Direct Methods for Frequency Filter Performance Analysis

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Additional information is available at the end of the chapter

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1. Introduction

Analysis methods based on determining system performance specifications by step response, as well as indirect methods: pole-zero plot, magnitude response and integral analysis methods are applied in automatic control theory for performance estimation of linear systems [1,2,3]. However, in many cases the mentioned methods result in crude performance estimation of a linear system (filter) operation. Furthermore, direct methods of linear system performance specifications (settling time, accuracy, overshoot etc.) characterization require a huge amount of calculations being performed.

Specification or estimation of signal processing performance criteria are usual tasks in frequency filter analysis. In some cases it is considered to be enough to examine a filter behavior at average statistical parameters of a useful signal and its disturbance. In other cases, for instance for robust filters, it is rather more complicated – one needs to determine the limit of variables for signal processing performance specifications at any possible input signal parameters variation.

The author offers to use the filter analysis methods, developed by him on the basis of spectral representations of the Laplace transform, to solve efficiently the problem of signal processing performance specifications determination by frequency filters at different variations of input signal parameter [4,5,6]. The mentioned methods are based on using consistent mathematical models for input signals and filter impulse characteristics by means of a set of continuous/ discrete semi-infinite or finite damped oscillatory components. Similar models can be applied for simple semi-infinite harmonic and aperiodic signals or filter impulse characteristics, compound signals of any form, including signals with composite envelopes, as well as pulse signals (radio and video pulses).

The application of signal/filter frequency and frequency-time representations, based on Laplace transform, allowed developing simple and effective direct methods for performance analysis of signal processing by analog and digital filters.

To simplify the task of analog and digital filter signal processing performance analysis the author offers two methods for performance express-analysis of signal processing by frequency filters using filter frequency responses based on Laplace transform: frequency and frequency-time analysis methods [7].

The frequency method from an indirect analysis method for signal processing, in fact, has transformed into a direct analysis method by means of Laplace transform spectral representations. This method is the most effective in cases, where only two main performance specifications: signal processing speed and accuracy – are required to be evaluated.

The frequency-time analysis method is being applied in cases, where there is a need to evaluate signal processing speed and accuracy, as well as the history of transient processes in a filter, for instance, to control oscillation of transient process in a filter. It is suggested to perform the analysis by using frequency responses based on filter transfer function, dependent on time, in sections of input signal complex frequencies.

In case of FIR filter an effective estimation of signal processing performance specifications can be carried out by using filter frequency response 3D analysis based on Laplace transform in sections of input signal complex frequencies, considering their change. To evaluate signal processing performance specifications for IIR filters one will need along with sections of 3D filter frequency responses to use sections of 3D signal spectrum on filter impulse response complex frequencies [5].

The issues about the application of the analog and digital filter analysis methods, developed by the author for signal processing performance analysis by frequency filters, are considered below.

2. IIR filters analysis

2.1. Signal processing performance analysis by analog IIR filters

Let us consider signal processing performance analysis by IIR filters at semi-infinite input signals on the basis of analysis methods based on the Laplace transform spectral representations.

Three methods of frequency filter analysis are suggested by the author for the time-and-frequency representations positions of signals and linear systems in coordinates of complex frequency [5,6]. Let us consider the first two methods for signal processing performance analysis by frequency filters.

The mathematical description for the generalized input signal and IIR filter at time and frequency domains for the first (item2) and the second (item 3) analysis methods, mathematical expres-

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sions for calculating forced and free components of a filter reaction by the first method (items 4,5), components of a filter reaction by the second method (item6) are given in the Table 1.

№ Name	Expression	Remark
1. Input signal	$x(t)=\mathrm{Re}(\dot{\boldsymbol{X}}^{\mathrm{T}} e^{\boldsymbol{p}t})$, $X(p)=\mathrm{Re}\left(\dot{\boldsymbol{X}}^{\mathrm{T}}\left[\frac{1}{p-p_n}\right]_N\right)$	$\dot{\boldsymbol{X}}=[\dot{X}_n]_N=\left[Xm_n e^{-j\varphi_n}\right]_N$, $\boldsymbol{p}=[p_n]_N=[-\beta_n+j\omega_n]_N$
2. Filter	$g(t)=\mathrm{Re}(\dot{\boldsymbol{G}}^{\mathrm{T}} e^{\boldsymbol{q}t})$, $K(p)=\mathrm{Re}\left(\dot{\boldsymbol{G}}^{\mathrm{T}}\left[\frac{1}{p-\rho_m}\right]_M\right)$	$\dot{\boldsymbol{G}}=[\dot{G}_m]_M=\left[k_m e^{-j\phi_m}\right]_M$, $\boldsymbol{q}=[\rho_m]_M=[-\alpha_m+jw_m]_M$
3. Time dependent transfer function	$K(p,\,t)=\mathrm{Re}\left(\dot{\boldsymbol{G}}^{\mathrm{T}}\left[\frac{1-e^{-(p-\rho_m)t}}{p-\rho_m}\right]_M\right)$	$K(p,\,t)=\int_0^t g(\tau)e^{-p\tau}d\tau$
4. Forced components	$y_1(t)=\mathrm{Re}(\dot{\boldsymbol{Y}}^{\mathrm{T}} e^{\boldsymbol{p}t})$	$\dot{\boldsymbol{Y}}=\mathrm{diag}(\dot{\boldsymbol{X}})K(\boldsymbol{p})$
5. Free components	$y_2(t)=\mathrm{Re}(\dot{\boldsymbol{V}}^{\mathrm{T}} e^{\boldsymbol{q}t})$	$\dot{\boldsymbol{V}}=\mathrm{diag}(\dot{\boldsymbol{G}})X(\boldsymbol{q})$
6. Filter reaction	$y(t)=\mathrm{Re}(\dot{\boldsymbol{Y}}(t)^{\mathrm{T}} e^{\boldsymbol{p}t})$	$\dot{\boldsymbol{Y}}(t)=\mathrm{diag}(\dot{\boldsymbol{X}})K(\boldsymbol{p},\,t)$

Table 1. IIR filters analysis

The operation of the real part extraction on the right side of the expression in the items 1,2,3 for $X(p)$, $K(p)$, $K(p,\,t)$ is solved in terms of the complex coefficients $\dot{G}_m$ and ρ_m with no relevance to the complex variable p.

The first method is a complex amplitude method generalization for definition of forced and free components for filter reaction at semi-infinite or finite input signals [6]. The advantages of this method are related to simple algebraic operations, which are used for determining the parameters of linear system reaction (filter, linear circuit) components to an input action described by a set of semi-infinite or finite damped oscillatory components. To analyze a filter it is needed to use simple algebraic operations and operate a set of complex amplitudes and frequencies of forced and free filter reaction components. In this case, there are simple relations between complex amplitudes of output signal forced components and complex amplitudes of an input signal (item 4 Table 1), between complex amplitudes of output signal free components and complex amplitudes of a filter impulse function (item5).

The time-and-frequency approach in the second analysis method applies to a filter transfer function, i.e. time dependent transfer function of the filter is used [6,8]. In that case, instead of two sets of filter reaction components only one of them may be used.

Analysis methods given in the Table 1 enable to reduce effectively the computational costs when performing a filter analysis by using simple algebraic operations to determine the forced and free components of a filter reaction to an input action as a set of damped oscillatory components. Therefore, the considered analysis methods for linear systems (filters) can be effectively applied for performance analysis of signal processing by frequency filters.

Let us consider a simple example of performance analysis of signal processing by a high-pass second-order filter relating to signal processing task in power system protection and automation devices [9,10]. The filter is used to extract a sinusoidal component of commercial frequency and eliminate disturbance as a free component of transient processes in a control object. In this case, a change of a useful signal initial phase is acceptable.

All the initial data and dependencies which are necessary for the analysis are represented in the Table 2. IIR filter parameters are specified, the mathematical description of an input signal with specified sizes of changing for useful signal and disturbance parameters affecting their spectrum is given in the Table 2 as well.

The impulse function of high-pass second-order filter contains a delta function of Dirac which is used for determining complex amplitudes of forced components when defining $K(p)$ by the impulse function (item 3 Table 2) and cannot be applied for determining complex amplitudes of filter reaction free components (item 5). To simplify the analysis the delta function can be represented as an extreme case of the exponential component $\alpha e^{-\alpha t}$ at $\alpha \to \infty$[6].

The analysis results should ensure the following performance criteria of signal processing by a filter:

1. a filter settling time should be less than 30 ms at 5% acceptable total error of signal processing at any value of disturbance parameters within the specified range,
2. an acceptable error at frequency deviation of useful signal from the nominal value of 50Hz within the range ±5 Hz should not be more than 5%,
3. an acceptable overshoot should not be more than 10%.

As it follows from the Table 1, simple algebraic operations are applied to determine complex amplitudes, as well as forced and free components of a filter output signal.

When using Mathematica, Mapple, Matlab, Mathcad and other state-of-art mathematical software for determining forced and free components of an output signal it is necessary to specify only complex amplitude vectors of an input signal and a filter impulse function, as well as complex frequency vectors of an input signal and a filter. In this case all the necessary calculations, related to a filter analysis, would be carried out automatically. If it is needed to determine complex amplitudes of a filter impulse function at specified transfer function the ready-made formulas may be used [6], which can be easily applied in the mathematical software mentioned above.

All the examples in the present chapter are given using the mathematical software Mathcad. Mathcad was chosen due to pragmatic considerations related to assuring the maximum visibility of the examples for filter analysis, as in Mathcad mathematical expressions are given in the form, closest to universally accepted mathematical notation [11,12].

An example of a filter computation using Mathcad at the specified filter parameters and the following input signal parameters: $Xm_2 = Xm_1 = 1$, $\omega_1 = 2\pi 50$rad/s, $\varphi_1 = 0$, $\beta_2 = 60$ s^{-1} is given on the Figure 1.

