretend to no advance information on the Hamiltonian, although some properties will be specified to make contact with existing models. We claim this freedom not to commit is a defining fact of basic quantum mechanics: but if that is not agreed, it is a fact of *our theory*.

Now proceed: Hamilton’s equations are invariant under symplectic (Sp) transformations. It is usually developed by combining (q_i, p_i) into a $2N$ dimensional multiplet $\Phi = (q_1 \dots q_N, p_1 \dots p_N)$. Hamilton’s equations become

$$\dot{\Phi} = J \frac{\partial \mathcal{H}}{\partial \Phi}. \quad (2)$$

Here J is a matrix with block representation

$$J = \begin{pmatrix} 0 & 1_{N \times N} \\ -1_{N \times N} & 0 \end{pmatrix}. \quad (3)$$

Under a real-valued $2N \times 2N$ transformation S , the equation transforms

$$\begin{aligned} \Phi &\rightarrow \Phi_S = S\Phi; \\ \dot{\Phi}_S &= S \cdot J \cdot S^T \frac{\partial \mathcal{H}}{\partial \Phi}. \end{aligned} \quad (4)$$

Super- T denotes the *transpose*. The *symplectic group* of $2N$ dimensions is the set of transformations such that

$$S \cdot J \cdot S^T = J. \quad (5)$$

It can be shown the determinant $\det(S) = 1$. Transformations satisfying Eq. 5 with determinant -1 will be called *Sp – parity changing*.

3.1.1. Simplistic Linear Theories Are Not Our Burden To Defend

By a linear theory we mean that the Hamiltonian H is bilinear in q_i, p_i :

$$\begin{aligned} \mathcal{H}(q, p) &= \frac{1}{2} \Phi^T H_\Phi \Phi; \\ H_\Phi &= \frac{1}{2} \begin{pmatrix} h_{qq} & h_{qp} \\ h_{qp}^T & h_{pp} \end{pmatrix} \end{aligned} \quad (6)$$

Matrix multiplication is implied, and $h_{qq}, h_{qp}...$ etc are $N \times N$ arrays of constant parameters. We are not writing linear terms like $\alpha q + \beta p$, which can be removed by translating coordinates. We have no commitment here to the bilinear form, which is presented to make contact with ordinary quantum mechanics.

The most general such theory has a familiar form, seen by writing¹

$$h_{qq} = K; \quad h_{pp} = M^{-1} \quad h_{qp} = -\Gamma^T M. \quad (7)$$

Complete the square:

$$\mathcal{H}(q, p) = \frac{1}{2} p M^{-1} p + \frac{1}{2} q K q + q M \Gamma p + p \Gamma^T M q, \quad (8)$$

$$= \frac{1}{2} (p - \mathcal{A}(q)) M^{-1} (p - \mathcal{A}(q)) + \mathcal{V}; \quad (9)$$

$$\text{where } \mathcal{A}(q) = \Gamma q; \quad \mathcal{V} = \frac{1}{2} q (K - \Gamma^T M^{-1} \Gamma) q.$$

¹ There's no loss of generality using these symbols, as M^{-1} is meant to be the inverse on the space M does not send to zero, i.e. the pseudoinverse.

The Hamiltonian of a classical, 3-dimensional, non-relativistic particle in an external electromagnetic field is $H_{em} = (\vec{p} - e\vec{A}(\vec{q}))^2/2m + V(\vec{q})$, where \vec{A} is the vector potential. Except for allowing a tensor mass, a quantum system with three *dof* (spin-1, say) is dynamically indistinguishable. For more *dof* the symbol $A(q)$ continues to serve as a vector potential with an associated curvature

$$\mathcal{F}_{ij} = \frac{\partial \mathcal{A}_i}{\partial q_j} - \frac{\partial \mathcal{A}_j}{\partial q_i} = \Gamma_{ij} - \Gamma_{ji}. \quad (10)$$

These theories have gauge symmetries. From Eq. 10 any symmetric part of Γ drops out of \mathcal{F}_{ij} and the equations of motion. That is equivalent to gauge transformation $\mathcal{A}(q) \rightarrow \mathcal{A}(q) + \Sigma q$, where $\Sigma = \Sigma^T$. As a rule gauge symmetries indicate a system that is being described with more coordinates than are truly dynamical: the redundant coordinates may be hard to eradicate, and easier to treat as “symmetries.” That will be a clue.

3.1.2. Diagonal Frame

Go to coordinates where the symplectic metric J is diagonal. Since J is antisymmetric, the transformation goes from real to complex numbers:

$$\Phi = \begin{pmatrix} q_1 \\ q_2 \dots \\ p_1 \\ p_2 \\ \dots \end{pmatrix} \rightarrow \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \dots \\ \psi_1^* \\ \psi_2^* \\ \dots \end{pmatrix} \quad (11)$$

An appropriate map is

$$\Psi = \mathcal{U}\Phi;$$

$$\mathcal{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1_{N \times N} & i 1_{N \times N} \\ 1_{N \times N} & -i 1_{N \times N} \end{pmatrix},$$

with

$$\mathcal{U}\mathcal{U}^\dagger = 1_{2N \times 2N}, \quad (12)$$

and then

$$\mathcal{U}\mathcal{J}\mathcal{U}^\dagger = - \begin{pmatrix} i 1_{N \times N} & 0 \\ 0 & -i 1_{N \times N} \end{pmatrix}. \quad (13)$$

The transformation produces a remarkable simplification of linear dynamical systems. Hamilton's first equation (Eq. 2) become

$$i \frac{\partial \Psi}{\partial t} = \frac{\partial H(\Psi, \Psi^*)}{\partial \Psi^*}. \quad (14)$$

Since it is an important point we show the algebra for one *dof*. We are given

$$\dot{q} = \frac{\partial H}{\partial p}; \quad \dot{p} = -\frac{\partial H}{\partial q}.$$

Combine two real numbers into one complex one:

$$\psi(q + ip)/\sqrt{2}. \quad (15)$$

We call this the "quantum map". It explains how complex numbers came to be essential in quantum theory. Compute

$$\dot{\psi} = (\dot{q} + i\dot{p})/\sqrt{2} = \left(\frac{\partial H}{\partial p} - i\frac{\partial H}{\partial q}\right)/\sqrt{2}. \quad (16)$$

The chain rule gives

$$\frac{\partial}{\partial p} - i\frac{\partial}{\partial q} = \sqrt{2}\frac{\partial}{\partial \psi^*},$$

and then Hamilton's equations are

$$i\dot{\psi} = \frac{\partial H}{\partial \psi^*}. \quad (17)$$

Continuing: When $H(q, p)$ is bilinear then $H(\Psi, \Psi^*)$ is bilinear. In quantum mechanics one always chooses parameters so that

$$H(\Psi, \Psi^*) = \Psi^* \Omega \Psi, \quad (18)$$

where Ω now contains the parameters. Eq. 14 and 18 give

$$i \frac{\partial \Psi}{\partial t} = \hat{\Omega} \cdot \Psi. \quad (19)$$

This is Schroedinger's equation, which is nothing more than Hamilton's equation in complex notation. We prefer symbol $\hat{\Omega}$ to \hat{H} for reasons to be explained soon.

3.1.3. Conventionally Assumed Properties

We do not have hermiticity in the form $\hat{\Omega} = \hat{\Omega}^\dagger$ automatically. First as Bender and collaborators has emphasized[8], the self-adjoint test does not have a magnificent degree of invariance. If an operator is self-adjoint it will be Hermitian, as defined by having real eigenvalues. It will also remain self-adjoint under unitary transformations. But if a Hermitian operator is subject to arbitrary similarity transformations it may cease to be self-adjoint, while its eigenvalues will not change.

Second, the class of eigenvalues of $\hat{\Omega}$ are a physics decision. The most general solution to Eq. 19 is an expansion in normal modes,

$$\Psi(t) = \sum_n \psi_n(0) e^{-i\omega_n t}, \quad (20)$$

where ψ_n are solutions to the eigenvalue problem

$$\hat{\Omega}\psi_n = \omega_n\psi_n. \quad (21)$$

This happens to eliminate a postulate, as we'll explain, and just to re-iterate, we're discussing a system of generalized classical coordinate where the assumption of units $\hbar = 1$ has definitely *not* been imposed. We have no need for \hbar as explained in Section 4. Eq. 21 is self explanatory. Frequencies are the eigenvalues of the *frequency operator* $\hat{\Omega}$. As Feynman must have said, the textbook business of multiplying $\omega_n \rightarrow \hbar\omega_n \equiv E_n$ so the time evolution appears as $e^{-iE_n t/\hbar} \equiv e^{-i\omega_n t}$ is a complete waste of time.

If ω_n has a complex part the time evolution contains exponentially growing or damped solutions, which were frowned upon by the authorities in charge of setting up quantum mechanics. That eliminates another postulate (the postulate of Hermiticity), replacing it by the Decision of Hermiticity. To conform with this reasonable decision we specify real ω_n , Hermitian $\hat{\Omega}$, which a short exercise shows is equivalent to Eq. 18.

To reiterate, in our approach we have available every freedom to consider non-linear or non-Hermitian systems, at least up to here. We have taken a *more general* framework and reduced it to the *less general* dynamical rules of quantum theory by identifying the restrictions assumed in standard lore. Classical mechanics is a vast general framework not at all the same as Newtonian physics. Understanding that, there is nothing but classical physics in the Schroedinger equation.

3.1.4. Discussion

Eq. 19 comes from mere algebraic manipulations. While developed a bit in Ref.[7] it is surprisingly unknown to most physicists. Discussions with many physicists find several frequently asked questions:

- *Every interesting quantum theory is non-linear. Right? Why is the linearization $H \rightarrow \langle \psi | \Omega | \psi \rangle$ relevant?* No, quantum dynamics is *always* linear. The mixup about what is linear comes from habitual sloppiness in physics discussions to mix operators with numbers, and then constructing Hamiltonians as non-linear combinations of operators. An operator appears in matrix elements $\hat{\Omega}_{ij} = \langle i | \hat{\Omega} | j \rangle$, which remains unspecified.

- *Does this method assume a finite-dimensional system, and why would that be relevant?* No. The whole point of applying linear algebra and Hilbert space methods to quantum mechanics is one unified notation. The usual infinite dimensional expression for the Hamiltonian is

$$H(\psi, \psi^*) = \int dx \psi^*(x) \hat{H} \psi(x). \quad (22)$$

Apply Hamilton's equations (Eq. 14) using a functional derivative and you are done.