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№	Name	Expression
1.	Input signal	$x(t)=Xm_1\cos(\omega_1 t-\varphi_1)-Xm_2 e^{-\beta_2 t}$, $\dot{\boldsymbol{X}}=\left[Xm_1 e^{-j\varphi_1} \quad Xm_2 e^{-j\pi}\right]^{\mathrm{T}}$, $\boldsymbol{p}=[j\omega_1 \quad -\beta_2]^{\mathrm{T}}$, $\mu_2=Xm_2/Xm_1=0\div 1$, $\omega_1=2\pi(45\div 55)$, $\varphi_1=0\div 2\pi$, $\beta_2=2\div 200$ $X(p)=Xm_1\frac{p\cos(\varphi_1)+\omega_1\sin(\varphi_1)}{p^2+\omega_1^2}-Xm_2\frac{1}{p+\beta_2}$
2.	High-pass filter	$K(p)=\frac{k_0 p^2}{p^2+2\alpha_1 p+w_2^2}$, $g(t)=k_0\delta(t)+\mathrm{Re}\left(\dot{G}_1 e^{\rho_1 t}\right)$,$k_0=1,\ 206$, $\dot{\boldsymbol{G}}=\left[k_1 e^{-j\phi_1}\right]=[-424,\ 5e^{j0.342}]$, $\boldsymbol{q}=[-\alpha_1+jw_1]=[-165,\ 9+j117,\ 1]$
3.	Forced components of complex amplitudes	$\dot{\boldsymbol{Y}}=\mathrm{diag}(\dot{\boldsymbol{X}})K(\boldsymbol{p})=\begin{bmatrix}\dot{X}_1 & 0\\ 0 & \dot{X}_2\end{bmatrix}\begin{bmatrix}K(j\omega_1)\\ K(-\beta_2)\end{bmatrix}=\begin{bmatrix}\dot{X}_1 K(j\omega_1)\\ \dot{X}_2 K(-\beta_2)\end{bmatrix}=\begin{bmatrix}\dot{Y}_1\\ \dot{Y}_2\end{bmatrix}=$ $=Xm_1\begin{bmatrix}\frac{-1,\ 044\omega_1^2 e^{-j\varphi_1}}{100^2-\omega_1^2+j166,\ 66\omega_1} & \frac{\mu_2\beta_2^2 e^{-j\pi}}{100^2+\beta_2^2-166,\ 66\beta_2}\end{bmatrix}^{\mathrm{T}}$
4.	Forced components	$y_1(t)=\mathrm{Re}\left(\dot{\boldsymbol{Y}}^{\mathrm{T}} e^{\boldsymbol{p}t}\right)=\mathrm{Re}\left(\begin{bmatrix}\dot{Y}_1\\ \dot{Y}_2\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}e^{j\omega_1 t}\\ e^{-\beta_2 t}\end{bmatrix}\right)=\mathrm{Re}\left(\dot{Y}_1 e^{j\omega_1 t}+\dot{Y}_2 e^{-\beta_2 t}\right)$, $y_1(t)=y_{11}(t)+y_{12}(t)$, $y_{11}(t)=\mathrm{Re}\left(\dot{Y}_1 e^{j\omega_1 t}\right)$, $y_{12}(t)=\mathrm{Re}\left(\dot{Y}_2 e^{-\beta_2 t}\right)$
5.	Complex amplitudes of free components	$\dot{\boldsymbol{V}}=[\dot{V}_1]=[X(\rho_1)\dot{G}_1]=Xm_1\left[\frac{\rho_1\cos(\varphi_1)+\omega_1\sin(\varphi_1)}{\rho_1^2+\omega_1^2}-\mu_2\frac{1}{\rho_1+\beta_2}\right]$
6.	Free components	$y_2(t)=\mathrm{Re}\left(\dot{V}_1 e^{\rho_1 t}\right)$
7.	Error	$\varepsilon(t)=y_2(t)+y_{12}(t)$

Table 2. IIR filter analysis

For determining or estimating performance specifications of signal processing by the investigated IIR filter one need either to improve the software (Figure 1) or to reduce the amount of calculations by simplifying the analysis task. Let us consider the second option first.

The easiest operation is to define the error level in signal processing by a filter at frequency deviation of useful sinusoidal signal within the range ±5 Hz from the nominal value of 50 Hz. This error, as it is known, may be determine by an average amplitude-frequency response of a filter. In this case the value of a filter amplitude-frequency response in the areas of frequency $2\pi(45\div 55)$ rad/s is between 0,95 and 1,038. Thus, the filter meets the signal processing performance requirement mentioned above.

A filter settling time can be defined by total damping of a free component τ_1 and a forced component of exponential disturbance τ_2 if the last component was not eliminated by the filter till the necessary level. Time τ_1 and τ_2 can be defined according to the Table 1 (item 6 and item 4).

A damping time of disturbance free component τ_1 and forced component τ_2 to the required level of 5% may be determined on the basis of the expressions given in the Table 2.

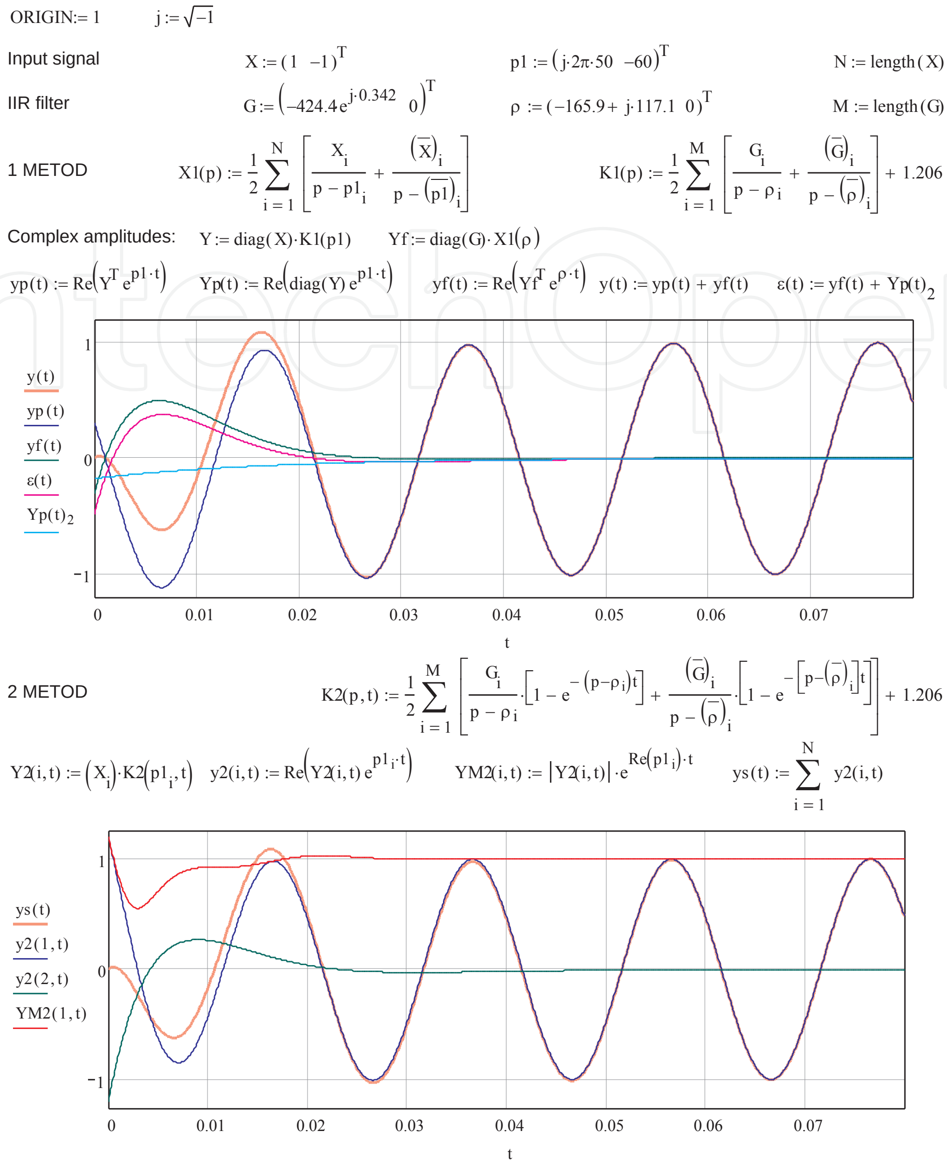


Figure 1. IIR filters analysis using Mathcad software

$$\tau_1 = -\frac{1}{\alpha_1}\ln\left(\frac{0{,}05}{k_1\left|X(\rho_1)\right|}\right), \tau_2 = -\frac{1}{\beta_2}\ln\left(\frac{0{,}05}{\mu_2\left|K(-\beta_2)\right|}\right). \tag{1}$$

Variables τ_1 and τ_2 depend not only on filter parameters which are constant, but also on signal parameters. Let us assume the worst option μ_2=1, φ_1=0, ω_1=2π50rad/s, taking into account the particularities of the controlled object [9,10]. Thus, let us consider the dependence $\tau_1$and τ_2 from β_2.

A presice settling time τ can be determined through the total error of signal processing $\varepsilon(t)$. In case of $t \geq \tau$the condition $|\varepsilon(t)| \leq \varepsilon_{\lim}$should be performed, when $\varepsilon_{\lim}$=0, 05.

Let us consider an estimation of total damping τ by the specified values of τ_1 and τ_2 5 %. In this case $\tau < \tau_1 + \tau_2$. If the filter is designed in a correct way, then $|y_{12}(t)| \le \varepsilon_{\lim}$ at $t \ge \tau_1$, that is when disturbance is eliminated to the specified level by the moment of the end of transient process in a filter, then $\tau \approx \tau_1$. An estimation of a filter settling time can be performed with some conservative value on the basis of a sum of modules of free component envelopes of transient process in a filter and disturbance forced components [13]. The dependence τ from β_2 can be quite easily determined by an insignificant improvement of the program on the Mathcad example, represented on the Figure 1.

Dependencies τ, τ_1 and τ_2, depending on the value of exponential disturbance damping coefficient β_2 are shown on the Figure 2.