- *Current physics of quantum field theory uses highly non-linear Hamiltonians, for which Eq. 22 fails. Why bother with beginning quantum mechanics?* Once again the question is about the matrix elements and dimension of the Hamiltonian operator, which we've left unspecified, and which is generally a non-linear function of the fields. That does not change (repeat) the linearity of the dynamics. It is most directly seen in the functional Schroedinger equation

$$i\dot{\Psi}(\Phi) = \hat{H}(\Phi, -i\delta/\delta\Phi)\Psi(\Phi).$$

This differential equation is equation is *linear* in the dynamical *dof* Ψ , while exactly equivalent to the non-linear operator relations of the usual approach. Feynman himself was very fond of the Schroedinger picture for the practical reason that wave function equations are easier to solve and approximate than operator equations. For us (up to here) quantum field theory is a very large classical dynamical system.

- *How can this be the same as the path integral formulation? Moreover, the field-theoretic path-integral is different from the one of beginning quantum mechanics.* We say that basic quantum mechanics is more fundamental than the path integral. *Given* the Schroedinger equation, the path integral comes to be derived as an integral representation of certain correlations. So we also have path integrals as (up to here) a representation of certain quantities evolving by generalized classical mechanics.
- *Where are the operator equations of motion? What role exists for operators?* It is interesting that the dis-ordering of material in the education of physicists is such that questions like these come up, while everyone knows the answer. *Given* the Schroedinger time evolution, and any arbitrary operator, the Heisenberg picture is developed as a *definition* of time-dependent operators. We must use the Schroedinger picture because it's not really true that Heisenberg operator equations of motion makes an "equivalent theory". The operators lack a wave function to encode a system's initial conditions and state, and which develops proper observables.
- *Where is Planck's constant? With classical mechanics and without Planck's constant how are you going to quantize the Hydrogen atom?* One of the advantages of our approach is the ability to discard deadwood. "Deriving the Hamiltonian" of the Hydrogen atom is schoolbook bunk: at least in our approach! Planck's constant deserves a separate discussion: the next topic.

4. A World without Planck's constant

Hamilton's equations in its three equivalent forms (Eq. 1, 2, 19) lack Planck's constant. Most physicists believe that Planck's constant is a fundamental feature of our Universe, cannot imagine a world without it, and also have no idea how \hbar could possible "emerge" from the (possible noise and chaos) of theory more fundamental. But a Universe without Planck's constant is not hard to imagine[12]. It is a Universe where human history would have gone differently.

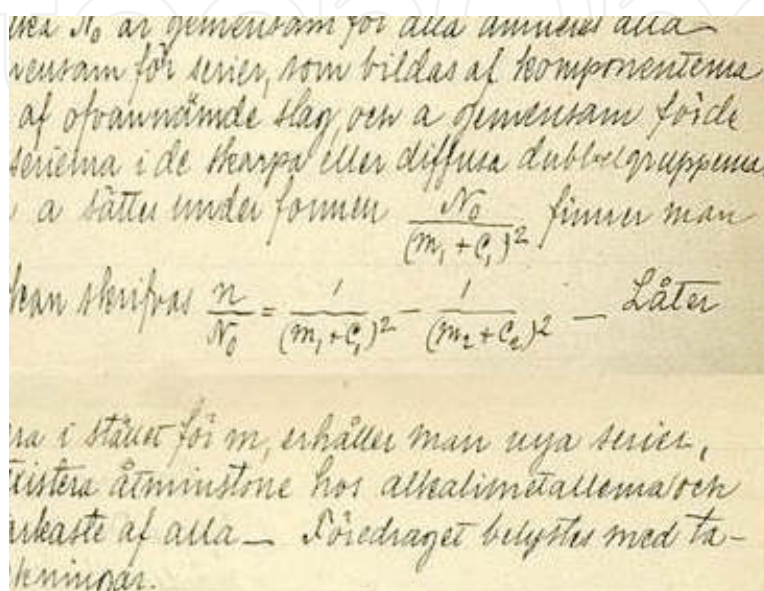


Figure 1. A record of physics from a Universe without Planck's constant. Rydberg's original fits to frequency (wavenumber) data for the Hydrogen spectrum did not need to be converted into Newtonian units and back again to frequency fix the parameters of quantum theory. As Feynman must have said, converting units and the associated conversion constant is a total waste of time.

4.1. The culprit is mass

Human history defined a notion of *mass* as a quantity of matter such as silver or butter long before physics defined mass. Imagine a history where Hamiltonian methods were developed first. Then *mass* would be more neutral, a particular "coupling constant" appearing in the Hamiltonian. We might find ourselves lacking the Newtonian intuitive picture of "mass," which might be a good thing. We would need to teach ourselves how to get the meaning of parameters from the theory where they appear.

Transformation properties are generally a key. Just as q 's and p 's transform under a change of variables, the parameters of a theory transform. However parameters do not transform automatically. Consider the constant c in an ordinary wave equation:

$$\frac{\partial^2 \phi}{\partial t^2} - c^2 \nabla^2 \phi = 0. \quad (23)$$

Under changes of scale $x \rightarrow x' = \lambda_x x$, $t \rightarrow t' = \lambda_t t$ the equation changes, and becomes false, unless c is transformed by hand. The equation is form-unchanged (has a symmetry) under

$c \rightarrow c' = \lambda_x c / \lambda_t$. That is read “ c has units of *length* over *time*.” Transforming constants is so familiar the need for derivation escapes notice.

Notice that transforming parameters is not treated as an ordinary symmetry. Under an ordinary symmetry, including space-time symmetries such as Lorentz transformations, the Hamiltonian is unchanged, including the parameters. The reason c must be changed by hand is that the value $c = 3 \times 10^8 m/s$ refers to a particular Universe where units of *length* and *time* measured it. Once c is measured and fixed in our particular universe, then changing its value (with fixed scales of space and time) is *definitely not* a physical symmetry. Educating the math about a passive scale change of coordinates requires we transform parameters measured in old units into new numbers so the Universe described remains the same.

Review such scale changes more generally. We noted that the group of canonical transformations, which preserve the action of our theory, is the symplectic group $Sp(2N)$. Every element $S \in Sp(2N)$ can be written locally as a product of three factors $S = K_1 \cdot \Lambda \cdot K_2$, where K_1, K_2 are “rotations” from the maximal compact subgroup of S . The elements Λ are scaling transformations of the form

$$\begin{aligned} q_i &\rightarrow q'_i = \lambda_i q_i; \\ p_i &\rightarrow p'_i = p_i / \lambda_i; \end{aligned} \tag{24}$$

$$dq_i dp_i = dq'_i dp'_i. \tag{25}$$

The phase-space volume on every pair (not just the entire space) is preserved by re-scaling and rotations. There are N parameters in Λ , and $2N^2$ in K_1, K_2 , accounting for all $N(2N+1)$ parameters of $Sp(2N)$. The decomposition is unique up to discrete row-swapping transformations maintaining $\det(S) = 1$.

The action-preserving transformations predict momenta p_i scale like $1/q_i$. Unless a different definition is made, that requires p_i to have units of $1/q_i$. This also follows from the momentum being the generator of translations. Similarly the energy E as the value of the Hamiltonian, and the time t are conjugate. Under scaling transformations preserving the action, they transform with $dt dE = dt' dE'$, as seen from $Ldt = p_i dq_i - Hdt$. The intrinsic units of energy are inverse time, as also seen using the action of a solved system directly: $H = -\partial S / \partial t$.

There is more than one way to apply this. It applies to our complexified dynamical wave functions, $\psi_i = (q_i + ip_i) / \sqrt{2}$. But it also applies at the most beginning level known as “high school physics.” It is astonishing that mixups at the level of high school physics might affect deep questions of quantum mechanics. But this is not as unlikely as it seems. High school students and their teachers are seldom given freedom to challenge what they are taught. Later on it is difficult to give up what we were taught as children.

4.1.1. Mass Paradox

Knowing the scaling information of Eq. 25, consider changing the units of a translational coordinate q , for example changing the units of length from meters to centimeters. There are 100 cm/m , hence $q_{cm} = \lambda_{cm/m} q_m$ with $\lambda_{cm/m} = 100$. Ordinary usage of Newtonian momentum predicts

$$\begin{aligned} p &= m\dot{q}; \\ q_i &\rightarrow q'_i = \lambda_i q_i; \\ p &\rightarrow p' = m\lambda\dot{q} = \lambda p. \end{aligned} \quad (26)$$

The transformation is precisely the inverse of Eq. 24. That is a good paradox.

The paradox comes because Eq. 26 has re-scaled coordinates without re-scaling parameters. Try it with the speed of light, $x = ct$. Under $t \rightarrow t' = \lambda_t t$, then $x \rightarrow x' = c\lambda_t t = \lambda_t x$. Changing units of one second to one hour changes the unit of a kilometer by a factor of 3600.

The correct parameter transformation properties can be found from the Hamiltonian. The Newtonian mass symbol m is defined by $H_N = p^2/2m$. The scaling transformation properties of time t , a spatial coordinate q , its conjugate momentum p , and mass m are:

$$\begin{aligned} t &\rightarrow t' = \lambda_t t; \\ q &\rightarrow q' = \lambda_q q; \\ p &= m \frac{dq}{dt} \rightarrow p' = m' \frac{\lambda_q}{\lambda_t} \frac{p}{m} = \frac{p}{\lambda_q} \\ m &\rightarrow m' = \frac{\lambda_t}{\lambda_q^2} m. \end{aligned} \quad (27)$$

The last relation tells us that Newtonian mass has the scaling properties of *time over length²*, or *seconds/meter²*.

Now just as we are accustomed to saying that the number for the speed of light is meaningless until it is expressed as a number of meters per second, or miles per hour, we need to get accustomed to mass as a number of seconds per square meter.

Example Consider a Newtonian object with mass $m = 3(\text{seconds}/\text{meter}^2)$ moving at $2\text{meter}/\text{second}$, carrying momentum $p = mv = 6/\text{meter}$ at position $q = 5\text{meter}$. Transform to $q' = 5\text{meter}(100\text{centimeter}/\text{meter}) = 500\text{centimeter}$ and $p' = 6/\text{meter}(1\text{meter}/100\text{centimeter}) = 0.06/\text{centimeter}$. The area of the initial phase space between the origin and the canonical coordinates is $\Delta A = \Delta p \Delta q = 5 \times 6 = 30$ and equals the area of the final phase space $\Delta p' \Delta q' = 500 \times 0.06 = 30$.