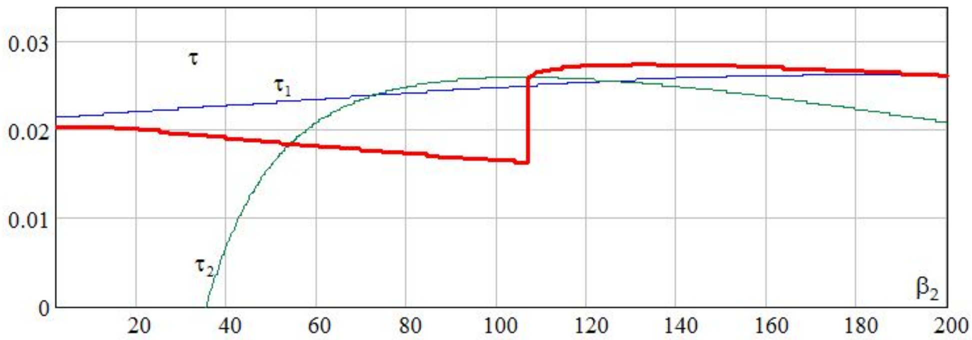


Figure 2. The dependence of a filter settling time from damping coefficient β_2

In case of $\beta_2 = 2 \div 35$ s^{-1} the initial level of a disturbance forced component is below the acceptable error, so $|\dot{Y}_2| \le \varepsilon_{\lim}$, and $\tau_2 = 0$. Then a filter settling time is mostly defined by damping transient process of its own in a filter, in other words by the value τ_1. Within the range $\beta_2 = 74 \div 125$ s^{-1} damping of a disturbance force component is longer than damping time of transient process of its own in a filter, that is $\tau_2 > \tau_1$. At $\beta_2 \le 107$ s^{-1} a filter settling time τ is less than values τ_1 and τ_2, and at $\beta_2 > 107$ s^{-1} is longer than any of the values mentioned above. It is due to plus-minus signs of filter reaction components, defining an error of signal processing, as well as to values of the components complex frequencies.

An overshoot level in a filter can be determined by the program improvement on the Mathcad example, shown on the Figure 1.

The performance analysis results for signal processing of the investigated IIR filter are represented in the Table 3. Performance specifications determined by using one traditional method - by step response of a second-order low-pass filter are given in the Table 3 as well. In this case the description for the low-pass filter was obtained on the basis of the investigated second-order low-pass filter by applying a well known frequency transformation [14].

№ Name	Step response	Direct estimation
1. Settling time, s	0,0172	0,0275
2. Maximum error in the steady-state mode, %	0	5
3. Maximum overshoot level, %	1,17	11,81
4. Additional error at a frequency variation of an useful signal	-	5

Table 3. Signal processing performance for a second-order high-pass filter

As it follows from the Table 3, in the considered example there are some substantial differences in the performance specifications, gained by the traditional method and on the basis of their direct determination.

2.2. IIR filters analysis at dissemination of time-and-frequency approach to transfer function filter

As it follows from the Table 1 (item 3 and item 6), estimation of functioning performance for filters can be performed at dissemination of time-and-frequency approach to filter transfer function, in other words by using time dependent transfer function of a filter [5,6].

In that case, comparing to the first method where two groups of filter reaction components are using – forced and free components, only the first group of components is using. The information about the transient process is containing in the time dependent complex amplitudes of a filter reaction $\dot{Y}(t)$. In the case of $t \to \infty$ the mentioned set of complex amplitudes will be equal to complex amplitudes of forced components $\dot{Y}(t) \to \dot{Y}$.

The necessary dependences for determination of output signal components of a filter are represented in the Table 1, an example for high-pass filter analysis at the specified parameters of an useful signal is given on the bottom part of the fig.1.

The advantage of the considered method is connected to determination envelopes for every component of a filter output signal, based on which the total envelope of a filter output signal and a variation law of initial phase can be defined. This information can be effectively used for performance analysis of signal processing by frequency filters. For instance, when determining an overshoot level (oscillativity) of a transient process in a filter.

2.3. IIR filter express-analysis method

It follows from the Table 1, that quality indexes estimation for filter operation can be carried out on the basis of interim calculation results $-K(\boldsymbol{p})$ and $X(\boldsymbol{q})$, that means - based on spectral representations of signals and filter impulse functions in complex frequency coordinates. Another, not less effective approach, is related to the usage of the interim results of the second analysis method – the transfer function $K(\boldsymbol{p}, t)$, which is dependent of time. Thus, the application of filter frequency characteristics and a signal spectrum in complex frequency coordinates increase significantly the effectiveness of using the frequency methods of performance analysis for frequency filter operation [7].

The express-analysis methods for filters, including performance analysis of signal processing, were developed based on investigation of 3D and 4D frequency responses [7]. It is enough to consider the sections $p = j\omega$ and $p = -\gamma$ of 3D frequency responses $K(\boldsymbol{p})$, as well as the section $p = -\alpha_1 + jw_1$ of a input signal spectrum according to the Laplace transform to estimate the settling time and accuracy of signal processing for the example given on the fig.1.

The express-analysis methods mentioned above can be effectively applied for FIR filters as well, the detailed explanation will be given further in the present chapter.

2.4. Digital IIR filters analysis

Under the definition of digital filters in the chapter discrete filters are ment. In many cases it is justified, for instance, in cases of using microcontrollers or digital signal processor with high digit capacity and especially for microprocessors with support for floating-point operations [15,16].

When using discrete filters their analysis has a lot of similarities with the analysis of analog filters-prototypes. There is a small difference only when it comes to transition from images to originals. The main expressions for determining components of a digital filter output signal when injecting on the filter input a signal as a set of discrete semi-infinite damped oscillatory components are given in the Table 4.

An example for digital filter analysis as a continuation of the example of the analog filter-prototype analysis (fig. 1) is represented on the fig. 3. The mathematical description for the digital filter was obtained by the method of invariant impulse responses at the discrete sampling step $T = 0,\ 0005$s.

№ Name	Expressions	Remark
1. Input signal	$x(k) = \mathrm{Re}\left(\dot{\boldsymbol{X}}^{\mathrm{T}} Z(\boldsymbol{P},\ k)\right)$, $X(z) = \mathrm{Re}\left(\dot{\boldsymbol{X}}^{\mathrm{T}}\left[\frac{z}{z - z_n}\right]_N\right)$	$\dot{\boldsymbol{X}} = \left[X m_n e^{-j\varphi_n}\right]_N$, $\boldsymbol{p} = \left[-\beta_n + j\omega_n\right]_N$ $\boldsymbol{z} = e^{\boldsymbol{p}T}$, $Z(p,\ k) = e^{pkT}$, T - discrete sampling step
2. Filter	$g(k) = \mathrm{Re}\left(\dot{\boldsymbol{G}}^{\mathrm{T}} Z(\boldsymbol{Q},\ k)\right)$, $K(z) = \mathrm{Re}\left(\dot{\boldsymbol{G}}^{\mathrm{T}}\left[\frac{z}{z - z_m}\right]_M\right)$	$\dot{\boldsymbol{G}} = \left[k_m e^{-j\phi_m}\right]_M$, $\boldsymbol{q} = \left[-\alpha_m + jw_m\right]_M$, $\boldsymbol{z} = e^{\boldsymbol{q}T}$
3. Time dependent transfer function	$K(z,\ k) = \mathrm{Re}\left(\dot{\boldsymbol{G}}^{\mathrm{T}}\left[\frac{z\left(1 - e^{p_m kT} z^{-k}\right)}{z - z_m}\right]_M\right)$	$K(z,\ k) = \sum_{i=0}^{k} g(i) z^{-i}$
4. Forced components	$y_1(k) = \mathrm{Re}\left(\dot{\boldsymbol{Y}}^{\mathrm{T}} Z(\boldsymbol{p},\ k)\right)$	$\dot{\boldsymbol{Y}} = \mathrm{diag}(\dot{\boldsymbol{X}}) K(\boldsymbol{z})$,
5. Free components	$y_2(k) = \mathrm{Re}\left(\dot{\boldsymbol{V}}^{\mathrm{T}} Z(\boldsymbol{q},\ k)\right)$	$\dot{\boldsymbol{V}} = \mathrm{diag}(\dot{\boldsymbol{G}}) X(\boldsymbol{z})$
6. Filter reaction	$y(k) = \mathrm{Re}\left(\dot{\boldsymbol{Y}}(k)^{\mathrm{T}} Z(\boldsymbol{p},\ k)\right)$	$\dot{\boldsymbol{Y}}(k) = \mathrm{diag}(\dot{\boldsymbol{X}}) K(\boldsymbol{z},\ k)$

Table 4. IIR digital filter analysis

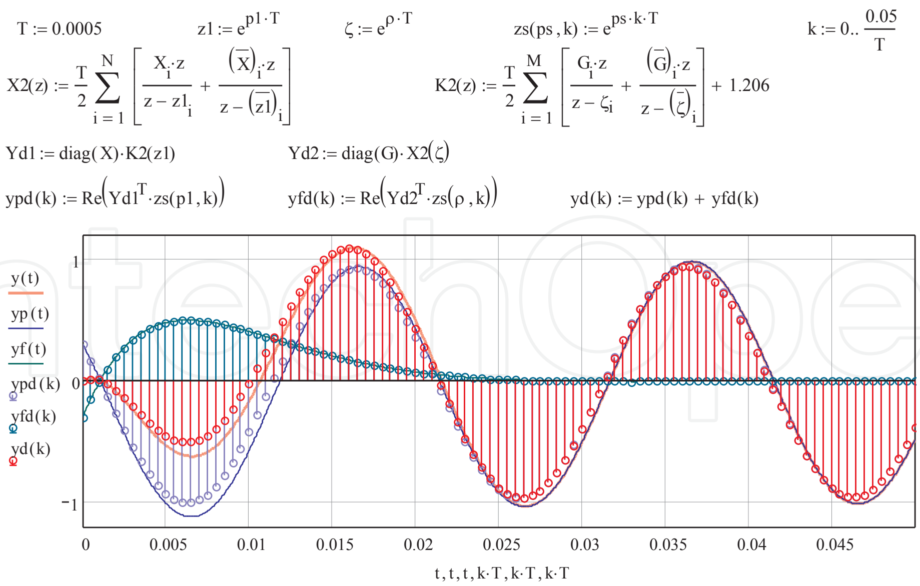


Figure 3. IIR digital filter analysis using Mathcad software

№ Name	Expressions	Remark
1. Input signal	$x(t)=\mathrm{Re}\left(\dot{\boldsymbol X}^{\mathrm T} e^{\boldsymbol P(\boldsymbol C t-\boldsymbol t)}-\dot{\boldsymbol X}'^{\mathrm T} e^{\boldsymbol P(\boldsymbol C t-\boldsymbol t')}\right)$, $X'(p)=\mathrm{Re}\left(\dot{\boldsymbol X}'^{\mathrm T}\left[\frac{1}{p-p_n}\right]_N\right)$, $X''(p)=\mathrm{Re}\left(\dot{\boldsymbol X}''^{\mathrm T}\left[\frac{1}{p-p_n}\right]_N\right)$, $X(p)=\mathrm{Re}\left(\dot{\boldsymbol X}'^{\mathrm T}\left[\frac{e^{-pt_n}}{p-p_n}\right]_N+\dot{\boldsymbol X}'^{\mathrm T}\left[\frac{e^{-pt'_n}}{p-p_n}\right]_N\right)$	$\dot{\boldsymbol X}=[\dot X_n]_N=\left[Xm_n e^{-j\varphi_n}\right]_N$, $\dot{\boldsymbol X}'=\mathrm{diag}(\dot{\boldsymbol X})e^{\boldsymbol P(\boldsymbol t-\boldsymbol t')}$, $\boldsymbol p=[p_n]_N=[-\beta_n+j\omega_n]_N$, $\boldsymbol P=\mathrm{diag}(\boldsymbol p)$, $\boldsymbol t=[t_n]_N$, $\boldsymbol t'=[t'_n]_N$, $\boldsymbol C=[1]_N$
2. Filter	$g(t)=\mathrm{Re}\left(\dot{\boldsymbol G}^{\mathrm T} e^{\boldsymbol q t}\right)$, $K(p)=\mathrm{Re}\left(\dot{\boldsymbol G}^{\mathrm T}\left[\frac{1}{p-\rho_m}\right]_M\right)$	$\dot{\boldsymbol G}=[\dot G_m]_M=\left[k_m e^{-j\phi_m}\right]_M$, $\boldsymbol q=[\rho_m]_M=[-\alpha_m+jw_m]_M$
3. Time dependent transfer function	$K(p,t)=\mathrm{Re}\left(\dot{\boldsymbol G}^{\mathrm T}\left[\frac{1-e^{-(p-\rho_m)t}}{p-\rho_m}\right]_M\right)$	$K(p,t)=\int_0^t g(\tau)e^{-p\tau}d\tau$
4. Forced components	$y_1(t)=\mathrm{Re}\left(\dot{\boldsymbol Y}^{\mathrm T} e^{\boldsymbol P(\boldsymbol C t-\boldsymbol t)}-\dot{\boldsymbol Y}'^{\mathrm T} e^{\boldsymbol P(\boldsymbol C t-\boldsymbol t')}\right)$	$\dot{\boldsymbol Y}=\mathrm{diag}(\dot{\boldsymbol X}')K(\boldsymbol p)$, $\dot{\boldsymbol Y}'=\mathrm{diag}(\dot{\boldsymbol X}'')K(\boldsymbol p)$
5. Free components	$y_2(t)=\mathrm{Re}\left(\sum_n \dot{\boldsymbol V}^{\langle n\rangle\mathrm T} e^{\boldsymbol q(t-t_n)}-\sum_n \dot{\boldsymbol V}'^{\langle n\rangle\mathrm T} e^{\boldsymbol q(t-t'_n)}\right)$	$\dot{\boldsymbol V}=[\dot G_m X(\rho_m)_n]_{M,N}$, $\dot{\boldsymbol V}'=[\dot G_m X'(\rho_m)_n]_{M,N}$
6 Filter reaction	$y(t)=\mathrm{Re}\left(\dot{\boldsymbol Y}(t)^{\mathrm T} e^{\boldsymbol P(\boldsymbol C t-\boldsymbol t)}-\dot{\boldsymbol Y}'(t)^{\mathrm T} e^{\boldsymbol P(\boldsymbol C t-\boldsymbol t')}\right)$	$\dot{\boldsymbol Y}(t)=\mathrm{diag}(\dot{\boldsymbol X}')K(\boldsymbol p, \boldsymbol C t-\boldsymbol t)$, $\dot{\boldsymbol Y}'(t)=\mathrm{diag}(\dot{\boldsymbol X}'')K(\boldsymbol p, \boldsymbol C t-\boldsymbol t')$

Table 5. IIR filter analysis at compound finite signals

2.5. IIR filters analysis at finite signals

Performance analysis of processing finite signals by IIR filters as a set of damped oscillatory components with finite duration may be performed on the basis of dependencies for IIR filters at semi-infinite signals [5,6].

All the needed expressions were obtained on the basis of the expressions from the Table 1 using a time shift and the principle of additivity.

Let us consider the IIR filter analysis at compound input signals as a set of sequentially adjacent finite signals [17]. The calculation of IIR filter reaction for this case is represented in the Table 5.

In this case every component of an input signal in a general way has a different shift and a different duration, where $\dot{V}^{\langle n \rangle}$ – n-th matrix column $\dot{V}$.

The expressions represented in the Table 5 can be significantly simplified, if the IIR filter analysis at a finite signal, for instance, at injection a finite signal on its input from N number of components with equal duration and the same time shift [6,17].

An example of filter calculation, analogous to the example on the fig.1, but using an input signal with finite duration is given on the fig.4.

The dependences given in the present section can be effectively used for not only analysis of passing through IIR filter one or another finite signal, but also for performance analysis of signal processing by filters.

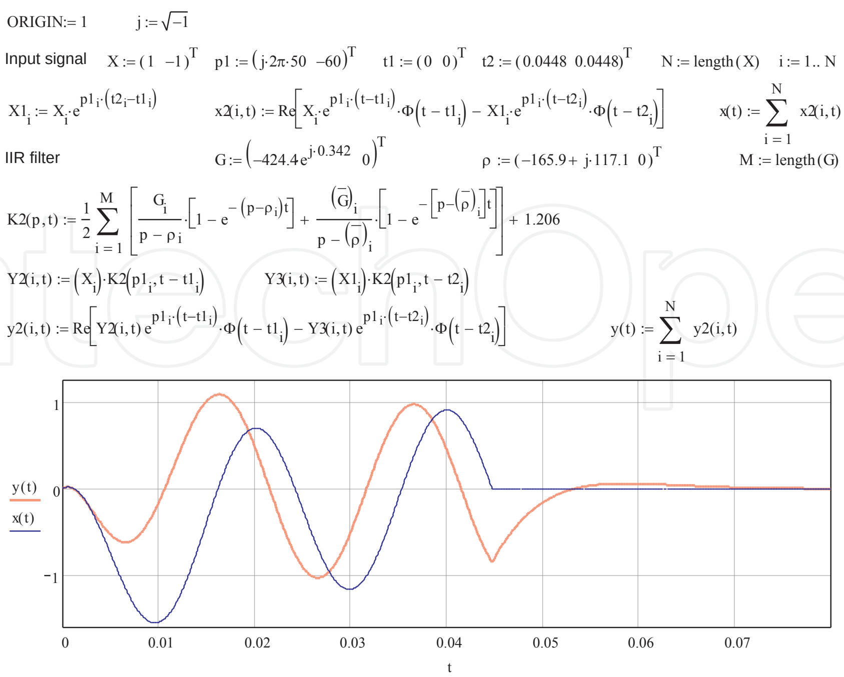


Figure 4. IIR filter analysis at a finite signal

For digital IIR filter analysis at injection finite signals on their inputs analogous mathematical operations are applied. All the necessary dependences may be obtained on the basis of the formulas from the Table 4 [13].

3. FIR filters performance analysis

3.1. Particularities of the analysis

Mathematical description for FIR filters can be obtained on the basis of the IIR filter description (Table 1) by using twice as many of filter impulse function components.

№ Name	Expressions	Remark
1. Input signal	$x(t)=\mathrm{Re}\left(\dot{\boldsymbol{X}}^{\mathrm{T}}e^{\boldsymbol{p}t}\right)$, $X(p)=\mathrm{Re}\left(\dot{\boldsymbol{X}}^{\mathrm{T}}\left[\frac{1}{p-p_n}\right]_N\right)$	$\dot{\boldsymbol{X}}=[\dot{X}_n]_N=\left[Xm_n e^{-j\varphi_n}\right]_N$, $\boldsymbol{p}=[p_n]_N=[-\beta_n+j\omega_n]_N$
2. FIR filter	$g(t)=\mathrm{Re}\left(\dot{\boldsymbol{G}}^{\mathrm{T}}e^{\boldsymbol{q}t}-\dot{\boldsymbol{G}}'^{\mathrm{T}}e^{\boldsymbol{Q}(\boldsymbol{C}t-\boldsymbol{T})}\right)$, $K_1(p)=\mathrm{Re}\left(\dot{\boldsymbol{G}}^{\mathrm{T}}\left[\frac{1}{p-\rho_m}\right]_M\right)$, $K_2(p)=\mathrm{Re}\left(\dot{\boldsymbol{G}}'^{\mathrm{T}}\left[\frac{1}{p-\rho_m}e^{-pT_m}\right]_M\right)$, $\boldsymbol{K_3(p)=\mathrm{Re}\left(\dot{G}'^{T}\left[\frac{1}{p-\rho_m}\right]_M\right)}$	$\dot{\boldsymbol{G}}=[\dot{G}_m]_M=\left[k_m e^{-j\phi_m}\right]_M$, $\dot{\boldsymbol{G}}'=\mathrm{diag}(\dot{\boldsymbol{G}})e^{\boldsymbol{QT}}$, $\boldsymbol{q}=[\rho_m]_M=[-\alpha_m+jw_m]_M$, $\boldsymbol{Q}=\mathrm{diag}(\boldsymbol{q})$, $\boldsymbol{T}=[T_m]_M$, $\boldsymbol{C}=[1]_M$, $K(p)=K_1(p)-K_2(p)$
3. Forced components	$y_1(t)=\mathrm{Re}\left(\dot{\boldsymbol{Y}}^{\mathrm{T}}e^{\boldsymbol{p}t}\right)$	$\dot{\boldsymbol{Y}}=\mathrm{diag}(\dot{\boldsymbol{X}})K(\boldsymbol{p})$
4. Free components	$y_2(t)=y_3(t)+y_4(t)$, $y_3(t)=\mathrm{Re}\left(\dot{\boldsymbol{V}}^{\mathrm{T}}e^{\boldsymbol{q}t}-\dot{\boldsymbol{V}}'^{\mathrm{T}}e^{\boldsymbol{Q}(\boldsymbol{C}t-\boldsymbol{T})}\right)$, $y_4(t)=y_1(t)-\mathrm{Re}\left(\dot{\boldsymbol{U}}^{\mathrm{T}}e^{\boldsymbol{p}t}-\sum_m \dot{\boldsymbol{U}}'^{\mathrm{T}}e^{\boldsymbol{p}(t-T_m)}\right)$	$\dot{\boldsymbol{V}}=\mathrm{diag}(\dot{\boldsymbol{G}})X(\boldsymbol{q})$, $\dot{\boldsymbol{V}}'=\mathrm{diag}(\dot{\boldsymbol{G}}')X(\boldsymbol{q})$, $\dot{\boldsymbol{U}}=\mathrm{diag}(\dot{\boldsymbol{X}})K_1(\boldsymbol{p})$, $\dot{\boldsymbol{U}}'=[\dot{X}_n K_3(p_n)_m]_{N,M}$
5. Filter reaction	$y(t)=y_1(t)+y_2(t)$	
6. Time dependent transfer function	$K_1(p,t)=\mathrm{Re}\left(\dot{\boldsymbol{G}}^{\mathrm{T}}\left[\frac{1-e^{-(p-\rho_m)t}}{p-\rho_m}\right]_M\right)$, $K_2(p,t)=\mathrm{Re}\left(\dot{\boldsymbol{G}}'^{\mathrm{T}}\left[\frac{1-e^{-(p-\rho_m)(t-T_m)}}{p-\rho_m}\right]_M\right)$, $K(p,t)=K_1(p,t)-K_2(p,t)e^{-pT_1}$	
7. Filter reaction	$y(t)=\mathrm{Re}\left(\dot{\boldsymbol{Y}}(t)^{\mathrm{T}}e^{\boldsymbol{p}t}-\dot{\boldsymbol{Y}}'(t)^{\mathrm{T}}e^{\boldsymbol{P}(\boldsymbol{C}t-\boldsymbol{T})}\right)$	$\dot{\boldsymbol{Y}}(t)=\mathrm{diag}(\dot{\boldsymbol{X}})K_1(\boldsymbol{p},t)$, $\dot{\boldsymbol{Y}}'(t)=\mathrm{diag}(\dot{\boldsymbol{X}})K_2(\boldsymbol{p},t)$