Example Under the force of gravity on Earth, an object falls with acceleration $g = 9.8\text{meter}/\text{second}^2$. The gravitational force F on a given mass $m_1 = 1\text{second}/\text{meter}^2$ is

$$F = m_1 g = 9.8 \frac{\text{seconds}}{\text{meter}^2} \frac{\text{meter}}{\text{seconds}^2} = \frac{9.8}{\text{meter} \cdot \text{second}}.$$

The work lifting the mass one meter is

$$work = m_1 g h = 9.8 \text{ meter} \frac{1}{\text{meter} \cdot \text{second}} = 9.8 \frac{1}{\text{second}}.$$

The power delivered by the force is

$$\begin{aligned} power = \vec{F} \cdot \vec{v} &= \frac{9.8}{\text{meter} \cdot \text{second}} \frac{\text{meter}}{\text{second}} \left(\frac{v}{\text{meter}/\text{second}} \right) \\ &= \frac{9.8}{\text{second}^2} v_{MS}, \end{aligned}$$

where v_{MS} is the dimensionless velocity measured in *meter/second*. Using a pulley or spring balance to apply the same force to a second object with mass $m_2 = 2 \text{ second}/\text{meter}^2$ produces an acceleration

$$a_2 = F/m_2 = \frac{9.8}{\text{meter} \cdot \text{second}} \frac{1}{2} \frac{\text{meter}^2}{\text{second}} = 4.9 \frac{\text{meter}}{\text{second}^2}.$$

4.1.2. How This is Related to Planck's Constant

To see how the discussion is related to quantum theory, use a theory that is relevant. Consider a standard wave equation:

$$\frac{\partial^2 \phi}{\partial t^2} - c^2 \vec{\nabla}^2 \phi + m^2 c^4 \phi = 0.$$

This equation comes from a ubiquitous linear Hamiltonian model. It is a trap to “derive” this equation using substitution rules of beginning quantum theory: they are circular. It is better to find the equation generic, as indeed it appears in the vibrations of any collection of oscillators that has an “optical” branch. By inspection the dimensions of the combination $m^2 c^4$ are seconds^{-2} . Then m scales like a Newtonian mass and we are entitled to call m a “mass parameter.”

Make the definition

$$\begin{aligned} \phi &= e^{-imc^2 t} \psi; \\ \frac{\partial \phi}{\partial t} &= e^{-imc^2 t} (-imc^2 \psi + \frac{\partial \psi}{\partial t}). \end{aligned}$$

This transformation removes the $m^2 c^4$ term. Continue to obtain the time evolution equation for ψ . Impose a low frequency approximation that drops the term proportional to $\ddot{\psi}$. The result is

$$i\dot{\psi} = -\frac{\vec{\nabla}^2}{2mc} \psi. \quad (28)$$

Here we have a model *frequency operator* $\hat{\Omega} = -\vec{\nabla}^2/2mc$ familiar from Schroedinger theory. We are still lacking \hbar , and in our approach, we will never find it in quantum theory.

Essentially the same analysis is done in Ref[12]. A more complicated frequency operator $\hat{\Omega} = -\vec{\nabla}^2/2m + U(x)$ represents an ansatz for interacting waves. We would not pretend to know the interaction function $U(x)$ from first principles. (The old predictive recipes, we noted, are just mnemonics and pedagogy.) We find U from data for electrons. Basic scattering theory allows one to invert the Born-level differential cross section of electron-atom scattering into $U(x)$. The same $U(x)$ predicts the observed Hydrogen frequency spectrum, which is quite non-trivial. Figure 1 shows an example of the frequency data of Rydberg[10]. Taken before 1900, the data was of surprisingly high quality. Several other data comparisons are consistent. The entire theory has only two parameters m and κ . The constant κ is dimensionless, as consistent with the results of data-fitting

$$U(x) = \frac{\kappa c}{|\vec{x}|},$$

and U correctly has dimensions of frequency.

We said we are only concerned with what is observable and testable. When using experimental data to fit the parameters of quantum theory \hbar is unobservable, and given up, *in our theory*. Figure 2 shows that we have done the work to fit parameters [12]. With basic information on the *frequencies* observed in the Rydberg spectrum, and the scattering *lengths* observed by Geiger and Marsden, etc. one derives κ and $\lambda_e = c/m_e$ directly. The numerical value of κ is about 1/137. By a natural coincidence Sommerfeld discovered the dimensionless constant κ and called it α , the fine structure constant. Dimensionless constants do not depend on the units used to compute them, so that the unobservable unit converter \hbar cancelled out for Sommerfeld. In none of this is the introduction of a conversion to archaic MKS units necessary, nor is it helpful. As Feynman must have said, “bothering to convert units with the meaningless constant known as \hbar is a total waste of time”.

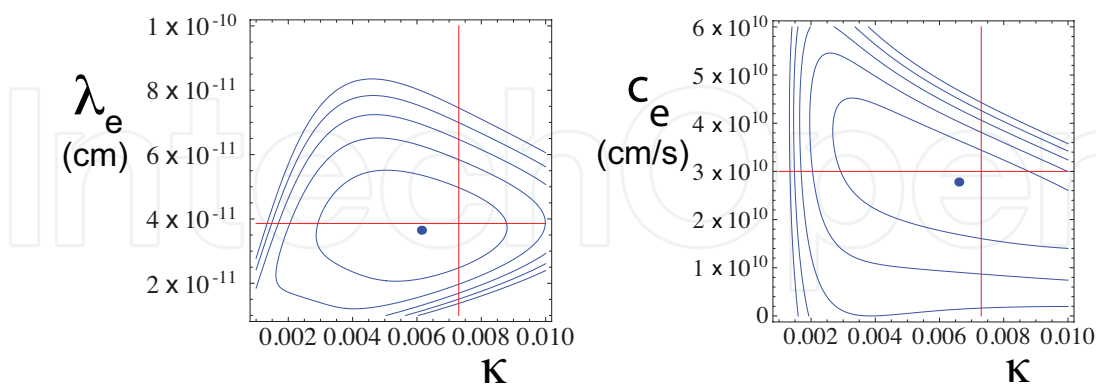


Figure 2. Contours of χ^2 , the summed-squared differences of data versus fit obtained from the data analysis of Ref. [12]. *Left panel:* As a function of parameters κ and λ_e with $c_e \rightarrow c$. *Right panel:* As a function of parameters κ and c_e with λ_e given by Compton’s 1922 experiment. Dots shows the points of minimum $\chi^2 \sim 0.24$ in both cases. Contours are $\chi^2 = 1, 2, 3, \dots$. Lines show modern values $c = 3 \times 10^{10} \text{ cm/s}$, $\lambda_e = 3.87 \times 10^{-11}$, $\kappa = 1/137$ all lie inside the range of $\chi^2 \lesssim 1$.

When one redundant unit is dropped, something is gained. The errors in the definition and inconsistent uses of the *kilogram* drop out. Continuing up the ladder, the system where mass

is measured in $\text{seconds}/\text{meter}^2$ is such that the accuracy of the best determinations of the electric charge and electron mass were improved[12] by a factor of order 100 compared to the official CODATA determinations[13].

4.2. A redundant convention

We come to see where Planck's constant entered human history. The space-time scaling properties of *mass* were overlooked early, which continues today, due to a Newtonian prejudice that *mass* is intuitively self-defined. For a long time *mass* was even thought to be a "constant quantity of matter, unchanging by the principle of conservation of mass." That led to a unit of mass unrelated to *meters* and *seconds*, and declared independent by defining a totally arbitrary unit known as the *kilogram*. Introducing an artificial reference standard was found acceptable for Newtonian physics: using artificial standards to weigh silver and butter was quite ancient and obvious.

Notice that introducing artificial unit conventions cannot be detected as faulty by math or logic. Intelligent technicians and business people use a huge array of unnecessary units daily, in many cases imagining that relations between them (such as 1 pascal = 0.000145 pounds-per-square-inch) must be "laws of physics." Once a redundant unit and its arithmetic enters the scaling laws it can stay around forever.

But as a price for these mistakes, a unit-conversion constant was needed in history to change black-body frequency in terms of temperature, which is energy, which is the frequency of the action, to black body frequency observed in the spectrum, which is frequency. *If and only if* one insists on measuring *mass* in kilograms, one needs a new symbol $m_{kg} = \eta m$, where *eta* has units of kgm^2/s . The value of the conversion constant η is arbitrary, just as the kilogram is arbitrary, and as Ref. [12] shows, fixing one predicts the other. (And that explains[7] the peculiar phenomenon of international unit standardization from global fits finding 100% correlation of the "experimental errors in the kilogram" with the "experimental errors in Planck's constant.")

There is a fast way to reach the same conclusions. The action principle is $\delta S = 0$. The right hand side "0" has no scale, and no units of S can be physically observable. By S we don't mean the action of some subsystem which can be compared to another to define an arbitrary fiducial unit, just like the kilogram. We mean the action of the Universe. That causes one overall constant that was defined in quantum pre-history "with the dimensions of action" to drop out.

At least in our approach, every quantum mechanical relation that involves \hbar is an ordinary relation not involving \hbar that has been multiplied by some power of \hbar on both sides: so that \hbar cancels out of everything observable.

Example: Although Planck is reported to have gotten his constant from black-body data, his original work shows otherwise. From his derivation and data fits [11] the most Planck could get was the ratio of \hbar/k_B , where k_B is Boltzman's constant. Planck thought k_B was extremely fundamental, although we now know it is nothing from Nature. It is a conversion constant of energy in temperature units to energy in energy units. The Newtonian convention for *mass* entered in k_B and \hbar both. The *kilogram* cancels out. When measuring quantum data with quantum data \hbar cannot be obtained [12], and it is nothing but convention to insert the *kilogram* so as to force a relation. The information was available in 1900. Indeed Wien's Law

used the classical adiabatic invariance of the action to relate the electromagnetic energy to its frequency. It was already known that the value of the Hamiltonian $H = -\partial S/\partial t$. Hence the clue of a redundant unit existed, and if the connection had been made, it might have led to getting rid of the *kilogram* before 1900.

Example: Due to pre-quantum nonsense everyone is obliged to learn, the quantization of angular momentum is blamed on the value of Planck's constant. One cannot possibly do without \hbar , it is claimed, due to the fundamental commutation relations of angular momentum

$$[J_i, J_j] = i\hbar\epsilon_{ijk}J_k. \quad (29)$$

Strictly deductive algebraic relations developed from those commutators produce the possible representations and the spectrum of observables $J_z = n\hbar$, where n must be an integer or half-integer. And this reproduces data. And in the limit of $\hbar \rightarrow 0$, the commutators go to zero, which is the classical limit of operators becoming c -numbers, etc.