Table 6. FIR filters analysis

The additional components have the same set of complex frequencies and differ by time shift and values of complex amplitudes in a way to ensure the finitude of a filter impulse characteristic [5,6]. According to this approach to mathematical description, IIR filters are special cases of FIR filters.

The input-output dependences for FIR filters can be obtained on the basis of analogous dependences of FIR filters by using time shift and principle of additivity operations [6].

Comparing to IIR filters, FIR filters have finite duration of transient processes of their own, which are defined by a filter length. This to a certain extent simplifies performance analysis for signal processing of this type of filters, especially when using the suggested express-analysis methods for signal processing performance by FIR filters.

3.2. Analog FIR filters

Basic expressions for FIR filter analysis at injection on the filter input a set of semi-infinite damped oscillatory components are given in the Table 6.

Due to characteristics of FIR filters, among forced and free components in an filter output signal there is the third group of components $y_4(t)$, which is conventionally referred to free components in the Table 6 [6].

№ Name	Expressions	Remark
1. Input signal	$x(t)=\mathrm{Re}\left(\dot{\boldsymbol{X}}^{\mathrm{T}} e^{\boldsymbol{P}(\boldsymbol{C}t-\boldsymbol{t})}-\dot{\boldsymbol{X}}'^{\mathrm{T}} e^{\boldsymbol{P}(\boldsymbol{C}t-\boldsymbol{t}')}\right)$	$\dot{\boldsymbol{X}}=[\dot{X}_n]_N$, $\boldsymbol{p}=[p_n]_N$, $\dot{\boldsymbol{X}}'=\mathrm{diag}(\dot{\boldsymbol{X}})e^{\boldsymbol{P}(\boldsymbol{t}-\boldsymbol{t}')}$, $\boldsymbol{P}=\mathrm{diag}(\boldsymbol{p})$, $\boldsymbol{C}=[1]_N$, $\boldsymbol{t}=[t_n]_N$, $\boldsymbol{t}'=[t_n']_N$
2. FIR filter: impulse function, time dependent transfer function	$g(t)=\mathrm{Re}\left(\dot{\boldsymbol{G}}^{\mathrm{T}} e^{\boldsymbol{q}t}-\dot{\boldsymbol{G}}'^{\mathrm{T}} e^{\boldsymbol{q}(t-T_1)}\right)$, $K_1(p,\ t)=\mathrm{Re}\left(\dot{\boldsymbol{G}}^{\mathrm{T}}\left[\dfrac{1-e^{-(p-\rho_m)t}}{p-\rho_m}\right]_M\right)$, $K_2(p,\ t)=\mathrm{Re}\left(\dot{\boldsymbol{G}}'^{\mathrm{T}}\left[\dfrac{1-e^{-(p-\rho_m)(t-T_1)}}{p-\rho_m}\right]_M\right)$, $K(p,\ t)=K_1(p,\ t)-K_2(p,\ t)e^{-pT_1}$	$\dot{\boldsymbol{G}}=[\dot{G}_m]_M$, $\boldsymbol{q}=[\rho_m]_M$, $\dot{\boldsymbol{G}}'=\mathrm{diag}(\dot{\boldsymbol{G}})e^{\boldsymbol{QT}}$, $\boldsymbol{Q}=\mathrm{diag}(\boldsymbol{q})$, $\boldsymbol{T}=[T_m]_M$
3. Filter reaction	$y(t)=\mathrm{Re}\left(\dot{\boldsymbol{Y}}_1(t)^{\mathrm{T}} e^{\boldsymbol{P}(\boldsymbol{C}t-\boldsymbol{t})}-\dot{\boldsymbol{Y}}_2(t)^{\mathrm{T}} e^{\boldsymbol{P}(\boldsymbol{C}t-\boldsymbol{t}')}-\right.$ $\left.-\dot{\boldsymbol{Y}}_3(t)^{\mathrm{T}} e^{\boldsymbol{P}(\boldsymbol{C}(t-T_1)-\boldsymbol{t})}+\dot{\boldsymbol{Y}}_3(t)^{\mathrm{T}} e^{\boldsymbol{P}(\boldsymbol{C}(t-T_1)-\boldsymbol{t}')}\right)$	$\dot{\boldsymbol{Y}}_1(t)=\mathrm{diag}(\dot{\boldsymbol{X}})K_1(\boldsymbol{p},\ \boldsymbol{C}t-\boldsymbol{t})$, $\dot{\boldsymbol{Y}}(t)=\mathrm{diag}(\dot{\boldsymbol{X}})K_1(\boldsymbol{p},\ \boldsymbol{C}t-\boldsymbol{t}')$, $\dot{\boldsymbol{Y}}_3(t)=\mathrm{diag}(\dot{\boldsymbol{X}})K_2(\boldsymbol{p},\ \boldsymbol{C}t-\boldsymbol{t})$, $\dot{\boldsymbol{Y}}_4(t)=\mathrm{diag}(\dot{\boldsymbol{X}})K_2(\boldsymbol{p},\ \boldsymbol{C}t-\boldsymbol{t}')$

Table 7. FIR filter analysis at input signal as a set of finite components

Basic expressions for FIR filter analysis at signal injection on a filter input as a set of damped oscillatory components with finite duration are represented in the Table 7. The calculation of a filter reaction is given in the Table 7 using only the second analysis method for the case when duration of all the components of a filter impulse function is equal T_1. The filter analysis based on the first method is represented in details in the author's papers [6,18].

Let us consider an analysis example of FIR filters which are used in one of the most perspective intelligent electronic devices (IED) - Phasor Measurement Units (PMU) [19].

A brief description of a basic algorithm for PMU signal processing on the example of an analog system-prototype is given in the item 1 of the Table 8 [20].

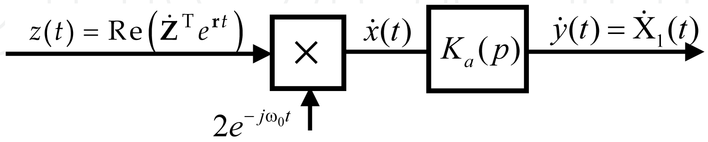
№	Name	Diagram/expressions
1.	Block scheme of algorithm	
2.	Signal description $z(t)$	Semi-infinite signal $z(t)=\mathrm{Re}\left(\dot{\boldsymbol{Z}}^{\mathrm{T}}e^{\boldsymbol{r}t}\right)$ or $z(t)=0,5\left(\dot{\boldsymbol{Z}}^{\mathrm{T}}e^{\boldsymbol{r}t}+\bar{\boldsymbol{Z}}^{\mathrm{T}}e^{\bar{\boldsymbol{r}}t}\right)$, $\dot{\boldsymbol{Z}}=[Z_0 \;\; \dot{Z}_1 \;\; \dot{Z}_2 \;\; \dots \;\; \dot{Z}_{K-1}]^{\mathrm{T}}$, $\boldsymbol{r}=[-\beta_0 \;\; j\omega_1 \;\; j2\omega_1 \;\; \dots \;\; j(K-1)\omega_1]^{\mathrm{T}}$, $\bar{\boldsymbol{Z}},\bar{\boldsymbol{r}}$ – are complex – adjoint vectors Signal as a set of finite components $z(t)=0,5\left(\dot{\boldsymbol{Z}}^{\mathrm{T}}e^{\boldsymbol{R}(\boldsymbol{C}t-\boldsymbol{t})}-\dot{\boldsymbol{Z}}'^{\mathrm{T}}e^{\boldsymbol{R}(\boldsymbol{C}t-\boldsymbol{t}')}+\bar{\boldsymbol{Z}}^{\mathrm{T}}e^{\bar{\boldsymbol{R}}(\boldsymbol{C}t-\boldsymbol{t})}-\bar{\boldsymbol{Z}}'^{\mathrm{T}}e^{\bar{\boldsymbol{R}}(\boldsymbol{C}t-\boldsymbol{t}')}\right)$ $\boldsymbol{R}=\mathrm{diag}(\boldsymbol{r})$
3.	Input signal of a filter	$\dot{x}(t)=2e^{-j\omega_0 t}z(t)$, At semi-infinite input signal device $\dot{x}(t)=\dot{\boldsymbol{Z}}^{\mathrm{T}}e^{(\boldsymbol{r}-\boldsymbol{C}j\omega_0 t)t}+\bar{\boldsymbol{Z}}^{\mathrm{T}}e^{(\bar{\boldsymbol{r}}-\boldsymbol{C}j\omega_0 t)t}$
4.	Algorithm	$\dot{X}_1(t)=\int_{t-T_1}^{t} z(\tau)e^{-j\omega_0\tau}g(t-\tau)d\tau=\int_{t-T_1}^{t}\dot{x}(\tau)g(t-\tau)d\tau$
5.	Average FIR filter	$\dot{\boldsymbol{G}}=[80.48e^{j4.273} \;\; 37.93e^{j0.5887}]^{\mathrm{T}}$, $\dot{\boldsymbol{G}}'=\mathrm{diag}(\dot{\boldsymbol{G}})e^{\boldsymbol{q}T_1}$ $\boldsymbol{q}=[-22.99+j62.30 \;\; -23.26+j186.9]^{\mathrm{T}}$, $\boldsymbol{T}=[T_1 \;\; T_1]^{\mathrm{T}}$, $T_1=0.051$c, $g(t)=\mathrm{Re}\left(\dot{\boldsymbol{G}}^{\mathrm{T}}e^{\boldsymbol{q}t}-\dot{\boldsymbol{G}}'^{\mathrm{T}}e^{\boldsymbol{q}(t-T_1)}\right)$, $K(p)=\mathrm{Re}\left(\dot{\boldsymbol{G}}^{\mathrm{T}}\left[\frac{1}{p-\rho_m}\right]_M-\dot{\boldsymbol{G}}'^{\mathrm{T}}\left[\frac{1}{p-\rho_m}e^{-pT_m}\right]_M\right)$