We say: that kind of argumentation fails the quality-control standards of the current millennium. Groups and representations are fine organizing devices that don't originate in claims about physical existence. The representations of $SU(2)$ had been worked out before quantum mechanics wanted them. To get Eq. 29 one takes a set of dimensionless $SU(2)$ generators \tilde{J}_i and arbitrarily multiplies them by \hbar :

$$[\hbar\tilde{J}_i, \hbar\tilde{J}_j] = i\hbar\epsilon_{ijk}(\hbar\tilde{J}_k). \quad (30)$$

Then $J_i = \hbar\tilde{J}_i$ come to obey Eq.29. But by remarkable rules of algebra, multiplying both sides of any equation by the same constant does not change the equation. In Eq. 30 one sees that \hbar^2 cancels out: including that magic limit that $\hbar \rightarrow 0$ or $\hbar \rightarrow -17.3$ or anything else, so it also cancels out in Eq. 29.

The quantization of angular momentum eigenvalues is a fact that has nothing to do with \hbar . The "classical limit" has nothing to do with \hbar , assuming one can count and distinguish low quantum numbers from huge ones. The reason that \hbar is artificially spliced into Eq. 29, and other algebras, is so that 21st century measurements of angular momentum will be cast into an MKS unit system designed for 17th century Newtonian physics. Which continues to assess and obstruct quantum mechanical data by interposing the universal *kilogram* which is not a feature of *Nature*.

5. Observables, quantization and bracket relations

We seek to get as much from the theory as possible without making unnecessary postulates:

5.0.0.1. Operators as Physical Observables

The postulate that operators are "physical observables" is redundant: *in our approach*. We find using it slyly abuses language that first defined observables as numbers, in order to slip in the operator as a philosophically transcendent realization of physics. That's too cheap. Very simply, the wave function is observable, and using operators to probe a wave function may be convenient, but that is not independent.

5.1. Observables as numbers

An *observable* is a number extracted from the wave function as a projective representation, meaning that $\psi \equiv z\psi$ for any complex z . This symmetry exists in the equation of motion. We wish to call attention to it as a feature of “quantum homogeneity.” It makes the information in $|\psi\rangle$ equivalent to the information in the density matrix $\rho_\psi = |\psi\rangle\langle\psi| / \langle\psi|\psi\rangle$. (The bracket notation, suppressed before to reduce clutter, is useful here.) Equivalence exists because if ρ_ψ is given then its unit eigenvector predicts $|\psi\rangle$ up to a constant z . (Later we discover that density matrices of rank-1 are too special to base the full theory upon, but that comes with development of quantum probability, which is not yet under discussion.)

It is convenient to extract a number using an “operator sandwich”. We define an observable $\langle \hat{A} \rangle$ as the number from the map

$$|\psi\rangle \rightarrow \langle \hat{A} \rangle = \frac{\langle \psi | \hat{A} | \psi \rangle}{\langle \psi | \psi \rangle} = \frac{\text{tr}(\hat{A}\rho_\psi)}{\text{tr}(\rho_\psi)}.$$

The last relation is general for cases where ρ is not so simple as *rank* – 1. Note we need nothing from quantum probability to make the map. Instead of prescribing $\langle \psi | \psi \rangle$ with a normalization postulate, we maintain it is simply unobservable, and drops out. We may then set $\langle \psi | \psi \rangle \rightarrow 1$ to simplify expressions.

The trace (symbol *tr*) acts as an inner product between operators. Let \hat{A}_i be a normalized complete set of operators, which is defined by $\text{tr}(\hat{A}_i^\dagger \hat{A}_j) = \delta_{ij}$. Since the set is complete,

$$\rho = \sum_j A_j \text{tr}(\hat{A}_j^\dagger \rho) = \sum_j \langle \hat{A}_j \rangle \hat{A}_j.$$

Thus ρ is equivalent to a number of observables, and $|\psi\rangle$ is observable to the exact extent it is defined. At some point this simple relation seems to have been re-packaged as “quantum holography.” The very late date of realizing the wave function is observable (to the extent it is defined) supports our case that quantum mechanics is still “emerging” from its history.

5.1.1. Eliminating More Postulates

There is no particular reason for us to postulate that \hat{A} must be Hermitian. If it is not Hermitian the operator sandwich gives a complex number, equivalent to two real numbers and two observables, because any operator is the sum of a Hermitian operator and i times a Hermitian operator. As for complex numbers being observable in the lab, mathematics tells us that complex numbers are real pairs with 2-vector addition and multiplication rules. It is not unusual to observe such number pairs that have phase relations like q_i, p_i which map directly into ψ_i , a complex number. So there is no reason for the 18th century trick of scaring people with complex numbers. And yet: given that ρ is Hermitian no harm is done by restricting the operators for observables to being Hermitian. One more grand postulate turns to clay.

At least in our approach, those matters of definition need no *foundation postulates*. Neither is there a good reason to insist that an observable be an eigenvalue of some especially known

operator. The interpolation of that rule seems designed to create conflict. Checking a bit of physical data, measuring the eigenvalue of an operator intended is rare. For example experimentalists have been measuring neutrinos for decades while the theorists argued about their operators. We think that measuring a number and calling it an eigenvalue of “some operator” is meaningless. If and when an eigenvalues of a known operator appear in a data set, one obtains that fact that $|\psi\rangle$ will then be the corresponding eigenvector of \hat{A} automatically. We also don’t need to discuss “compatible and incompatible” observers in terms of operators that commute. In this century everyone knows how that math works.

5.1.2. Bracket Relations Are Kinematic

The quantum map shows that complex ψ_i are canonical coordinates, up to i :

$$\psi_i = (q_i + ip_i)/\sqrt{2}.$$

A factor of i is tolerable, in that Hamilton’s equations are recognizable including it:

$$i\dot{\psi} = \frac{\partial H}{\partial \psi_i^*}, \quad i\dot{\psi}^* = -\frac{\partial H}{\partial \psi_i}.$$

Thus $i\psi_i^*$ is the canonical momentum conjugate to ψ_i . Poisson brackets (PB) are canonically invariant, so that transcribing them to ψ, ψ^* involves only a factor of i .

In fact the PB relations among our observable are rather simple due to the decision to make observables bilinear in ψ, ψ^* . The most famous application was the early desire to project the wave function into a few particle-like observables \vec{Q}, \vec{P} , mistakenly thought to be important from Newtonian bias. For \vec{Q} to represent a translational coordinate it must transform properly:

$$\begin{aligned} \text{given } \psi(x) &\rightarrow \psi_{\vec{a}}(\vec{x}) = \psi(\vec{x} - \vec{a}), \\ \text{then } \vec{Q} = d^3x \psi^* \hat{Q} \psi &\rightarrow \vec{Q} \rightarrow \vec{Q} + \vec{a}. \end{aligned}$$

There are few choices but $\vec{Q} = \vec{x}$. The test that a candidate variable P_i is conjugate to these Q_i needs the bracket $\{Q_i, P_j\}_{PB} = \delta_{ij}$.

Write this out, assuming operator sandwiches:

$$\{Q_i, P_j\}_{PB} = -i \sum_x \left(\frac{\delta Q_i}{\delta \psi_x} \frac{\delta P_j}{\delta \psi_x^*} - \frac{\delta P_j}{\delta \psi_x} \frac{\delta Q_i}{\delta \psi_x^*} \right).$$

Computing the derivatives gives

$$\{Q_i, P_j\}_{PB} = -i \int dx \psi_x^* [\hat{Q}_i, \hat{P}_j] \psi_x.$$

This must be true for all ψ . Quantum homogeneity (the irrelevance of $\langle \psi | \psi \rangle$) then obtains the map between the operator algebra and Poisson bracket

$$\{Q_i, P_j\}_{PB} = \delta_{ij} \rightarrow [\hat{Q}_i, \hat{P}_j] = i\delta_{ij} \quad (31)$$

The other consistency relations of “quantization” are similar. A PB algebra predicts a commutator algebra, as follows: If $\{A, B\}_{PB} = C$ is true for general $|\psi\rangle$, and all quantities are operator sandwiches, then $[\hat{A}, \hat{B}] = i\hat{C}$ follows by identity. An interesting application comes from the lack of any non-zero commutator with the unit operator “1.” It tells us that $\langle \psi | \psi \rangle$ does not transform, time evolve, nor give a non-trivial result, so it is a conserved “momentum” of the theory that drops out as unobservable.²

The early history of quantum theory found the map between Poisson brackets and commutators profound, and it tends to still be viewed that way. In our approach it emerges on its own as useful, but automatic. If one chooses operators that satisfy the bracket-commutator rules, then their observables transform as they are intended, and vice-versa. It is not really necessary to cast around and discover operators by trial and error. Noether’s Theorem will manufacture any number of generalized conjugate P_i from point transformations on Q_i as the “charges” of conserved (or un-conserved) currents[7].

It is interesting that in retrospect the original Heisenberg program, based on Poisson brackets, guaranteed such an outcome. The virtue of bracket relations lies in generating Lie algebra and related relations that are inherently coordinate-free. Once a given algebra is transcribed to a different *notation*, it is not going to produce new results. Thus when Heisenberg transcribed the Poisson bracket algebra of Hamiltonian time evolution to a different notation he was building up a classical Hamiltonian dynamics of ordinary kind, if it was not recognized at the time.

The big advance, as mentioned before, comes with the physical model of electrons having an infinite number of *dof*, as Eq. 31 requires, and as found in a *wave theory*. That fact was supposed to be evident in the spectrum of atoms showing a (practically) infinite number of normal mode frequencies. That in turn could have been done by 19th century classical physicists, who knew about spectra and normal modes. And indeed Stokes, Kelvin and Lorentz[9] all deduced the facts that atoms are vibrating jello from such clues before 1900. Lacking any technology to test the speculation, they made little of it which is a pity. Immediately *after* 1900, the cult of the quantum of action went the way of postulating mistakes that could not be expressed without Planck’s constant. *All of that can be dropped.*

5.1.2.1. Quantization

The PB -commutator relation of Eq. 31 is commonly called the “quantization” principle according to the recipe of Heisenberg or Dirac, which (being a postulate) cannot be explained. While that is what those gentlemen believed, it is not our approach.

Once the physicists have committed to a linear, Hamiltonian theory, there is very little left to determine except its dimension. As already mentioned Eq. 31 realized with $x_i, \partial_j = -i\partial/\partial x_i$ requires a space of a continuously infinite number of degrees of freedom: waves. Less

² Similarly, the center of mass momentum of the Universe in Newtonian physics is unobservable.

information exists in the algebra than in the direct and simple model of waves. For one thing, the algebra is kinematic, and will work for any Hamiltonian, including non-local ones that do not seem to be observed. In comparison the wave model predicts the algebra, because it contains everything, so it is superior.

It is sometimes thought that “field quantization” proves that quantization principles are a golden road. But what’s kinematic on one space is kinematic on a larger space. If one believes there should be wave functions for classical fields, one defines quantum field theory straightforwardly. It also happens to be equivalent to the space made from products of an arbitrary number of beginning quantum systems, which is neat, but which again shows that invoking the quantization principle was redundant.

Finally, we find there is a perception that abstract operator methods are superior just because they are difficult. It is seldom noticed that an unlimited amount of tortuous and clever operator manipulation can never have more information than just solving the differential equation, which predicts everything, and (in fact) all the differential equations of quantum mechanics are already “solved” by Eq. 20.

These are reason we wrote in the Introduction that the viable models are a higher accomplishment than framework.

In our approach the state space is not going to be predicted by a simplistic algebraic transcription. Finite dimensional quantum models are known and hardly useless: they are models of spin, and molecular rotational dynamics. Finite dimensional models of quantum field theory are known. They are called “lattice theories.” The dimensionality of quantum models has no restriction. Leaving the dimension free to grow without limit is what transpired, and one of the reasons the subject is so flexible it cannot fail.

6. Quantum probability

In this Section we explore the origins of probability in our approach. Quantum probability is an old subject with many contributions we cannot possibly review. There is some agreement that the Born rule should be “a Theorem, not a Principle”[14], although our approach is not quite the same. In the first place we must dismiss a common misconception that quantum probability contradicts classical probability, or is inconsistent with it, by defining each.

6.1. Define terms

Probability itself is a subtle topic. It is not well-defined until “objects” are categorized for the purposes of probability. Classical probability (*CP*) of frequentist kind is about classifying objects into mutually exclusive (*me*) equivalence classes³, and assigning numbers to the information by counting. Distributions are a useful tool of *CP*. Quantum probability (*QP*) allows such classifications but does not insist on them. Instead *QP* is a projective map from a system’s state, represented by density matrix, into a number. Distributions sometimes exist in *QP*, but are not always compatible.

³ Although “equivalence classes” are often mutually exclusive sets by definition, we use the term more broadly, and add *me* when the term is intended.

The breakthrough of quantum probability, we believe, lies in generalizing the notion of probability so as not to *insist* on pre-ordained equivalence classes. Vectors are categorized so they have a great chance of being nearly equivalent. Physicists seem to believe that “quantum objects” from Nature are needed to make sense of quantum probability, and vice-versa. But nothing from physics is involved in developing an accounting system where different vectors are not automatically treated as mutually exclusive. While we will extensively use physics and its examples, the ultimate goal of this Section is not to depend on physics.

6.2. Discovering quantum probability with a hidden-variable map

Begin with a remarkably simple map from classical to quantum probability which illustrates the necessary class ideas. Let $|D\rangle$ be a big vector we call “data”. By means which are quite arbitrary it is partitioned in a collection of smaller vectors $D_i^J = \langle i|D\rangle$ with names $J = 1 \dots J_{max}$ and components $i = 1 \dots i_{max}$. We tentatively interpret J as labeling the sample number taken from some (deterministic or random) process. The other index is interpreted as describing “objects”. The sample space and object space are tentative because certain operations will mix them, as we will see. Formally D_i^J exists on the direct product of spaces of dimension $i_{max} \otimes J_{max}$, upon which there are certain transformation groups and invariants. For convenience the record is normalized $\langle D|D\rangle = 1$ in the usual way, removing one number set aside.

Now we seek a notion of orderliness or physical regularity. We will expand the vectors in an orthonormal basis set $\{|e^\alpha\rangle\}$, where $e_i^\alpha = \langle i|\alpha\rangle$, and seek some form of statistical repetition. The basis matters, so which basis is used? Every data record actually has two preferred bases, in which the expansion is diagonal:

$$|D\rangle = |e^\alpha\rangle \Lambda^\alpha |s^\alpha\rangle. \quad (32)$$

This is the singular value decomposition (*svd*), which is unique. It is proven by diagonalizing two correlations (matrices) that automatically have positive real eigenvalues:

$$\begin{aligned} DD^\dagger &= \sum_\alpha |e^\alpha\rangle (\Lambda^\alpha)^2 \langle e^\alpha|; \\ D^\dagger D &= \sum_\alpha |s^\alpha\rangle (\Lambda^\alpha)^2 \langle s^\alpha|. \end{aligned} \quad (33)$$

Notice that the decomposition yields vectors which are orthonormal on their respective spaces. Notice that the vectors are defined up to a symmetry:

$$\begin{aligned} |e_\alpha\rangle &\rightarrow z_e |e_\alpha\rangle; \\ |s^\alpha\rangle &\rightarrow z_s |s^\alpha\rangle; \\ |D\rangle &\rightarrow |D\rangle. \end{aligned}$$

Here z_e, z_s are arbitrary complex numbers. The factor vectors are eigenvectors which have no normalization except the normalization given by convention. By phase convention the

singular values are positive real numbers, $\Lambda_\alpha > 0$. The singular values are invariant with the data is transformed by arbitrary and different unitary transformations on the object and sample spaces.

The interpretation of each term $|s^\alpha\rangle |e_\alpha\rangle$ summed in Eq. 32 is a *strict correlation* of a unique object vector labeled α with a unique sample vector labeled α .

Example Suppose the data consists of integer numbers of objects that are $|apple\rangle$ or $|orange\rangle$. This is classical *me* data: by existing in different spaces, $\langle apple|orange\rangle = 0$. Sufficiently fine sampling will produce samples which are either 1 or 0. Typical data is then

$$|D\rangle = (|apple\rangle, 0, |orange\rangle, |apple\rangle, |orange\rangle, \dots |apple\rangle).$$

The expression only makes sense if these *me* objects are normalized, $\langle apple|apple\rangle = 1$, and so on, else the normalization would conflict with the number of apples. Expand in the natural basis we started with,

$$|D\rangle = |apple\rangle (1, 0, 0, 1, 0, \dots 1) + |orange\rangle (0, 0, 1, 0, 1, \dots 0). \quad (34)$$

The diagonal form of *svd* has appeared, up to a normalization. Whenever data consists of disjoint *me* objects, one can show those same objects are *automatically* the *svd* factors. The fact of strict correlation comes with projecting onto one object such as $|apple\rangle$ and producing its sample vector, which is automatically orthogonal to all the other sample vectors:

$$\langle apple|D\rangle = (1, 0, 0, 1, 0, \dots 1).$$

Conversely, selecting one of the *me* sampling histories automatically selects a unique *object*. These are features of classical “events.”

To reach the *svd* form implies samples that are normalized: $\langle s^\alpha | s^\beta \rangle = \delta^{\alpha\beta}$. Let N_{apple} be the total number of apples observed. Let N_{tot} be the total of apples and oranges. Remember that we normalized our data. Then

$$\begin{aligned} (1, 0, 0, 1, 0, \dots 1) &\rightarrow \\ &\sqrt{N_{apple}/N_{tot}}(1, 0, 0, 1, 0, \dots 1) / \sqrt{N_{tot}/N_{apple}} \\ &= \sqrt{N_{apple}/N_{tot}} |s^{apple}\rangle. \end{aligned} \quad (35)$$

Once normalized we can read off the singular values:

$$\begin{aligned} |D\rangle &= \sqrt{N_{apple}/N_{tot}} |s^{apple}\rangle |apple\rangle \\ &+ \sqrt{N_{orange}/N_{tot}} |s^{orange}\rangle |orange\rangle. \end{aligned}$$

Now suppose there is a unitary transformation of our data on either object of sample or both spaces - but not mixing them. The result will involve linear combinations of the form $\alpha|apple\rangle + \beta|orange\rangle$, which is classically “taboo.” For better or worse, we cannot stop linear transformations from being used or being useful. Whatever the coordinate system, we can construct the *svd* factors and singular values as invariants.

In many cases (and always in physics) we are actually forced to suppress some detail by summing over unwanted or unrecorded details of the sampling history. That was already done in Eq. 33. We use that observation as the first example in constructing the *density matrix* ρ_{object} of the object system:

$$\rho_{object} = tr_s(|D\rangle\langle D|) = \sum_{\alpha} |e^{\alpha}\rangle (\Lambda^{\alpha})^2 \langle e^{\alpha}|.$$

The form ρ_{object} we traced-out the sample history. This identity now *defines* the *me* $|object_{\alpha}\rangle = |e_{\alpha}\rangle$. By construction, whenever *me* data is used, we have a convenient invariant formula for the probability of finding such an object:

$$P(|e_{\alpha}\rangle | \rho_{object}) = \langle e_{\alpha} | \rho_{object} | e_{\alpha} \rangle = tr(\rho_{object} | e_{\alpha} \rangle \langle e_{\alpha} |).$$

The symbol $P(|e_{\alpha}\rangle | \rho_{object})$ is read as the probability of $|e_{\alpha}\rangle$ given ρ_{object} , and exactly coincides with counting numbers: thus

$$P(|apple\rangle | \rho) = (\sqrt{N_{apple}/N_{tot}})^2 = N_{apple}/N_{tot}.$$

In general form the probability P to get an observable $\langle \hat{A} \rangle$ is

$$P = tr(\rho \hat{A}) / tr(\rho). \quad (36)$$

6.3. The quantum-style agreement

We propose an agreement on how data will be managed: *we agree to describe a system using its density matrix*. We give up the possibility of keeping more information, because it is efficient not to have it.

Review the *apples – oranges* discussion with a physical example where transformations are natural. Suppose observations consist of events with 3-vector polarizations $|\mathcal{E}\rangle$. Moreover, only two orthogonal components $|e_1\rangle, |e_2\rangle$ are measured. A generic data set including the sampling history will not generally consist of *me* events, but combinations of the form

$$|D\rangle = (|\mathcal{E}_1\rangle, 0|\mathcal{E}_2\rangle, \dots |\mathcal{E}_n\rangle)$$

which is expanded in the basis

$$\begin{aligned} |D\rangle = & |e_1\rangle (\langle e_1 | \mathcal{E}_1 \rangle, \langle e_1 | \mathcal{E}_{12} \rangle, \dots, \langle e_1 | \mathcal{E}_n \rangle) \\ & + |e_2\rangle (\langle e_2 | \mathcal{E}_1 \rangle, \langle e_2 | \mathcal{E}_2 \rangle, \dots, \langle e_2 | \mathcal{E}_n \rangle). \end{aligned}$$

In the form above we have both the events and the sample history. The samples are not mutually exclusive, but naturally fall into the corresponding projections. That can be called the “underlying reality.” Meanwhile there exists a unitary transformation on the objects and samples where this arbitrary data will be a sum of strictly correlated, *me* elements which are indistinguishable from classical events. The difference between that interpretation and the quantum-style one is a coordinate transformation not available from the density matrix, so it becomes meaningless.