Table 8. IED algorithm

An input signal of intelligent electronic devices is represented by a set of complex amplitudes $\dot{\boldsymbol{Z}}$ and frequencies $\boldsymbol{r}$, as well as time parameters when using a signal model as a set of finite components. Let us constrain the signal models to one model only as an exponential component, useful sinusoidal component of commercial frequency ω_1 (nominal value is $\omega_0=2\pi 50$rad/sec) and higher harmonics. More complicated signal models are considered in the papers [6,20].

An average FIR filter $K_a(p)$ should isolate the constant ($\dot{y}(t)=\dot{X}_1$ at $\omega_1=\omega_0$) or low-frequency component ($\dot{y}(t)=\dot{X}_1(t)$ at $\omega_1\neq\omega_0$). The filters should suppress higher harmonics and a damped oscillatory component with the complex frequency $p=-\beta_0+j\omega_0$.

The considered filters should have low sensitivity to a change of damping coefficient β_0 in the range from 10÷200 sec^{-1} and the frequency $\omega_1=2\pi(50\pm5)$ rad/sec. The acceptable static error of signal processing should not be more than 0,5%, and the acceptable dynamic error at $t\geq T_1$ should not be higher than 3%.

FIR filter analysis is performed at input signal of a device as a set of semi-infinite or finite damped oscillatory components according to the algebraic expressions from the Table 7 and the Table 8.

An example of FIR filter analysis using Mathcad at compound input signals as a set of sequentially adjacent finite signals is given on the fig.6.

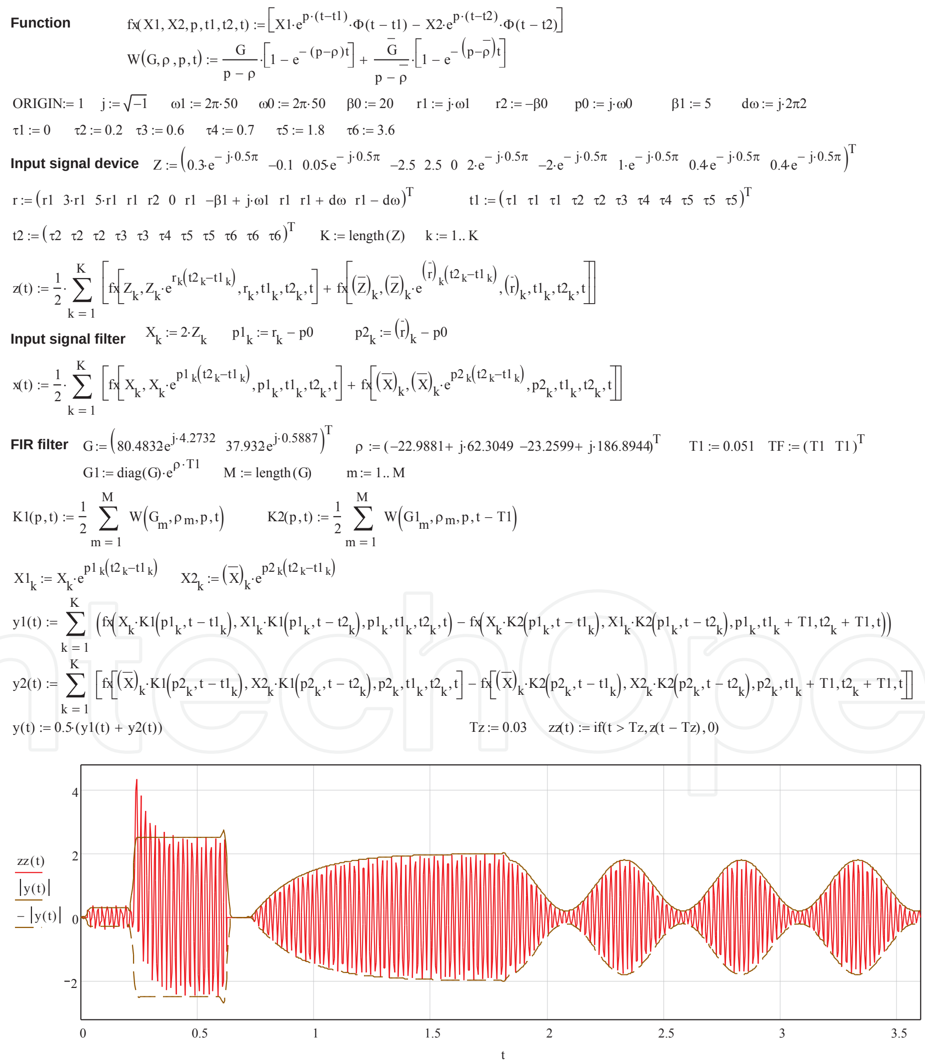


Figure 5. FIR filter analysis using Mathcad software

Each of five sets of finite components represented comply with a particular power system regime. The first set is related to a normal regime of a power system and is represented by sections of sinusoidal component of industrial frequency and higher harmonics. The second set of finite signals corresponds to an accidental regime and contains of finite sinusoidal and exponential component, the third set shows no-current regime, the fourth set is connected to a change of PMU input signals envelope due to automation operation, the fifth set represents a swings regime.

A filter processes a complex signal $\dot{x}(t)$, formed after multiplication of PMU input signal $z(t)$ by a reference signal $e^{-j\omega_0 t}$ to shift the signal spectrum to the left for the formation of orthogonal components of complex amplitude of sinusoidal signal with industrial frequency ω_1.

A plot for PMU input signal $z(t)$and a plot for a module of filter output signal $|\dot{y}(t)|$ taking into account filter group delay time are represented on the fig.6. As it follows from the fig.1, the plot $|\dot{y}(t)|$ is close to the envelope of sinusoidal component of PMU input signal.

It appears from the plot that the investigated filter has a feature which is connected to the absence of overshoot (oscillation) of transient process in the filter in its traditional definition, in other words at a stepwise growth of a signal. and presence of overshoot at a stepwise reduction of signal.

In fact, due to particular characteristics of impulse function, from the traditional point of view, an overshoot is absent in the considered FIR filter, as at the end of transient process in the filter signal behavior is close to aperiodic process. In this case oscillation is noticeable on the initial stage of transient process.

Analysis of FIR filter and signal processing algorithm in total is carried out only at fixed values of input signal parametres in the example. With some further ordinary improvement, analog to the example on the fig.1, it is possible to determine performance specifications for FIR filter signal processing at any variation of useful signal and disturbance parameters. However, it is much easier to use the specific express-analysis methods.

3.3. Express-analysis methods for signal processing performance

Let us consider an example of express-analysis of signal processing performance of a FIR filter, which mathematical description is given in the Table 7. In addition, let us consider performance analysis for signal processing of a filter with the impulse function $g_2(t) = g_1(T_1 - t)$ (Fig.7).

To check the adherence to the mentioned conditions, it is enough to consider amplitude-frequency response of the filters in complex frequency coordinates.

The amplitude-frequency response for the first and second filters is given in the Fig.8. The multiplier $e^{-\beta T_1}$ at $\beta \neq 0$ allows for attenuation of forced damped components by the moment of the transient process ending in the filter. The speed of the FIR filter is determining by its length T_1 in the case of noise damped components are suppressed to the level of the acceptable dynamic error during the time, that is equal to the filter length.

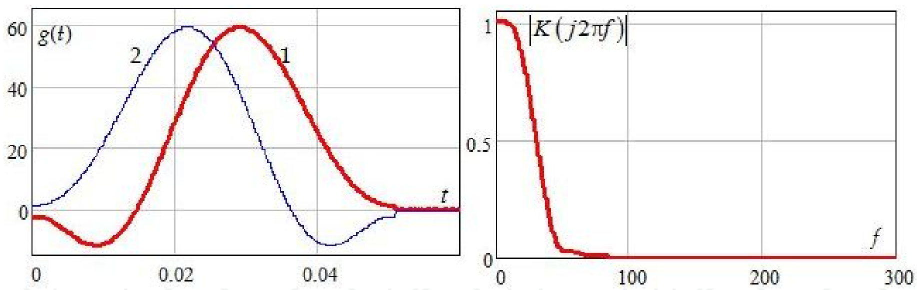


Figure 6. Impulse characteristics of the filters

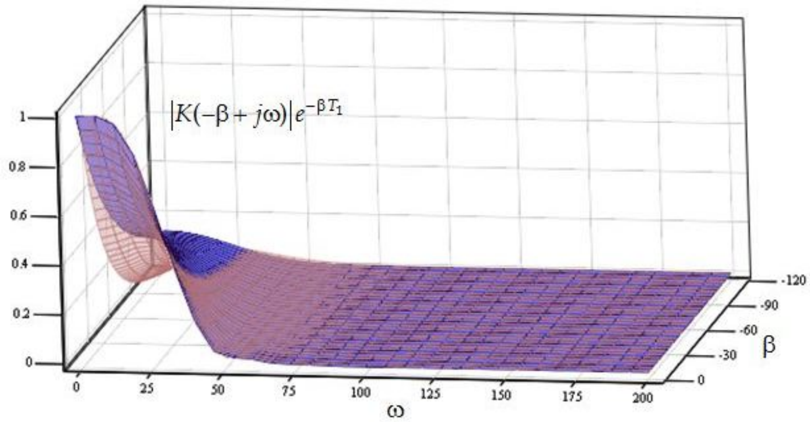


Figure 7. Amplitude-frequency response of the filter

To check the compliance with the requirements for filters to the specified quality indexes of signal processing for the specified input signal, it would be enough to consider two sections of 3D amplitude-frequency response: in the sections $p = j2\pi f$ (Fig.6) and $p = -\beta + j\omega_0$ (Fig.8). As it follows from the Fig.6, the examined filters have the same amplitude-frequency response in the section $p = j2\pi f$ and ensure the required quality of useful signal processing and suppressing higher harmonics. Absolutely different situation appears in the case of eliminating by filters a damped oscillatory component with frequency $p_0 = -\beta_0 + j\omega_0$.

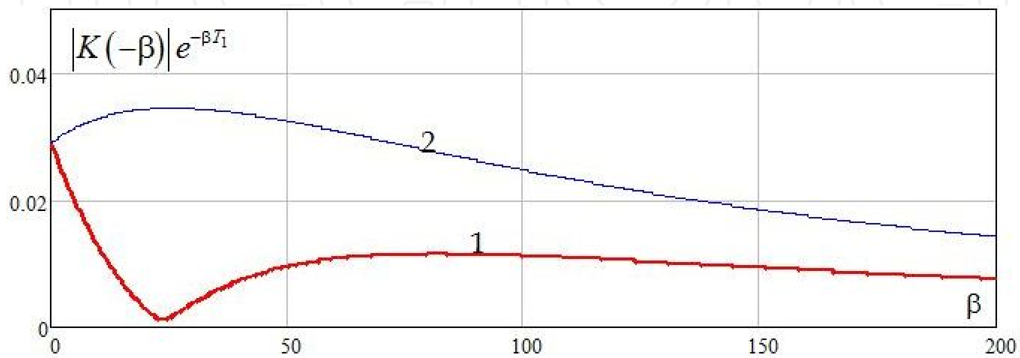


Figure 8. Frequency response of the filters in the section $p = -\beta + j\omega_0$

The filters 1 and 2 in the section of 3D frequency response $p = -\beta + j\omega_0$ (Fig.7) have different characteristics, and the second filter does not ensure compliance with the specified requirements to suppress the component of an input signal with the complex frequency $p_0 = -\beta_0 + j\omega_0$. It results in ambiguity of using the traditional frequency response of filters FIR filters with asymmetric form of the impulse response for analysis aperiodic signals.

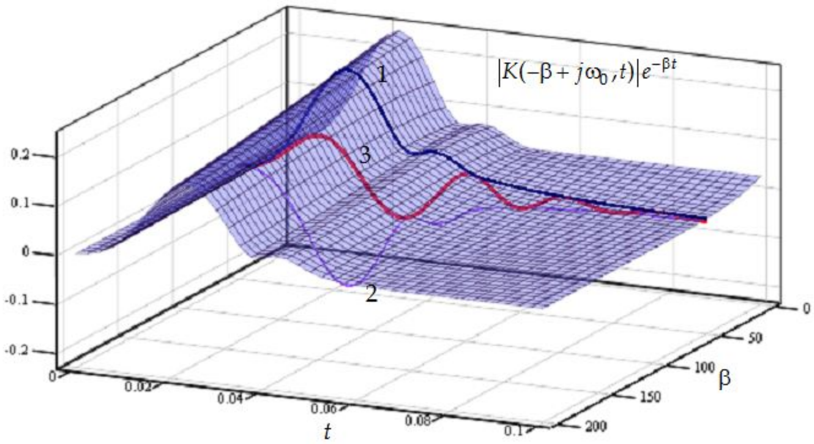


Figure 9. Amplitude-frequency response of the filter

In the case of a need along with estimating the speed and accuracy of signal processing to appraise the history of transient processes in a filter, for instance, to control oscillation of transient process in a filter, it is eligible to conduct the analysis using amplitude-frequency response based on the filter transfer function, which depends on time.

It is necessary for analysis to depress a damped oscillatory component with the complex frequency $p_0 = -\beta_0 + j\omega_0$ to fix the imaginary part, due to ω_0=const. The plot, proportional to the product $|K_2(-\beta + j\omega_0, t)| e^{-\beta t}$, is given in the Fig.10. In the case of complex frequency $p = -\beta + j\omega_0$ the plot will be equal to the envelope (curves 1 and 2) of the reaction of the second filter (curve 3) to the examined input impact at β_0=50 sec^{-1}.

The graph presented describes graphically the fact, that the second filter does not ensure the suppressing of an input signal component with the complex frequency $p_0 = -\beta_0 + j\omega_0$ by the moment of free components completion in the filter at the most specified values of β_0.

3.3. Digital FIR filters

As for IIR filters (Table 1 and 4), input-output dependencies for digital (discrete) FIR filter may be obtained by discretization of expressions for analog FIR filter and transition from the Laplace transform to Z transform [13].

In case of using the mathematical software Matlab, Mathcad etc. for digital FIR filter analysis it is enough to get the data about complex amplitudes and input signal frequencies $\dot{X}$ ($\dot{X}'$ at finite signals), $\boldsymbol{p}$ (or $z = e^{pT}$) and a filter $\dot{\boldsymbol{G}}$, $\dot{\boldsymbol{G}}'$, $\boldsymbol{q}$ (or $z = e^{qT}$), as well as a set of parameters, which define the duration of filter impulse function components and the beginning and duration of input signal components.

In some cases only mathematical description of analog filter-prototypeis specified. In case of FIR filters, different methods of transition from analog filter-prototype description to digital filter description, for instance, method of differential equation discretization, method of invariant impulseresponses, bilinear transformation are applied [14]. The mentioned methods expand also for the case of FIR filters in the author's paper [13].

The transfer function of analog filter-prototype with finite impulse response according to the Table 6 can be described in the following way

$$K(p) = \mathrm{Re}\left(\dot{\mathbf{G}}^{\mathrm{T}}\left[\frac{1}{p-\rho_m}\right]_M - \dot{\mathbf{G}}'^{\mathrm{T}}\left[\frac{1}{p-\rho_m}e^{-pT_m}\right]_M\right) \tag{2}$$

The expression for a digital filter transfer function

$$K(z) = \mathrm{Re}\left(\dot{\mathbf{G}}^{\mathrm{T}}\left[\frac{k_m(z+a)}{z-z_m}\right]_M - \dot{\mathbf{G}}'^{\mathrm{T}}\left[\frac{k_m(z+a)}{z-z_m}z^{-N_m}\right]_M\right) \tag{3}$$

where k_m, a - coefficients; z_m- m-th pole of system function, N_mT - duration of the m-th impulse function component, T - discrete sampling step.

All the mentioned constants, except N_m, depend on the transition method being applied. The values of the given coefficients for the three transition methods mentioned above are represented in the Table 9

№	Method	Meaning
1.	Method of differential equation discretization	$k_m = T$, $a = 0$, $z_{1m} = 1/(1-\rho_{1m})$, $\rho_{1_m} = \ln(z_m)/T$
2.	Method of invariant impulse responses	$k_m = T$, $a = 0$, $z_{2m} = e^{\rho_m T}$, $\rho_{2_m} = \rho_m$
3.	Bilinear transformation	$k_m = T/(2-\rho_m T)$, $a = 1$, $z_{3m} = (2+\rho_m T)/(2-\rho_m T)$, $\rho_{3_m} = \ln(z_{3m})/T$

Table 9. Transition methods

The additional indexes of the constants in the Table 9, corresponding to order numbers of transition methods, are given for the constants, which do not coincide with the analog filter-prototype parameters.

4. Performance express-analysis at modulated signal

4.1. Mathematical description of signals

The signals as a set of semi-infinite or finite damped oscillatory components were considered above. Compound signals of different forms, including compound periodical and quasi-periodic signals, nonstationary signals and signals with compound envelopes can be synthesized on the basis of the collection of components mentioned above [5,6]. The mentioned models also make it possible to describe the majority of impulse signals, which are widely applicable in radio engineering (radio pulse and video pulse).

The analysis methods considered above can be applied for that kind of signals.

However, the more general case is obviously more interesting, when signals with compound dependencies of envelopes and signal total phase are applied. Semi-infinite or finite signals with compound envelopes, the most frequently used signals in radio engineering, can be described by the following model

$$x(t) = \mathrm{Re}\left(\dot{X}_1(t)e^{p_1(t)t}\right) \tag{4}$$

or in general case it would be

$$x(t) = \mathrm{Re}\left(\dot{\mathbf{X}}(t)^{\mathrm{T}}\, e^{\mathbf{P}(t)(\mathbf{C}t-\mathbf{t})}\right) \tag{5}$$

For the considered signal models, for instance, of the signal (1),their finite duration can be specified by the finite duration of a complex amplitude $\dot{X}_1(t)$.

Using the model (1), signals with amplitude, phase and frequency modulation which are commonly used in radio engineering signals can be described. Among with the similar models input signals in sound processing systems, automation devices of power systems at electro-mechanical transients process can be described.

It is known, that using the signal (1), in case when amplitude, frequency or initial phase are time functions, it is impossible to define the law of amplitude variation and a filter initial phase by values of amplitude-frequency response and phase-frequency response on an input signal frequency [21,22]. Thus, the analysis methods considered above cannot be applied directly.

There are exceptional cases when at some variation laws $\dot{X}_1(t)$ and $p_1(t)$ signals of the type (2) can be transformed into signals, described by a set of semi-infinite or finite damped oscillatory components [5,6].

However, signals (1) and (2) can be decomposed into components of a set of damped oscillatory components using the Prony's method and its modifications [23]. Leaving behind the issue concerning decomposition into “real” components by using the mentioned methods it is important to note that even general signal approximation (1) and (2) by Prony's method for calculations enables to apply the considered filter analysis methods.

Another option for solution of the filter analysis at signal type (1) is connected to modification of the suggested analysis methods. Modification of the second analysis method is represented below.