Normalize the sample history and take the trace over the sample space to form the density matrix. Upon reaching that level, one cannot distinguish the system from one where every event actually was a product of mutually exclusive object and sample vectors. This information is deliberately lost in forming the data categories. That makes it consistent and unique to define probability using the Born rule. No subsequent experiment can make it false. Probabilities defined by naive counting of integer-valued data will agree exactly.

6.3.1. Outcome Dependence

“Outcome dependence” is the name given to statistics that depend on the order of measurement. Our procedure has outcome dependence. Suppose one is selecting channels by simple filters. A “measurement” β uses a state $|\beta\rangle$ and an associated projector $\pi_\beta = |\beta\rangle\langle\beta|$. A series of measurements $\beta, \gamma, \eta\dots$ in that order, yields

$$P(\beta, \gamma, \dots, \eta|\rho) = \text{tr}(\pi_\eta \dots \pi_\gamma \pi_\beta \rho \pi_\beta \pi_\gamma \dots \pi_\eta).$$

The projective and non-commutative nature of this kind of probability is self-evident. It follows immediately that no classical distributions can reproduce this kind of probability in general.

The famous Bell inequalities[15] dramatize this fact, yet there was nothing new in finding that distributions fail in general. It would have been extraordinary for probability based on density matrix projections to be equivalent to distributions in the first place. Years after Bell, Werner[16] formulated the criteria called “separability” of density matrices, which when true allows QP to coincide with CP as formulated with distributions. Thus the classical probability rules exist inside of QP, when and if a special case happens to occur. Conversely QP in our approach is a more general extension of the concepts and rules of CP that does not contradict any of it. We will argue that QP is so general there are no restrictions on how it might be used.

About Disturbing Measurements: It is a geometrical fact that any vector can be considered to be any other vector, plus the difference. It takes one step to make the difference orthogonal, by writing

$$|a\rangle = |b\rangle\langle b|a\rangle + (|a\rangle - |b\rangle\langle b|a\rangle).$$

The term in braces is orthogonal to $|b\rangle$, given normalized vectors. Up to an overall scale, the vector $|b\rangle$ that is literally pre-existing in vector $|a\rangle$ is $|b\rangle\langle b|a\rangle$. This decomposition, of course, does not come from physics. We see that agreeing vectors are “equivalent” up to a scale is a prelude to counting them as equivalent for probability purposes.

It is quite important (we repeat) that the coefficient $\langle b|a \rangle$ happens to measure the amount of vector $|a \rangle$ that is pre-existing and pre-aligned with $|b \rangle$. It is very reasonable that a quiet, non-disturbing “physical measurement” of a system will filter out a pre-existing component of a vector variable, and pass it through undisturbed. Undisturbed filtering is precisely what happens in textbook discussions of polarizers, Stern-Gerlach, and diffraction gratings. We wish it would be mentioned it has nothing to do with \hbar (to repeat). Using the undisturbed, pre-existing projection also totally contradicts the old line of thinking that ‘measurement’ involves an uncontrollable disturbance of the system due to the finite quantum of action. Indeed if one believed that, the overlaps would change by the process of measurement, and the Born rule would fail.

6.3.2. Division and Reduction

The conventional approach to “quantum interacting systems” holds that system A “exists on” space A , system B “exists on” space B , and when they interact the joint system exists on the direct product of spaces $C = A \otimes B$. This is physically puzzling, and we think backwards. Instead we use our idea of equivalence classes developed by partitioning information, or “division”.

Given any vector on a space C , we may partition it into factor vectors on procedurally-defined spaces A and B , as used above. For example a vector of 40 dimensions can be written as the product of vectors on 4 dimensions and vectors on 10 dimensions, or products of 5×8 , etc.

Division is particularly well-developed with Clebsch-series done in the inverse direction: discovering what smaller group representations can be composed to make a given bigger one. A more straightforward division groups a data vector’s components into adjacent bins of sub-dimension i_{max} and names J :

$$D_a \rightarrow D_{ij} \quad J = \text{int}(a/i_{max}); \quad i = \text{mod}(a, J),$$

where int takes the integer part, and $\text{mod}(a, J)$ returns the remainder of a/J . Arnold’s famous “cat map” is an example. The freedom to choose the bins and dimensions is very important.

Division is quite coordinate-dependent. Division can be repeated to divide the factors, and make subdivisions. The process of vector “division” is not profound mathematics, but the arbitrariness is important for physics and data manipulation. Given a particular division, the physicist (knowingly or not) inspects the decomposition searching for simplicity and regularity to “emerge”. The factor-states that turn out to make physics easy become well known under many terms...electrons, photons, quarks, etc. The interesting question of the ultimate meaning of such entities is discussed in Section 7.2.

Given a density matrix ρ_{AB} defined on C , the decision not to study an observable with a non-trivial operator on space B allows us to prepare the density matrix

$$\rho_A = \text{tr}_B(\rho_{AB}).$$

Here tr_B sums the diagonal elements of the labels on space B . This defines reduction in the conventional way. Reduction is inevitable in physics because physics measures very little.

The rank of a density matrix obtained from reduction depends on how the reduction was done. Obtaining a $rank - 1$ reduced matrix is exceptional. Finding such systems in the laboratory requires great ingenuity. That is why we treat systems that can be described with wave functions as special cases. That may seem to put our dynamical framework discussing wave functions somewhat askew relative to the probabilistic one. Section 6.3.4 explains why and how Hamiltonian time evolution remains relevant.

6.3.3. What Are Those Hidden Variables?

It is remarkable that Bell's artful introduction of distribution theory[15], which is inappropriate for quantum-style systems, led to a false perception that "hidden variables" had been excluded. The hidden variables of ordinary quantum mechanics are the physical degrees of freedom (wave function or density matrix projections) on those spaces the physicist ignored in setting up his oversimplified model. There are always such spaces in Nature.

There is no limit to the number of products or their dimensionality that can be used as "data" or "sample vectors." Partitioning vectors on any given space into a number of factors is a highly coordinate-dependent business. It is essentially a map from one sort of index to a number of composite indices, which can always be done linearly: $D_k \rightarrow D^{abc\dots} = \Gamma_k^{abc\dots} D_k$, where Γ is an array of constants. In a sufficiently large data vector the "sample" and "object" spaces can be re-configured in practically infinite variations. There is very little that is invariant about entanglement when we allow such freedom. Nature cannot possibly care about these coordinate conventions: Pause to consider how it affects physics.

In early days the Hilbert spaces of single electrons or single photons were considered utterly fundamental. They were building blocks for lofty postulates that could not be explained. Yet quantum mechanics was long ago enlarged to develop quantum field theory (*QFT*). For some purposes *QFT* is considered to have no new information on quantum mechanics itself, while defining very complicated quantum models. Yet Nature has subtleties. Basic non-relativistic quantum mechanics is incapable of dealing with the very questions of causality and non-locality that cloud interpretation of measurements. Relativistic *QFT* deals with issues of causality directly in terms of correlation functions with very well-defined properties. On that basis *QFT* is the more fundamental topic: it is big enough to support realistic models.

The relation of ordinary quantum mechanics to field theory is then developed by reduction, where unobserved *dof* are integrated out. As a result all of the phenomenology of ordinary quantum mechanics is subject to the hidden variables known to have been integrated over in developing density matrices that actually occur. This is ignored in ordinary quantum mechanics seeking by itself to be "fundamental." It is not logically consistent to ignore what is known. Now we have shown how naive probabilities of counting emerge from density matrix constructions integrating over quite arbitrary sample spaces. It is hard to escape the inference that the probability interpretation – which beginning quantum mechanics could not explain about its own framework – must certainly originate in reduction of interacting systems of *QFT* down to the experimentally crude probes developed in beginning quantum mechanics.

Infrared Example: There exists certain pure states of the *QFT* called bare electrons: the quanta of a free field theory. If such a state actually participated in an experiment we

doubt we'd have a statistical explanation for its behavior in the same free field theory. But that dynamics is too trivial to describe anything, or even permit participation, because free electrons are free. The so-called bare electron of free field theory has never been observed and cannot in principle be observed. All electrons and all observable states are "dressed." Calculations addressing infrared divergences find that electrons with zero photons are unobservable, or have zero probability to participate in reactions, as known from the ancient time of the Bloch-Nordseik analysis[21]. When the experimenter finally specifies his experimental resolution adequately, the probabilities of events emerge from density matrix steps integrating over unobserved quanta, exactly as we have discussed, *yet in such a technical fashion that its relation to beginning questions of quantum probability is never recognized.* The same facts also occur for all degrees of freedom not directly pinned down by experimental probes, which is most of them, that are *much more difficult* to categorize. Every single physical experiment involves so many uncontrolled variables that a statistical description via density matrices cannot be avoided. It is a rare experiment that even finds a single wave function will model the data: and a rare experimenter who can tune his instruments to make that expectation come out.

6.3.4. Invariants Under Time Evolution

At the level of *QFT* one can assert certain wave functions exist, and model them with the Hamiltonian time evolution we cited earlier. That is the state of the art, and returns to how the Hamiltonian dynamics is relevant once again.

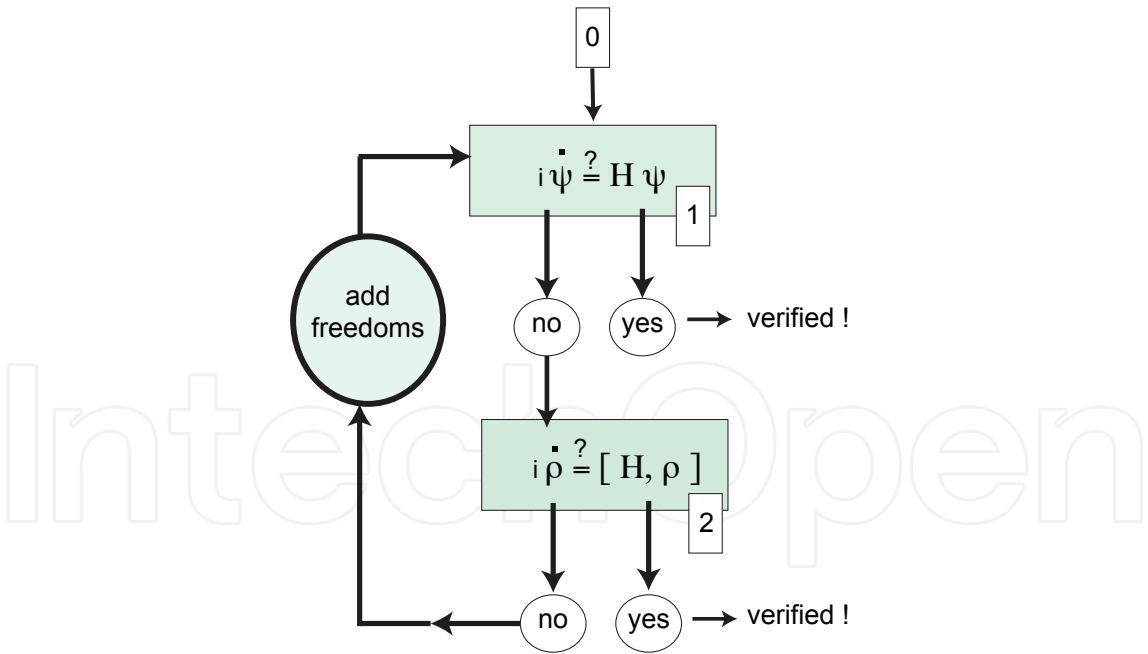


Figure 3. *How to test Quantum Dynamics:* Tests begin at point 0 with maximal optimism, and assuming the Schroedinger equation applies. At stages 1, 2 tests of dynamics and statistics against observations are made. Dynamical freedoms will be added by the user when a test fails: The system dimension is increased. The procedure flows around a closed loop. The framework cannot be falsified; All outputs verify quantum dynamics.

When probabilities refer to physical measurements, it is important that they be invariants of the system being measured. Invariants refer to some definite transformation group. The

symmetries of (our approach) to quantum dynamics are $Sp(2N)$. There is no precedent to define quantum probability with that symmetry. By adopting a linear dynamical model with Hermitian $\hat{\Omega}$ the dynamical symmetry group is no worse than $U(N)$. This is easier to work with.

The relation of the groups in our approach is very intimate. Consider the largest set of transformations preserving the unobservable $\langle \Psi | \Psi \rangle = \sum_j^N (p_j^2 + q_j^2)/2$. That is $O(2N)$, a relatively large group overlooked in ordinary quantum discussions. The intersection with the actual symmetries $O(2N) \cap SP(2N) \sim U(N)$, where \sim means isomorphic after complexification. Thus invariance of a probability notion under $U(N)$ is enough for consistency.

Once more our motivation differs from the traditional one. Tradition asserts - blindly and falsely - that since $U(N)$ is the symmetry group of Schroedinger's equation, the notion of total probability must be preserved under $U(N)$. We can't buy that. We don't have a reason to preserve the precious *notation* of the Schroedinger equation. (It is *less general* than Hamilton's equations.) What we buy is the fact that $U(N)$ time evolution will not destroy a $U(N)$ invariant probability, if and when the time evolution is that simple. It seems unwise to expect more than that from physics.

Supposing the system is so orderly on a chosen space, one cannot be sure of its effective dynamics for a density matrix on a reduced space. This contradicts lore of the Von Neumann (vN) equation, which is "derived" by methods hoping it might be correct[18]. The equation is equivalent to predicting the time evolution $\rho(t) = U(t)\rho(0)U^\dagger(t)$, where $U(t) = \exp(-i\hat{\Omega}t)$. Notice the traditional context assumes that symbol " ρ " refers to a unique object space, while we recognize that concept is procedural. Once a particular division and reduction has been done, it is enough for a single eigenvalue of ρ to be time dependent for the vN equation to fail. While the Von Neumann equation is true by definition in textbooks, it is seldom true in experimental practice. That is because almost all physical systems which are "dirty" enough to need a density matrix are also dirty enough to interact with the environment and spoil the assumptions. There are schemes ("Lindblad theory")[19] to cover the gap. If sometimes a good phenomenology, the cannot be considered general. When energy and interactions leave a subsystem they go into the larger system to return on any number of different time scales. It is not possible in principle for a first order dynamical system to contain enough initial conditions to parameterize all possible cases.

Once every system is a subsystem of a larger system, we should never expect to *always* predict dynamics.

How the Framework Never Fails: Nevertheless physicists put great faith in the fundamental existence of a wave function on the largest space they are thinking about. That makes a puzzle of why their faith persists. There is a question of whether that framework can be falsified. We do not believe it is possible to falsify the framework. Any system that fails the test of Hamiltonian evolution can be embedded in a larger system. On the larger system it's always possible to "unitarize" any transformation. One method is the "unitary dilation" found by Sz.-Nagy. Figure 3 illustrates the more painful process that physicists follow. By now particle physicists have added numerous quantum fields to the early quantum theory of electrons and photons following the process the figure illustrates. The infinite capacity of theory to expand practically terminates questions of whether such a theory could fail.

7. Applications

As mentioned in the Introduction, our main goal is to take advantage of the efficiency and flexibility of quantum-style data descriptions. We have remarked that description by quantum-style methods is deliberately incomplete. At the same time a great deal of practical experimental information from physics is encoded in wave functions and density matrices by default usage. That is a clue on how to proceed.

7.1. The optimal patterns defined by data

Given any “data” as a set of numbers, real or complex, and choosing a method of division, one can start making density matrices and classifying data. But in what sense is this intrinsic and why should it be powerful?

As before let D_x^J stand for the J th instance of data D_x . The index x now may stand for multiple labels x_1, x_2, \dots, x_n of any dimension. We are interested in the *patterns* of fluctuations which tend to occur in the entire data set. We define a “pattern” $|e\rangle$ as a normalized vector with projections $e(x) = \langle x|e\rangle$. To quantify the importance of a particular pattern, we calculate its overlap-squared summed over the entire data set:

$$E = \sum_J^{J_{tot}} |\langle D^J | e \rangle|^2.$$

Define the optimal pattern as having the largest possible overlap, subject to the normalization constraint. Algebra gives

$$E = \langle e | \rho | e \rangle; \quad \rho = \sum_J^{J_{tot}} |D^J\rangle \langle D^J| \quad (37)$$

$$\frac{\delta}{\delta |e\rangle} \left(\frac{\langle e | \rho | e \rangle}{\langle e | e \rangle} \right) = 0;$$

$$\rho |e_\alpha\rangle = \Lambda_\alpha^2 |e_\alpha\rangle$$

The eigenvalue problem tells us that a *complete set* of solutions $|e_\alpha\rangle$ generally exists; since $\rho = \rho^\dagger$, the patterns are automatically orthogonal, and eigenvalues Λ_α^2 are real and positive. To interpret the eigenvalues note the overlap $\Omega_\beta = \langle e_\beta | \rho | e_\beta \rangle = \Lambda_\beta^2$. Sorting the eigenvalues $\Lambda_1^2 > \Lambda_2^2 > \dots \Lambda_N^2$ to makes an optimally convergent expansion of the data expressed in its own patterns.

This result attributed to *Karhunen-Loeve*[17] is a foundation point of modern signal and image-processing schemes of great effectiveness. Optimally compressing data while retaining a given overlap is done by truncating the expansion of Eq. 37. We think it is no accident that the density matrix appears in Eq. 37.

7.2. The principle of minimum entropy

How does one evaluate a system, and in particular the *division* of data we mentioned earlier has tremendous flexibility? For that we consider the system's *entropy* \mathcal{S} . The definition is

$$\mathcal{S} = -\text{tr}(\rho \log(\rho)). \quad (38)$$

For a normalized $N \times N$ density matrix the absolute minimum entropy is $\mathcal{S} = 0$, if and only if ρ has *rank* -1 . The minimum entropy state is the most orderly, least complicated, and is hardly typical. Developing $\mathcal{S} = 0$ from samples requires every single sample pattern to be a multiple of every other. The maximum value of $\mathcal{S} = \log(N)$ comes when $\rho \rightarrow 1_{N \times N}/N$ is completely isotropic, and has no preferred basis or pattern associated with it.

In thermodynamics the entropy is recognized as a logarithmic measure of the phase space-volume occupied by the system. The equilibrium distribution is defined by maximizing that volume, the entropy, subject to all constraints such as a fixed total energy or particle number. Experimental science seeks order where it can be found, under which we express the principle of *minimum* entropy. The principle predicts we should actively use our freedom to partition data to discover *low-entropy* divisions. They are simple.

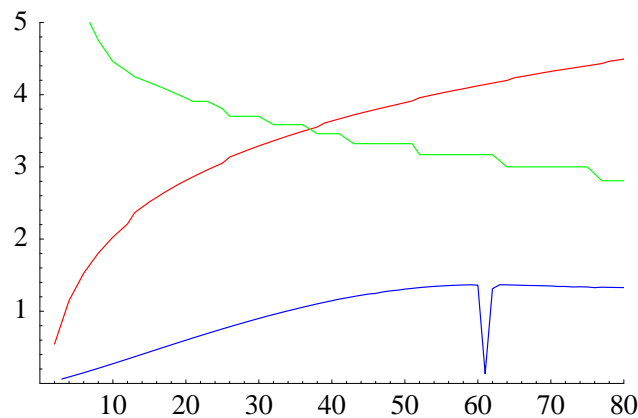


Figure 4. Entropy (in bits) of a coherent data record partitioned on dimension D . Bottom curve (blue online): the quantum entropy $-\text{tr}(\rho \log_2(\rho))$ is comparatively small, and detects the optimal division with a sharp dip. Middle curve (red online): the classical entropy on the same data resolved to 2^D bit accuracy. Curve highest on left (green online): the entropy of Huffman compression can be less than the classical unprocessed entropy, but it is larger than the quantum entropy.

An Experiment: Figure 4 shows an experiment with Monte Carlo simulation. Two dimensional distributions were defined using sums of Gaussians with randomly generated parameters. A sample from the distribution was extracted from 0.1 unit bins over the interval $-10 < x < 10$, $-3 < y < 3$, making a 12261-point data record D_{ij} , which was normalized. Re-partitioning the record on intervals of length L created arrays⁴ Now compare classical and quantum methods. The empirical marginal distribution on index i is $d_i = \sum_j d_{ij}$. That leads to the *classical entropy* $\mathcal{S}_{cl} = -\sum_i d_i \log(d_i)$. Compare the density matrix on the same space, $\rho_{ii'} = d_{ij} d_{ij'}$, with quantum entropy $\mathcal{S}_q = -\text{tr}(\rho \log(\rho))$. The figure shows that $\mathcal{S}_q \ll \mathcal{S}_{cl}$ unless the data is very noisy. A dip in \mathcal{S}_q occurs at favored divisions (Fig. 4). Indeed if one

⁴ Data were rescaled by a constant to be resolved on exactly L dimensions.

makes data with random linear combinations of specified patterns, the entropy will dip at the division of the pattern's periodicity, or approximate periodicity. Minimum entropy finds simplicity.

Information theory is like experimental physics in manipulating the encoding of repeated patterns to lower the effective entropy. As L increases longer patterns or “words” can be compressed into symbols. Huffman compression is a method to optimize classical information content towards minimum entropy. Huffman coding is a procedure based on *me* class definitions and therefore classical characterization. Figure 4 shows that using Huffman coding produces $S_{huffman} < S_{cl}$. Yet across the board both classical entropies exceed the quantum value. Both miss the optimal division, because no notion of dividing a product space exists. Similar features have been seen in dozens of different types of data.