4.2. Filter analysis

Let us consider IIR filter analysis at input signals (1), as well as for special cases, when amplitude $\dot{X}(t)=X_m(t)e^{-j\varphi}$ and phase $\dot{X}(t)=X_m e^{-j\varphi(t)}$ are corresponding to a modulation signal.

To develop the necessary dependences one can use expressions, obtained for IIR filter at injection on its input a set of finite damped oscillatory components. Let us decompose the signal into time steps, during which signal time dependent parameters are mostly constant. In this case time step Δt can be even and uneven.

In case of an even step of signal decomposition one will obtain the following expression to determine a filter ouput signal

$$y(t) = \mathrm{Re}\left(\sum_{n=0}^{N-1} \dot{X}_{n1} e^{p_{n1} n\Delta t}\left(K\left(p_{n1}, t-n\Delta t\right)e^{p_{n1}\left(t-n\Delta t\right)} - e^{p_{n1}\Delta t}K\left(p_{n1}, t-\left(n+1\right)\Delta t\right)e^{p_{n1}\left(t-\left(n+1\right)\Delta t\right)}\right)\right) \qquad (6)$$

where $\dot{X}_{n1}=\dot{X}_1(n\Delta t)$, $p_{n1}=p_1(n\Delta t)$, $N=t_k / \Delta t$, t_k- duration of a finite signal (beginning of the signal coincides with zero reading on time).

As before, only algebraic operations with complex amplitudes, frequencies and values of time dependent transfer function on complex frequency of an input signal are applied for the filter analysis at the input signals type (1).

The same approach can be used for digital filters as well – one should replace continuous time t to discrete time kT, and instead of transfer function $K(p,\ t)$ of an analog filter one should use a transfer function $K(z,\ k)$in the expression. In this case, if one assumes Δt to be equal to discrete sampling step T, it enables to take into account the errors of analog-to-digital converter and finite digit capacity microprocessor in filter analysis.

The filter analysis on the basis of the expression (3) is approximate and can be considered as numerical method. To determine explicit dependencies one need to perform passage to the limit $\Delta t \rightarrow 0$.

The required dependency of input-output can be defined also by using the convolution integral by substitution of the expression for the input signal (1). Performing discretization with the passage to the limit, the following input-output dependency for an analog filter-prototype can be obtained

$$y(t) = \mathrm{Re}\left(\int_0^t \dot{X}_1\left(t-\tau\right) e^{p_1(\tau)t} K^{'}\left(p_1\left(\tau\right),\tau\right) d\tau \right) \tag{7}$$

where

$$K^{'}\left(p_1\left(t\right),t\right) = \frac{dK\left(p_1\left(t\right),t\right)}{d\tau} \tag{8}$$

In case of amplitude, phase modulation or their combination, the expression for input-output would significantly simplify

$$y(t) = \mathrm{Re}\left(\left(\int_0^t \dot{X}_1\left(t-\tau\right) K^{'}\left(p_1,\tau\right) d\tau \right) e^{p_1 t} \right) \tag{9}$$

For IIR filter the following expression for a derivative of time dependent transfer function takes place

$$K^{'}\left(p_1,t\right) = \frac{dK\left(p_1,t\right)}{d\tau} = \mathrm{Re}\left(\left[\dot{G}_m e^{-(p_1-\rho_m)t} \right]_M \right) \tag{10}$$

In case of $\dot{X}_1(t) = \dot{X}_1$ the input-output dependency coincides with input-output dependency for IIR filter obtained before (item 6 Table 1)

$$y(t) = \mathrm{Re}\left(\dot{X}_1 K\left(p_1,t\right) e^{p_1 t} \right) \tag{11}$$

It follows from the comparison of the input-output dependencies (4) and (5), that in the second case complex amplitude of IIR filter output signal on the input injection as a damped oscillatory component is determined by multiplication of a signal complex amplitude by the value of time dependent transfer function on the input signal frequency $\dot{Y}(t) = \dot{X}_1 K(p_1, t)$, in the first case more complicated dependency takes place $\dot{Y}(t) = \int_0^t \dot{X}_1(t-\tau) K^{'}(p_1, \tau) d\tau$.

The dependencies enable to perform filter analysis, including performance analysis of signal processing, at input signals with a composite form. Solving problems of this kind is relevamt not only for radio engineering and communication systems, but also for other industries.

For the example of considered in the sections 3.2 and 3.3 analysis of frequency filters, which are used in PMU, according to a new version of IEEE C37.118.2 standard it is necessary to perform testing for the mentioned devices not only at stepwise change of amplitude and initial phase of sinusoidal component of input signal with commercial frequency, but also at PMU input signals at electromechanical transient process in a power system. In the mentioned regimes of power system operation in the controlled currents and voltage envelopes and total phases are time functions.

For FIR filter analysis at input actions (1) the input - output expression can be obtained on the basis of the dependences given in the Table 6. Such-like dependences for IIR filters and FIR filters can be obtained based on the dependences given before for the first analysis method.

5. The application of the analysis methods

Let us consider possible areas of application for the suggested performance analysis methods for signal processing by frequency filters, which are used in intelligent electronic devices of power systems, in automation devices, in radio engineering and communication systems, as well as in other fields of engineering where digital signal processing is commonly applied.

The prospectives of power system development in nearest future are related to technology Smart Grid implementation and the application of automatic control and regulation systems of a new generation. Power system control improvement involves a wide application of fast action IED based on synchronized measurement of current and voltage phasors of a fundamental harmonic on the basis of IEEE C37.118-2011 and IEC 61850-90-5 standards.

Up to date IED should ensure performance signal processing in conditions of an intense electromagnetic and electromechanical transients process. Mathematical model of an input signal IED in a normal and an accidental regimes of power systems in some cases can be represented by a set of semi-infinite or finite damped oscillatory components, in other cases – by analogous models, in which complex amplitudes and frequencies of mentioned components are time functions.

Most of IED power systems should ensure performance of signal processing at any possible combination of input signal parameters. The suggested analysis methods enable to solve effectively problems of determination of performance specifications for frequency filters, used in IED power systems. The examples for the performance analysis of signal processing by frequency filters and algorithms of signal processing for IED of power systems based on the phasor measurement technology are considered in the sections 3 and 4 of the present chapter. The example for IIR filter analysis for general devices of relay protection and automation is given in the section 2.

The analysis methods for analog IIR filters considered in the chapter can be applied also for the linear circuit analysis. It is important to note that for absolute majority of microprocessor control systems and measuring systems the information sources have analogue nature. In this case for analysis of controlled objects equivalent circuits based on linear electric circuits are used. The illustrative examples: equivalent circuit of power systems, power plants, electric grids and power-supply systems. Application of the suggested analysis methods enables a consistent approach for the regime analysis of the controlled object operation, analysis of analog filters-prototype and digital filters [5,6].

The suggested analysis methods may be applied for performance analysis of signal processing by frequency filters in up-to-date measuring devices, automation devices, radio devices, in communications systems, sound processing systems and other devices, where digital signal processing is commonly used.

Advantages and particularities of the suggested analysis methods are related to uniform analysis methods for analog filter-prototypes (IIR filters and FIR filters), as well as for digital filters.

The majority of pulse signals (radio and video pulses), which are commonly used in radio engineering, can be described by a set of finite damped oscillatory components. For performance analysis of pulse signal processing by IIR filters and FIR filters the analysis methods considered in sections 2 and 3 of the chapter can be applied.

The author suggests to use the methods, considered in the section 4 of the present chapter for performance analysis of modulated signal processing by filters.

The synthesis methods mentioned above can be also effectively applied for typical signal filtering problems, including problems of a useful signal extraction against the white noise.

In this case, the white noise realizations can be described as a set of time-shifted fast damping exponents of different digits. The initial values and appearance time of the mentioned exponential components are random variables, which variation law ensures the white noise to have specified spectral characteristics.

6. Performance analysis of signal processing as a step of filter synthesis

Guaranteeing the necessary quality of signal processing is utterly important at frequency filter synthesis. The application of the considered above approaches for the signal type (1) and the filter impulse characteristics (2) enables to reject the traditional approach, related to formulation the requirements to a filter amplitude-frequency response in different fields (band pass, stop band, transition region). In the case of filter synthesis, it is enough to lay down the requirements to the filter frequency characteristics on the basis of Laplace transform on complex frequencies of an input signal with the allowance for their change. Thus, according to the approach described above, the signal processing performance analysis is one of the steps of filter synthesis.

This approach enables to formalize the task of filter synthesis and to gain optimal solutions in combination with methods of multicriterion optimization with limitations[5]. Using methods of multicriterion optimization, by limitations values of filter frequency responses are ment and in some cases values of input signal spectrum based on the Laplace transform spectral representations on complex frequencies of impulse function of a filter and input signal [5,20].

Synthesized in that way filters ensure the specified performance specifications for signal processing at any possible variation of input signal parameters.

Among with frequency filter synthesis this approach gives an opportunity to perform time window synthesis for hort-time Fourier transform, as well as synthesis of father and mother wavelets for cases when an input signal can be represented by a set of semi-infinite or finite damped oscillatory components [5,20].

7. Conclusion

The effective methods of performance analysis for signal processing based on the Laplace transform spectral representations and the uniformity of mathematical models of input signals as well as the filter impulse functions as a sets of continuous/discrete semi-infinite or finite damped oscillatory components were developed for a direct determination of IIR filters and FIR filters performance specification. input signals as well as the filter impulse functions as a set of continuous/discrete semi-infinite or finite damped oscillatory components were developed. Simple semi-infinite harmonic, aperiodic signals, compound signals and impulse characteristics of any form can be synthesized on the basis of components set mentioned above, including signal composite envelopes, as well as pulse signals (radio and video pulses). The uniformity of the mathematical description of the signals and filters enables, on one hand, allows to employ both a consistent and a compact form of their characterization in the configuration of a set of complex amplitudes, complex frequencies and time parameters. On the other hand, it simplifies significantly performance analysis for signal processing by analog or digital filters at any possible useful signal and disturbance parameters variation by reducing significantly the amount of calculations.

The analysis methods can be used in case of mathematical models as well - where complex amplitudes and/or complex frequencies are time functions.

To simplify the task of the analysis, two methods are suggested for performance express-analysis of signal processing by frequency filters using filter frequency responses based on Laplace transform: frequency and frequency-time analysis methods.

The application of the suggested methods for performance analysis of signal processing as one of the steps of filters synthesis enables to automatize filter design for filters with low sensitivity to a signal parameter change within the specified range.

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