An Experiment: A *symmetry* of a correlation means it is unchanged under a transformation. A symmetry of the density matrix implies it commutes with the generator of the transformation, and then shares eigenstates. Figure 5 shows an experiment in self-organizing or “auto-quantization” of the eigenvectors of the density matrix. To make the figure the first 60 decimal digits of π were collected as an array D_{1i} . Random cyclic permutations of the same list produced the samples $D_{2i}, D_{3i}...D_{ji}$ for $J = 1...1000$. The eigenvectors of $\rho_{ii'}$ are found to be nearly pure momentum eigenstates: Each shows a peak in Fourier power at a single wave number. The probability to find $\cos(\pi x/2)$ sampled on the first 60 integers is about 2.3%.

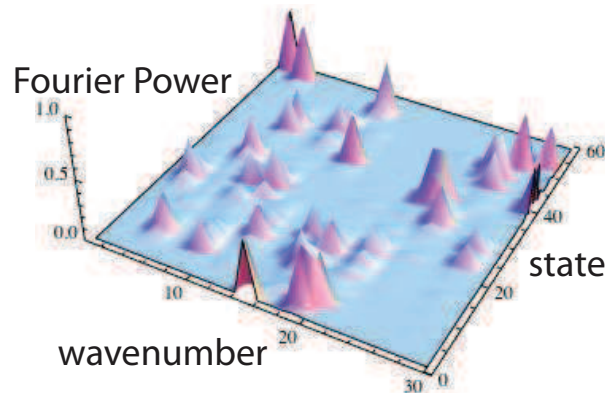


Figure 5. “Auto-quantization” by symmetry. The Fourier power of density matrix eigenvectors made from the first 60 digits of π sampled over 1000 random circular permutations is quantized.

Similar results are found with more structured group operations, such as rotations, unitary transformations, or more complicated group operations with elements $R(\omega)$, where $\omega = \alpha_J, \beta_J, \gamma_J...$ are group parameters of sample J . Let $|l, m...\rangle$ be an irreducible basis of group representations. Let the objects in the sample be copies of a single object transformed under the group, and let $\sum_J \rightarrow d\omega$, the invariant group volume. The density matrix approaches a limit:

$$\rho = \int d\omega \sum_{j,j',l,l',m,m'...} R(\omega) |l, m...\rangle \langle l, m...| \\ \times |a\rangle \langle a| |l', m'...\rangle \langle l', m'...| R^\dagger(\omega).$$

Use the completeness property of matrix representations $D_{m''m'''}^{j',l'...}$:

$$\int d\omega D_{mm'}^{j,l...}(\omega) \bar{D}_{m''m'''}^{j',l'...}(\omega) = \delta^{jj'} \delta^{ll'} \delta^{mm''} \dots,$$

Only the diagonal elements survive the sums, with the weights just as dictated by quantum rules:

$$\rho \rightarrow |j, l, m... \rangle \langle j, l, m... | a \rangle^2 \langle j, l, m... |$$

Data which is many copies of some complicated scalar function of angles, such as the shape of a paramecium or the map of the Earth permuted over all orientations will give a density matrix a diagonal sum of spherical harmonics, with probabilities given by the standard formula. It is kinematic because the machinery defining probability was constructed to expedite linear transformations.

Time Evolution: Turn to time evolution. Earlier we commented that physics predicts little else, and that our Hamiltonian model was a toy. Bounded linear Hamiltonian time evolution is unitary time evolution, also a toy. We offer the idea these features emerge from the Agreement to categorize things with density matrices. A typical correlation in QFT is written

$$\begin{aligned} C(x_1, t_1, x_2, t_2, \dots, x_n, t_n) &= \langle 0 | \phi(x_1, t_1) \dots \phi(x_n, t_n) | 0 \rangle, \\ &= \text{tr}(\rho_0 \phi(x_1, t_1) \dots \phi(x_n, t_n)). \end{aligned}$$

The state of the art of physics consists of reverting all known correlations into a model for the density matrix ρ_0 . It is very beautiful that the model is relatively simple, but whatever was obtained in the lab, we cannot see how the representation itself could fail.

From space - and time-translational symmetry, which physicists desire to arrange, the correlation is a function of differences $C = C(x_2 - x_1, t_2 - t_1, \dots, t_n - t_{n-1})$. Any function of $x_2 - x_1$ has a Fourier representation in terms of $\exp^{i\vec{k}_{12}(\vec{x}_2 - \vec{x}_1)} \exp^{-i\omega(t_2 - t_1)}$. The ultimate time evolution is unitary. By standard steps of the convolution theorem the law of Conservation of Frequency (energy) is kinematic, and inventing a Hamiltonian with enough degrees of freedom cannot fail to describe it⁵, so long as the eigenvalues of ρ_0 are conserved: otherwise, add freedoms. Thus we don't need to be too embarrassed about the toy dynamics, whose success ought to be judged on the effectiveness and simplicity of the *model*, not needing higher principles: in our approach.

Many Experiments: In the Introduction we mentioned that the restriction of the quantum methods to describing micro-physical objects of fundamental physical character is obsolete. We've done many experiments to extend the scope. • In Ref. [20] the radio frequency emissions of relativistic protons were used to construct density matrices from signal data and noise data. A combination of techniques improved the signal to noise ratio by about a factor of more than 100, producing the first detection of virtual Cherenkov radiation from protons.

⁵ One might discover a time-dependent Hamiltonian, which does occur in physics, yet which is invariably embedded in a time-independent larger system.

- In Refs. [22] data from the cosmic microwave background (CMB) was analyzed to test the “cosmological principle” requiring isotropy. The alignment of spherical harmonic multipoles and the entropy of their power distribution contradicts isotropy at a high degree of statistical significance. The origin is unknown, while it cannot be explained by galactic foreground subtractions [23].
- In Ref.[24] the density matrix was constructed from high-dimensional spectroscopic data of a pharmaceutical protein. The principal values were sorted to make projections onto certain subspaces from which the phases of the protein could be determined by inspection. Ref. [25] reviews subsequent progress. By now the method has been used to make empirical phase diagrams towards characterizing the active states, phase transitions and shelf-life of about 100 pharmaceuticals.

8. Concluding remarks

Our discussion began with conventional observations that “the framework of quantum mechanics is thought the perfection of fundamental theory” that “predicts an absolute and unvarying law of time evolution.” Those observations have been found false in general.

Our Universe seems to need many degrees of freedom for its description. Physics has greatly progressed in the details of models, while giving away credit to a quantum-style framework we claim is independent. The important part of the framework is its extension and enlargement of the definition of probability. In Section 6.1 we stated that the breakthrough of quantum probability lies in generalizing the notion of probability so as not to *insist* on pre-ordained equivalence classes.

We showed that a map exists going from macroscopic information to quantum probability. The map maintains a hidden variable description for quantum systems, and extends the scope of subjects of quantum information theory. We showed that the probabilistic features of quantum mechanics itself come from the process of reducing hidden freedoms. It is no more profound than a certain method of counting. We believe that quantum physics operates by the same procedures, but so long as parts of physics are unknown, that cannot be “derived.” Rather than argue with opinions about what the Universe is, we call it “our approach.”

No relation to microphysics is needed. Quantum probability can be viewed as an efficient data-management device, a branch of information theory, reversing a perception that quantum information theory should be a science of microphysical objects.

8.0.1. What do we mean by the probability of a vector?

Both classical probability and quantum probability have a feature that certain independent probabilities multiply, using direct product spaces to organize the mathematics. The difference is illustrated by the different way to discriminate between “different” vectors, which finishes our discussion.

Both approaches will decompose a state space into coordinates $v_i = \langle i|v \rangle$, for $i = 1 \dots N$. The typical approach to classical probability defines a distribution $f(v) = dN/dv_1 dv_2 \dots dv_N$. Break each dimension into K equal bins of resolution Δv_i . The distribution for any situation is a list among K^N *mutually exclusive* possibilities. Completely sampling the distribution needs K^N pieces of information, which may well be impossible. There are practical consequences. The classical device made with 10 2-state q – bits has 2^{10} possible states, all declared distinct.

One such state is written $v_i = (0, 0, 1, 1, 0, 0, 1, 0, 0, 1)$. Another vector is $u_i = (0, 1, 0, 0, 1, 1, 0, 0, 1, 1)$. These are not the same so their probability of being the same is *zero*. The probability for any random 2-state vector is of order $2^{-10} \sim 9 \times 10^{-4}$, which is very small.

Compare the quantum probability that one vector can serve for another. It is not based on a distribution. The normalized inner product $\langle u|v \rangle = 1/(2\sqrt{5})$. The probability the vector $|u\rangle$ defined above can serve for $|v\rangle$ is $|\langle u|v \rangle|^2 = 1/20$, which is a much larger probability of coincidence than *zero*. A numerical calculation finds the average overlap-squared of such random vectors with another one is about 0.3. If one generates 1000 normalized vectors, there will be around 180 with a Born probability exceeding 0.5 that look like a given vector.

Two facts are so basic they tend to escapes notice. First, it is not actually possible in principle to sample and then categorize the spaces of most distributions cited for classical physics. If the data uses 16-bit accuracy and 10-element vectors there are $(2^{16})^{10} = 1.46 \times 10^{48}$ mutually exclusive class labels set up in the first step. The entire 19th century conception of multidimensional phase spaces for many particles (“moles” of atoms) does not exist in any physically realizable form. While ignoring all dynamics was a success of thermodynamics, the ambition to keep track of some fragment of the vast *experimental* complexity of many degrees of freedom cannot really be maintained within the framework of distributions: a great deal of classical theory notwithstanding.

Second, the quantum description of any given data is not more complicated, it is vastly less. The language abuse of “quantum particles” greatly confuses this. The 10-dimensional space cited above has no more than $10-1=9$ vectors that are mutually-exclusive of any given vector. The simplification of quantum-style data characterization also occurs with infinitely fine resolution, and with no great sensitivity to the resolution. It is grossly misleading to compare a quantum space of 2^{10} dimensions, from spin products $1/2 \otimes 1/2 \otimes \dots 1/2$ with a classical space of K^{10} mutually-exclusive categories.

That is why we end reiterating the principle of *minimum* entropy from Section 7.2. Whether or not physics must do it by principle, there is a practical fact that large data sets should be reduced to correlations, and that correlations should be reduced to sub-correlations. When the entropy of dividing data is as low as possible, the experimenter has found order, and statistical regularity, which is the purpose of science. The applications of a new science seeking minimum entropy as defined on flexible quantum-style categories are unlimited.

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